A DATA-RATE LIMITED VIEW OF ADAPTIVE CONTROL

Geordie Z. Zhang * **Girish N. Nair** *
Robin J. Evans * **Björn Wittenmark** **

* *Department of Electrical and Electronics Engineering, the University of Melbourne*
** *Department of Automatic Control, Lund University*

Abstract: This paper addresses the problem of adaptively controlling a plant with unknown parameters using communication-limited feedback. Assuming known dynamics, expressions have recently been obtained for the minimum average feedback data rate required for asymptotic stabilisability. The main purpose of this work is to demonstrate that this minimum rate does not increase if the plant parameters are unknown, and the key elements of a stabilizing, minimum-rate policy are explicitly discussed. By regarding the uncertain plant as a higher-dimensional, nonlinear plant with unknown initial condition, it is shown that this result agrees with the recent concept of local topological feedback entropy. Extensions to the case of uncertain nonlinear plants are discussed.

Keywords: adaptive control, data-rate limited control, entropy

1. INTRODUCTION

Adaptive control is an important discipline in engineering. Often times, given a control system, we are presented with a problem of knowing the plant model but not some of the parameters, and are subjected to the situation of having to achieve certain control objectives while trying to figure out the parameters at the same time. In this paper, we look at adaptive control within a system that has data-rate limited communication channels.

Data-rate limited control became an emerging field of research in the 1990s. In this field, we are concerned with control systems that have within them data-rate limited communication channels, and how to communicate efficiently (from a bandwidth perspective) while still achieving the required control objectives. Sev-

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eral factors have contributed to the progress of data-rate limited control. From a practical perspective, the emergence of technologies such as wireless sensor networks, which are complex control networks where bandwidth is at a premium, have pushed engineers to look at this kind of problems. From a theoretic perspective, it would seem that this is the next step in the evolution of communication and control theory, and given that they both have a dynamical systems framework and traditionally studied as separate disciplines, it is natural that someday a new field would be born in the intersection of these areas.

One problem that particularly interests people in data-rate limited control is bandwidth efficiency. Given a control system, what is the minimum data rate that we need for the control objectives to be achievable? There are several papers currently in the literature that answer this question for classes of systems and control criteria. See, for example, (Wong and Brockett, 1999), (Baillieul, 2001) and (Nair and Evans, 2003). In this paper, we are interested in looking at data-rate limited *adaptive control* for linear systems. The key difference between our problem and those discussed in the papers above is that in our problem, the control system not only have to perform certain control tasks, but also must identify the system parameters and transmit this knowledge through the control system. The main theorem we show is that for a linear scalar plant, the minimum data rate required for adaptive control is the *same* as the minimum data rate required for control if we knew the parameter already. Two approaches will be discussed for proof of this result. We first present a full proof using a "bottom up" approach, building the proof from first principles and showing explicitly how to encode the parameter information in such a way that the minimum data-rate remains the same. This is followed by a sketch of a "top down" approach, where we use a theorem regarding *local topological feedback entropy (LTFE)* to derive the minimum data rate for the adaptive system. We then discuss generalizations to vector state spaces (state spaces in $\mathbb{R}^n$) and nonlinear systems.

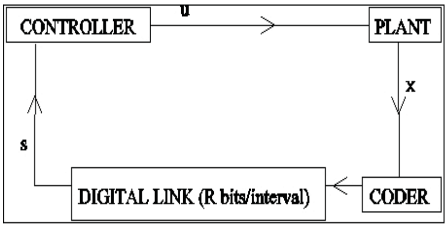

Fig. 1. General Set up of a Data-rate Limited Control System

2. PROBLEM FORMULATION

As described in the introduction, the key problem we look at is adaptive data-rate limited control. In particular, what is the minimum data rate required for the control system to achieve its objectives. We now formally introduce the general set up under which data-rate limited control problems are analysed, a schematic of which can be found in Figure 1.

In this set up there is a discrete time, time invariant plant, a sensor/coder that is able to observe fully the plant state and transmits it to the controller, and the controller decodes the symbols and transmits control inputs to the plant. We assume that the channel between the coder and the controller is data-rate limited noiseless and has no delay, while in the rest of the system we assume that there is perfect communication. The noiseless assumption is actually quite general, since we can trade off bandwidth for noise reduction by the usage of error-correction (if we had a channel with noise, then the lower bound we calculate on the data-rate should be adjusted to take into effect the extra bandwidth needed for error-correction). With the no delay assumption, it turns out that the argument of the proof works identically as long as we have a fixed, known delay in the channel.

Using the set up above, we now specify the system that we want to study. The main result of the paper concerns scalar linear systems. Denote the plant state space by $X = \mathbb{R}$, the input space $U = \mathbb{R}$, and suppose that the plant evolves according to the law

$$x_{k+1} = \alpha x_k + u_k$$

where α is a parameter.

The control objective is to drive the plant state asymptotically to the origin (i.e. $x_k \to 0$).

An important issue in the set up is the interaction between the coder and the controller, and this deserves some more explanation. Since the channel between them is data-rate limited, the coder in general must quantize the plant state before transmission. After quantizing, we encode the state space at every point in time by $s_k = \gamma_k(x_0, \ldots, x_k, s_0, \ldots, s_{k-1})$ (an infinite-memory, causal encoding). The function γ must be injective as decoding plus control happens at the controller via $u_k = \delta_k(s_0, \ldots, s_{k-1})$. In the case where α is known, the functions γ_k and δ_k may depend on α. In the adaptive case, α is *unknown* and the coder-controller must be able to communicate and drive the plant state to the origin *without full knowledge of* α.

Let R be the data-rate of the link between the coder and the controller. The term data-rate used in this paper refers to the average asymptotic data rate. Suppose that μ_k denotes the size of the time-varying alphabet which the coder sends to the controller. Then the average asymptotic data-rate is defined to be:

$$\liminf_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \log_2 \mu_j$$

At this stage we may ask two related questions: What is the minimum value of R for which there exists a coder-controller that will drive the plant state asymptotically to the origin, when α is known? What about in the adaptive case when α is unknown.

Let Q denote the minimum data rate needed for control to work in the case of nonadaptive control and Q' the minimum data rate in the case of adaptive control. The relationship between Q and Q' is given by the key theorem in this paper:

Theorem 1. Given the problem set up as above, then $Q = Q'$.

3. PROOF OF THE MAIN THEOREM: BOTTOM UP APPROACH

In this section, we prove Theorem 1 by first principles. Now in (Nair and Evans, 2003), it was shown that $Q = \max\{0, \log_2 |\alpha|\}$. We need to show that Q' is also equal to this quantity. It follows easily that $Q' \geq Q$, because any coder-controller that works for the case when we do not know α will also work for the case when we do know α. To show the converse, we explicitly construct a coder-controller where the codec (γ_k and δ_k) does not depend on full knowledge of α, but the asymptotic average bit rate is still $\max\{0, \log_2 |\alpha|\}$.

Step 1: Run the plant in open-loop (set δ_k to map identically to zero for the first few points in time) and calculate α in finite time. Note that since we have a pretty simple plant, measurement of x_2 and x_1 is sufficient to calculate α. More explicitly, $\alpha = (x_2 - u_1)/x_1 = x_2/x_1$ (since $u_1 = 0$). Since this takes place in a finite amount of time involving a finite number of bits transmitted through the coder-controller channel, step 1 does not affect the average asymptotic data rate for our coder-controller.

Step 2: After Step 1, the coder has figured out what α is, but since the controller does not know what α is, the codec (γ_k, δ_k) cannot depend on full knowledge of α. The trick now is that when we send information from the coder to the controller, encode each symbol s_k so that it contains two parts, y_k and z_k. The first part y_k gives the controller more and more information about α, while the second part z_k encodes where the current plant state is (up to resolution of the quantization used). Intuitively what we want is for z_k to be more or less the plant information we would transmit if we know apriori what α is, and with y_k, send it in such a way so that after infinite time the controller knows α although the transmission required zero asymptotic average date rate. We want the average data rate required to send y_k to go to zero, but the total number of bits sent about α to go to infinity (this is like a harmonic series; the terms go to zero but the series adds to infinity).

Step 3: Step 2 has provided us a strategy and we now explicitly construct a family of codec schemes for the coder-controller and show that their asymptotic average data-rates arbitrarily approach the bound Q, thereby showing that $Q = Q'$.

We first show how we'd encode the y_k. Suppose that the coder-controller we want has average asymptotic data rate of R. This is equivalent to that per sample we can transmit one letter from an alphabet consisting of $\mu = 2^R$. Let $\alpha_0\alpha_1\ldots$ denote the base-μ expansion for α. Transmit α_j at time μ^j for the y_j part of the symbol s_k, and transmit nothing at all other times. Then the average bit rate used to transmit information about α is $A_k = \frac{\log_\mu k}{k} \to 0$. But at the same time, $\sum_{k=1}^{\infty} A_k = \infty$, since eventually each term in the series is greater than the corresponding term in the harmonic series.

Next we construct z_k. Assume that x_0 is bounded to start with, which is not unreasonable for all practical purposes. Let $\hat{\alpha}_k$ denote the largest-magnitude estimate of α by the controller based on all of the information it has received up to time k. Let l_0 be a positive real satisfying $|x_0| \leq l_0$. This is allowed as we assumed that x_0 is bounded. We now inductive construct bounds l_k. Suppose that at time k we have $|x_k| \leq l_k$. Since $|\hat{\alpha}_k| \geq |\alpha|$, it follows that $|\alpha x_k| \leq |\hat{\alpha}_k| l_k$. Now αx_k is the unforced plant state at time $k+1$. Due to the bound we just calculated, $\alpha x_k \in I_k := [-|\hat{\alpha}_k| l_k, |\hat{\alpha}_k| l_k]$. Cut this up into μ identical subintervals and set the index of the one containing αx_k to be z_k.

Once we have y_k and z_k this defines our function γ_k. The controller is able to decipher s_k (which is the concatenation of y_k and z_k) since it knows $\hat{\alpha}_k$, and it decodes by setting $u_k = -$ (midpoint of subinterval containing z_k). This gives the function δ_k. If we now put u_k into the control law we have:

$$|x_{k+1}| = |\alpha x_k + u_k| \leq \frac{|\hat{\alpha}_k l_k|}{\mu} =: l_{k+1}$$

and this recursively defines bounds l_k for x_k. From the definition we can see immediately that $l_k \to 0$ if $\limsup \frac{|\hat{\alpha}_k|}{\mu} < 1$, and since $\hat{\alpha}_k \to \alpha$, the required inequality holds whenever $\mu > |\alpha| \Leftrightarrow R > \log_2 |\alpha|$. This shows that we can construct coder-controllers with data rate arbitrarily close to $\log_2 |\alpha|$, which in turn proves that $Q \geq Q'$. This concludes the proof.

4. TOP DOWN APPROACH OF MAIN THEOREM

In this section we sketch a different proof of Theorem 1, using local topological feedback entropy. This section presents the heuristic of the proof only. There are several technical details that need to be taken care of for the full proof and is beyond the scope of this paper. The full proof will be presented in a longer journal paper, at a later date.

The reader is asked to refer to (G. N. Nair and Moran, 2004), in which a result says if we want to drive a nonlinear system to a point x^* in the state space, then the minimum data rate is give by the local topological feedback entropy, which equals to the sum of the logs of the magnitudes of the unstable eigenvalues of the Jacobian (with respect to the state variables) of the plant evolution map at x^*. In our adaptive control problem, we can rewrite the system as

$$x_{k+1} = a_k x_k + u_k$$
$$a_{k+1} = a_k$$

By introducing an extra state variable a and setting $a_0 = \alpha$, we have converted the scalar linear adaptive control system into a two state nonlinear nonadaptive control system. In effect, we've absorbed the adaptive process into the measurement of the extra state. In this problem we want to drive the system to $(0, \alpha)$. The plant evolves according to (using dummy variables for clarity) $f(p, q) = (pq + u_k, q)$, and the Jacobian at $(0, \alpha)$ is

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

This matrix has two eigenvalues, α and 1. Therefore provided that we choose a scheme that sends the right amount of information to the controller about α, we can achieve stabilisation with data-rate $\log_2 |\alpha|$, by the LTFE theorem.

5. VECTOR STATES ADAPTIVE CONTROL

Let us begin by reexamining the scalar case. We have identified that in the coding scheme used, the symbols s_k had two parts y_k and z_k, which respectively encoded information about α and the plant state at time k. The average data rate required to transmit y_k dropped to zero while the average data rate required to transmit z_k approached Q (the controller learns more and more about α) and since s_k is just a concatenation of these two symbols, the average data rate used to transmit s_k approached $0 + Q = Q$. This is why the argument worked.

The key issue to notice with the proof of Theorem 1 is that even though we do not know α exactly for any finite time horizon, *we are still able to drive the system towards the origin in a continuous manner, and at each stage we are guaranteed to be closer to the origin than before*. In another words, the essence of why the adaptive and nonadaptive data rate limits are the same is due to the fact that the plant evolution depends Lipschitz continuously on the value of α and the plant state. It is relatively easy to

(1) Come up with a scheme in the nonadaptive case to drive the plant state to the origin.
(2) Devise a scheme that transmits α from the coder to the controller in infinite time, with an average data rate that decays to zero.

However, it does not immediately follow that just because we can do (1) and (2), then we can do (1) and (2) in such a way that there is enough information available at the controller during the finite time horizons that we can actually drive the plant state asymptotically to the origin. If we examine the construction for the adaptive coder-controller above, the fact that I_k shrinks down to zero relies on the fact that the function $f(\alpha, x) = \alpha x$ depends Lipschitz continuously on α and x, so if we can bound α and x, then this forces a bound on $f(\alpha, x)$.

With this observation we see that the proof for the scalar case is actually quite general. In fact, it carries over immediately to prove the following generalization of Theorem 1.

Theorem 2. Suppose that in the general data-rate limited control set up we have $X = \mathbb{R}^n$, $U = \mathbb{R}^n$ and that the plant evolves according to the equation

$$x_{k+1} = Ax_k + Bu_k$$

where A is a diagonal n by n matrix and B is the identity. Then $Q = Q'$.

To prove this result it is sufficient to observe that if we let $\alpha_1, \dots, \alpha_n$ denote the diagonal entries of A, then we just have to construct a y_k sequence for each of the α_i, and repeat the proof for the scalar case.

The next generalization to this would be to look at the problem for the case when A is not diagonal. Making B not surjective would an interesting path to go down too. We have the following result.

Theorem 3. Suppose that in the general data-rate limited control set up we have $X = \mathbb{R}^n$, $U = \mathbb{R}^m$ and that the plant evolves according to the equation

$$x_{k+1} = Ax_k + Bu_k$$

where (A, B) is controllable. Then for the problem of driving the plant state to the origin, $Q = Q'$.

From our work so far, it seems either approach used for the proof of the scalar case can be extended to a proof for the vector case, although the proof becomes more technical as the dimension of the state space increases. A more extensive discussion this result will be presented in a journal paper.

6. CONCLUSION

In this paper, we have presented results concerning the minimum data-rate required to achieve adaptive control. In particular, we have shown that for linear systems where the control objective is to drive the plant state to the origin, the minimum data rate needed in the adaptive case is the same as the nonadaptive case. For the case of the scalar system we presented two contrasting approaches to the problem. In terms of future work, it would be interesting to look at if the theorems in this paper can be generalised to nonlinear systems. In fact, we suspect that given a general evolution equation $x_{k+1} = f(\alpha, x_k, u_k)$, if f depends jointly Lipschitz continuously with respect to the first two variables (that is, the function $f(.,.,u_k)$ is a Lipschitz continuous map from $\mathbb{R}^2$ to $\mathbb{R}$ for any u_k), and the system has certain controllability properties, then $Q = Q'$. Looking at this problem for a stochastic system also has been suggested as a possible extension.

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