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Stabilization with data-rate-limited feedback: tightest attainable bounds

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Abstract

This paper investigates the stabilizability of a linear, discrete-time plant with a real-valued output when the controller, which may be nonlinear, receives observation data at a known rate. It is first shown that, under a finite horizon cost equal to the *m*th output moment, the problem reduces to quantizing the initial output. Asymptotic quantization theory is then applied to directly obtain the limiting coding and control scheme as the horizon approaches infinity. This is proven to minimize a particular infinite horizon cost, the value of which is derived. A necessary and sufficient condition then follows for there to exist a coding and control scheme with the specified data rate that takes the *m*th output moment to zero asymptotically with time. If the open-loop plant is finite-dimensional and time-invariant, this condition simplifies to an inequality involving the data rate and the unstable plant pole with greatest magnitude. Analagous results automatically hold for the related problem of state estimation with a finite data rate. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

As large, digitally interconnected control systems become more common, it is increasingly important to understand how the communication and control objectives of such systems are related. The traditional assumption that plant observations are available to controllers with infinite precision is clearly unrealistic when the various parts of a system are connected by a network that has finite information-carrying capacity. More than merely introducing delay and quantization, this limited capacity forces the question of how to choose the bits of information that would be most useful for control. In this paper we study the simplest type of networked system, a plant and controller linked by a channel with a fixed data rate. In particular, we seek to determine the smallest data rate needed to stabilize the output of a linear plant when no structural constraints are imposed on the coder or controller.

In recent years, a number of researchers have proposed and analyzed various versions of this problem. In general, the focus has been on *memoryless* coding, in which the plant output is quantized without reference to its past. In [4], it was shown that if the output of an unstable, deterministic, discrete-time, linear, time-invariant (LTI) system is passed through a fixed, memoryless quantizer, then controllability in the sense of being able to make the trajectory approach any arbitrary point is impossible. In [9], the communication delays in this model were explicitly included and sufficient conditions given for

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the state to eventually remain within a given bound. Wong and Brockett proved that if the initial condition of a continuous-time LTI system is within a given bounded set then memoryless coding *and* control suffice to bound the state under certain conditions [16]. Investigating discrete-time, Gaussian LTI systems, Borkar and Mitter derived a separation principle and a memoryless, certainty equivalent controller, under the constraint that the coder was a memoryless quantizer acting on the *innovations* of a Kalman filter [1].

The only case in which schemes with memory have been dealt with in communication-limited control appears to be in [14], in which a noisy, analog channel was considered. In this paper, we assume a noiseless, digital channel and permit the coder and controller to have potentially unlimited memory. The motivation for this comes from the closely related field of communication-limited state estimation, in which recursive coding schemes have been studied extensively [3,15,12,13]. Moreover, this divorces the effects of the memory constraint from those of the communication constraint proper, thereby allowing the latter to be analyzed more clearly. In the same spirit we ignore finite word-length issues, by assuming that the measurements are available to the coder with infinite precision, and focus solely on the fundamental performance limitations imposed by the finite data rate.

We consider a discrete-time, linear, time-varying and infinite-dimensional plant with no process or measurement noise and with a directly observed, real-valued output which is zero before time zero and governed by a probability density p at time zero. Our primary aim is to find the smallest data rate needed to asymptotically stabilize the output of this plant in mth moment, i.e. to take the mth moment of the output to zero with time, when no structural constraints save causality are placed on the coder or controller. With this in mind, we first consider a finite horizon cost proportional to the mth output moment and show that the problem then reduces to causally reformulating an optimal quantizer for the initial output.

The insights gained from the finite horizon analysis are then combined with asymptotic quantization theory [10,5,2,8] to derive the limiting scheme as the horizon tends to infinity, *without having to solve the finite horizon problem first*. We prove that this limiting coding and control scheme is optimal with respect to a certain infinite horizon cost, provided that p is continuous and satisfies certain technical conditions, and derive the optimal cost. This then leads to a necessary and sufficient condition for the plant to be asymptotically stabilizable at a data rate R. If the open-loop dynamics are finite-dimensional and time-invariant, this simplifies to

$$R > \log_2|\lambda|, \tag{1}$$

where λ is the unstable open-loop pole with largest magnitude. It is then observed that analogous results automatically hold for the problem of state estimation with a finite data rate.

2. Formulation of control problem

Consider the infinite-dimensional, time-varying, ARMA model

$$x_{k+1} = \sum_{j=0}^{\infty} a_{k,j}x_{k-j} + b_{k,j}u_{k-j}, \quad k = 0, 1, 2, \ldots, \tag{2}$$

where $x_k, u_k \in \mathbb{R}$ are the output and control, respectively, at time k, with $x_k = u_k = 0$ when $k < 0$, and $a_{k,j}, b_{k,j} \in \mathbb{R}$, $j, k \geqslant 0$, are known parameters. We assume that $b_{k,0} \neq 0$, $\forall k \geqslant 0$, so that the control always affects the output at the next time instant. In addition, we further assume that x_0 is a realization of a random variable X_0 on the probability space $(\mathbb{R}, \mathscr{L}, P)$, where $\mathscr{L}$ is the σ-algebra of Lebesgue sets on the real line and P is a known probability measure such that $E|X_0|^m < \infty$ for some $m > 0$.

Suppose a coder observes the outputs and then sends real-time data to a distant controller over a digital channel that can carry only one symbol s_k from an alphabet $\mathbb{Z}_M \triangleq [0, 1, \ldots, M-1]$ during each sampling interval. The corresponding *data rate* is $R \triangleq \log_2 M$ bits per interval. Neglecting the propagation delay and transmission errors, the finite data rate implies that each symbol takes one sampling interval to reach the other end of the channel. Hence, at time k the controller has $s_0, \ldots, s_{k-1}$ available and generates

$$u_k = v_k(\tilde{s}_{k-1}), \quad \forall k \geqslant 0, \tag{3}$$

where the notation $\tilde{y}_k$ denotes a sequence $\{y_j\}_{j=0}^{k}$ and $v_k : \mathbb{Z}_M^k \to \mathbb{R}$ is the controller function at time k.

If no restrictions save causality are placed on the structure of the coder, each symbol s_k which it transmits may be a function of the sequences of past and present outputs $\tilde{x}_k$ and past symbols $\tilde{s}_{k-1}$. However, (2) and (3) imply that x_k in its turn is completely determined by the initial condition and $\tilde{s}_{k-2}$, so s_k is consequently a function of x_0 and $\tilde{s}_{k-1}$,

$$s_k = \gamma_k(x_0, \tilde{s}_{k-1}), \quad \forall k \geqslant 0, \tag{4}$$

where $\gamma_k : \mathbb{R} \times \mathbb{Z}_M^k \to \mathbb{Z}_M$ is the coder function at time k.

Note that there is no explicit communication constraint between the controller and actuator. This is obviously a reasonable assumption if they are colocated, but even otherwise the formulation above would be unchanged, since the location of the controller is purely nominal. The symbols that would be transmitted by it over an additional link to the actuator would have to be translated once again into control signals, making intermediate calculations redundant. The number R should thus be regarded as the *overall* rate of the complete link from sensor to actuator. In addition, we remark that there are slightly different ways in which the digital link can be defined (see [16,1] for details).

We call the pair of sequences $(\gamma, v) \triangleq (\{\gamma_k\}_{k\geqslant 0}, \{v_k\}_{k\geqslant 0})$ a *coder–controller* and our objective is to find one that minimizes an infinite horizon cost of the form

$$J_m \triangleq \limsup_{k\to\infty} \rho_k^{-1} E|X_k|^m, \tag{5}$$

where X_k is the random variable corresponding to the output x_k. This compares the asymptotic behaviour of the mth output moment against some positive sequence $\{\rho_k\}_{k\geqslant 0}$, which serves as a rough benchmark of the desired output behaviour. Although it does not attach a cost to the magnitudes of the controls or intermediate outputs, it does succeed in capturing the asymptotic stochastic behaviour of the closed-loop system. However, before addressing this objective, we first investigate the finite horizon cost

$$J_{m,N} \triangleq E|X_{N+1}|^m. \tag{6}$$

The insights gained then lead to an optimal solution of the infinite horizon problem and hence a necessary and sufficient condition for the system to be asymptotically stabilizable in mth moment.

3. Finite horizon cost

If the initial condition were known with complete accuracy by the controller, it could easily use the system equations to generate controls that yield $x_1 = x_2 = \cdots = 0$. However, in the presence of the data rate constraint this is evidently impossible. One way around this is for the coder to transmit a progressively more accurate estimate of x_0 to the controller, by using the bits available at each time instant to quantize it recursively. The controller can then use these estimates to generate the controls. It is shown in this section that for linear systems of form (2), the optimal finite horizon scheme has exactly this structure. More precisely, the core of the $J_{m,N}$-optimal coder-controller is shown to be a causally reformulated, optimal quantizer for X_0 with M^N levels.

Observe that by using downward induction on k, the output of the system can be expressed in terms of the initial condition and past controls,

$$x_{k+1} = \alpha_k x_0 + \sum_{j=0}^{k} \beta_{k,j} u_{k-j}, \quad \forall k \geqslant 0, \tag{7}$$

where $\alpha_{-1} \triangleq 1$ and α_k, $\beta_{k,j}$ are given recursively by

$$\alpha_{k+1} \triangleq \sum_{j=0}^{k+1} a_{k+1,j}\alpha_{k-j},$$

$$\beta_{k+1,j} \triangleq b_{k+1,j} + \sum_{i=0}^{j-1} a_{k+1,i}\beta_{k-i,j-i-1}, \tag{8}$$

$$\forall k \geqslant 0,\ j \in [0,\ldots,k+1].$$

Notice that the problem becomes trivial if $\alpha_k = 0$ for some $k \in [0,\ldots,N]$, since from (7) and (2) controls $u_k,\ldots,u_N$ can then be found which yield $x_{k+1} = \cdots = x_{N+1} = 0$. As such, we focus on the nontrivial case when $\alpha_k \neq 0$, $\forall k \in [0,\ldots,N]$.

The next step is to change variables. Define the mappings by

$$\eta_k(\tilde{t}_{k-1}) \triangleq -\alpha_k^{-1} \sum_{j=0}^{k} \beta_{k,j} v_{k-j}(\tilde{t}_{k-j-1}),$$

$$\forall k \in [0,\ldots,N],\ \tilde{t}_{k-1} \in \mathbb{Z}_M^k, \tag{9}$$

so that $-\alpha_k\eta_k(\tilde{s}_{k-1})$ represents the cumulative effect of past controls on the output at time $k+1$. As $\beta_{k,0} = b_{k,0}$, which by hypothesis is nonzero, the matrix of coefficients $\{-\alpha_k^{-1}\beta_{k,j}\}_{k,j}$ is triangular with nonzero diagonal elements. Hence, the equation above can easily be inverted to express the coder function sequence $\tilde{v}_N$ in terms of $\tilde{\eta}_N$,

$$v_k(\tilde{t}_{k-1}) \triangleq \sum_{j=0}^{k} \tau_{k,j}\eta_{k-j}(\tilde{t}_{k-j-1}),$$

$$\forall k \in [0,\ldots,N],\ \tilde{t}_{k-1} \in \mathbb{Z}_M^k, \tag{10}$$

where $\tau_{k,j}$ is given by the recursion

$$\tau_{k,0} \triangleq -\beta_{k,0}^{-1}\alpha_k, \quad \tau_{k,j} \triangleq -\beta_{k,0}^{-1} \sum_{i=1}^{j} \beta_{k,i}\tau_{k-i,j-i},$$

$$\forall k \in [0,\ldots,N],\ j \in [0,\ldots,k]. \tag{11}$$

This one-to-one correspondence between $\tilde{\eta}_N$ and $\tilde{v}_N$ implies that it is perfectly equivalent to optimize $J_{m,N}$ with respect to either. Substituting (7) and (9) into (6), we obtain

$$J_{m,N} = |\alpha_N|^m E|X_0 - \eta_N(\tilde{S}_{N-1})|^m, \tag{12}$$

where $\tilde{S}_{N-1}$ is the random variable corresponding to the symbol sequence $\tilde{s}_{N-1}$. Now, $\eta_N(\tilde{s}_{N-1})$ is a function of x_0 with up to M^N distinct values, one for each possible symbol sequence, so the function R_N defined by $R_N(x_0) \triangleq \eta_N(\tilde{s}_{N-1})$ can be regarded as an M^N-level quantizer for x_0. The RHS of (12) is then simply its mean mth power error (MmPE), scaled by $|\alpha_N|^m$. As such, if we can find a pair (γ^*, η^*) that achieves the minimum MmPE for an M^N-level quantizer for X_0, it will automatically be $J_{m,N}$-optimal. We show next how to construct such a pair.

Denote the points of the MmPE-optimal, M^N-level quantizer for X_0 by $q_N(0), q_N(1), \ldots, q_N(M^N - 1)$, ranked from least to greatest, and set

$$\eta_N^*(\tilde{t}_{N-1}) \triangleq q_N\left(\sum_{k=0}^{N-1} t_k M^{N-1-k}\right), \quad \forall \tilde{t}_{N-1} \in \mathbb{Z}_M^N, \tag{13}$$

i.e. the argument of η_N^* is the M-ary representation of that of q_N. By the *nearest-neighbour rule* [6], the optimal quantizer selects the point closest to x_0, so choose the symbol sequence by

$$\tilde{s}_{N-1} = \arg \min_{\tilde{t}_{N-1} \in \mathbb{Z}_M^N} |x_0 - \eta_N^*(\tilde{t}_{N-1})|^m, \tag{14}$$

breaking possible ties by selecting the greater point. As this equation can be expressed recursively, $\forall k \in [0, \ldots, N-1]$, as

$$\begin{aligned} s_k &= \gamma_k^*(x_0, \tilde{s}_{k-1}) \\ &\triangleq \arg \min_{t_k \in \mathbb{Z}_M} \left\{ \min_{t_{k+1},\ldots,t_{N-1} \in \mathbb{Z}_M} |x_0 - \eta_N^*(\tilde{s}_{k-1}, t_k, t_{k+1}, \ldots, t_{N-1})|^m \right\}, \end{aligned} \tag{15}$$

it is realizable within our framework. Substituting (14) into (12), we obtain

$$\begin{aligned} |\alpha_N|^{-m} J_{m,N}^* &= \int \min_{\tilde{t}_{N-1} \in \mathbb{Z}_M^N} |x_0 - \eta_N^*(\tilde{t}_{N-1})|^m \, \mathrm{d}P(x_0) \\ &= \int \min_{j \in \mathbb{Z}_{M^N}} |x_0 - q_N(j)|^m \, \mathrm{d}P(x_0). \end{aligned} \tag{16}$$

As the last integral is simply the expression for the MmPE of the optimal M^N-level quantizer for X_0 [6], the pair (γ^*, η^*) is $J_{m,N}$-optimal.

We now recast the coder equation (5) in a somewhat simpler form. First extend q_N over the real interval $[-1, M^N]$, so that it remains increasing and is furthermore continuous, and for convenience set $q_N(-1) \triangleq -\infty$, $q_N(M^N) \triangleq \infty$. Next, define

$$e_N(z) \triangleq (q_N(M^N z - 1) + q_N(M^N z))/2, \quad \forall z \in [0, 1], \tag{17}$$

$$\zeta_k \triangleq \sum_{j=0}^{k} s_j M^{-j-1}, \quad \forall k \geqslant 0, \tag{18}$$

so that $e_N(\zeta_{N-1})$ is half-way between the neighbouring quantizer points $q_N(M^N \zeta_{N-1})$ and $q_N(M^N \zeta_{N-1} + 1)$. From (14), the sequence $\tilde{s}_{N-1}$ is transmitted iff the quantizer point closest to x_0 is $q_N(M^N \zeta_{N-1})$, equivalent to x_0 lying inside the interval $[e_N(\zeta_{N-1}), e_N(\zeta_{N-1} + M^{-N}))$. As e_N is increasing and continuous, x_0 lies in this interval iff $e_N^{-1}(x_0) \in [\zeta_{N-1}, \zeta_{N-1} + M^{-N})$. Referring to (18), this in turn is equivalent to $\tilde{s}_{N-1}$ being the first N digits of the M-ary representation for $e_N^{-1}(x_0)$. That is, the optimal coder simply applies a transformation e_N^{-1} to the initial condition and then transmits the first N digits of its M-ary expansion. This and the previous results are encapsulated below:

Coder 1. *First, transform the initial condition x_0 of system* (2) *to obtain $\zeta \triangleq c_N(x_0) \triangleq e_N^{-1}(x_0)$, where e_N is given by* (17). *Then at time k, transmit the $(k+1)$th most significant digit in the M-ary representation of ζ as the symbol s_k.*

Controller 1. *Upon receiving the symbol s_{k-1} at time k, calculate the number ζ_k using* (18). *Set*

$$\begin{aligned} &\eta_N^*(\tilde{s}_{N-1}) \triangleq q_N(M^N \zeta_{N-1}), \\ &\eta_k^*(\tilde{s}_{k-1}) \triangleq c_N^{-1}(\zeta_{k-1} + 0.5/M^k), \\ &\quad \forall k \in [0, \ldots, N-1], \end{aligned} \tag{19}$$

where $q_N(0) < q_N(1) < \cdots < q_N(M^N - 1)$ are the points of the MmPE-optimal, M^N-level quantizer for X_0, and use (10) *to calculate the control $v_k(\tilde{s}_{k-1})$.*

We make several comments here. Firstly, the optimal coder–controller is basically a *compander*, i.e. it consists of a *compressor* c_N which maps $x_0 \in \mathbb{R}$ to $\zeta \in [0, 1]$, followed by a uniform, M^N-level quantizer which maps this to ζ_{N-1} and then an *expander* $q_N(M^N \cdot)$ which transforms ζ_{N-1} into an estimate of x_0 [6]. Secondly, the mappings $\eta_0^*, \ldots, \eta_{N-1}^*$ are actually completely arbitrary, since they do not affect the integrand in (16). However, the choice above is intuitively appealing, as $\zeta_k + (1/2M^{k+1})$ is the mid-point of the interval of length M^{-k-1} which the controller knows that ζ lies in, from the sequence $\tilde{s}_k$. Furthermore, this makes the infinite horizon analysis of the next section somewhat easier. Thirdly, although the optimal quantizer may be unique, η_N^* can be defined in as many different ways as there are one-to-one maps from the integers $\mathbb{Z}_{M^N}$ to the M-ary sequences $\mathbb{Z}_M^N$. The choice of mapping taken here, as implied in Eq. (13), is one of the more tractable ones. Finally, explicit expressions for the optimal coder–controller are generally impossible to derive, since q_N can normally only be obtained numerically for a given p and N [10,6]. One of the few exceptions is when X_0 is *Laplacian* and $m=1$, for which case a closed form solution parametrized by N and the mean and variance of X_0 has been obtained [11].

4. Infinite horizon cost

In the previous section, we observed that $J_{m,N}$-optimal coder–controllers are usually impossible to derive in closed form. However, we demonstrate here that when $N \to \infty$ the limiting coder–controller can be obtained directly, *without explicitly solving the finite horizon problem*. We then prove that this limiting scheme is in fact optimal with respect to an infinite horizon cost of form (5), under certain mild conditions on the probability density p governing the initial output x_0.

The key is the classic result that as the number of MmPE-optimal quantizer points approaches infinity, their normalized density per unit x_0 approaches

$$v(x_0) \triangleq \left(\int p(y)^{1/(m+1)}\,dy \right)^{-1} p(x_0)^{1/(m+1)}, \quad \forall x_0 \in \mathbb{R}, \tag{20}$$

under certain technical conditions on p [5,2]. As $q_N(M^N\zeta_{N-1})$ is the $(M^N\zeta_{N-1}+1)$th quantizer point by (13), the nearest-neighbour rule implies that there are $M^N\zeta_{N-1} + \mathrm{O}(1)$ points less than or equal to x_0. Observing that ζ_{N-1}, being a sum of exponentially decaying terms, must converge to a number $\bar{\zeta} \in [0,1]$ as $N \to \infty$, we may define

$$c(x_0) \triangleq \bar{\zeta} = \lim_{N\to\infty} \frac{M^N\zeta_{N-1} + \mathrm{O}(1)}{M^N} = \int_{y\leqslant x_0} v(y)\,\mathrm{d}y, \quad \forall x_0 \in \mathbb{R}. \tag{21}$$

By analogy with Coder–Controller 1, the following scheme as the horizon N becomes unbounded is suggested:

Coder 2. *First, transform the initial condition x_0 of system* (2) *to yield $\bar{\zeta} \triangleq c(x_0)$, where c is given by* (21). *Then at time k transmit the* $(k+1)$*th most significant digit in the M-ary representation of $\bar{\zeta}$ as the symbol s_k.*

Controller 2. *Upon receiving the symbol s_{k-1} at time k, calculate*

$$\eta_k(\tilde{s}_{k-1}) = c^{-1}(\zeta_{k-1} + 0.5/M^k), \tag{22}$$

where ζ_{k-1} is defined by (18), *and use* (10) *to generate the control signal $v_k(\tilde{s}_{k-1})$.*

For instance, for *Laplacian* X_0 with mean μ and mean absolute deviation ε, it can be shown that

$$c(x_0) = 0.5 + \mathrm{sign}(x_0-\mu)(1-\mathrm{e}^{-|x_0-\mu|/((m+1)\varepsilon)})/2,$$

and for Gaussian X_0 with mean μ and standard deviation ε, we have

$$c(x_0) = F\left(\frac{x_0-\mu}{\varepsilon\sqrt{m+1}}\right),$$

where F is the unit normal distribution function.

We now consider whether the coder–controller above is actually optimal with respect to an infinite horizon cost of form (5). First we need to fix the weights ρ_k, $k \geqslant 0$. Observe that as $N \to \infty$,

$$\min_{\gamma,v} M^{mN}|\alpha_N|^{-m}E|X_{N+1}|^m = \min_{\gamma,\eta} M^{mN}E|X_0 - \eta_N(\tilde{S}_{N-1})|^m \to (m+1)^{-1}2^{-m}\|p\|_{1/(m+1)}, \tag{23}$$

where the limit is a well-known result of asymptotic quantization theory [2] and $\|p\|_r \triangleq (\int p(x_0)^r\,\mathrm{d}x_0)^{1/r}$. This is nearly what we want, except that in (5) the minimization is to be performed after the limit is taken. This suggests that an appropriate choice of weighting sequence is

$$\rho_k = |\alpha_{k-1}|^m/M^{m(k-1)}, \quad \forall k \geqslant 0. \tag{24}$$

In order to prove that Coder–Controller 2 is J_m-optimal, we make use of the fact that it is essentially a compander and apply a result of [8]. Let $G : \mathbb{R} \to [0,1]$ be a compressor function with a continuous and nonnegative derivative g such that $g(x)$ decreases monotonically with $|x|$ for sufficiently large $|x|$. Suppose G is applied to a real random variable X with a probability density function π such that $E\{g(X)^{-m}\} < \infty$ and such that, for some $\delta > 0$, both

$$\int_0^{\delta} s(z)^m h(2z)\,\mathrm{d}z, \quad \int_{1-\delta}^{1} s(z)^m h(2z-1)\,\mathrm{d}z < \infty,$$

where $s \triangleq 1/gG^{-1}(\cdot)$ and $h \triangleq \pi G^{-1}(\cdot)/gG^{-1}(\cdot)$. In [8], it is shown that if $G(x)$ is quantized by f_Q, a Q-level, midpoint-based, uniform quantizer on $[0,1]$, and G^{-1} subsequently applied to form an estimate of x, then

$$Q^m E|X_0 - G^{-1} f_Q G(X)|^m \rightarrow (m+1)^{-1} 2^{-m} E\{g(X)^{-m}\} \quad \text{as } Q \rightarrow \infty. \quad (25)$$

We can now prove the main result of this section:

Theorem 1. *Let the initial output x_0 of system* (2) *be governed by a continuous probability density function p which decreases with $|x_0|$ for sufficiently large $|x_0|$ and satisfies $E|X_0|^{m+n} < \infty$, for some $m, n > 0$. Suppose further that*

$$pc^{-1}(2z-1) \leqslant A\,pc^{-1}(z), \quad \forall z \in [1-\delta, 1],$$
$$pc^{-1}(2z) \leqslant A\,pc^{-1}(z), \quad \forall z \in [0, \delta]$$

for some $A, \delta > 0$, where c is given by (21). *Then Coder–Controller* 2 *is J_m-optimal and achieves*

$$J_m^* = \min_{\gamma,v} \lim_{k\rightarrow\infty} |\alpha_{k-1}|^{-m} M^{m(k-1)} E|X_k|^m = \frac{1}{(m+1)2^m}\left(\int p(x_0)^{1/(m+1)}\,\mathrm{d}x_0\right)^{m+1}, \quad (26)$$

where α_k, $k \geqslant 0$, are given by (8). *Furthermore, for a given coding alphabet size M, a coder–controller that takes $E|X_k|^m \rightarrow 0$ exists if and only if*

$$\alpha_k / M^k \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (27)$$

Proof. Note that $\rho_k^{-1} E|X_k|^m = M^{m(k-1)} E|X_0 - \eta_{k-1}(\tilde{S}_{k-2})|^m$. As $\eta_{k-1}(\tilde{s}_{k-2})$ may be expressed as the compander output $c^{-1} f_{M^{k-1}} c(x_0)$, with $f_{M^{k-1}}$ being the M^{k-1}-level, midpoint-based, uniform quantizer on $[0,1]$, our first step is to show that the compressor c satisfies the conditions of [8]. Its derivative $c' = v = \kappa^{-1} p(\cdot)^{1/(m+1)}$, where $\kappa \triangleq \int p(x_0)^{1/(m+1)}\,\mathrm{d}x_0$, so c' is evidently continuous, nonnegative and monotonically decreasing for large enough x_0, by hypothesis on p. Furthermore, $E\{g(X_0)^{-m}\} = \|p\|_{1/(m+1)}$ which, as remarked in [2], is guaranteed to be bounded by Hölder's inequality if $E|X_0|^{m+n} < \infty$. Next, note that $h = \kappa pc^{-1}(\cdot)^{m/(m+1)}$, so that $h(2z) \leqslant A' h(z)$, $\forall z \in [0, \delta]$, where $A' \triangleq A^{m/(m+1)}$. Hence,

$$\int_0^{\delta} s(z)^m h(2z)\,\mathrm{d}z \leqslant A' \int_0^{\delta} s(z)^m h(\zeta)\,\mathrm{d}z = A' \kappa^m \int_0^{c^{-1}(\delta)} p(x_0)^{1/(m+1)}\,\mathrm{d}x_0,$$

which is finite. The boundedness of the remaining integral can be proven in exactly the same way. Hence, all the preconditions for (25) hold, so that as $k \rightarrow \infty$,

$$M^{m(k-1)} E|X_0 - \eta_{k-1}(\tilde{S}_{k-2})|^m \rightarrow (m+1)^{-1} 2^{-m} E\{v(X_0)^{-m}\} = (m+1)^{-1} 2^{-m} \|p\|_{1/(m+1)}.$$

Now observe that for any coder–controller (γ', η'),

$$\limsup_{k\rightarrow\infty} M^{m(k-1)} E|X_0 - \eta'_{k-1}(\tilde{S}_{k-2})|^m \geqslant \lim_{k\rightarrow\infty} \min_{\gamma',\eta'} M^{m(k-1)} E|X_0 - \eta'_{k-1}(\tilde{S}_{k-2})|^m = \frac{\|p\|_{1/(m+1)}}{(m+1)2^m},$$

by (23). As the scheme above achieves this lower bound, it is optimal.

To prove the sufficiency of (27), suppose it holds. Eq. (26) indicates that if Coder–Controller 2 is applied then the sequence $\{|\alpha_{k-1}|^{-m} M^{m(k-1)} E|X_k|^m\}_{k\geqslant 0}$ is bounded, which forces $E|X_k|^m$ to approach zero. To prove necessity, suppose that (27) does not hold. For any coder–controller,

$$\frac{M^{m(k-1)}}{|\alpha_{k-1}|^m} E|X_k|^m = M^{m(k-1)} E|X_0 - \eta'_{k-1}(\tilde{S}_{k-2})|^m \geqslant \min_{\gamma',\eta'} M^{m(k-1)} E|X_0 - \eta'_{k-1}(\tilde{S}_{k-2})|^m \rightarrow \frac{\|p\|_{1/(m+1)}}{(m+1)2^m} \quad \text{as } k \rightarrow \infty,$$

where we have again made use of the limit from (23). Hence, for any coder–controller and $\varepsilon > 0$, $\exists k' > 0$ such that

$$E|X_k|^m \geqslant \frac{|\alpha_{k-1}|^m}{M^{m(k-1)}}\left(\frac{\|p\|_{1/(m+1)}}{(m+1)2^m} - \varepsilon\right), \quad \forall k \geqslant k'.$$

By hypothesis the RHS cannot approach zero, so no coder–controller exists which takes $E|X_k|^m \rightarrow 0$. □

Condition (27) compares the accuracy of the initial condition estimate, proportional to M^k, with the dynamical coefficient α_k which propagates the uncertainty in x_0 and states that the system can be stabilized if and only if the former increases more rapidly than the latter. Note that it is trivially satisfied by asymptotically stable systems, for which $\alpha_k \to 0$. When the open-loop portion of the system is d-dimensional, linear and time-invariant (LTI), the first equation in (8) becomes

$$\alpha_{k+1} = \sum_{j=0}^{d-1} a_j \alpha_{k-j},$$

where $\alpha_{-1} = 1$ and $\alpha_{-2} = \cdots = \alpha_{-d} = 0$. The solution to this is of the form

$$\alpha_k = \sum_{j=0}^{d-1} h_j \theta_j^k,$$

where $\theta_0, \ldots, \theta_{d-1}$ are the poles of the system and $h_0, \ldots, h_{d-1}$ are constants, with possibly a polynomial dependence on k if a pole is repeated. Hence, if λ is the pole with largest magnitude then $\alpha_k \sim \lambda^k$ for large k, to within a polynomial factor in k. Substituting this into (27), we see that this system is asymptotically stabilizable in mth moment iff $M > |\lambda|$. As the data rate $R = \log_2 M$, we obtain the condition $R > \log_2 |\lambda|$. This makes precise the notion that the more unstable a system is, the higher the data rate needed to stabilize it. This result is summarized below.

Corollary 1. *Consider the linear system below, with time-invariant, d-dimensional open-loop dynamics,*

$$x_{k+1} = \sum_{j=0}^{d-1} a_j x_{k-j} + \sum_{j=0}^{\infty} b_{k,j} u_{k-j}, \quad \forall k \geqslant 0,$$

where $x_k, u_k \in \mathbb{R}$ are the output and control, respectively, at time k, a_j, $b_{k,j} \in \mathbb{R}$ with $b_{k,0} \neq 0$, $\forall j, k \geqslant 0$, and $x_k = u_k = 0$ when $k < 0$. If the probability density function governing x_0 satisfies the conditions in Theorem 1, then a coder–controller with data rate R that takes $E|X_k|^m \to 0$ exists if and only

$$R > \log_2 |\lambda|,$$

where λ is the unstable system pole with largest magnitude.

The technical conditions on p in Theorem 1 effectively limit the speed of decay of $pc^{-1}(z)$ as z approaches 0 and 1. They can be shown to be satisfied by any p such that $p(y) \sim |y|^v \exp(-B|y|^w)$ for large $|y|$ and parameters $B, w > 0$, $v \in \mathbb{R}$, which includes densities such as the Gaussian and Laplacian. We conjecture that the infimum of J_m assuming only that p is Lebesgue-integrable is also given by (23), since it should be possible to construct compressors c_i, $i \geqslant 0$, which satisfy the conditions of [8] and approach c in an appropriate integral sense as $i \to \infty$. However, it is much more difficult to prove that Coder–Controller 2 actually achieves this lower bound for any such general p, despite being the limiting form of the optimal finite horizon scheme.

We remark that the results above automatically apply to the problem of output estimation under a data rate constraint [15,12]. The only differences in the problem formulation are that the controls in the system equation (2) are set to zero, the controller is replaced by an estimator

$$\hat{x}_k = \delta_k(\tilde{s}_{k-1}), \quad \forall k \geqslant 0 \tag{28}$$

and the objective is to find a *coder–estimator* $(\gamma, \delta) \triangleq (\{\gamma_k\}_{k\geqslant 0}, \{\delta_k\}_{k\geqslant 0})$ that minimizes the distortion

$$D_m \triangleq \limsup_{k\to\infty} \rho_k^{-1} E|X_k - \hat{X}_k|^m. \tag{29}$$

The optimal coder–estimator is the same as Coder–Controller 2, except that $\eta_k(\tilde{s}_{k-1})$ is used to generate an estimate $\delta_k(\tilde{s}_{k-1}) = \alpha_{k-1}\eta_k(\tilde{s}_{k-1})$ rather than a control.

Finally, note that the results of this section indicate that *one-bit mean coding schemes* [15,7] are suboptimal with respect to the infinite horizon, mean-square-error cost J_2. Such schemes have intuitive appeal, as they proceed by simply partitioning a coding interval I containing x_0 at the conditional mean $E\{X_0 | x_0 \in I\}$ to form two new candidate intervals. However, it is easy to show from the discussion at the beginning of this section that the intervals formed by Coder 2 always contain equal proportions of optimal, infinite-level quantizer points. For a one-bit scheme, this means that each coding interval $[a, b)$ should be divided at the point u such that

$$\int_a^u v(x_0)\, dx_0 = \int_u^b v(x_0)\, dx_0,$$

which in general does not coincide with $E\{X_0 | x_0 \in [a, b)\}$. Hence, although the conditional mean is the mean-square-error-optimal reconstruction point, it is not the J_2-optimal partition point.

5. Conclusion

In this paper, the asymptotic stabilizability of a linear, discrete-time system with a communication constraint was investigated. Finite and infinite horizon control objectives were formulated and it was shown that the optimal finite horizon coder–controller is essentially an optimal quantizer for the initial output. Asymptotic quantization theory was then used to directly obtain the limiting scheme as the horizon approaches infinity. Under certain technical conditions on the probability density p governing the initial output, this scheme was shown to be optimal in the infinite horizon sense and an expression for the optimal cost was derived. This led to a necessary and sufficient condition for the system to be asymptotically stabilizable in mth moment at a given data rate. Further work is currently being undertaken on relaxing the conditions on p and extending the results presented here to stochastic and nonlinear systems.

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