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Exponential stabilisability of finite-dimensional linear systems with limited data rates☆

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Abstract

A critical notion in the field of communication-limited control is the smallest data rate above which there exists a stabilising coding and control law for a given plant. This quantity measures the lowest rate at which information can circulate in a stable feedback loop and provides a practical guideline for the allocation of communication resources. In this paper, the exponential stabilisability of finite-dimensional LTI plants with limited feedback data rates is investigated. By placing a probability density on the initial state and casting the objective in terms of state moments, the problem is shown to be similar to one in asymptotic quantisation. Quantisation theory is then applied to obtain the infimum stabilising data rate over all causal coding and control laws, under mild requirements on the initial state density.

Keywords: Stabilisability; Exponentially stable; Communication channels; Quantisation

1. Introduction

A common assumption in control theory is that measurements and control signals are available with arbitrarily high accuracy to the various components of a system. However, in an increasing number of applications such as micro-electromechanical systems, power control in mobile networks and decentralised tracking, this premise is unsound, as communication occurs over digital channels with limited data rates. An obvious effect of this is the introduction of quantisation errors, since signals transmitted over such channels must first be converted into discrete-valued, digital symbols. As each symbol requires a finite amount of time for complete transmission, a delay is also introduced which further degrades system performance. Although these related effects may be traded off against each other, there is an underlying, fundamental limit to the amount of information that may be exchanged per unit time. In order to fully exploit this limited resource, the components of the system must be able to choose and transmit the most important information at the time for the task at hand. The achievable control performance will inevitably be affected by the communication resources available and conversely, the communication resources required will be dictated by the performance level desired.

Historically, the issue of limited communication rates was first considered in the context of estimation. Important results were derived within the framework of *rate-distortion theory* (Shannon, 1959; Berger, 1971), but its reliance on arbitrarily long *block* coders implied long delays that were undesirable in many real-time situations. Recursive coders such as *delta* and *differential pulse code* modulators overcame this drawback but were restricted to stationary or ergodic systems (see e.g. Kieffer, 1982). Though a suitable assumption in communications, this did not agree with the unstable dynamics often encountered in control systems.

The first rigorous analysis of an unstable system was in (Delchamps, 1990), in which it was proved that if a memorylessly quantised, noiseless, *linear time-invariant* (LTI) plant has an eigenvalue exceeding two in magnitude then it is not *controllable*, regardless of the fineness of quantisation. Other coding and control schemes for deterministic LTI

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systems were subsequently proposed and analysed in Wong and Brockett (1999), Baillieul (1999), Baillieul (2001), Elia and Mitter (1999), Brockett and Liberzon (2000), Petersen and Savkin (2001), Ishii and Francis (2002). A feature common to many these articles was the derivation of data rate inequalities that were sufficient to ensure various notions of stability but tight only for one-dimensional states. In Nair and Evans (2000), *asymptotic quantisation theory* (see e.g. Gersho, 1979; Bucklew & Wise, 1982; Graf & Luschgy, 2000) was used to explicitly derive a necessary and sufficient data rate bound for the asymptotic stabilisability of a given *auto-regressive moving average* (ARMA) system of arbitrary dimension. However, in a sense this work was not far from the one-dimensional case, as the initial condition was scalar. See Borkar and Mitter (1997), Tatikonda, Sahai, and Mitter (1998), Sahai (2000), Nair and Evans (2002) for results pertaining to stochastic linear systems.

In this paper, quantisation theory is applied to the problem of exponentially stabilising a general noiseless, discrete-time, LTI system in $\mathbb{R}^n$. A precise formulation is given in the next section. Assuming that the initial state is random, the objective is to find the least data rate above which there exists a coding and control law which takes the rth absolute state moment to zero faster than a specified exponential decay. The main result, Theorem 1, is stated there and the remainder of the paper essentially constitutes its proof. A similar result was outlined in Tatikonda and Mitter (2000) in a deterministic setting and with a known bound on the initial state. In contrast, the initial state here is random and governed by a density with possibly infinite support. In Section 3, the system dynamics are explicitly decoupled and the control objective is shown to be equivalent to recursively quantising the initial state so that a *mean rth power error* term approaches zero exponentially fast. In the section after, the necessity of the data rate specified in Theorem 1 is established by means of a non-asymptotic quantiser distortion lower bound. Its sufficiency is then confirmed in the penultimate section by explicitly constructing a coding and control scheme and using a result on scalar asymptotic quantisation (Linder, 1991).

2. Formulation

Certain conventions are followed in this paper. Sequences $\{a_j\}_{j=0}^k$ are denoted $\tilde{a}_k$ and $\|\cdot\|$ represents either the Euclidean norm on a real vector space or the matrix norm induced by it. However when subscripted as in $\|f\|_\theta$ it denotes the $\mathscr{L}_\theta$ norm $(\int |f(\mathbf{x})|^\theta \, \mathrm{d}\lambda(\mathbf{x}))^{1/\theta}$, where λ denotes Lebesgue measure on the domain of f, assumed to be a real vector space. Vectors are written in bold-face type, matrices in bold-face upper-case, random variables in upper-case and their realisations in corresponding lower-case. All random variables are assumed to exist in a common probability space. The absolute continuity of the distribution of a random variable $\mathbf{X}$ is understood to be with respect to (w.r.t.) λ and its density is written as $p_{\mathbf{X}}$. Expectation is denoted by E, the $n \times n$ identity matrix $\mathbf{I}_n$, the $m \times n$ 0 matrix $\mathbf{0}_{m\times n}$, the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, positive integers $\mathbb{N}$, non-negative integers $\mathbb{Z}_+$ and the finite set $[0, 1, \ldots, \mu-1]$ by $\mathbb{Z}_\mu$.

Consider the discrete-time, linear time-invariant system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \quad \mathbf{y}_k = \mathbf{C}\mathbf{x}_k, \quad \forall k \in \mathbb{Z}_+, \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ is the state, $\mathbf{y}_k \in \mathbb{R}^p$ the sensor measurement and $\mathbf{u}_k \in \mathbb{R}^m$ the control vector, $\forall k \in \mathbb{Z}_+$. It is assumed that $(\mathbf{A}, \mathbf{B})$ is *reachable*, $(\mathbf{C}, \mathbf{A})$ is *observable* and that $\mathbf{x}_0$ is a realisation of a random variable $\mathbf{X}_0$ that satisfies $\mathrm{E}\|\mathbf{X}_0\|^{r+\varepsilon} < \infty$ for some $r, \varepsilon > 0$. The probability distribution of $\mathbf{X}_0$ will also be required to be absolutely continuous on $\mathbb{R}^n$, so that it has a density $p_{\mathbf{X}_0}$.

Suppose the sensor is connected to a distant controller by a digital channel that can carry only one symbol s_k from a coding alphabet $\mathbb{Z}_\mu \triangleq [0, 1, \ldots, \mu-1]$ during each sampling interval. The data rate of the channel may be defined as $R \triangleq \log_2 \mu$ bits per interval and, clearly, the continuous-valued measurements must be encoded into digital symbols prior to transmission. In practice, this analog to digital conversion may have computational constraints such as zero or finite memory. However, as the objective here is to address communications and not computational limitations, each symbol transmitted by the coder is permitted to depend on all past and present measurements and past symbols, i.e.

$$s_k = \gamma_k(\tilde{\mathbf{y}}_k, \tilde{s}_{k-1}), \quad \forall k \in \mathbb{Z}_+, \tag{2}$$

where $\gamma_k : \mathbb{R}^{p\times(k+1)} \times \mathbb{Z}_\mu^k : \to \mathbb{Z}_\mu$ is the coder mapping at time k. Neglecting the propagation delay and transmission errors, the finite data rate implies that each symbol takes one sampling interval to be completely transmitted. Hence at time k the controller has $s_0, \ldots, s_{k-1}$ available and generates

$$\mathbf{u}_k = \delta_k(\tilde{s}_{k-1}), \quad \forall k \in \mathbb{Z}_+, \tag{3}$$

where $\delta_k : \mathbb{Z}_\mu^k \to \mathbb{R}^m$ is the controller function at time k.

The definitions of the coder, controller and digital link above are essentially the same as in Nair and Evans (2000). As discussed there, if an additional data rate constraint is imposed between the controller and a separate actuator then the formulation above still applies, with a feedback data rate given by the smaller of the coder-to-controller and controller-to-actuator rates. The number R should thus be regarded as the *overall* rate of the complete link from sensor to actuator.

Define the *coder-controller* as the pair of mapping sequences $(\tilde{\gamma}_\infty, \tilde{\delta}_\infty)$. Given a data rate $R > 0$ and constant $\varrho > 0$, the objective is to investigate whether there exists a coder-controller that ϱ-*exponentially* stabilises the plant (1), in the sense that

$$\varrho^{-kr}\mathrm{E}\|\mathbf{X}_k\|^r \to 0 \quad \text{as } k \to \infty. \tag{4}$$

Note that this limit implies that $\varrho^{-kr}\|\mathbf{x}_k\|^r \to 0$ for almost all $\mathbf{x}_0$ such that $p_{\mathbf{X}_0} > 0$, so it is essentially stronger than the usual, deterministic notion of exponential stability. On the other hand, exponential stability uniform over all initial states is easily shown to be more stringent than the above. The main motivation for formulating the objective as one of moment stabilisation is to permit the application of ideas from quantisation theory. It also serves as a contact point for extensions of this problem to stochastic systems.

For a noiseless ARMA system with a scalar initial condition, it is known that there is a critical data rate which determines whether closed-loop asymptotic moment stability is possible or not (Nair & Evans, 2000). It may therefore be expected that a critical rate will also exist for the case of an LTI state-space system. The main result of this paper is now stated:

Theorem 1. *Assume that the linear time-invariant plant* (1) *is reachable and observable, with (possibly repeated and conjugate) eigenvalues* $\eta_1, \ldots, \eta_n$. *Further assume that its initial state* $\mathbf{x}_0 \in \mathbb{R}^n$ *is random, with a distribution which is absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}^n$ *and has finite* $(r+\varepsilon)$*th absolute moment* $\mathrm{E}\|\mathbf{X}_0\|^{r+\varepsilon} < \infty$, *for some* $r, \varepsilon > 0$.

Then for a given data rate R (bits per interval), a coder-controller (2)–(3) *that ϱ-exponentially stabilises the system in the rth absolute moment sense* (4) *exists* if and only if

$$R > \sum_{|\eta_j| \geqslant \varrho} \log_2 \left| \frac{\eta_j}{\varrho} \right|. \tag{5}$$

As nothing is assumed about the coding and control laws but causality, this result measures the smallest rate at which information can circulate in an exponentially stable feedback loop. It thus draws a fundamental line of demarcation between what is and is not achievable when communication bandwidth is limited and, as remarked in Baillieul (1999), in some sense quantifies the intrinsic complexity of stabilising a given plant.

Further insight into its meaning can be gained by rewriting (5) as $\mu > \prod_{|\eta_j| \geqslant \varrho} |\eta_j/\varrho|$. The right hand side (RHS) is the factor by which a volume in the state subspace corresponding to eigenvalues with $|\eta_j| \geqslant \varrho$ increases at each time step, relative to ϱ, while the left hand side (LHS) is the number of disjoint regions into which a μ-point quantiser can divide it. In other words, the system is exponentially stabilisable if and only if the relative increase in state 'uncertainty volume' due to the system dynamics can be counteracted by the decrease due to quantisation.

Note that the critical data rate for asymptotic stabilisability, corresponding to $\varrho = 1$, differs from that derived in Nair and Evans (2000) for ARMA systems, i.e. $\log_2 \max_j |\eta_j|$. This may appear inconsistent, since any ARMA model can be put into state-space form by stacking its outputs. However, as remarked in the introduction, the analysis in Nair and Evans (2000) hinges on a scalar initial condition, which restricts the initial state to a one-dimensional subspace. This makes Theorem 1 inapplicable, since the initial state distribution is then singular. As all subsequent states also lie in one-dimensional subspaces, it is reasonable to expect that the growth of the state uncertainty length will be dominated by the eigenvalue with maximum magnitude. However, if the initial condition of an ARMA system does have full dimension, then the result here applies.[1]

Although eigenvalues with $|\eta_j| \leqslant \varrho$ play no role in (5), the sum includes the case $|\eta_j| = \varrho$ to stress that the corresponding dynamical modes still need to be coded and controlled. The additional data rate required can, however, be arbitrarily small. Finally, note that the requirement of absolutely continuity of the initial state distribution on $\mathbb{R}^n$ can be relaxed to absolute continuity on the subspace of $\mathbb{R}^n$ corresponding to the eigenvalues $|\eta_j| \geqslant \varrho$.

The remainder of this paper is devoted to proving Theorem 1 in three stages. In the next section the problem is transformed in order to simplify its analysis and clarify its connection to asymptotic quantisation. In Section 4, the necessity of (5) is then established via a quantiser distortion lower bound. Finally, its sufficiency is established in Section 5 by explicitly constructing a coder-controller that achieves ϱ-exponential stability for any data rate satisfying it.

3. Reformulation

Instead of dealing with the state $\mathbf{x}_k$ directly, it is convenient to transform it so as to simplify the system dynamics. The obvious approach of putting the matrix $\mathbf{A}$ in *Jordan canonical form* requires a transformation matrix which is generally complex-valued. In order to avoid the complications that would ensue, the *real Jordan canonical form* (Horn & Johnson, 1985) is used instead.

Let $\lambda_1, \ldots, \lambda_v$, $v \leqslant n$, be the *distinct* eigenvalues of $\mathbf{A}$, not counting conjugates, and let the algebraic multiplicity of each λ_i be κ_i. The real Jordan canonical form $\mathbf{J}$ of $\mathbf{A}$ has the block diagonal structure

$$\mathbf{J} \triangleq \operatorname{diag}(\mathbf{J}_1, \ldots, \mathbf{J}_v) \in \mathbb{R}^{n \times n}, \tag{6}$$

where $\mathbf{J}_i \in \mathbb{R}^{n_i \times n_i}$ has either exactly one eigenvalue λ_i or a pair of complex conjugate eigenvalues λ_i, λ_i^*, each with multiplicity κ_i, and where

$$n_i \triangleq \begin{cases} \kappa_i & \text{if } \lambda_i \in \mathbb{R}, \\ 2\kappa_i & \text{otherwise.} \end{cases} \tag{7}$$

Furthermore there exists a *similarity* matrix $\mathbf{T} \in \mathbb{R}^{n \times n}$ such that $\mathbf{T}^{-1}\mathbf{J}\mathbf{T} = \mathbf{A}$. Defining the transformed state

$$\acute{\mathbf{x}}_k \triangleq \mathbf{T}\mathbf{x}_k \in \mathbb{R}^n, \quad \forall k \in \mathbb{Z}_+, \tag{8}$$

[1] Between these two extremes, it may be conjectured that if the intersection of the initial condition and unstable subspaces has dimension n', then the infimum stabilising rate is the $\log_2|\cdot|$ sum of those n' eigenvalues which dominate it.

the system equations (1) can then be written

$$\acute{\mathbf{x}}_{k+1} = \mathbf{J}\acute{\mathbf{x}}_k + \mathbf{T}\mathbf{B}\mathbf{u}_k, \quad \mathbf{y}_k = \mathbf{C}\mathbf{T}^{-1}\acute{\mathbf{x}}_k. \tag{9}$$

By partitioning the transformed state vector into the vectors $\acute{\mathbf{x}}_k^{(1)}, \ldots, \acute{\mathbf{x}}_k^{(v)}$ corresponding to each subsystem, the dynamical equation above can be rewritten more explicitly $\forall k \in \mathbb{Z}_+$, $i \in [1, \ldots, v]$ as

$$\acute{\mathbf{x}}_{k+1}^{(i)} = \mathbf{J}_i \acute{\mathbf{x}}_k^{(i)} + (\mathbf{T}\mathbf{B}\mathbf{u}_k)^{(i)} \in \mathbb{R}^{n_i}, \tag{10}$$

where $(\mathbf{T}\mathbf{B}\mathbf{u}_k)^{(i)}$ denotes that portion of the $\mathbf{T}\mathbf{B}\mathbf{u}_k$ control vector that feeds into the ith subsystem. In the form (10), it can be seen that the original system has been decoupled into v real subsystems, with open-loop dynamics characterised by either a single eigenvalue or a pair of complex conjugate eigenvalues, possibly repeated. As $\mathbf{T}$ is invertible, the problems of exponentially stabilising (9) and (1) are equivalent.

The key idea in the proof of Theorem 1 is the equivalence of the control problem to recursive quantisation of the initial state. Expanding (9) out and using (3),

$$\acute{\mathbf{x}}_k = \mathbf{J}^k \acute{\mathbf{x}}_0 + \sum_{j=0}^{k-1} \mathbf{J}^{k-1-j} \mathbf{T}\mathbf{B}\delta_j(\tilde{s}_{j-1}), \quad \forall k \in \mathbb{Z}_+. \tag{11}$$

Observe that the sum above is a function of the symbol sequence $\tilde{s}_{k-2}$ which is in turn completely determined by $\acute{\mathbf{x}}_0$, for a given coder. Thus $\forall k \in \mathbb{Z}_+$, $\acute{\mathbf{x}}_0 \in \mathbb{R}^n$, a function $q_{k-1} : \mathbb{R}^n \to \mathbb{R}^n$ may be defined by

$$q_{k-1}(\acute{\mathbf{x}}_0) \triangleq -\sum_{j=0}^{k-1} \mathbf{J}^{k-1-j} \mathbf{T}\mathbf{B}\delta_j(\tilde{s}_{j-1}). \tag{12}$$

As the symbol sequence $\tilde{s}_{k-2} \in \mathbb{Z}_\mu^{k-1}$ can take μ^{k-1} distinct values, q_{k-1} can only assume up to μ^{k-1} distinct values in $\mathbb{R}^n$, i.e. it can be viewed as a μ^{k-1}-point quantiser for the initial state. Hence finding a coder-controller that ϱ-exponentially stabilises the system is equivalent to finding a sequence $\{q_k\}_{k \in \mathbb{Z}_+}$ of vector quantisers which (i) are expressible in the form (12) for some sequence of coder and controller mappings and (ii) achieve

$$\varrho^{-kr} \mathrm{E}\|\mathbf{J}^k \acute{\mathbf{X}}_0 - q_{k-1}(\acute{\mathbf{X}}_0)\|^r \to 0 \quad \text{as } k \to \infty. \tag{13}$$

Note that by (12) this sequence of quantisers is generally recursive, since $q_{k-1}(\acute{\mathbf{x}}_0)$ depends on past quantiser outputs $q_{k-2}(\acute{\mathbf{x}}_0), \ldots, q_0(\acute{\mathbf{x}}_0)$ via its dependence on $\tilde{s}_{k-2}$.

Apart from the time-dependent coefficient and matrix, the term on the LHS above is the mean rth power error (MrPE) distortion criterion that is commonly used in asymptotic quantisation theory (e.g., Gersho, 1979; Bucklew & Wise, 1982). This equivalence to a quantisation problem is not surprising, since the initial state is the only unknown quantity in the system. However the presence of the time-varying matrix $\mathbf{J}^k$ makes achieving (13) somewhat more complicated than in standard asymptotic vector quantisation, primarily because it makes the quantiser errors of the subsystems grow at different speeds. It is intuitively obvious that the recursive quantiser for the initial state should somehow allocate more quantisation levels—and hence a larger proportion of the data rate—to more unstable subsystems, so as to balance the MrPE's of all. In Section 5, it is shown how this can be done. First however, very general quantisation arguments are used to show that the data rate must satisfy (5) for exponential stability in the sense of (4), irrespective of the coder-controller used.

4. Necessity

For the sake of notational simplicity, it is assumed in this and the following section that all dynamical modes satisfy $|\lambda_i| \geqslant \varrho$. No generality is lost in doing so, since if modes with $|\lambda_i| < \varrho$ exist then they are ϱ-exponentially stable without requiring control. Thus it is necessary and sufficient to consider only those modes which are not.

The first step is to find a lower bound for the expectation in (13) which is independent of the coder-controller and possesses a clearer dependence on the dynamics and data rate. One such bound can be obtained using the following result:

Lemma 2. *Let the distribution of a random variable* $\mathbf{X} \in \mathbb{R}^n$ *be absolutely continuous with respect to Lebesgue measure, with density* $p_\mathbf{X}$, *and have rth absolute moment* $\mathrm{E}\|\mathbf{X}\|^r < \infty$.

If $c_v : \mathbb{R}^n \to \mathbb{R}^n$ *is a quantiser with not greater than v distinct points, then* $\forall \theta \in (n/[n+r], 1)$, $v \in \mathbb{N}$,

$$\mathrm{E}\|\mathbf{X} - c_v(\mathbf{X})\|^r \geqslant \beta v^{-r/n} \|p_\mathbf{X}\|_\theta^{r\theta/[n(1-\theta)]}, \tag{14}$$

where β *is a parameter determined only by* r, n *and* θ.

Proof. See Appendix A.

The quantity $\|p_\mathbf{X}\|_\theta^{2\theta/[n(1-\theta)]}$ is known in the information theory literature as the *Rényi differential entropy power of order* θ (see e.g. Cover & Thomas, 1991) and is a measure of the radius squared of the effective support volume of the density. The inequality above then simply states that the MrPE is always bounded below by the effective support volume of $p_\mathbf{X}$ divided into v disjoint partitions, with the ratio raised to the r/nth power so that dimensions agree and with a density-independent constant in front. Note that by absolute continuity, $\|p_\mathbf{X}\|_\theta > 0$, $\forall \theta \in (0, 1)$, otherwise the lower bound is the trivial one of 0. What this indicates is that a faster rate of decrease than $v^{-r/n}$ is possible when quantising singular distributions, which agrees with limit results in (Bucklew & Wise, 1982; Graf & Luschgy, 2000).

Asymptotic lower bounds with a $v^{-r/n}$ decay are well-known but crucially, the inequality above is valid for finite v with a density-independent coefficient β. Applying it to the LHS of (13) with $\mathbf{X} = (\varrho^{-1}\mathbf{J})^k \acute{\mathbf{X}}_0$, $v = \mu^{k-1}$ and

$c_v(\mathbf{X}) = \varrho^{-k} q_{k-1}(\acute{\mathbf{X}}_0)$, it follows that

$$\mathrm{E}\|(\varrho^{-1}\mathbf{J})^k \acute{\mathbf{X}}_0 - \varrho^{-k} q_{k-1}(\acute{\mathbf{X}}_0)\|^r \geqslant \frac{\beta}{\mu^{(k-1)r/n}} \left(\int p_{(\varrho^{-1}\mathbf{J})^k \acute{\mathbf{X}}_0}(\mathbf{x})^\theta \mathrm{d}\lambda(\mathbf{x}) \right)^{r/[n(1-\theta)]}. \quad (15)$$

Noting that $\mathbf{J}$ is invertible, change the integration variable in (15) to $\mathbf{y} = (\varrho \mathbf{J}^{-1})^k \mathbf{x}$. Observe that

$$\mathrm{d}\lambda(\mathbf{x}) = |\det(\varrho^{-1}\mathbf{J})|^k \mathrm{d}\lambda(\mathbf{y}),$$

$$p_{(\varrho^{-1}\mathbf{J})^k \acute{\mathbf{X}}_0}(\mathbf{x}) = p_{(\varrho^{-1}\mathbf{J})^k \acute{\mathbf{X}}_0}((\varrho^{-1}\mathbf{J})^k \mathbf{y}) = |\det(\varrho^{-1}\mathbf{J})^k|^{-1} p_{\acute{\mathbf{X}}_0}(\mathbf{y}),$$

$$|\det(\varrho^{-1}\mathbf{J})| = \varrho^{-n} \prod_i |\det \mathbf{J}_i| = \varrho^{-n} \prod_i |\lambda_i|^{n_i},$$

where the last line follows from the fact that $\mathbf{J} = \mathrm{diag}(\mathbf{J}_1, \dots, \mathbf{J}_v) \in \mathbb{R}^{n \times n}$. Substituting into (15),

$$\mathrm{E}\|(\varrho^{-1}\mathbf{J})^k \acute{\mathbf{X}}_0 - \varrho^{-k} q_{k-1}(\acute{\mathbf{X}}_0)\|^r \geqslant \frac{\beta |\det(\varrho^{-1}\mathbf{J})|^{kr/n}}{\mu^{(k-1)r/n}} \left(\int p_{\acute{\mathbf{X}}_0}(\mathbf{y})^\theta \mathrm{d}\lambda(\mathbf{y}) \right)^{r/[n(1-\theta)]}$$

$$= \frac{\beta}{\mu^{(k-1)r/n}} \left(\varrho^{-n} \prod_i |\lambda_i|^{n_i} \right)^{kr/n} \| p_{\acute{\mathbf{X}}_0} \|_\theta^{r\theta/[n(1-\theta)]}.$$

If ϱ-exponential stability has been achieved, then (13) implies that the RHS $\to 0$ as $k \to \infty$. As $\|p_{\acute{\mathbf{X}}_0}\|_\theta > 0$, it is necessary that

$$2^R \equiv \mu > \frac{\prod_i |\lambda_i|^{n_i}}{\varrho^n} = \prod_i \left| \frac{\lambda_i}{\varrho} \right|^{n_i}$$

$$\Leftrightarrow R > \sum_i n_i \log_2 \left| \frac{\lambda_i}{\varrho} \right| = \sum_{j : |\eta_j| \geqslant \varrho} \log_2 \left| \frac{\eta_j}{\varrho} \right|,$$

where the second sum includes conjugate and repeated eigenvalues. This proves the necessity of (5).

5. Sufficiency

The final step is to establish the sufficiency of (5), by constructing and analysing a specific coding and control scheme. Though this scheme is not a candidate for use in practice, it demonstrates that exponential stabilisability is preserved for data rates satisfying (5). As in the previous section it is assumed w.l.o.g. that all subsystems (10) satisfy $|\lambda_i| \geqslant \varrho$.

The basic idea is to recursively quantise the initial state. By means of a time-sharing protocol, each scalar component of the ith subsystem is allocated an effective data rate roughly proportional to $\log_2|\lambda_i/\varrho|$, which is used to quantise and transmit it with increasing accuracy. At the other end of the channel, the progressively more accurate initial state estimates are recovered and converted into control signals. This scheme is formally defined below.

Coder 1. Divide times $k \geqslant n$ into *epochs* $[n+j\tau, \dots, n+j\tau+\tau-1]$, $j \in \mathbb{Z}_+$, of uniform integer duration $\tau \geqslant n$ and subdivide each epoch into v *subepochs* of duration $n_i \tau_i$, where

$$\tau_i \triangleq \lfloor \tau R^{-1} \log_2 |\lambda_i/\varrho| \rfloor + 1, \quad (16)$$

with $\lfloor \cdot \rfloor$ denoting rounding down. Define, $\forall x \in \mathbb{R}$,

$$\alpha(x) \triangleq \frac{\xi}{(1+|x|)^{1+\frac{\varepsilon}{r}}},$$

$$c(x) \triangleq \int_{-\infty}^{x} \alpha(y) \mathrm{d}\lambda(y) \in (0,1), \quad (17)$$

where $\xi > 0$ is a normalisation factor such that α integrates to unity. At times $k \in [0, \dots, n-1]$, record the measurements of the system (1). At time $k = n$, calculate the (transformed) initial state $\acute{\mathbf{x}}_0$ by solving

$$\mathbf{W}_\mathrm{o} \mathbf{T}^{-1} \acute{\mathbf{x}}_0 = [\mathbf{y}_0^\mathrm{T} \quad \mathbf{y}_1^\mathrm{T} \quad \cdots \quad \mathbf{y}_{n-1}^\mathrm{T}]^\mathrm{T},$$

where the observability matrix $\mathbf{W}_\mathrm{o} \triangleq [\mathbf{C}^\mathrm{T} \ (\mathbf{CA})^\mathrm{T} \ \cdots \ (\mathbf{CA}^{n-1})^\mathrm{T}]^\mathrm{T}$ has rank n. Indicating each scalar component of $\acute{\mathbf{x}}_0^{(i)}$ with an additional superscript $h \in [1, \dots, n_i]$, apply the *compressor* c component-wise and expand in base-μ,

$$c(\acute{x}_0^{(i,h)}) \equiv \sum_{l=1}^{\infty} \frac{z_l^{(i,h)}}{\mu^l}, \quad (18)$$

where $z_l^{(i,h)} \in \mathbb{Z}_\mu$. During the ith sub-epoch of the $(j+1)$th epoch, transmit the $(j+1)$th block of τ_i successive digits in this expansion, $\forall h \in [1, \dots, n_i]$.

Controller 1. Set $\mathbf{u}_0, \dots, \mathbf{u}_{n+\tau-1} = 0$. At time $k = n + j\tau$, $j \in \mathbb{N}$, the symbol sequence $\tilde{s}_{n+j\tau-1}$ has been received, comprising the first $j\tau_i$ digits $z_1^{(i,h)}, \dots z_{j\tau_i}^{(i,h)}$ in the base-μ expansion of each compressed scalar component $c(\acute{x}_0^{(i,h)})$. Estimate each transformed initial state component $\acute{x}_0^{(i,h)}$ by

$$e_{n+j\tau}^{(i,h)} \triangleq c^{-1} \left(\frac{1}{2\mu^{j\tau_i}} + \sum_{l=1}^{j\tau_i} \frac{z_l^{(i,h)}}{\mu^l} \right). \quad (19)$$

Then calculate control signals $\mathbf{u}_{n+j\tau}, \dots, \mathbf{u}_{2n+j\tau-1}$ by using the reachability of $(\mathbf{J}, \mathbf{TB})$ to solve

$$\sum_{k=n+j\tau}^{2n+j\tau-1} \mathbf{J}^{2n+j\tau-1-k} \mathbf{TB} \mathbf{u}_k \equiv \mathbf{W} \mathbf{v}_j = \mathbf{J}^{2n+j\tau}(\mathbf{e}_{n+(j-1)\tau} - \mathbf{e}_{n+j\tau}), \quad \forall j \in \mathbb{N}, \quad (20)$$

where $\mathbf{W} \triangleq [\mathbf{TB} \ \mathbf{JTB} \ \cdots \ \mathbf{J}^{n-1}\mathbf{TB}] \in \mathbb{R}^{n \times nm}$, $\mathbf{v}_j \triangleq [\mathbf{u}_{2n+j\tau-1}^\mathrm{T} \ \mathbf{u}_{2n+j\tau-2}^\mathrm{T} \ \cdots \ \mathbf{u}_{n+j\tau}^\mathrm{T}]^\mathrm{T} \in \mathbb{R}^{nm}$ and $\mathbf{e}_n \triangleq 0$. Set the remaining control signals $\mathbf{u}_{2n+j\tau}, \dots, \mathbf{u}_{n+(j+1)\tau-1}$ in the epoch to $\mathbf{0}$.

The function c is called a *compressor* in the quantisation literature and α the *asymptotic quantiser point density*. From (17), the number of quantiser points per unit length of x drops off according to a power law. In fact, the precise form of α is immaterial. It suffices to use any quantiser point density with tails that decay rapidly enough that quantiser points are not wasted on the low-probability regions but not so rapidly that these regions dominate the overall quantiser error.

Coder-Controller 1 is evidently an open-loop scheme, since the symbols to be transmitted are computed in advance as explicit functions of the initial state. In this form it is easy to apply results from asymptotic quantisation and quickly establish the sufficiency of (5), but from a practical point of view a feedback formulation is naturally more desirable. One possible structure is to first pass the measurements through a linear state observer prior to the coder. Each symbol is then calculated as a function of the current observer output and an internal, finite-dimensional *coder state*, which encapsulates the effect of past symbols and is itself updated according to the next τ symbols transmitted. The controller, which has a local version of the coder state, updates its version according to the latest batch of symbols received and generates a corresponding control signal. Such a scheme is proposed for stochastic linear systems in (Nair and Evans, 2002) and can be readily modified for noiseless systems. As the proof relies on rather different techniques, it is not expanded on here.

Upper bounds on the rth state absolute moments generated by this coder-controller will now be derived under the assumption that (5) is satisfied. This will first be done at times $k \equiv j\tau + n$ corresponding to epoch beginnings and then be extended to times within epochs. First though, it needs to be verified that the sub-epoch durations $n_i\tau_i$ can be selected so that their sum does not exceed the epoch duration τ, for otherwise the time-sharing protocol above is infeasible. Observe that

$$\sum_i n_i\tau_i \leqslant \sum_i n_i\left(\tau\frac{\log_2|\lambda_i/\varrho|}{R}+1\right) = \tau\frac{\sum_i n_i\log_2|\lambda_i/\varrho|}{R}+\sum_i n_i = \tau\frac{\sum_{|\eta_j|>\varrho}\log_2|\eta_j/\varrho|}{R}+n, \quad \forall\tau\in\mathbb{N},$$

where the final sum includes repeated and conjugate eigenvalues. As the coefficient of τ is <1 by the hypothesis (5), it follows that the LHS $\leqslant$ any sufficiently large epoch length τ.

Next, observe that by (11) and (20), $\forall j\in\mathbb{N}$,

$$\acute{\mathbf{x}}_{n+j\tau} = \mathbf{J}^{n+j\tau}\acute{\mathbf{x}}_0 + \sum_{k=n+\tau}^{n+j\tau-1}\mathbf{J}^{n+j\tau-1-k}\mathbf{TBu}_k$$
$$= \mathbf{J}^{n+j\tau}\acute{\mathbf{x}}_0 + \sum_{l=1}^{j-1}\mathbf{J}^{(j-l)\tau-n}\sum_{k=n+l\tau}^{2n+l\tau-1}\mathbf{J}^{2n+l\tau-1-k}\mathbf{TBu}_k$$
$$= \mathbf{J}^{n+j\tau}\acute{\mathbf{x}}_0 + \sum_{l=1}^{j-1}\mathbf{J}^{(j-l)\tau-n}\mathbf{J}^{2n+l\tau}(\mathbf{e}_{n+(l-1)\tau}-\mathbf{e}_{n+l\tau})$$
$$= \mathbf{J}^{n+j\tau}\acute{\mathbf{x}}_0 - \mathbf{J}^{n+j\tau}\mathbf{e}_{n+(j-1)\tau}. \quad (21)$$

$$\Rightarrow \|\acute{\mathbf{x}}_{n+j\tau}\|^r = \left(\sum_{i=1}^{v}\|\mathbf{J}_i^{n+j\tau}(\acute{\mathbf{x}}_0^{(i)}-\mathbf{e}_{n+(j-1)\tau}^{(i)})\|^2\right)^{r/2}$$
$$\leqslant v^{r/2}\sum_i\|\mathbf{J}_i^{n+j\tau}\|^r\|\acute{\mathbf{x}}_0^{(i)}-\mathbf{e}_{n+(j-1)\tau}^{(i)}\|^r$$
$$\leqslant v^{r/2}\sum_i\|\mathbf{J}_i^{n+j\tau}\|^r n_i^{r/2}\sum_{h=1}^{n_i}|\acute{x}_0^{(i,h)}-e_{n+(j-1)\tau}^{(i,h)}|^r, \quad (22)$$

using the diagonal block structure (6) of the real Jordan form $\mathbf{J}$.

Now, by (18) and (19), the sequence of base-μ digits $\tilde{s}_{n+(j-1)\tau-1}$ received by the controller at time $n+(j-1)\tau$ identify the unique interval

$$\left[\frac{z}{\mu^{(j-1)\tau_i}}, \frac{z+1}{\mu^{(j-1)\tau_i}}\right),$$

where $z \triangleq \sum_{l=1}^{(j-1)\tau_i} z_l^{(i,h)}\mu^{(j-1)\tau_i-l}\in\mathbb{Z}_{\mu^{(j-1)\tau_i}}$, which contains each component $c(\acute{x}_0^{(i,h)})$ of the transformed and compressed initial state, From the initial state estimate equation (19), the transformed initial state component estimate $e_{n+(j-1)\tau}^{(i,h)}$ is then simply c^{-1} applied to the midpoint of this interval and can be represented as

$$e_{n+(j-1)\tau}^{(i,h)} \equiv c^{-1}Q^{\mathrm{u}}_{\mu^{(j-1)\tau_i}}c(\acute{x}_0^{(i,h)}), \quad (23)$$

where $Q_v^{\mathrm{u}}:[0,1]\to\{\frac{1}{2v},\frac{3}{2v},\dots,\frac{2v-1}{2v}\}$ is the uniform, v-level, mid-point quantiser on the unit interval.

The composite function on the RHS above describes a so-called *companding* quantiser for a scalar random variable. A fairly general result concerning the asymptotic properties of such quantisers was established in Linder (1991) and, for convenience, is restated below:

Lemma 3 (Linder, 1991). *Let $X\in\mathbb{R}$ be a random variable with absolutely continuous distribution and let $c:\mathbb{R}\to(0,1)$ be a function with derivative α. If*

(1) *α is continuous and positive,*
(2) *$\alpha(x)$ decreases monotonically for $|x|$ sufficiently large,*
(3) *$\mathrm{E}\{\alpha(X)^{-r}\}<\infty$ for some $r>0$,*
(4) *$\int_{x<-t}[\alpha c^{-1}(0.5c(x))]^{-r}p_X(x)\mathrm{d}\lambda(x)<\infty$ and*
(5) *$\int_{x>t}[\alpha c^{-1}(0.5[1+c(x)])]^{-r}p_X(x)\mathrm{d}\lambda(x)<\infty$, for some $t>0$,*

then as $v\to\infty$,

$$v^r\mathrm{E}|X-c^{-1}Q_v^{\mathrm{u}}c(X)|^r\to(r+1)2^{-r}\mathrm{E}\{\alpha(X)^{-r}\}.$$

The requirement that α be positive is equivalent to the differentiable compressor c being invertible. The continuity of the former is assumed for technical reasons and can readily be relaxed to almost-everywhere continuity. Recalling that α is also the asymptotic proportion per unit length of the points of $c^{-1}Q_v^{\mathrm{u}}c$, condition (2) ensures that quantiser points are not wasted on large values of x that occur rarely. On the other hand, the remaining conditions guarantee that the point density does not decay so rapidly with large x that the tails of the distribution end up dominating the quantisation error. [2]

Setting $X = \acute{X}_0^{(i,h)}$ and $v = \mu^{(j-1)\tau_i}$, conditions (1) and (2) are obviously satisfied and condition (3) is ensured by requirement that $\mathrm{E}\|\mathbf{X}_0\|^{r+\varepsilon} < \infty$. Looking at next condition (5), observe that c can be calculated explicitly as

$$c(x) = 1 - \frac{1}{2(1+x)^{\varepsilon/r}} \geqslant 0.5, \quad \forall x \geqslant 0.$$

$$\Rightarrow c^{-1}(y) = \left(\frac{0.5}{1-y}\right)^{r/\varepsilon} - 1 \geqslant 0, \quad \forall y \geqslant 0.5.$$

Hence $\forall x \geqslant 0$,

$$\alpha(c^{-1}(0.5[c(x)+1]))^{-r} = \left(\frac{2^{r/\varepsilon+2}r}{\varepsilon}\right)^r (1+x)^{r+\varepsilon}.$$

It is then evident that condition (5) is also enforced by the boundedness of $\mathrm{E}\|\mathbf{X}_0\|^{r+\varepsilon}$. A similar calculation applies to condition (4).

Consequently, $\forall w > 0$, $h \in [1,\dots,n_i]$ and $i \in [1,\dots,v]$, $\exists j_*^{(i,h)} \in \mathbb{N}$ such that $\forall j \geqslant j_*^{(i,h)}$,

$$\mathrm{E}|\acute{X}_0^{(i,h)} - E_{n+(j-1)\tau}^{(i,h)}|^r \leqslant \frac{(1+w)\mathrm{E}\{\alpha(\acute{X}_0^{(i,h)})^{-r}\}}{\mu^{(j-1)\tau_i r}(r+1)2^r}.$$

Furthermore, as $\mathbf{J}_i$ is *similar* to the block diagonal matrix consisting of all Jordan blocks associated with λ_i, a trivial adaptation of a result in Horn and Johnson (1985, p. 138) states that $\exists \phi > 0$ such that

$$\|\mathbf{J}_i^k\| \leqslant \phi k^{n_i-1}|\lambda_i|^k, \quad \forall i \in [1,\dots,v], \;\; k \in \mathbb{Z}_+. \tag{24}$$

Taking the expectation of (22), dividing by $\varrho^{(n+j\tau)r}$ and applying the two inequalities above,

$$\begin{aligned}\frac{\mathrm{E}\|\acute{\mathbf{X}}_{n+j\tau}\|^r}{\varrho^{(n+j\tau)r}} &\leqslant \frac{v^{r/2}\phi^r(1+w)}{(r+1)2^r}\sum_i n_i^{r/2}\left[\frac{(n+j\tau)^{n_i-1}|\lambda_i|^{n+j\tau}}{\varrho^{n+j\tau}\mu^{(j-1)\tau_i}}\right]^r \\ &\quad \times \sum_h \mathrm{E}\{\alpha(\acute{X}_0^{(i,h)})^{-r}\}, \quad \forall j \geqslant j_* \triangleq \max_{i,h} j_*^{(i,h)}.\end{aligned} \tag{25}$$

Looking at the RHS, observe that by (16), $\tau_i > \tau R^{-1}\log_2|\lambda_i/\varrho|$ with strict inequality. This is equivalent to $\varrho^\tau\mu^{\tau_i} > |\lambda_i|^\tau$, which implies that as $j \to \infty$,

$$\begin{aligned}&\frac{(n+j\tau)^{n_i-1}|\lambda_i|^{n+j\tau}}{\varrho^{n+j\tau_i}\mu^{(j-1)\tau_i}} \\ &\quad = \frac{|\lambda_i|^n\mu^{\tau_i}}{\varrho^n}(n+j\tau)^{n_i-1}\left(\frac{|\lambda_i|^\tau}{\varrho^\tau\mu^{\tau_i}}\right)^j \to 0.\end{aligned}$$

Hence each summand on the RHS of (25) approaches zero. As the number of summands is finite, the LHS of (25) also $\to 0$ as the epoch number $j \to \infty$.

The only task remaining is to show that $\varrho^{-kr}\mathrm{E}\|\acute{\mathbf{X}}_k\|^r \to 0$ as the *integer* time $k \to \infty$. First note that any integer $k \geqslant \tau + n$ is uniquely representable as $k = j\tau + n + t$ with $j = \lfloor (k-n)/\tau \rfloor \in \mathbb{N}$ and for some $t \in \mathbb{Z}_\tau$. Iterating the dynamic equation (9),

$$\begin{aligned}\acute{\mathbf{x}}_k &= \mathbf{J}^t\acute{\mathbf{x}}_{j\tau+n} + \sum_{l=0}^{t-1}\mathbf{J}^{t-1-l}\mathbf{TBu}_{j\tau+n+l} \\ &\equiv \mathbf{J}^t\acute{\mathbf{x}}_{j\tau+n} + \mathbf{W}_t\mathbf{v}_j,\end{aligned}$$

where $\mathbf{v}_j \triangleq [\mathbf{u}_{2n+j\tau-1}^{\mathrm{T}} \dots \mathbf{u}_{n+j\tau}^{\mathrm{T}}]^{\mathrm{T}}$

$$\begin{aligned}\Rightarrow \frac{\|\acute{\mathbf{x}}_k\|^r}{\varrho^{kr}} &\leqslant \frac{(\|\mathbf{J}^t\acute{\mathbf{x}}_{j\tau+n}\| + \|\mathbf{W}_t\mathbf{v}_j\|)^r}{\varrho^{(n+j\tau+t)r}} \\ &\leqslant 2^r\max_{t\in\mathbb{Z}_\tau}\|\mathbf{J}^t\|^r\frac{\|\acute{\mathbf{x}}_{j\tau+n}\|^r}{\varrho^{(n+j\tau)r}} \\ &\quad + 2^r\max_{t\in\mathbb{Z}_\tau}\|\mathbf{W}_t\|^r\frac{\|\mathbf{v}_j\|^r}{\varrho^{(n+j\tau)r}}.\end{aligned} \tag{26}$$

Now, as the controllability matrix $\mathbf{W}$ has rank n, it possesses n linearly independent columns $\in \mathbb{R}^n$ and only the corresponding n scalar components of the stacked control signal vector $\mathbf{v}_j$ are needed. If the inverse of the matrix formed by these columns is padded with $nm - n$ null rows (corresponding to the unnecessary components of $\mathbf{v}_j$) to form $\mathbf{W}_* \in \mathbb{R}^{nm\times n}$, then the control vector may be expressed explicitly as

$$\begin{aligned}\mathbf{v}_j &= \mathbf{W}_*\mathbf{J}^{2n+j\tau}(\mathbf{e}_{n+(j-1)\tau} - \mathbf{e}_{n+j\tau}) \\ &= \mathbf{W}_*(\mathbf{J}^{2n+j\tau}(\mathbf{e}_{n+(j-1)\tau} - \acute{\mathbf{x}}_0) \\ &\quad + \mathbf{J}^{2n+j\tau}(\acute{\mathbf{x}}_0 - \mathbf{e}_{n+j\tau})) \\ &= \mathbf{W}_*(-\mathbf{J}^n\acute{\mathbf{x}}_{n+j\tau} + \mathbf{J}^{n-\tau}\acute{\mathbf{x}}_{n+(j+1)\tau}),\end{aligned}$$

using (21) and noting that $\mathbf{J}^{n-\tau}$ exists since $\mathbf{J}$ is invertible. As each term on the RHS is ϱ-exponentially stable in rth absolute moment, the triangle inequality guarantees that the LHS is too. Thus the RHS of (26) $\to 0$ as the integer time k and epoch number $j = \lfloor (k-n)/\tau \rfloor \to \infty$, and Coder-Controller 1 achieves exponential stability in the sense of (4) for any data rate R satisfying (5). This completes the proof of Theorem 1. □

[2] This result had been presented in Bucklew and Wise (1982) with (3)–(5) replaced by the weaker assumption $\mathrm{E}\{\alpha(X)^{-r-\delta}\} < \infty$ for some $r, \delta > 0$. However, it is asserted in Linder (1991) that this is not sufficient.

6. Conclusion

In this paper the problem of exponentially stabilising a noiseless, finite-dimensional, linear time-invariant system under a feedback data rate constraint was posed. Particular attention was given to the question of determining the smallest data rate above which stability with a specified exponential decay can be achieved, when no restrictions apart from causality are placed on the coder and controller. By casting the problem as one of moment stabilisation, it was shown to be equivalent to finding a recursive quantiser for the initial state which yielded exponentially diminishing error moments. Quantisation theory was then used to derive the infimum data rate in terms of the decay constant and open-loop eigenvalues. Analogous results for jump Markov and stochastic linear systems are reported in Nair, Dey, and Evans (2002), Nair and Evans (2002) and extensions of these techniques to non-linear plants and systems connected by multiple digital links are being investigated. In this context, an interesting problem that remains open is the precise characterisation of the set of all stabilising data rate combinations for a given networked dynamical system.

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Appendix A. Proof of Lemma 2 in Section 4

Let $\mathscr{V}_1,\ldots,\mathscr{V}_v \subset \mathbb{R}^n$ and $t_1,\ldots,t_v \in \mathbb{R}^n$ be the quantisation regions and corresponding (not necessarily distinct) quantiser points for c_v. The first step is to find a lower bound for the conditional MrPE $\mathrm{E}_{\mathbf{X}\in\mathscr{V}_i}\|\mathbf{X}-t_i\|^r =: l_i^r$. For conciseness, let $\pi_i \triangleq p_{\mathbf{X}|\mathbf{X}\in\mathscr{V}_i}$ and assume that all integrals are over $\mathbb{R}^n$ unless indicated otherwise. By Hölder's inequality,

$$\int \pi_i(\mathbf{x})^\theta \mathrm{d}\lambda(\mathbf{x}) = \int \frac{(\|\mathbf{x}-t_i\|^r + l_i^r)^\theta}{(\|\mathbf{x}-t_i\|^r + l_i^r)^\theta}\pi_i(\mathbf{x})^\theta \mathrm{d}\lambda(\mathbf{x})$$
$$\leqslant \left(\int (\|\mathbf{x}-t_i\|^r + l_i^r)\pi_i(\mathbf{x})\mathrm{d}\lambda(\mathbf{x})\right)^\theta$$
$$\times\left(\int \frac{\mathrm{d}\lambda(\mathbf{x})}{(\|\mathbf{x}-t_i\|^r + l_i^r)^{\theta/(1-\theta)}}\right)^{1-\theta}$$
$$= (2l_i^r)^\theta\left(\int \frac{\mathrm{d}\lambda(\mathbf{x})}{(\|\mathbf{x}-t_i\|^r + l_i^r)^{\theta/(1-\theta)}}\right)^{1-\theta}$$
$$= (2l_i^r)^\theta\left(\int \frac{\mathrm{d}\lambda(\mathbf{x})}{(\|\mathbf{x}\|^r + l_i^r)^{\theta/(1-\theta)}}\right)^{1-\theta},$$

by virtue of the invariance of Lebesgue measure to translations. Now change integration variables to $\mathbf{w} = l_i^{-1}\mathbf{x}$. As $\mathbf{x}\in\mathbb{R}^n$, it then follows that $\mathrm{d}\lambda(\mathbf{w}) = l_i^{-n}\mathrm{d}\lambda(\mathbf{x})$. Hence

$$\int \pi_i(\mathbf{x})^\theta \mathrm{d}\lambda(\mathbf{x}) \leqslant (2l_i^r)^\theta\left(\int \frac{l_i^n \mathrm{d}\lambda(\mathbf{w})}{(\|l_i\mathbf{w}\|^r + l_i^r)^{\frac{\theta}{1-\theta}}}\right)^{1-\theta}$$
$$= 2^\theta l_i^{n(1-\theta)}\left(\int \frac{\mathrm{d}\lambda(\mathbf{w})}{(\|\mathbf{w}\|^r + 1)^{\frac{\theta}{1-\theta}}}\right)^{1-\theta}.$$

Observe that the integrand on the RHS is a function only of the radial distance $\|\mathbf{w}\| =: z$. As the n-dimensional Lebesgue measure or volume of a spherical shell of radius z and infinitesimal thickness $\mathrm{d}\lambda(z)$ is $\kappa z^{n-1}\mathrm{d}\lambda(z)$, where κ depends only on d, this inequality can be rewritten

$$\int \pi_i(\mathbf{x})^\theta \mathrm{d}\lambda(\mathbf{x}) \leqslant 2^\theta l_i^{n(1-\theta)}\left(\int_{z\geqslant 0} \frac{\kappa z^{n-1}\mathrm{d}\lambda(z)}{(z^r + 1)^{\frac{\theta}{1-\theta}}}\right)^{1-\theta}.$$

The integral on the RHS is non-zero and finite when $r\theta/(1-\theta) > n$ i.e. $\theta > n/(n+r)$. Reversing the inequality above and taking θth roots, it then follows that

$$l_i^{n(1-\theta)/\theta} \geqslant \phi\left(\int \pi_i(\mathbf{x})^\theta \mathrm{d}\lambda(\mathbf{x})\right)^{1/\theta}$$
$$= \phi \mathrm{P}\{\mathbf{X}\in\mathscr{V}_i\}^{-1}\left(\int_{\mathscr{V}_i} p_{\mathbf{X}}(\mathbf{x})^\theta \mathrm{d}\lambda(\mathbf{x})\right)^{1/\theta}$$

where ϕ depends only on θ, n and r. Hence

$$\sum_{i=1}^{v} l_i^{n(1-\theta)/\theta}\mathrm{P}\{\mathbf{X}\in\mathscr{V}_i\} \geqslant \phi\sum_{i=1}^{v}\left(\int_{\mathscr{V}_i} p_{\mathbf{X}}(\mathbf{x})^\theta \mathrm{d}\lambda(\mathbf{x})\right)^{1/\theta}$$
$$\equiv \phi\sum_{i=1}^{v} a_i^{1/\theta},$$

where $a_i \triangleq \int_{\mathscr{V}_i} p_{\mathbf{X}}(\mathbf{x})^\theta \mathrm{d}\lambda(\mathbf{x})$. Observe that as $\theta < 1$ the RHS is a convex function of $a_1,\ldots,a_v$, with the constraints $a_i \geqslant 0$ and $\sum_{i=1}^{v} a_i = \int p_{\mathbf{X}}(\mathbf{x})^\theta \mathrm{d}\lambda(\mathbf{x}) = \|p_{\mathbf{X}}\|_\theta^\theta$. Hence

$$\sum_{i=1}^{v} l_i^{n(1-\theta)/\theta}\mathrm{P}\{\mathbf{X}\in\mathscr{V}_i\}$$
$$\geqslant \phi v\sum_{i=1}^{v}\frac{a_i^{1/\theta}}{v} \geqslant \phi v\left(\sum_{i=1}^{v}\frac{a_i}{v}\right)^{1/\theta} = \frac{\phi\|p_{\mathbf{X}}\|_\theta}{v^{1/\theta-1}}. \qquad \text{(A.1)}$$

However by the definition of l_i and the convexity of $(\cdot)^{r\theta/[n(1-\theta)]}$ (since the exponent > 1),

$$\mathrm{E}\|\mathbf{X}-c_v(\mathbf{X})\|^r = \mathrm{E}\{\mathrm{E}_{\mathbf{X}\in\mathscr{V}_i}\|\mathbf{X}-t_i\|^r\}$$
$$= \sum_{i=1}^{v} l_i^r \mathrm{P}\{\mathbf{X}\in\mathscr{V}_i\}$$
$$= \sum_{i=1}^{v}(l_i^{\frac{n(1-\theta)}{r}})^{\frac{r\theta}{n(1-\theta)}}\mathrm{P}\{\mathbf{X}\in\mathscr{V}_i\}$$
$$\geqslant \left(\sum_{i=1}^{v} l_i^{n(1-\theta)/\theta}\mathrm{P}\{\mathbf{X}\in\mathscr{V}_i\}\right)^{r\theta/[n(1-\theta)]}.$$

Substituting the inequality (A.1) into the RHS,

$$\mathrm{E}\|\mathbf{X} - c_v(\mathbf{X})\|^r \geqslant \left(\frac{\phi\|p_{\mathbf{X}}\|_\theta}{v^{1/\theta-1}}\right)^{r\theta/[n(1-\theta)]}$$

$$= \frac{\phi^{r\theta/[n(1-\theta)]}}{v^{r/n}}\,\|p_{\mathbf{X}}\|_\theta^{r\theta/[n(1-\theta)]}.$$

This completes the proof. □

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