Communication-Limited Stabilisability of Jump Markov Linear Systems

Girish N. Nair, Subhrakanti Dey and Robin J. Evans [1]
Department of Electrical and Electronic Engineering
University of Melbourne
VIC 3010, Australia

Abstract

This paper investigates the control of fully observed, scalar jump Markov linear systems in which feedback is transmitted at finite data rates over noiseless digital channels. In particular, the objective is to find the infimum data rate, over all causal coding and control laws, at which such a system may be asymptotically stabilised in mth absolute output moment. The control problem is first shown to be equivalent to quantisation of the initial output under a mean-mth-power-error criterion. Quantisation methods are then applied to derive the smallest stabilising data rate in terms of the *Perron-Frobenius* eigenvalue of a matrix defined by the transition probabilities, the dynamic parameters and the power m.

1 Introduction

The question of how to control dynamical systems that have feedback communication constraints has recently begun to receive much attention. This may be traced to the fact that in many cutting edge applications, such as mobile telephone power control, micro-electromechanical systems and distributed tracking, communication capacity is expensive or limited by physical constraints. In such situations it important to understand how the control objectives are affected by the limited communication resources, particularly if the system dynamics are unstable.

In this paper, we focus on the basic set-up in which output data from a single dynamical system are transmitted to a controller over a noiseless digital channel. In recent years significant progress has been made with regard to this set-up, particularly for deterministic linear systems. Beginning with [4] and continuing with [15, 1, 2, 6], a number of different coding and control schemes have been proposed and analysed. A common feature of many of these schemes are data rate inequalities that are sufficient for stability, in various senses, but necessary only for one-dimensional states. Necessary *and* sufficient data rate lower bounds for stabilising noiseless *auto-regressive moving average* (ARMA) [12] and multivariable linear systems [14, 13] were subsequently derived. In [14] this was done via a deterministic approach for an unknown initial state with a known bound. In [13] the initial state is governed by some probability density, with possibly infinite support, and the asymptotic quantisation methods of [12] were extended to derive the infimum data rate for exponential stabilisability in mth absolute output moment.

In this paper, the asymptotic quantisation approach is applied to jump Markov linear systems (JMLS), a class of system that is often encountered in telecommunications, manufacturing and defence. As in [12], the objective is to find the infimum data rate permitting asymptotic stabilisability in mth absolute moment. In the next section, the communication channel, coder and controller are formulated and the control objective is shown to be equivalent to recursively quantising the initial output so that a term resembling the *mean mth power error* of quantisation theory approaches zero with time. The main result of this paper, Theorem 2.1, is stated here and expresses the smallest achievable data rate in terms of the *Perron-Frobenius* eigenvalue of a matrix defined by the transition probabilities and dynamic parameters. The remainder of the paper essentially constitutes its proof.

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2 Formulation and Statement of Result

Certain conventions are followed in this paper. Sequences $\{a_j\}_{j=0}^k$ are denoted $\tilde{a}_k$, with $\tilde{a}_{-1}$ the null sequence. The symbol $\|\cdot\|$ denotes a norm on a Euclidean space but $\|f\|_\theta$ denotes the L_θ-norm $\left(\int \|f(x)\|^\theta \mathrm{d}\lambda(x)\right)^{1/\theta}$ of a function f with respect to Lebesgue measure λ on the domain of f. Vectors are written in boldface, matrices in upper-case boldface, random variables in upper-case and their realisations by the corresponding lower-case letters. All random variable are assumed to exist in a common probability space, the probability density of a random variable X is written p_X, $\mathrm{P}\{S = s\}$ denotes the probability of the event $S = s$ and E_s denotes conditional expectation with respect to $S = s$.

Consider the fully observed jump Markov linear system

$$x_{k+1} = a(z_k)x_k + b(z_k)u_k \in \mathbb{R}, \quad \forall k \in \mathbb{W}, \tag{2.1}$$

where $x_k, u_k \in \mathbb{R}$ and $z_k \in [1, 2, \ldots, z]$ are the output, control signal and discrete mode respectively at time k. We assume that $a(i), b(i) \neq 0$, $\forall i \in [1, \ldots, z]$ and that $\{z_k\}_{k\in\mathbb{W}}$ is a realisation of a control-independent Markov chain $\{Z_k\}_{k\in\mathbb{W}}$ such that

$$\mathrm{P}\{Z_{k+1} = i | Z_k = j\} \triangleq t_{ij}, \quad \mathrm{P}\{Z_0 = i\} \triangleq t_i \quad \forall i, j \in [1, \ldots, z], k \in \mathbb{W}.$$

The $z \times z$ matrix with t_{ij} in the ith row and jth column is denoted by $\mathbf{T}$. We further assume that x_0 is a realisation of a random variable X_0 that is independent of $Z_0, Z_1, \ldots$, possesses a density that is non-singular with respect to λ and has a finite $(m + \varepsilon)$th absolute moment for some $m, \varepsilon > 0$.

We now formulate the coder, controller and digital channel, along the same lines as [12]. Suppose the sensor observing the outputs and Markov modes of the plant above is connected to a distant controller by a digital channel that can carry only one symbol s_k from a coding alphabet $\mathbb{Z}_\mu \triangleq [0, 1, \ldots, \mu - 1]$ during each sampling interval. Clearly the discrete-valued modes and continuous-valued outputs must then be encoded into digital symbols prior to transmission. Each of these may generally depend on all past and present outputs and modes. As, for a given coder-controller the output at time k is completely determined by the inital output x_0 and the mode sequence $\tilde{z}_{k-1}$, the coding law may without loss of generality be written

$$s_k = \gamma_k(x_0, \tilde{z}_k) \in \mathbb{Z}_\mu, \quad \forall k \geq 0, \tag{2.2}$$

where γ_k is the coder mapping at time k.

The data rate of the channel is defined as $R \triangleq \log_2 \mu$ bits per sampling interval. Neglecting the propagation delay and transmission errors, the finite data rate implies that each symbol takes one sampling interval to be completely transmitted, so that at time k the controller has received $s_0, \ldots, s_{k-1}$. Under the assumption that the controller is able to observe the Markov modes directly, the control signal it generates will most generally be of the form

$$u_k = \delta_k(\tilde{s}_{k-1}, \tilde{z}_k) \in \mathbb{R}, \quad \forall k \geq 0, \tag{2.3}$$

where δ_k is the controller function at time k.

Define the *coder-controller* as the pair $(\gamma, \delta) \triangleq (\{\gamma_k\}_{k=0}^\infty, \{\delta_k\}_{k=0}^\infty)$. The objective is to construct one that asymptotically stabilises the system in the sense that

$$\mathrm{E}|X_k|^m \to 0 \text{ as } k \to \infty, \tag{2.4}$$

without using a higher data rate than necessary. Note that the expectation is over both the initial output and the Markov modes.

We now transform the control problem into an asymptotic quantisation problem. Expanding (2.1) out and using (2.3), it follows that $\forall k \in \mathbb{N}$,

$$\begin{aligned} x_k &= \left(\prod_{i=0}^{k-1} a(z_i)\right) x_0 + \sum_{j=0}^{k-1} \left(\prod_{i=j+1}^{k-1} a(z_i)\right) b(z_j)\delta_j(\tilde{s}_{j-1}, \tilde{z}_j), \\ &= \left(\prod_{i=0}^{k-1} a(z_i)\right) \left(x_0 + \sum_{j=0}^{k-1} \frac{b(z_j)\delta_j(\tilde{s}_{j-1}, \tilde{z}_j)}{\prod_{i=0}^{j} a(z_i)}\right). \end{aligned}$$

Note that the right-most sum is a function of the symbol and mode sequences $\tilde{s}_{k-2}$ and $\tilde{z}_{k-1}$. As $\tilde{s}_{k-2}$ can only take μ^{k-1} distinct values exhausting $\mathbb{Z}_\mu^{k-1}$, it follows that for each mode sequence $\tilde{z}_{k-1}$, this sum has a maximum of μ^{k-1} distinct values in $\mathbb{R}$. Furthermore, for a given coder γ the symbol sequence is completely determined by x_0 and the mode sequence. Hence we may define

$$q_{k-1}(x_0, \tilde{z}_{k-1}) \triangleq -\sum_{j=0}^{k-1} \frac{b(z_j)\delta_j(\tilde{s}_{j-1}, \tilde{z}_j)}{\prod_{i=0}^{j} a(z_i)}, \qquad \forall k \in \mathbb{W},$$

where $q_{k-1}(\cdot, \tilde{z}_{k-1}) : \mathbb{R} \to \mathbb{R}$ is a function with up to μ^{k-1} distinct levels, i.e. a quantiser. By inverting the expression above, it can be seen that for a given quantiser sequence $\{q_k\}_{k\in\mathbb{W}}$ and coder, the controller mappings are uniquely given by

$$\delta_k(\tilde{s}_{k-1}, \tilde{z}_k) = \frac{1}{b(z_k)} \left(\prod_{i=0}^{k} a(z_i)\right) (q_{k-1}(x_0, \tilde{z}_{k-1}) - q_k(x_0, \tilde{z}_k)) \quad \forall k \in \mathbb{W}. \tag{2.5}$$

In other words, the problem of finding coder and controller mappings that achieve (2.4) is exactly equivalent to that of finding coder and *quantiser* mappings such that

$$\mathrm{E}\left\{\prod_{i=0}^{k-1} |a(Z_i)|^m |X_0 - q_{k-1}(X_0, \tilde{Z}_{k-1})|^m\right\} \to 0 \text{ as } k \to \infty. \tag{2.6}$$

The communication-limited control problem is now in a form that bears a strong resemblance to standard *mean-mth-power-error (MmPE) asymptotic quantisation theory* [7, 3, 8]. What distinguishes it are the facts that the quantisers q_k, $k \in \mathbb{W}$ are recursive and that they and the multiplying product term are dependent on the Markov chain states. Nevertheless, quantisation arguments may still be applied to yield the following result:

Theorem 2.1. *Suppose that the initial output of the jump Markov linear system (2.1) is governed by a probability density that is non-singular with respect to Lebesgue measure on $\mathbb{R}$ and has finite $(m+\varepsilon)$th absolute moment for some $m, \varepsilon > 0$, and that the modes are governed by an irreducible $z \times z$ transition probability matrix $\mathbf{T}$. Then for a given data rate R (bits per interval), a coder-controller that asymptotically stabilises the system in the sense (2.4) exists* if and only if

$$R > m^{-1} \log_2 \varrho(\mathbf{A}^m \mathbf{T}), \tag{2.7}$$

where $\mathbf{A} \triangleq \mathrm{diag}(|a(1)|, \ldots, |a(z)|)$ and $\varrho(\cdot)$ denotes the Perron-Frobenius eigenvalue of the argument.

This result specifies the infimum data rate that suffices to be able to stabilise a JMLS and the remainder of this paper is devoted to its proof. In the next section a new quantisation lower bound will be applied to establish the necessity of (2.7). Its sufficiency is subsequently proven by proposing a specific coder-controller and using asymptotic quantisation theory to analyse its performance.

3 Necessity

The first step towards proving the necessity of (2.7) is to lower-bound $\mathrm{E}|X_k|^m$ for arbitrary k, given an asymptotically stabilising coder-controller. The basic tool used is the following new quantisation inequality:

Lemma 3.1. *Let $\mathbf{X} \in \mathbb{R}^f$ be a random variable with non-singular density $p_{\mathbf{X}}$ and $c_\nu : \mathbb{R}^f \to \mathbb{R}^f$ a quantiser with up to ν distinct points. Then*

$$\mathrm{E}\|\mathbf{X} - c_\nu(\mathbf{X})\|^m \geq \frac{\beta}{\nu^{m/f}} \|p_{\mathbf{X}}\|_r^{mr/[f(1-r)]}, \quad \forall r \in (f/[f+m], 1), \nu \in \mathbb{N}, \tag{3.1}$$

where β is a parameter determined only by m, f and r.

Proof: See Appendix.

This explicitly bounds from below the MmPE that can be achieved with ν quantiser points and states that it cannot decrease faster than $\nu^{-m/f}$ if the L_r norm of the probability density is nonzero (guaranteed if $p_{\mathbf{X}}$ is non-singular with respect to Lebesgue measure). The quantity $\|p_{\mathbf{X}}\|_r^{r/(1-r)}$ is known in the information theory literature as *exponential entropy* or *Renyi differential entropy power* of order r and is a measure of the effective support volume of the probability density. The inequality above then simply states that the MmPE is always bounded below by the effective support volume of $p_{\mathbf{X}}$ divided into ν dijoint partitions, with the ratio raised to the m/fth power so that dimensions agree and with a density-independent constant in front.

Applying this lemma $\forall k \in \mathbb{W}$ to (2.6) with $f = 1$, $p_{\mathbf{X}} = p_{X_0|\tilde{z}_{k-1}}$, $\nu = \mu^{k-1}$ and $c_\nu = q_{k-1}(\cdot, \tilde{z}_{k-1})$,

$$\begin{aligned}
\mathrm{E}|X_k|^m &= \mathrm{E}\left\{\prod_{i=0}^{k-1} |a(Z_i)|^m \mathrm{E}_{\tilde{Z}_{k-1}} |X_0 - q_{k-1}(X_0, \tilde{Z}_{k-1})|^m\right\}, \\
&\geq \mathrm{E}\left\{\prod_{i=0}^{k-1} |a(Z_i)|^m \frac{\beta \|p_{X_0|\tilde{Z}_{k-1}}\|_r^{mr/(1-r)}}{\mu^{m(k-1)}}\right\}, \\
&= \mathrm{E}\left\{\prod_{i=0}^{k-1} |a(Z_i)|^m \frac{\beta \|p_{X_0}\|_r^{mr/(1-r)}}{\mu^{m(k-1)}}\right\} = \mathrm{E}\left\{\prod_{i=0}^{k-1} |a(Z_i)|^m\right\} \frac{\beta \|p_{X_0}\|_r^{mr/(1-r)}}{\mu^{m(k-1)}},
\end{aligned} \tag{3.2}$$

where we have used the fact that the initial output and modes are mutually independent. Now the non-singularity of p_{X_0} with respect to Lebesgue measure implies that $\|p_{X_0}\|_r > 0$, $\forall r \in (0, 1]$. Thus if the closed-loop system is asymptotically stable, i.e. the LHS approaches 0 with time, it is necessary that

$$\frac{1}{\mu^{mk}} \mathrm{E}\left\{\prod_{i=0}^{k-1} |a(Z_i)|^m\right\} \to 0 \quad \text{as } k \to \infty. \tag{3.3}$$

We now simplify this condition using a slight adaptation of the Perron-Frobenius theorem as given in [5]. Recalling that $\mathbf{T}$ is the transition probability matrix and $\mathbf{t}$ the column vector of initial Markov mode probabilities,

$$\begin{aligned}
\mathrm{E}\left\{\prod_{j=0}^{k} |a(Z_j)|^m\right\} &= \sum_{\tilde{z}_k} \prod_{j=0}^{k} |a(z_j)|^m \mathrm{P}\{\tilde{Z}_k = \tilde{z}_k\}, \\
&= \sum_{1 \leq z_0, z_1, \ldots, z_k \leq z} |a(z_0)|^m t_{z_0} \prod_{j=1}^{k} |a(z_j)|^m t_{z_j z_{j-1}}, \\
&= \sum_{1 \leq z_0, z_k \leq z} \mathbf{H}^k_{z_k z_0} h_{z_0} = [1, \ldots, 1]\mathbf{H}^k \mathbf{h}, \quad \forall k \in \mathbb{W},
\end{aligned}$$

where $\mathbf{H} \triangleq \text{diag}(|a(1)|,\ldots,|a(z)|)^m\mathbf{T}$, $\mathbf{H}^k_{ij}$ is the element in the ith row and jth column of $\mathbf{H}^k$, $h_i \triangleq |a(i)|^m t_i$ and $\mathbf{h} \triangleq [h_1,\ldots h_z]^{\mathrm{T}}$. Observe that the irreducibility of $\mathbf{H}$ follows from that of $\mathbf{T}$, since $|a(1)|,\ldots,|a(z)| > 0$. By the Perron-Frobenius theorem, $\mathbf{H}$ then possesses a real eigenvalue $\varrho(\mathbf{H}) > 0$ of maximum magnitude, which furthermore has a corresponding left eigenvector $\mathbf{e}^{\mathrm{T}} = (e_1,\ldots,e_z)$ with strictly positive components. Let e_u be the largest component of $\mathbf{e}$ and e_l the smallest. Then

$$e_u^{-1}\varrho(\mathbf{H})^k\mathbf{e}^{\mathrm{T}}\mathbf{h} = e_u^{-1}\mathbf{e}^{\mathrm{T}}\mathbf{H}^k\mathbf{h} \le [1,\ldots,1]\mathbf{H}^k\mathbf{h} \le e_l^{-1}\mathbf{e}^{\mathrm{T}}\mathbf{H}^k\mathbf{h} = e_l^{-1}\varrho(\mathbf{H})^k\mathbf{e}^{\mathrm{T}}\mathbf{h},\ \forall k \in \mathbb{W}.$$

As the column vector $\mathbf{h}$ is nonnegative with at least one positive element, $\mathbf{e}^{\mathrm{T}}\mathbf{h} > 0$. Dividing the above expression by μ^{mk}, it immediately follows that

$$\frac{1}{\mu^{mk}}\mathrm{E}\left\{\prod_{j=0}^{k}|a(Z_j)|^m\right\} = \frac{[1,\ldots,1]\mathbf{H}^k\mathbf{h}}{\mu^{mk}} \to 0 \ \Leftrightarrow\ \frac{\varrho(\mathbf{H})^k}{\mu^{mk}} \to 0, \quad \text{as } k \to \infty, \tag{3.4}$$

which is equivalent to $\mu^m > \varrho(\mathbf{H})$ or $R > m^{-1}\log_2\varrho(\mathbf{H})$. Thus (2.7) is necessary for any asymptotically mth-moment-stabilising coder-controller.

4 Sufficiency

In order to prove that (2.7) is also sufficient for asymptotic stabilisability, a coder-controller will be explicitly constructed and its convergence properties analysed. We stress that this scheme is not necessarily a practical control design. Its primary purpose is to demonstrate as directly as possible that the data rate lower bound of Theorem 2.1 is achievable.

Coder 1.

$$\alpha(x) \triangleq \xi(1+|x|)^{-1-\varepsilon/m}, \quad c(x) \triangleq \int_{y\le x}\alpha(y)\mathrm{d}\lambda(y) \in (0,1),\ \forall x \in \mathbb{R}, \tag{4.1}$$

where $\xi > 0$ is such that α integrates to unity. Apply c to x_0 and expand in base-μ digits, i.e. $c(x_0) \equiv \sum_{l=0}^{\infty} d_l\mu^{-l-1}$, where $d_l \in \mathbb{Z}_\mu$. At time k, transmit the symbol $s_k = d_k$.

Controller 1. *At time k the controller has received the symbol sequence $\tilde{s}_{k-1}$, comprising the first k digits in the base-μ expansion of $c(x_0)$. It estimates the initial output via*

$$q_{k-1}(x_0,\tilde{z}_{k-1}) = q'_{k-1}(x_0) \triangleq c^{-1}Q^u_{\mu^{k-1}}c(x_0) = c^{-1}\left(\frac{1}{2\mu^k} + \sum_{l=0}^{k-1}\frac{d_l}{\mu^{l+1}}\right), \quad \forall k \in \mathbb{W}, \tag{4.2}$$

where Q^u_ν is the ν-level, uniform midpoint quantiser on $[0,1]$. It then calculates control signals by using (2.5).

In the quantisation literature, the function c is called a *compressor*, its inverse an *expander* and the composition $c^{-1}Q^u_{\mu^{k-1}}c$ a *compander*. In order to analyse the output moment behaviour of the feedback loop with this coder controller, we use the following lemma:

Lemma 4.1 (Linder, [9]). *Let $X \in \mathbb{R}$ be a random variable with probability density p_X and $c : \mathbb{R} \to (0,1)$ a compressor with derivative α. If*

1. *p_X is absolutely continuous with respect to Lebesgue measure λ*
2. *α is continuous and positive*
3. *α decreases monotonically with the magnitude of sufficiently large arguments*

4. $\mathrm{E}\{\alpha(X)^{-m}\} < \infty$ *for some* $m > 0$

5. $\int_{x<-n} \left[\alpha c^{-1}\left(0.5c(x)\right)\right]^{-m} p_X(x)\mathrm{d}\lambda(x)$, $\int_{x>n} \left[\alpha c^{-1}\left(0.5[1-c(x)]\right)\right]^{-m} p_X(x)\mathrm{d}\lambda(x) < \infty$ *for some* $n > 0$

then

$$\nu^m \mathrm{E}|X - c^{-1}Q_\nu^u c(X)|^m \to (m+1)2^{-m}\mathrm{E}\{\alpha(X)^{-m}\}, \quad \textit{as } \nu \to \infty.$$

By the construction of α in this instance and the fact that $\mathrm{E}|X_0|^{m+\varepsilon} < \infty$, all the prerequisites for this result can be shown to hold here with $X = X_0$ and $\nu = \mu^{k-1}$. Hence

$$\mathrm{E}|X_0 - q'_{k-1}(X_0)|^m \sim (m+1)^{-1}2^{-m}\mu^{-m(k-1)}\mathrm{E}\{\alpha(X_0)^{-m}\}, \quad \text{as } k \to \infty. \tag{4.3}$$

As X_0 and the Markov modes are mutually independent, substitution of this into (2.6) then yields

$$\begin{aligned} \mathrm{E}\left\{\prod_{i=0}^{k-1}|a(Z_i)|^m|X_0 - q'_{k-1}(X_0)|^m\right\} &= \mathrm{E}\left\{\prod_{i=0}^{k-1}|a(Z_i)|^m\right\}\mathrm{E}|X_0 - q'_{k-1}(X_0)|^m, \\ &\sim \frac{\mathrm{E}\{\alpha(X_0)^{-m}\}}{(m+1)2^m\mu^{(m-1)k}}\mathrm{E}\left\{\prod_{i=0}^{k-1}|a(Z_i)|^m\right\}, \text{ as } k \to \infty. \end{aligned} \tag{4.4}$$

If (2.7) holds, then by (3.4) and (2.6) the RHS and hence (2.4) approach zero as time progresses. This completes the proof of Theorem 2.1.

The formulation of Coder-Controller 1 as an initial state compander permits convenient analysis by asymptotic quantisation methods, as demonstrated. However from an engineering point of view it is far from ideal, since the symbols are determined in open-loop. A scheme which is more suitable for feedback implementation can be easily constructed by adapting the scheme for finite bit rate state estimation in [11]. Its stability can be verified either by the methods of ch. 3, sec. 3 in [10] or by obtaining its α function and applying a version of the lemma above.

Finally, we remark on the dependence of the infimum stabilising data rate on the moment order m. This is in interesting contrast to a noiseless linear time-invariant system, for which the smallest data rate is independent of the particular notion of asymptotic stability desired, whether mean-absolute, -square or stronger [13]. The m-dependence emerges quite explicitly when the Markov modes are i.i.d. The necessary and sufficient limit condition (4.4) then simplifies to

$$\begin{aligned} \frac{\mathrm{E}\left\{\prod_{i=0}^{k-1}|a(Z_i)|^m\right\}}{\mu^{mk}} = \frac{(\mathrm{E}|a(Z_0)|^m)^k}{\mu^{mk}} \to 0 \quad &\Leftrightarrow \quad \mu > (\mathrm{E}|a(Z_0)|^m)^{1/m} = \left(\sum_{i=1}^{z}|a(i)|^m t_i\right)^{1/m}, \\ &\Leftrightarrow \quad R > \frac{1}{m}\log_2\sum_{i=1}^{z}|a(i)|^m t_i. \end{aligned}$$

It can be seen that if the mode probabilities t_i are nonzero then the infimum data rate increases with m and approaches $\log_2 \max_{1\le i\le z}|a(i)|$ as $m \to \infty$.

5 Conclusion

In this paper the problem of data-rate-limited, asymptotic stabilisation of a fully observed, scalar jump Markov linear system with no process noise was formulated. By casting the problem as one of moment stabilisation, the control problem was shown to be equivalent to finding a sequence of quantisers for the initial state that yielded an asymptotic mean mth power error of zero. A new quantiser error lower bound and asymptotic quantisation techniques were then used to derive the smallest data rate at which the system is stabilisable, in terms of the Perron-Frobenius eigenvalue of a matrix defined by the transition probabilities and dynamic parameters. Similar techniques are currently being investigated for stochastic linear systems and multidimensional jump Markov linear systems.

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Appendix. Proof of Lemma 3.1

Let $\mathcal{V}_1, \ldots, \mathcal{V}_\nu \subset \mathbb{R}^f$ and $t_1, \ldots, t_\nu \in \mathbb{R}^f$ be the Voronoi regions and corresponding points for the quantiser c_ν. The first step is to find a lower bound for the conditional MmPE $\mathrm{E}_{\mathbf{X} \in \mathcal{V}_i} \|\mathbf{X} - t_i\|^m =: l_i^m$. For conciseness, let $\pi_i \stackrel{\Delta}{=} p_{\mathbf{X}|\mathbf{X}\in\mathcal{V}_i}$. By Holder's inequality,

$$\begin{aligned}
\int_{\mathbb{R}^f} \pi_i(\mathbf{x})^r \mathrm{d}\lambda(\mathbf{x}) &= \int_{\mathbb{R}^f} \frac{(\|\mathbf{x}-t_i\|^m + l_i^m)^r}{(\|\mathbf{x}-t_i\|^m + l_i^m)^r} \pi_i(\mathbf{x})^r \mathrm{d}\lambda(\mathbf{x}), \\
&\le \left(\int_{\mathbb{R}^f} (\|\mathbf{x}-t_i\|^m + l_i^m)\pi_i(\mathbf{x})\mathrm{d}\lambda(\mathbf{x})\right)^r \left(\int_{\mathbb{R}^f} \frac{\mathrm{d}\lambda(\mathbf{x})}{(\|\mathbf{x}-t_i\|^m + l_i^m)^{r/(1-r)}}\right)^{1-r}, \\
&= (2l_i^m)^r \left(\int_{\mathbb{R}^f} \frac{\mathrm{d}\lambda(\mathbf{x})}{(\|\mathbf{x}-t_i\|^m + l_i^m)^{r/(1-r)}}\right)^{1-r}, \\
&= (2l_i^m)^r \left(\int_{\mathbb{R}^f} \frac{\mathrm{d}\lambda(\mathbf{x})}{(\|\mathbf{x}\|^m + l_i^m)^{r/(1-r)}}\right)^{1-r},
\end{aligned}$$

by virtue of the invariance of Lebesgue measure to translations. Now change integration variables to $\mathbf{w} = l_i^{-1}\mathbf{x}$. As $\mathbf{x} \in \mathbb{R}^f$, it then follows that $\mathrm{d}\lambda(\mathbf{w}) = l_i^{-f}\mathrm{d}\lambda(\mathbf{x})$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^f} \pi_i(\mathbf{x})^r \mathrm{d}\lambda(\mathbf{x}) &\le (2l_i^m)^r \left(\int_{\mathbb{R}^f} \frac{l_i^f \mathrm{d}\lambda(\mathbf{w})}{(\|l_i\mathbf{w}\|^m + l_i^m)^{r/(1-r)}}\right)^{1-r}, \\
&= 2^r l_i^{f(1-r)} \left(\int_{\mathbb{R}^f} \frac{\mathrm{d}\lambda(\mathbf{w})}{(\|\mathbf{w}\|^m + 1)^{r/(1-r)}}\right)^{1-r}.
\end{aligned}$$

Observe that the integrand in the third integral is only a function of the radial distance $\|w\| =: z$. As the f-dimensional Lebesgue measure or volume of a spherical shell of radius z and infinitesimal thickness $\mathrm{d}\lambda(z)$ is $\kappa z^{f-1}\mathrm{d}\lambda(z)$, where κ depends only on f, this integral can be rewritten to yield

$$\int_{\mathbb{R}^f} \pi_i(\mathbf{x})^r \mathrm{d}\lambda(\mathbf{x}) \le 2^r l_i^{f(1-r)} \left(\int_{z\ge 0} \frac{\kappa z^{f-1}\mathrm{d}\lambda(z)}{(z^m+1)^{r/(1-r)}}\right)^{1-r}.$$

The integral on the RHS is non-zero and in this form can clearly be seen to be finite when $mr/(1-r) > f$ i.e. $r > f/(f+m)$. Reversing the inequality above and taking the rth root, it then follows that

$$l_i^{f(1-r)/r} \ge \theta \left(\int_{\mathbb{R}^f} \pi_i(\mathbf{x})^r \mathrm{d}\lambda(\mathbf{x})\right)^{1/r} = \theta \left(\int_{\mathbb{R}^f} p_{\mathbf{X}|\mathbf{X}\in\mathcal{V}_i}(\mathbf{x})^r \mathrm{d}\lambda(\mathbf{x})\right)^{1/r} = \frac{\theta \left(\int_{\mathcal{V}_i} p_{\mathbf{X}}(\mathbf{x})^r \mathrm{d}\lambda(\mathbf{x})\right)^{1/r}}{\mathrm{P}\{\mathbf{X}\in\mathcal{V}_i\}},$$

where θ depends only on r, f and m. Taking expectations,

$$\sum_{i=1}^{\nu} l_i^{f(1-r)/r}\mathrm{P}\{\mathbf{X}\in\mathcal{V}_i\} \ge \theta \sum_{i=1}^{\nu}\left(\int_{\mathcal{V}_i} p_{\mathbf{X}}(\mathbf{x})^r \mathrm{d}\lambda(\mathbf{x})\right)^{1/r} \equiv \theta \sum_{i=1}^{\nu} a_i^{1/r},$$

where $a_i \stackrel{\Delta}{=} \int_{\mathcal{V}_i} p_{\mathbf{X}}(\mathbf{x})^r \mathrm{d}\lambda(\mathbf{x})$. Observe that as $r < 1$ the RHS is a convex function of $a_1, \ldots, a_\nu$, with the constraints $a_i \ge 0$ and $\sum_{i=1}^{\nu} a_i = \int_{\mathbb{R}^f} p_{\mathbf{X}}(\mathbf{x})^r \mathrm{d}\lambda(\mathbf{x}) = \|p_{\mathbf{X}}\|_r^r$. Hence

$$\sum_{i=1}^{\nu} l_i^{f(1-r)/r}\mathrm{P}\{\mathbf{X}\in\mathcal{V}_i^*\} \ge \theta\nu \sum_{i=1}^{\nu}\frac{a_i^{1/r}}{\nu} \ge \theta\nu\left(\sum_{i=1}^{\nu}\frac{a_i}{\nu}\right)^{1/r} = \frac{\theta\|p_{\mathbf{X}}\|_r}{\nu^{1/r-1}}.$$

However by the definition of l_i and the convexity of $(\cdot)^{mr/[f(1-r)]}$ (since the exponent > 1),

$$\begin{aligned}
\mathrm{E}\|\mathbf{X} - c_\nu(\mathbf{X})\|^m = \mathrm{E}\{\mathrm{E}_{\mathbf{X}\in\mathcal{V}_i}\|\mathbf{X}-t_i\|^m\} &= \sum_{i=1}^{\nu} l_i^m \mathrm{P}\{\mathbf{X}\in\mathcal{V}_i\}, \\
&= \sum_{i=1}^{\nu} (l_i^{f(1-r)/r})^{mr/[f(1-r)]}\mathrm{P}\{\mathbf{X}\in\mathcal{V}_i\}, \\
&\ge \left(\sum_{i=1}^{\nu} l_i^{f(1-r)/r}\mathrm{P}\{\mathbf{X}\in\mathcal{V}_i\}\right)^{mr/[f(1-r)]}.
\end{aligned}$$

Substituting the previous inequality into the RHS,

$$\mathrm{E}\|\mathbf{X} - c_\nu(\mathbf{X})\|^m \geq \left(\frac{\theta\|p_\mathbf{X}\|_r}{\nu^{1/r-1}}\right)^{mr/[f(1-r)]} = \frac{\theta^{mr/[f(1-r)]}}{\nu^{m/f}}\|p_\mathbf{X}\|_r^{mr/[f(1-r)]}, \quad \forall \nu \in \mathbb{N}.$$

This completes the proof of the lemma.