STRUCTURAL RESULTS FOR FINITE BIT-RATE STATE ESTIMATION

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ABSTRACT

This paper considers the problem of estimating the state of a dynamic system from measurements obtained via a digital link with finite bit-rate R. It is shown that, under a quadratic cost, the problem reduces to coding and estimating the sequence of expected states conditioned on past measurements. The existence of deterministic, optimal coder-estimators for Markovian processes in $\mathbf{R}^D$ is then established and their structure derived. These results are then combined to prove that the optimal coder for a Gauss-Markov system consists of a Kalman filter followed by a stage which encodes the latest Kalman estimate according to the symbols previously transmitted.

1. INTRODUCTION

In many problems in estimation theory, the implicit assumption is that the estimator has direct access to a sequence of possibly noise-corrupted measurements. That is, if $\{X_k\}_{k\geq 0}$ is some process to be estimated from measurements of a related process $\{Y_k\}_{k\geq 0}$, then in most instances the estimator would have direct knowledge of each Y_k. However, in certain situations there is a constraint on the amount of information available to the estimator. In particular, if the estimator is not at the same location as the measurement sensor and receives information via a digital communication channel with a finite bit-rate, then it can only have partial knowledge of $\{Y_k\}_{k\geq 0}$, since at each time k the output of the system must be mapped to a finite set of symbols before transmission to the estimator. This causes a loss of resolution and also introduces an additional delay into the system, since the bits that constitute each symbol require a finite amount of time for complete transmission. Natural questions to then ask are how to perform the coding so as to minimize the estimation error and how high a bit-rate is needed to achieve a specified estimation accuracy.

Similar problems are addressed in the fields of rate distortion theory [1] and estimation via compressed information [2, 3]. However, the techniques used in these areas typically require the measurements to be collected into long blocks before coding, thereby causing long delays that are impractical in many situations. In such cases, a causal coding and estimation scheme which improves in accuracy incrementally would be more appropriate. The field of estimation with quantized measurements [4, 5] is also related, but the fixed quantizer that is generally assumed there makes it significantly different from the problem discussed here. By allowing the coder output to be a time-varying function of all past observations, the estimator in our problem effectively has the freedom to choose what to measure. This has the potential to significantly increase the estimation accuracy, if done appropriately.

The idea of causal *coder-estimators* for dynamical systems was first investigated in [6]. Subsequently, a number of different coder-estimators have been proposed, under various conditions on the noise and initial condition distributions, leading to a number of convergence conditions and estimation error bounds [7, 8, 9, 10]. The aim of this paper is to investigate the structural questions underlying this problem. In the next section, a general framework is developed and in the subsequent section various results on the structure of optimal coder-estimators are derived.

2. GENERAL FORMULATION

Consider any random, discrete-time process $\{X_k\}_{k\geq 0}$ in $\mathbf{R}^D$ observed via a sequence of measurements $\{Y_k\}_{k\geq 0}$ in $\mathbf{R}^{D'}$. Suppose estimates of the current state X_k are required at a distant location that is linked to the measurement sensor by a digital communication channel with a capacity of R bits of information per sampling interval [11]. For simplicity, we convert this into the more restrictive condition that only R bits of data may be sent per sampling pe-

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riod, where R is now constrained to be an integer. Hence at each time k, one symbol from an alphabet of size $M = 2^R$ is transmitted without error to the estimator. Assuming that there is no restriction on the complexity of the coder, the transmitted symbol may depend on all past and present measurements as well as all previous symbols, i.e.

$$S_k = \Gamma_k(\tilde{Y}_k, \tilde{S}_{k-1}) \in \mathbf{Z}_M, \quad \forall k \geq 0, \qquad (1)$$

where s_k is the symbol transmitted at time k and the notation $\tilde{z}_k$ denotes the sequence $\{z_j\}_{j=0}^{k}$, with the convention that $\tilde{z}_{-1}$ is the empty sequence. For generality, the mapping $\Gamma_k : \mathbf{R}^{D' \times (k+1)} \times \mathbf{Z}_M^k \to \mathbf{Z}_M$ is allowed to be probabilistic, that is given $\tilde{Y}_k, \tilde{S}_{k-1}$ the symbol S_k occurs according to a prescribed probability mass function $\mathrm{P}\{S_k|\tilde{Y}_k, \tilde{S}_{k-1}\}$. Due to the finite bit-rate, each transmitted symbol takes one sampling interval to reach the estimator, neglecting the propagation delay. Hence at time k the estimator has the sequence $\tilde{s}_{k-1}$ available and estimates the current state x_k via

$$\hat{X}_k = \Delta_k(\tilde{S}_{k-1}), \qquad \forall k \geq 0, \qquad (2)$$

where $\Delta_k : \mathbf{Z}_M^k \to \mathbf{R}^D$ is another probabilistic mapping, defined by the conditional density $p_{\hat{X}_k|\tilde{S}_{k-1}}$.

The objective is to find a causal coder-estimator, defined by $\{\Gamma_k, \Delta_k\}_{k\geq 0}$, such that some measure of estimation error is as small as possible. Unlike the classical situation, corresponding to $R \to \infty$, minimizing $\mathrm{E}\{\|X_k - \hat{X}_k\|^2\}$ for each $k \geq 0$ is not appropriate. The reason for this is that $\hat{X}_k$ still depends on past choices of Γ_j, $j \leq k$, so a coder that minimizes $\mathrm{E}\{\|X_k - \hat{X}_k\|^2\}$ will not, in general, minimize $\mathrm{E}\{\|X_{k+1} - \hat{X}_{k+1}\|^2\}$. If estimates are needed only up to time N, then a more suitable measure of error is a weighted average such as

$$D_N = \sum_{k=0}^{N} \alpha_k \mathrm{E}\{\|X_k - \hat{X}_k\|^2\}, \qquad (3)$$

where $\{\alpha_k\}_{k=0}^{N}$, is a sequence of nonnegative numbers. Possible choices might be $\alpha_k = 1/N$ or $\alpha_k = \beta^{N-k}$, with $0 < \beta < 1$. If, on the other hand, the asymptotic performance of the coder-estimator is of interest, an appropriate measure would be $\limsup_{N\to\infty} D_N$, assuming it exists. In the next section, various structural properties of optimal coder-estimators under a quadratic cost will be derived.

3. STRUCTURAL RESULTS

Our starting point is an extension of a result proven in [12] in the context of block source coding.

Theorem 1 *Let $\{X_k\}_{k\geq 0}$ be a random process in a separable Hilbert space, with measurements $\{Y_k\}_{k\geq 0}$. Suppose the process is coded and estimated using (1) and (2) under the quadratic cost*

$$D = L\left(\left\{\mathrm{E}\{\|X_k - \hat{X}_k\|^2\}\right\}_{k\geq 0}\right), \qquad (4)$$

where $L : \mathbf{R}^\infty \to [-\infty, +\infty]$ is linear and nonnegative for nonnegative arguments. Then for a given coder $\{\Gamma_k\}_{k\geq 0}$, an optimal estimator is

$$\hat{X}_k^* \triangleq \mathrm{E}\{\bar{X}_k|\tilde{S}_{k-1}\}, \quad \forall k \geq 0, \qquad (5)$$

where $\bar{X}_k = \mathrm{E}\{X_k|\tilde{Y}_{k-1}\}$, and

$$\inf_{\{\Gamma_k,\Delta_k\}_{k\geq 0}} D = L\left(\{\mathrm{E}\{\|X_k - \bar{X}_k\|^2\}\}_{k\geq 0}\right) + \inf_{\{\Gamma_k,\Delta_k\}_{k\geq 0}} L\left(\left\{\mathrm{E}\{\|\bar{X}_k - \hat{X}_k\|^2\}\right\}_{k\geq 0}\right). \qquad (6)$$

Proof: The logic of our argument is essentially the same as in [12]. First note that, given $\tilde{Y}_{k-1}$, $\hat{X}_k$ is independent of X_k, so

$$\mathrm{E}_{X_k,\tilde{Y}_{k-1}}\{\hat{X}_k\} = \mathrm{E}_{\tilde{Y}_{k-1}}\{\hat{X}_k\}, \qquad (7)$$

where the notation $\mathrm{E}_A\{\cdot\}$ is short-hand for $\mathrm{E}\{\cdot|A\}$. Now $\forall k \geq 0$,

$$\begin{aligned}
&\mathrm{E}\{\|X_k - \hat{X}_k\|^2\} \\
&= \mathrm{E}\{\|X_k - \bar{X}_k + \bar{X}_k - \hat{X}_k\|^2\}, \\
&= \mathrm{E}\{\|X_k - \bar{X}_k\|^2\} + \mathrm{E}\{\|\bar{X}_k - \hat{X}_k\|^2\} \\
&\quad + 2\mathrm{E}\{\langle X_k - \bar{X}_k, \bar{X}_k - \hat{X}_k\rangle\}. \qquad (8)
\end{aligned}$$

Rewrite the expected inner product as

$$\mathrm{E}\mathrm{E}_{\tilde{Y}_{k-1}}\mathrm{E}_{X_k,\tilde{Y}_{k-1}}\{\langle X_k - \bar{X}_k, \bar{X}_k - \hat{X}_k\rangle\},$$

and note that, since $\bar{X}_k = \mathrm{E}\{X_k|\tilde{Y}_{k-1}\}$, $X_k - \bar{X}_k$ is a function of $X_k, \tilde{Y}_{k-1}$. Using the fact that if f is any function and A, B random variables such that $f(A), B$ lie in a separable Hilbert space, then

$$\mathrm{E}_A\{\langle f(A), B\rangle\} = \langle f(A), \mathrm{E}_A\{B\}\rangle, \qquad (9)$$

we obtain

$$\begin{aligned}
&\mathrm{E}\{\langle X_k - \bar{X}_k, \bar{X}_k - \hat{X}_k\rangle\} \\
&= \mathrm{E}\mathrm{E}_{\tilde{Y}_{k-1}}\left\{\langle X_k - \bar{X}_k, \mathrm{E}_{X_k,\tilde{Y}_{k-1}}\{\bar{X}_k - \hat{X}_k\}\rangle\right\}, \\
&= \mathrm{E}\mathrm{E}_{\tilde{Y}_{k-1}}\left\{\langle X_k - \bar{X}_k, \bar{X}_k - \mathrm{E}_{X_k,\tilde{Y}_{k-1}}\{\hat{X}_k\}\rangle\right\}, \\
&= \mathrm{E}\mathrm{E}_{\tilde{Y}_{k-1}}\left\{\langle X_k - \bar{X}_k, \bar{X}_k - \mathrm{E}_{\tilde{Y}_{k-1}}\{\hat{X}_k\}\rangle\right\}, \\
&\quad \text{(using (7))}, \\
&= \mathrm{E}\left\{\langle \mathrm{E}_{\tilde{Y}_{k-1}}\{X_k - \bar{X}_k\}, \bar{X}_k - \mathrm{E}_{\tilde{Y}_{k-1}}\{\hat{X}_k\}\rangle\right\}, \\
&\quad \text{(using property (9) again)}, \\
&= \mathrm{E}\left\{\langle 0, \bar{X}_k - \mathrm{E}_{\tilde{Y}_{k-1}}\{\hat{X}_k\}\rangle\right\}, \\
&= 0.
\end{aligned}$$

Substituting this into (8) and using the linearity of L, we obtain

$$D = L\left(\{\mathrm{E}\{\|X_k - \bar{X}_k\|^2\}\}_{k\geq 0}\right) + L\left(\left\{\mathrm{E}\{\|\bar{X}_k - \hat{X}_k\|^2\}\right\}_{k\geq 0}\right), \quad (10)$$

which leads directly to (6) since the first term on the R.H.S. is independent of the choice of coder-estimator.

In order to prove (5), note that $\forall k \geq 0$,

$$\begin{aligned}&\mathrm{E}\{\|\bar{X}_k - \hat{X}_k\|^2\} \\ &= \mathrm{E}\{\|\bar{X}_k - \hat{X}_k^*\|^2\} + \mathrm{E}\{\|\hat{X}_k^* - \hat{X}_k\|^2\} \\ &\quad + 2\mathrm{E}\{\langle \bar{X}_k - \hat{X}_k^*, \hat{X}_k^* - \hat{X}_k\rangle\}. \quad (11)\end{aligned}$$

Now, given $\tilde{S}_{k-1}$, $\hat{X}_k$ is independent of $\bar{X}_k$, so that

$$\mathrm{E}_{\bar{X}_k,\tilde{S}_{k-1}}\{\hat{X}_k\} = \mathrm{E}_{\tilde{S}_{k-1}}\{\hat{X}_k\}. \quad (12)$$

Looking at the expected inner product in (11),

$$\begin{aligned}&\mathrm{E}\{\langle \bar{X}_k - \hat{X}_k^*, \hat{X}_k^* - \hat{X}_k\rangle\} \\ &= \mathrm{E}\mathrm{E}_{\tilde{S}_{k-1}}\mathrm{E}_{\bar{X}_k,\tilde{S}_{k-1}}\{\langle \bar{X}_k - \hat{X}_k^*, \hat{X}_k^* - \hat{X}_k\rangle\}, \\ &= \mathrm{E}\mathrm{E}_{\tilde{S}_{k-1}}\left\{\langle \bar{X}_k - \hat{X}_k^*, \mathrm{E}_{\bar{X}_k,\tilde{S}_{k-1}}\{\hat{X}_k^* - \hat{X}_k\}\rangle\right\}, \\ &\quad \text{(using property (9))}, \\ &= \mathrm{E}\mathrm{E}_{\tilde{S}_{k-1}}\left\{\langle \bar{X}_k - \hat{X}_k^*, \hat{X}_k^* - \mathrm{E}_{\bar{X}_k,\tilde{S}_{k-1}}\{\hat{X}_k\}\rangle\right\}, \\ &= \mathrm{E}\mathrm{E}_{\tilde{S}_{k-1}}\left\{\langle \bar{X}_k - \hat{X}_k^*, \hat{X}_k^* - \mathrm{E}_{\tilde{S}_{k-1}}\{\hat{X}_k\}\rangle\right\}, \\ &\quad \text{(using (12))}, \\ &= \mathrm{E}\left\{\langle \mathrm{E}_{\tilde{S}_{k-1}}\{\bar{X}_k - \hat{X}_k^*\}, \hat{X}_k^* - \mathrm{E}_{\tilde{S}_{k-1}}\{\hat{X}_k\}\rangle\right\}, \\ &\quad \text{(using property (9) again)}, \\ &= \mathrm{E}\left\{\langle 0, \hat{X}_k^* - \mathrm{E}_{\tilde{S}_{k-1}}\{\hat{X}_k\}\rangle\right\}, \\ &\quad \text{(by definition of } \hat{X}_k^*\text{)}, \\ &= 0.\end{aligned}$$

Substituting this into (11) and using (10) and the linearity of L, we obtain

$$\begin{aligned}D &= L\left(\{\mathrm{E}\{\|X_k - \bar{X}_k\|^2\}\}_{k\geq 0}\right) \\ &\quad + L\left(\left\{\mathrm{E}\{\|\bar{X}_k - \hat{X}_k^*\|^2\}\right\}_{k\geq 0}\right) \\ &\quad + L\left(\left\{\mathrm{E}\{\|\hat{X}_k^* - \hat{X}_k\|^2\}\right\}_{k\geq 0}\right).\end{aligned}$$

Since L is nonnegative for nonnegative arguments and the first two terms on the R.H.S. are independent of the choice of estimator, D is minimized w.r.t. the estimator by setting $\hat{X}_k = \hat{X}_k^*$, $\forall k \geq 0$. □

From equation (5), it can be seen that the probabilistic estimator mappings $\{\Delta_k\}_{k\geq 0}$ may be constrained to be deterministic without any loss of optimality. Equation (6) implies that the general problem of coding and estimating a process $\{X_k\}_{k\geq 0}$ with measurements $\{Y_k\}_{k\geq 0}$ under a quadratic cost is equivalent to coding and estimating the sequence of conditional means $\left\{\mathrm{E}\{X_k|\tilde{Y}_{k-1}\}\right\}_{k\geq 0}$. Hence the first stage of an optimal coder, if one exists, consists of a causal filter that transforms the measurements into the sequence $\{\bar{X}_k\}_{k\geq 0}$. The sequence $\{\bar{X}_k\}_{k\geq 0}$ is then causally encoded by a second stage before transmission on the digital link. This procedure agrees with intuition. As the coding step is where information is lost, it is sensible to preprocess the measurements in order to extract the best possible state estimates and to then encode them, rather than the raw measurements. In addition, note that the first term on the R.H.S. of (6) is just the classical cost that would be obtained without a bit-rate constraint. Hence the classical cost is a lower bound for the rate-constrained cost, as expected.

At each time k, the coder has R bits with which to encode $\{\bar{X}_j\}_{j\leq k}$. The next question that arises is how it should best allocate those bits. If the filtered process is noisy, i.e. $\bar{X}_k$ has a low correlation with $\{\bar{X}_j\}_{j\leq k-1}$, then the available bits should be used to encode only the more recent outputs of the filter. On the other hand, if it has very little noise, then it might be supposed that a certain number of bits should still be allocated to earlier filter outputs, since they continue to influence the evolution of the filtered process. However, if $\{\bar{X}_k\}_{k\geq 0}$ is Markovian in $\mathbf{R}^D$, then the following theorem states that all R bits at time k, which together constitute the symbol $s_k \in \mathbf{Z}_M$, should be used to encode only the current filter output.

Theorem 2 *Let $\{X_k\}_{k\geq 0}$ be a Markovian process in $\mathbf{R}^D$. Suppose that the states X_k are observed and then coded and estimated via causal probabilistic mappings of the form*

$$\begin{aligned}S_k &= \Gamma_k(\tilde{X}_k, \tilde{S}_{k-1}) \in \mathbf{Z}_M, \\ \hat{X}_k &= \Delta_k(\tilde{S}_k), \quad \forall k \in [0, \ldots, N],\end{aligned}$$

so as to minimize a quadratic, finite-time cost of the form (3). Then deterministic, optimal coder-estimators exist and have the coder structure

$$S_k = \gamma_k^*(X_k, \tilde{S}_{k-1}), \quad \forall k \in [0, \ldots, N]. \quad (13)$$

In addition,

$$\inf_{\{\Gamma_k,\Delta_k\}_{0\leq k\leq N}} D_N = \min_{\{\delta_k\}_{0\leq k\leq N}} f_0^*, \quad (14)$$

where $\forall k \in [0, \ldots, N]$, the functions $f_k : \mathbf{R}^D \times \mathbf{Z}_M^k \to \mathbf{R}_+ \cup \{0\}$ are defined by the recursion

$$f_k^*(x_{k-1}, \tilde{s}_{k-1}) \triangleq \int p_{X_k|X_{k-1}}(x_k|\, x_{k-1}) \times$$

$$\min_{s_k \in \mathbf{Z}_M} \{\alpha_k \|x_k - \delta_k(\tilde{s}_k)\|^2 + f^*_{k+1}(x_k, \tilde{s}_k)\}\, dx_k, \tag{15}$$

with $f^*_{N+1} \triangleq 0$.

Proof: From the preceding theorem, for each probabilistic coder there exists a deterministic, optimal estimator $\hat{X} = \mathrm{E}\{X_k|\tilde{S}_k\}$, so the estimator may be safely constrained to be deterministic. Let $\{\delta_k\}_{k\geq 0}$ be any deterministic estimator. For all $k \in [-1,\ldots,N]$, define $f_{k+1} : \mathbf{R}^{D\times(k+1)} \times \mathbf{Z}_M^{k+1} \to \mathbf{R}_+ \cup \{0\}$ by

$$f_{k+1}(\tilde{x}_k, \tilde{s}_k) = \mathrm{E}_{\tilde{X}_k,\tilde{S}_k} \left\{ \sum_{j=k+1}^{N} \alpha_j \|X_j - \hat{X}_j\|^2 \right\},$$

where the notation E_A is shorthand for $\mathrm{E}\{\cdot|A\}$. From this definition, $f_0 \triangleq f_0(\tilde{x}_{-1}, \tilde{s}_{-1}) = D_N$, $f_{N+1} = 0$ and $\forall k \in [-1,\ldots,N]$, we have the recursion

$$\begin{aligned} & f_k(\tilde{x}_{k-1}, \tilde{s}_{k-1}) \\ &= \mathrm{E}_{\tilde{X}_{k-1},\tilde{S}_{k-1}} \left\{ \sum_{j=k}^{N} \alpha_j \|X_j - \hat{X}_j\|^2 \right\}, \\ &= \int p_{X_k|X_{k-1}}(x_k|x_{k-1}) \sum_{s_k} \mathrm{P}\{s_k|\tilde{x}_k, \tilde{s}_{k-1}\} \times \\ & \quad [\alpha_k \|x_k - \delta_k(\tilde{s}_k)\|^2 + f_{k+1}(\tilde{x}_k, \tilde{s}_k)]\, dx_k. \end{aligned} \tag{16}$$

Now suppose that $\exists k \in [-1,\ldots,N]$ such that

$$\min_{\{\Gamma_j\}_{j\geq k+1}} f_{k+1}(\tilde{x}_k, \tilde{s}_k) \equiv f^*_{k+1}(x_k, \tilde{s}_k), \tag{17}$$

i.e. the minimum exists and is independent of $\tilde{x}_{k-1}$. Clearly this is true when $k = N$. Since $p_{X_k|X_{k-1}}$, δ_k and $\mathrm{P}\{S_k|\tilde{X}_k, \tilde{S}_{k-1}\}$ $(\equiv \Gamma_k)$ are independent of the choice of $\{\Gamma_j\}_{j\geq k+1}$, we have from (16)

$$\begin{aligned} & \min_{\{\Gamma_j\}_{j\geq k+1}} f_k(\tilde{x}_{k-1}, \tilde{s}_{k-1}) \\ &= \min_{\{\Gamma_j\}_{j\geq k+1}} \int p_{X_k|X_{k-1}}(x_k|x_{k-1}) \times \\ & \quad \sum_{s_k} \mathrm{P}\{s_k|\tilde{x}_k, \tilde{s}_{k-1}\} \times \\ & \quad [\alpha_k \|x_k - \delta_k(\tilde{s}_k)\|^2 + f_{k+1}(\tilde{x}_k, \tilde{s}_k)]\, dx_k, \\ &= \int p_{X_k|X_{k-1}}(x_k|x_{k-1}) \sum_{s_k} \mathrm{P}\{s_k|\tilde{x}_k, \tilde{s}_{k-1}\} \times \\ & \quad \left[\alpha_k \|x_k - \delta_k(\tilde{s}_k)\|^2 + \min_{\{\Gamma_j\}_{j\geq k+1}} f_{k+1}(\tilde{x}_k, \tilde{s}_k)\right] dx_k. \end{aligned}$$

Observe that, given $\tilde{X}_k, \tilde{S}_k$, $f_{k+1}(\tilde{X}_k, \tilde{S}_k)$ is independent of the choice of $\mathrm{P}\{S_k|\tilde{X}_k, \tilde{S}_{k-1}\}$. As such the R.H.S. is at a minimum with respect to Γ_k iff

$$\mathrm{P}\{s_k|\tilde{x}_k, \tilde{s}_{k-1}\} = 0,$$

for all s_k not in $\operatorname{Arg\,min}_{s\in\mathbf{Z}_M} \{\alpha_k\|x_k - \delta_k(\tilde{s}_{k-1}, s)\|^2 + f^*_{k+1}(x_k, (\tilde{s}_{k-1}, s))\}$, which depends only on x_k and $\tilde{s}_{k-1}$. A deterministic, optimal coder can be constructed by using some arbitrary rule to select a single minimizing s as the value of s_k, Furthermore, we have

$$\begin{aligned} & \min_{\{\Gamma_j\}_{j\geq k}} f_k(\tilde{x}_{k-1}, \tilde{s}_{k-1}) \\ &= \int p_{X_k|X_{k-1}}(x_k|x_{k-1}) \times \\ & \quad \min_{s_k\in\mathbf{Z}_M} \{\alpha_k \|x_k - \delta_k(\tilde{s}_k)\|^2 + f^*_{k+1}(x_k, \tilde{s}_k)\}\, dx_k \\ & \triangleq f^*_k(x_{k-1}, \tilde{s}_{k-1}), \end{aligned} \tag{18}$$

i.e. (17) holds for $k-1$ as well. By induction, it then holds $\forall k \in [-1,\ldots,N]$, which proves (13) and (15).

To prove the existence of optimal coder-estimators, define the finite-dimensional vector

$$\delta \triangleq \{\delta_k(\tilde{s}_k)\}_{\tilde{s}_k\in\mathbf{Z}_M^{k+1}, 0\leq k\leq N} \in \mathbf{R}^{M(M^N-1)/(M-1)}.$$

It is easy to see that f^*_0 is continuous w.r.t. δ, so that its level sets are closed. In addition, note that $\forall k \in [0,\ldots,N]$,

$$f^*_0 \geq \int p_{X_k}(x_k) \min_{\tilde{s}_k\in\mathbf{Z}_M^{k+1}} \alpha_k \|x_k - \delta_k(\tilde{s}_k)\|^2\, dx_k,$$

from which it can be shown that $f^*_0 \to \infty$ as $\|\delta\| \to \infty$. Hence the level sets of f^*_0 are also bounded, and therefore compact, so a minimizing δ^* exists. □

What this says is quite intuitive. If the process is Markovian, then the current state already encapsulates all past information and so no bits need to be wasted on encoding past states. In addition, it confirms that deterministic, optimal coder-estimators exist, so that nothing is lost by focusing on deterministic rather than probabilistic mappings. Furthermore, the problem of finding the infimum of D_N over all causal, probabilistic coder and estimator mappings has been reduced to the simpler, finite-dimensional problem of finding the minimum of f^*_0 with respect to the vector δ.

We next consider Gauss-Markov systems. The corollary below follows directly from the previous theorems:

Corollary 1 *Consider the linear system*

$$\begin{aligned} X_{k+1} &= A_k X_k + V_k, \\ Y_k &= H_k X_k + W_k, \quad \forall k \geq 0, \end{aligned}$$

where $X_k \in \mathbf{R}^D$, $Y_k \in \mathbf{R}^{D'}$ *and* $\{X_0, V_k, W_k\}_{k\geq 0}$ *are mutually independent and Gaussian. Deterministic optimal coder-estimators for this system, under the cost (3), exist and have the following structure:*

Coder: *At time k, a Kalman one-step ahead predictor recursively processes the measurements* $\tilde{Y}_k$ *to yield*

$$\bar{X}_{k+1} = \mathrm{E}\{X_{k+1}|\tilde{Y}_k\}. \quad (19)$$

The conditional mean $\bar{X}_{k+1}$ *is then encoded according to the symbols previously transmitted,*

$$S_k = \gamma_k^*(\bar{X}_{k+1}, \tilde{S}_{k-1}). \quad (20)$$

Estimator: *Upon receiving the symbol sequence* $\tilde{S}_{k-1}$ *at time k,* X_k *is estimated via*

$$\hat{X}_k = \delta_k^*(\tilde{S}_{k-1}) = \mathrm{E}\{\bar{X}_k|\tilde{S}_{k-1}\}. \quad (21)$$

Proof: Equation (19) follows directly from Theorem 1. In the innovations representation,

$$\bar{X}_{k+1} = A_k\bar{X}_k + Z_k, \quad \forall k \geq 0,$$

where $\{\bar{X}_0, Z_k\}_{k\geq 0}$ are uncorrelated and therefore, by the Gaussian assumption, mutually independent. Hence the process $\{\bar{X}_k\}_{k\geq 0}$ is Markovian. The rest of the corollary then follows from Theorem 2. □

In general, it is difficult to obtain explicit formulae for the optimal coder-estimator, even for the case of a one-dimensional system. The Generalized Lloyd Algorithm [13] or some other method could be used to derive them numerically for a given system and bit-rate, but the entire procedure would have to be repeated if the system dynamics, noise distributions, duration N or bit-rate changed. Moreover, if we are interested in minimizing D_N as $N \to \infty$, a numerical approach is impossible since the dimension of the problem becomes unbounded. Nevertheless, the structural properties of optimal coder-estimators are a useful foundation upon which to construct a class of suboptimal coder-estimators that are easier to analyze [10, 14].

4. CONCLUSION

In summary, this paper presents a number of structural results for coding and estimating dynamical systems with noise. Under very general conditions and assuming a quadratic cost, it is shown that prior to coding, the measurements should be filtered to yield the sequence of conditional means. It is next shown that deterministic, optimal coder-estimators exist for Markovian processes in $\mathbf{R}^D$. An expression for the minimum cost is derived and the general structure of the optimal coder is obtained. As a corollary of these results, the optimal coder for a Gauss-Markov system consists of a Kalman filter followed by a stage that encodes the latest Kalman estimate according to the symbols previously transmitted. Although explicit expressions for the optimal coder-estimator are difficult to obtain, these results provide valuable structural insights for constructing suboptimal coder-estimators that are more tractable. Work is currently in progress on extending the results in this paper to situations with multiple sensors and links.

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