OPTIMAL CONTROL UNDER A DATA RATE CONSTRAINT

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ABSTRACT

This paper investigates the problem of controlling a one-dimensional, time-varying, linear system over a communication link with a finite data rate. It is assumed that the initial condition is governed by a known probability density p and that there is no process or measurement noise. Finite and infinite horizon costs involving m-th state moments are specified and the optimal finite horizon *coder-controller* is explicitly derived when $m = 1$ and p is *Laplacian*. Asymptotic quantization theory is then used to obtain the limiting solution directly for any m and p. It is shown that if p is continuous and satisfies certain technical conditions, then this limiting coder-controller is optimal. This leads to a necessary and sufficient condition for the system to be *asymptotically stochastically stabilizable* in the presence of a data rate constraint.

INTRODUCTION

In recent years, a number of researchers have begun to investigate the effects of finite communication rates in control problems. Traditionally, the communications channel between the plant and controller goes unmodelled, for it is simply assumed that the outputs of the former are available to the latter completely and with infinite precision. Clearly, this is untrue if the link between them can only carry a limited number of digital symbols per unit time. In addition to introducing both delay and quantization, this finite data rate forces the question of how to choose the bits of information that would be most useful for control.

So far the focus in this area has generally been on *memoryless* coding. In Delchamps [1], it was shown that if the output of an unstable, deterministic, dicrete-time, linear time-invariant (LTI) system is passed through a fixed memoryless quantizer, then controllability in the sense of being able to make the trajectory approach any arbitrary point is generally impossible. In Liu and Wong [2], the communication delays in this model were explicitly included and sufficient conditions given for any trajectory to eventually lie within a given bounded set. Continuous-time LTI systems were studied by Wong and Brockett [3], who proved that if the initial condition is within a given bounded set then memoryless coding *and* control suffice to bound the trajectory, provided certain conditions are met. Investigating discrete-time, Gaussian LTI systems, Borkar and Mitter [4] showed that if the output is filtered and the resulting *innovations* process quantized without memory, then a separation principle holds which yields a memoryless, certainty equivalent controller.

The only case in which coders with memory have been studied in communication-limited control appears to be in Tatikonda et al [5], in which a noisy, analog channel was considered. In this paper, we assume a noiseless, digital channel and permit the coder to have potentially unlimited memory, i.e. each symbol may depend on all past symbols and past and present system outputs. The motivation for this comes from the the closely related field of communication-limited state estimation, in which recursive coding schemes have been shown to perform well; see Wong and Brockett [6] and Nair and Evans [7, 8]. Moreover, by relaxing the structural constraint of memoryless coding and/or control, we can isolate its effects from those of the actual communication constraint.

So as to better demonstrate the effectiveness of recursive coding and control schemes, we focus here on one-dimensional, linear systems with a random initial condition but no noise. We show that the optimal *coder-controller* under a certain finite horizon, mean m-th power cost is given by a causally reformulated optimal quantizer acting on the initial condition. This observation leads to explicit, closed form equations for the special case of a *Laplacian* initial condition distribution and $m = 1$. As the horizon tends to infinity in these equations, a limiting coder-controller is obtained.

A way of deriving this limiting scheme for any given m and initial condition distribution, *without having to solve the finite horizon problem first*, is then presented, using the insights gained from the finite horizon

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analysis and the powerful theory of asymptotic quantization. It is shown that this limiting coder-controller is optimal in an infinite horizon sense, provided that the initial condition probability density is continuous and satisfies certain technical conditions. Furthermore, an expression for the infimum of the infinite horizon cost is derived. This immediately leads to a necessary and sufficient condition for the system to be *asymptotically stochastically stabilizable* by a coder-controller, in the sense that the m-th moment of the state approaches zero with time.

FORMULATION

Consider the discrete-time control system

$$x_{k+1} = a_k x_k + u_k \ \in \mathbf{R}, \quad \forall k \geq 0, \qquad (1)$$

where x_k and u_k are the system state and control respectively at time k. We assume that X_0 is governed by a distribution function P such that $\mathrm{E}|X_0|^m < \infty$ for some $m > 0$. Suppose a coder observes the states and then sends real-time data to a controller over a digital channel that can carry only one symbol s_k from an alphabet $\mathbf{Z}_M \triangleq [0, 1, \ldots, M-1]$ during each sampling interval. The corresponding *data rate* is then $R \triangleq \log_2 M$ bits per interval. Neglecting the propagation delay and transmission errors, each symbol takes one sampling interval to reach the controller, due to the finite data rate. Using the notation $\tilde{y}_k$ to denote a sequence $\{y_j\}_{j=0}^k$, this means that at time k the controller has $\tilde{s}_{k-1}$ available and produces a signal

$$u_k = \upsilon_k(\tilde{s}_{k-1}), \quad \forall k \geq 0, \qquad (2)$$

where $\upsilon_k : \mathbf{Z}_M^k \to \mathbf{R}$ is the controller function. If no restrictions save causality are placed on the structure of the coder, each transmitted symbol s_k may be a function of the sequences of past and present states $\tilde{x}_k$ and past symbols $\tilde{s}_{k-1}$. However, (1) and (2) imply that x_k in its turn is completely determined by x_0 and $\tilde{s}_{k-2}$, so s_k is really just a function of x_0 and $\tilde{s}_{k-1}$. By downward induction on k, it is then apparent that s_k may generally be expressed as a function of x_0 alone,

$$s_k = \gamma_k(x_0), \quad \forall k \geq 0, \qquad (3)$$

where $\gamma_k : \mathbf{R} \to \mathbf{Z}_M$ is the coder function.

Note that there is no explicit communication constraint between the controller and actuator. This is obviously a reasonable assumption if they are colocated, but even otherwise such a constraint would not change the formulation above, since the location of the controller is purely nominal. The symbols that would be transmitted by it over an additional link to the actuator would have to be translated once again into control signals, making intermediate calculations redundant. The number R should thus be regarded as the *overall* rate of the complete link from sensor to actuator. In addition, we remark that there are other ways in which the digital link can be defined [3, 4].

We call the pair of sequences $(\gamma, \upsilon) \triangleq (\{\gamma_k\}_{k\geq 0}, \{\upsilon_k\}_{k\geq 0})$ a *coder-controller* and our objective is to find one that minimizes an infinite horizon cost of the form

$$J_{m,\infty} \triangleq \limsup_{k\to\infty} \rho_k^{-1} \mathrm{E}|X_k|^m. \qquad (4)$$

This compares the asymptotic behaviour of the m-th moment of the state against some positive sequence $\{\rho_k\}_{k\geq 0}$ which serves as a rough benchmark of the desired system behaviour. Although this does not attach a cost to the magnitudes of the control signals or the transient state moments, it does succeed in capturing the asymptotic stochastic behaviour of the closed-loop system. However, before addressing it we first investigate the finite horizon criterion

$$J_{m,N} \triangleq \mathrm{E}|X_{N+1}|^m. \qquad (5)$$

This is done in the next section, with particular attention to the case of *Laplacian* X_0 with $m = 1$. The insights gained then lead to an optimal solution of the infinite horizon problem.

THE FINITE HORIZON PROBLEM

In this section, we show that the coder-controller that minimizes (5) is given by reformulating an optimal, M^N-level quantizer for X_0. Assuming trivially that $a_k \neq 0, \forall k \in [0, \ldots, N]$, define the mapping $\eta_k : \mathbf{Z}_M^k \to \mathbf{R}$ by

$$\eta_k(\tilde{t}_{k-1}) \triangleq -\sum_{j=0}^{k} \frac{\upsilon_j(\tilde{t}_{j-1})}{\prod_{i=0}^{j} a_i}, \ \ \forall k \in [0, \ldots, N]. \qquad (6)$$

Observe that there is a one-to-one correspondence between the function sequences $\tilde{\eta}_N$ and $\tilde{\upsilon}_N$, since this equation can be inverted to yield

$$\upsilon_k(\tilde{t}_{k-1}) = \left(\prod_{i=0}^{k} a_i\right)\left(\eta_{k-1}(\tilde{t}_{k-2}) - \eta_k(\tilde{t}_{k-1})\right), \quad (7)$$

for all $k \in [0,\ldots,N]$ and $\tilde{t}_{k-1} \in \mathbf{Z}_M^k$. Using (6) and (1), the cost (5) becomes

$$J_{m,N} = \left|\prod_{k=0}^{N} a_k\right|^m \mathrm{E}|X_0 - \eta_N(\tilde{S}_{N-1})|^m. \quad (8)$$

As the symbol sequence $\tilde{s}_{N-1}$ is completely determined by x_0, the R.H.S. is simply the mean m-th power error (MmPE) of an M^N-level quantizer for X_0, apart from the dynamical constants in front. The finite horizon problem is thus solved by setting $\{\eta_N(\tilde{t}_{N-1})\}_{\tilde{t}_{N-1}\in\mathbf{Z}_M^N}$ equal to the points of an optimal quantizer. The symbols s_k, $k = 0,\ldots,N-1$, are then given by the *nearest neighbour rule* $\tilde{s}_{N-1} = \arg\min_{\tilde{t}_{N-1}\in\mathbf{Z}_M^N} |x_0 - \eta_N^*(\tilde{t}_{N-1})|$, whereby the optimal quantizer point closest to x_0 is selected [9]. The expectation in (8) then simplifies to

$$\int \min_{\tilde{t}_{N-1}\in\mathbf{Z}_M^N} |x_0 - \eta_N^*(\tilde{t}_{N-1})|^m \, dP(x_0). \quad (9)$$

Note that the mappings η_k, $k = 0,\ldots,N-1$, may be chosen arbitrarily without affecting this integral. Equation (7) can then be used to yield optimal controller functions.

Unfortunately, optimal quantizers are usually impossible to obtain in closed form. Often they can only be obtained numerically for a given number of levels, using iterative techniques such as the Lloyd and Nitadori algorithms; see Lloyd [10], Noll and Zelinski [11] and Gersho and Gray [9] for details. One of the few exceptions to this rule occurs when X_0 has a *Laplacian* probability density $p(x_0) = \mathrm{e}^{-|x_0-\mu|/\epsilon}/(2\epsilon)$ and is quantized under the mean absolute error (MAE) criterion. In Fu and Wise [12], it was shown that the optimal, Q-level quantizer for X_0 is then defined by using the nearest neighbour rule on the points

$$q(j) \triangleq \mu + \epsilon\,\mathrm{sign}(2j+1-Q)\times \ln\left(\frac{\rho(Q)}{(Q+1-|2j+1-Q|)^2}\right), \quad (10)$$

where $j = 0,1,\ldots,Q-1$ and

$$\rho(Q) \triangleq \begin{cases} Q(Q+2) & \text{if } Q \text{ is even} \\ (Q+1)^2 & \text{if } Q \text{ is odd} \end{cases}. \quad (11)$$

The corresponding optimal MAE is given by

$$\mathrm{MAE}_Q^* = \begin{cases} \epsilon\ln(1+2/Q) & \text{if } Q \text{ is even} \\ 2\epsilon/(Q+1) & \text{if } Q \text{ is odd} \end{cases}. \quad (12)$$

The closed form of these equations allows the corresponding coder-controller to be explicitly derived:

Theorem 1 *If the initial state X_0 of the system (1) has a Laplacian distribution with mean μ and mean absolute deviation ϵ, then*

$$\min_{\gamma,\upsilon} J_{1,N} = \begin{cases} |\prod_{j=0}^{N} a_j|\epsilon\ln(1+\frac{2}{M^N}), \\ \text{if } M \text{ is even} \\ |\prod_{j=0}^{N} a_j|\frac{2\epsilon}{M^N+1}, \\ \text{if } M \text{ is odd} \end{cases} \quad (13)$$

and an optimal coder-controller is given by the following:

Coder: *Define $e_N : (0,1) \to \mathbf{R}$ by $e_N(y) \triangleq \frac{1}{2}[q(M^N y - 1) + q(M^N y)]$, where $q : (-1, M^N) \to \mathbf{R}$ is given by (10), with $Q = M^N$. Letting $c_N \triangleq e_N^{-1}$, the symbol s_k is then the $(k+1)$-th digit in the M-ary representation of $\zeta \triangleq c_N(x_0)$, where it can be shown that $\forall x_0 \in \mathbf{R}$,*

$$c_N(x_0) = \frac{1}{2} + \mathrm{sign}(x_0-\mu)\times \frac{M^N + 1 - \sqrt{1+\rho(M^N)\mathrm{e}^{-|x_0-\mu|/\epsilon}}}{2M^N}, \quad (14)$$

where ρ is defined by (11).

Controller: *Set*

$$\eta_N^*(\tilde{s}_{N-1}) = q(M^N\zeta_{N-1}), \quad (15)$$
$$\eta_k^*(\tilde{s}_{k-1}) = c_N^{-1}(\zeta_{k-1} + 1/(2M^k)), \quad (16)$$

for all $k < N$, where

$$\zeta_k \triangleq \sum_{j=0}^{k} s_j M^{-j-1}, \ \forall k \geq 0. \quad (17)$$

The controls are then given by (7).

Proof: As discussed above, equation (8) implies that the minimum cost is achieved by setting $\{\eta_N(\tilde{t}_{N-1})\}_{\tilde{t}_{N-1}\in\mathbf{Z}_M^N}$ equal to the optimal quantizer points $\{q(j)\}_{j\in\mathbf{Z}_Q}$ of (10). The expression (13) then follows immediately from (12). As the points $\eta_N^*(\tilde{t}_{N-1})$, $\tilde{t}_{N-1} \in \mathbf{Z}_M^N$, can be ordered arbitrarily without changing the integrand in (9), we fix the optimal mapping $\eta_N^* : \mathbf{Z}_M^N \to \mathbf{R}$ as

$$\eta_N^*(\tilde{t}_{N-1}) \triangleq q\left(\sum_{k=0}^{N-1} t_k M^{N-1-k}\right), \quad (18)$$

i.e. the argument of η_N^* is the M-ary representation of that of q. By the nearest neighbour rule, the sequence $\tilde{s}_{N-1}$ is transmitted iff one of the possibly two quantizer points closest to x_0 is $q\left(\sum_{k=0}^{N-1} s_k M^{N-1-k}\right) =$

$q(M^N\zeta_{N-1})$. Breaking ties by choosing the greater quantizer point, this means that $\tilde{s}_{N-1}$ is transmitted iff x_0 lies inside the interval

$$\left[e_N(\zeta_{N-1}),\, e_N(\zeta_{N-1}+M^{-N})\right) = \left[\frac{1}{2}\left(q(M^N\zeta_{N-1}-1)+q(M^N\zeta_{N-1})\right),\, \frac{1}{2}\left(q(M^N\zeta_{N-1})+q(M^N\zeta_{N-1}+1)\right)\right)$$

As e_N is continuous (except at $y=\frac{1}{2}\pm\frac{1}{2M^N}$ when M is even) and strictly increasing, this happens iff $e_N^{-1}(x_0)\in[\zeta_{N-1},\zeta_{N-1}+M^{-N})$. Referring to the expression above for ζ_{N-1}, this in turn is exactly equivalent to $\tilde{s}_{N-1}$ being the first N digits of the M-ary representation for $e_N^{-1}(x_0)=\zeta$. □

Several observations can be made here. Firstly, the optimal coder-controller has a classical *compander* as its core [9]. The *compressor* c_N maps $x_0\in\mathbf{R}$ to $\zeta\in(0,1)$ and a uniform, M^N-level quantizer maps this to ζ_{N-1}, which is then transformed by an *expander* $q(M^N\cdot)$ into an estimate $\hat{x}_{0|N}$ of the initial condition. Secondly, the choice of the mappings η_k^*, $k=0,\ldots,N-1$, is completely arbitrary, since they do not affect the finite horizon cost. However, the choice above makes the infinite horizon analysis of the next section a little easier. It is also intuitively appealing, since $\zeta_k+\frac{1}{2M^{k+1}}$ is the midpoint of the interval of length M^{-k-1} which the controller knows contains ζ, from the symbols $\tilde{s}_k$. Finally, although the MAE-optimal, Laplacian quantizer is unique, η_N^* can be defined in as many different ways as there are to map $\mathbf{Z}_{M^N}$ to $\mathbf{Z}_M^N$. The choice of mapping taken here, as implied in equation (18), is one of the more tractable ones. Other mappings are possible, but most of these can be shown to have disconnected coding regions that are difficult to implement.

INFINITE HORIZON COST

As the horizon N becomes arbitrarily large, the compressor (14) of the $J_{1,N}$-optimal, Laplacian coder-controller converges for each $x\in\mathbf{R}$ to

$$c(x)=\frac{1}{2}+\mathrm{sign}(x-\mu)\frac{1-\mathrm{e}^{-|x-\mu|/(2\epsilon)}}{2}. \quad (19)$$

In this section, we derive the limiting compressor for a broad class of initial condition distributions and any $m\geq 1$, without first solving the finite horizon problem.

The key is the classic result, proven rigorously in Bucklew and Wise [13] and derived more loosely in Gersho [14], that as the number of MmPE-optimal quantizer points approaches infinity, their normalized density approaches

$$\nu \triangleq \left(\int p^{1/(m+1)}(x_0)\,dx_0\right)^{-1} p^{1/(m+1)}, \quad (20)$$

under certain technical conditions on p. As $q(M^N\zeta_{N-1})$ is the $(M^N\zeta_{N-1}+1)$-th quantizer point, by (18), the nearest neighbour rule implies that there are $M^N\zeta_{N-1}+O(1)$ of them less than or equal to x_0. From definition (17) ζ_{N-1} must converge to a number $\zeta\in[0,1]$, hence

$$c(x_0)\triangleq\zeta=\lim_{N\to\infty}\frac{M^N\zeta_{N-1}+O(1)}{M^N} = \int_{y\leq x_0}\nu(y)\,dy,\quad \forall x_0\in\mathbf{R}. \quad (21)$$

This immediately leads to the following limiting scheme as the horizon N becomes unbounded:

Coder-Controller 1

Coder: *The symbol s_k is the $(k+1)$-th digit in the M-ary representation of $\zeta\triangleq c(x_0)$, where c is given by (21).*

Controller: *At time k, calculate*

$$\eta_k^*(\tilde{s}_{k-1})=c^{-1}(\zeta_{k-1}+1/(2M^k)), \quad (22)$$

where ζ_{k-1} is defined by (17), and use (7) to generate the control signal.

For instance, for a Laplacian initial condition with mean μ and mean absolute deviation ϵ, it can be shown that

$$c(x_0)=\frac{1}{2}+\mathrm{sign}(x_0-\mu)\frac{1-\mathrm{e}^{-|x_0-\mu|/((m+1)\epsilon)}}{2}.$$

which agrees with equation (19) derived for the special case of $m=1$. For a Gaussian initial condition with mean μ and standard deviation ϵ, we immediately have

$$c(x_0)=F\left((x_0-\mu)/(\epsilon\sqrt{m+1})\right),$$

where F is the unit normal distribution function.

We now consider whether the coder-controller above is actually optimal with respect to an infinite horizon cost of the form (4). First we need to fix the weights ρ_k, $k\geq 0$. Observe that

$$\begin{aligned}\lim_{N\to\infty}\min_{\gamma,v}&\frac{M^{mN}}{\prod_{k=0}^{N}|a_k|^m}\mathrm{E}|X_{N+1}|^m\\ &= \lim_{N\to\infty}\min_{\gamma,\eta}M^{mN}\mathrm{E}|X_0-\eta_N(\tilde{S}_{N-1})|^m,\\ &= (m+1)^{-1}2^{-m}\|p\|_{1/(m+1)}, \quad (23)\end{aligned}$$

where the last line is a well-known result of asymptotic quantization theory, requiring only that $\mathrm{E}|X_0|^{m+n} < \infty$, for some $n > 0$, and $\|p\|_{1/(m+1)} \triangleq \left(\int p(x_0)^{1/(m+1)}\, dx_0\right)^{m+1}$ [13]. This is nearly what we want, except that in (4) the minimization is to be performed after the limit is taken. This suggests that an appropriate choice of weighting sequence is

$$\rho_k = M^{-m(k-1)} \prod_{j=0}^{k-1} |a_j|^m, \ \forall k \geq 0. \quad (24)$$

In order to prove that the scheme above is optimal, we make use of the fact that it is essentially a compander. In Linder [15], it is shown that if a compressor $G : \mathbf{R} \to [0,1]$ satisfies the conditions

1. $G' \triangleq g \geq 0$ and is continuous,
2. $g(x_0)$ decreases with $|x_0|$ when $|x_0| > t$, for some $t > 0$,
3. $\mathrm{E}\{g(X_0)^{-m}\} < \infty$,
4. $\exists \delta > 0$: both $\int_0^\delta s(\zeta)^m h(2\zeta) d\zeta$ and $\int_{1-\delta}^1 s(\zeta)^m h(2\zeta - 1) d\zeta < \infty$, where $s \triangleq \frac{1}{gG^{-1}(\cdot)}$ and $h \triangleq \frac{pG^{-1}(\cdot)}{gG^{-1}(\cdot)}$,

then

$$Q^m \mathrm{E}|X_0 - G^{-1} f_Q G(X_0)|^m \to \frac{\mathrm{E}\{g(X_0)^{-m}\}}{(m+1)2^m} \quad (25)$$

as $Q \to \infty$, where f_Q is the Q-level, midpoint-based, uniform quantizer on $[0,1]$. We now state the main result of this section:

Theorem 2 *Let X_0 be governed by a continuous probability density function p which is decreasing with $|x_0|$ if $|x_0| > t$ and satisfies $\mathrm{E}|X_0|^{m+n} < \infty$, for some $n, t > 0$. Suppose further that*

$$\begin{aligned} pc^{-1}(2\zeta - 1) &\leq Apc^{-1}(\zeta), \ \forall \zeta \in [1-\delta, 1], \\ pc^{-1}(2\zeta) &\leq Apc^{-1}(\zeta), \ \forall \zeta \in [0, \delta], \end{aligned}$$

for some $A > 0, \delta \in (0, \frac{1}{2}]$, where c is given by (21). Then Coder-Controller 1 is $J_{m,\infty}$-optimal, with minimal cost

$$J^*_{m,\infty} = (m+1)^{-1} 2^{-m} \|p\|_{1/(m+1)}. \quad (26)$$

Hence the system (1) is asymptotically stochastically stabilizable, in the sense that there exists a coder-controller that takes $\mathrm{E}|X_k|^m \to 0$, if and only if

$$\lim_{k\to\infty} M^{-k} \prod_{j=0}^{k-1} |a_j| = 0. \quad (27)$$

Proof: Note that $\rho_k^{-1} \mathrm{E}|X_k|^m = M^{m(k-1)} \mathrm{E}|X_0 - \eta^*_{k-1}(\tilde{S}_{k-2})|^m$, where $\eta^*_{k-1}(\tilde{s}_{k-2}) = c^{-1} f_{M^{k-1}} c(x_0)$, with $f_{M^{k-1}}$ the M^{k-1}-level, midpoint-based, uniform quantizer on $[0,1]$. We show that p and the compressor c satisfy the conditions 1-4 of [15]. As $g = c' = \nu = \kappa^{-1} p^{1/(m+1)}$, where $\kappa \triangleq \int p^{1/(m+1)}(x_0) dx_0$, $G = c$ immediately satisfies conditions 1-3.[1] We now verify the first integral bound in condition 4, the second following by similar reasoning. Note that $h = \kappa p c^{-1}(\cdot)^{m/(m+1)}$, so that $h(2\zeta) \leq A' h(\zeta)$, $\forall \zeta \in [0, \delta]$, where $A' \triangleq A^{m/(m+1)}$. Hence

$$\begin{aligned} \int_0^\delta s(\zeta)^m h(2\zeta) d\zeta &\leq A' \int_0^\delta s(\zeta)^m h(\zeta) d\zeta, \\ &= A' \kappa^m \int_0^{c^{-1}(\delta)} p(x_0)^{1/(m+1)} dx_0, \end{aligned}$$

which is finite. Therefore (25) holds, yielding

$$\begin{aligned} &M^{m(k-1)} \mathrm{E}|X_0 - \eta^*_{k-1}(\tilde{(S)}_{k-2})|^m \\ &\to \quad \frac{\mathrm{E}\{\nu(X_0)^{-m}\}}{(m+1)2^m} = \frac{\|p\|_{1/(m+1)}}{(m+1)2^m}. \end{aligned}$$

Now observe that for any coder-controller (γ, η),

$$\begin{aligned} &\limsup_{k\to\infty} M^{m(k-1)} \mathrm{E}|X_0 - \eta_{k-1}(\tilde{S}_{k-2})|^m \\ &\quad \geq \lim_{k\to\infty} \min_{\gamma,\eta} M^{m(k-1)} \mathrm{E}|X_0 - \eta_{k-1}(\tilde{S}_{k-2})|^m \\ &\quad = (m+1)^{-1} 2^{-m} \|p_{X_0}\|_{1/(m+1)}, \end{aligned}$$

by (23). As the scheme above achieves this lower bound, it is optimal. The necessity and sufficiency of (27) for the existence of a coder-controller such that $\mathrm{E}|X_k|^m \to 0$ then follows immediately, using (24). □

Condition (27) can be interpreted as comparing the speed with which the the initial condition estimate becomes more accurate and that with which the state of the open-loop system changes. Note that it is automatically satisfied for asymptotically stable systems. The assumptions of this theorem are fairly restrictive but can be shown to be satisfied by Gaussian and Laplacian densities. We conjecture that the infimum of $J_{m,\infty}$ for Lebesgue-integrable p is given by (26), since it should be possible to construct companders G_i, $i \geq 0$, which satisfy conditions 1-4 above and approach c in an appropriate integral sense as $i \to \infty$. However, it is much more difficult to prove that Coder-Controller 1 actually achieves this lower bound for any such

[1] As remarked in [13], $\mathrm{E}|X_0|^{m+n} < \infty$ implies condition 3 is satisfied, by Hölder's inequality.

general p, despite being the limiting form of the optimal finite horizon scheme.

CONCLUSION

In this paper, a data-rate-constrained optimal control problem was formulated for one-dimensional linear systems with initial state probability density p. The optimal finite horizon coder-controller under a Laplacian p was explicitly derived and asymptotic quantization theory then used to directly obtain the limiting infinite horizon scheme for any p. Under certain technical conditions this scheme was shown to be optimal in an infinite horizon sense, yielding a necessary and sufficient condition for a system to be asymptotically stochastically stabilizable. Further work is currently being undertaken on relaxing the conditions on p and extending the results presented here to linear stochastic and nonlinear systems.

REFERENCES

[1] Delchamps, D. F., 1990, IEEE Trans. Autom. Contr., 35, 916–24.

[2] Liu, X. and Wong, W. S., 1997, Proc. 36th IEEE Conf. Dec. Contr., 60–5.

[3] Wong, W. S. and Brockett, R. W., 1999, IEEE Trans. Autom. Contr., 44, 1049–53.

[4] Borkar, V. S. and Mitter, S. K., 1997, "Communications, Computation, Control and Signal Processing", Dordrecht, Boston, 365–73.

[5] Tatikonda, S., Sahai, A. and Mitter, S. K., 1998, Proc. 37th IEEE Conf. Dec. Contr., 1165–70.

[6] Wong, W. S. and Brockett, R. W., 1997, IEEE Trans. Autom. Contr., 42, 1294–9.

[7] Nair, G. N. and Evans, R. J., 1997, Proc. 36th IEEE Conf. Dec. Contr., 866–71.

[8] Nair, G. N. and Evans, R. J., 1999, Proc. 14th IFAC World Cong., I, 19–24.

[9] Gersho, A. and Gray, R. M., 1993, "Vector Quantization and Signal Compression", Kluwer, Boston.

[10] Lloyd, S. P., 1982, IEEE Trans. Info. The., 28, 129–37.

[11] Noll, P. and Zelinski, R., 1979, IEEE Trans. Comm., 27, 1259–60.

[12] Fu, S. L. and Wise, G. L., 1984, Proc. 27th Midw. Symp. Circ. Syst., 369–372.

[13] Bucklew, J. A. and Wise, G. L., 1982, IEEE Trans. Info. The., 28, 239–47.

[14] Gersho, A., 1979, IEEE Trans. Info. The., 25, 373–80.

[15] Linder, T., 1991, Prob. Contr. Info. The., 20, 475–84.