Data rate theorem for stabilization over fading channels

(Invited Paper)

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***Abstract*—In this paper, we present a data rate theorem for stabilization of a linear, discrete-time, unstable dynamical system with arbitrarily large disturbances, over a noiseless communication channel with time-varying rates. Necessary and sufficient conditions for stabilization are derived, their implications and relationships with related results in the literature are discussed.**

I. Introduction

In modern control theory, the *data rate theorem* refers to the smallest feedback data rate above which an unstable dynamical system can be stabilized. In its scalar form, it states that a discrete linear plant of unstable mode $|\lambda| > 1$ can be stabilized if and only if the data rate R over the (noiseless) digital feedback link satisfies the inequality $R > \log_2 |\lambda|$ bits per sample, where $\tilde{H} = \log_2 |\lambda|$ is called the intrinsic entropy rate of the plant. From its first appearance, this result has been generalized to different notions of stability and system models, and has also been extended to multi-dimensional systems with time-varying rates [1] [3] [4] [7] [10] [15] [18] . The survey papers [2] and [11] give an historical and technical account of the various formulations.

In this paper, we are concerned with the formulation of the data rate theorem over a digital *wireless* channel. This poses a number of challenges unforseen in the classical wired scenario. In wireless communication, the quality of the communication link between transmitter and receiver varies over time because of random fading in the received signal. In the case of digital communication, this can reflect in a time variation of the rate supported by the wireless channel. The coherence time indicates the time interval over which the channel can be considered constant. If the coherence time is long enough, then transmitter and receiver can estimate the quality of the link by sending a known sequence called pilot, and can adapt the communication scheme to the channel condition. We ask the following question: is it possible to design a communication scheme that changes dynamically according to the channel condition and, at the same time, is guaranteed to stabilize the system?

To answer the above question we assume that the communication channel is noiseless and, at any given time k, allows transmission of R_k bits without error, where R_k fluctuates randomly over time. We model the coherence time by assuming that R_k remains constant in blocks of n consecutive channel uses and then varies according to an i.i.d process across blocks. Channel state information at the transmitter and receiver is reinterpreted here as causal knowledge, at both encoder and decoder, of the rate supported by the communication link, see Figure 1.

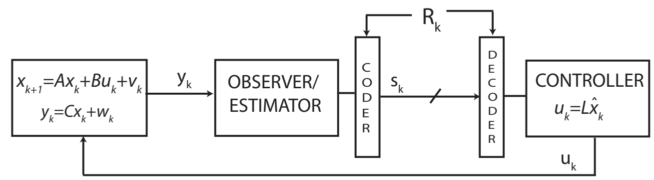

Fig. 1. Feedback loop model. The (encoded) estimated state s_k is sent to the controller over a wireless digital link that supports a rate of R_k bits per second.

This formulation can be placed in the context of the related literature. In an influential paper, Tatikonda and Mitter [15] have studied a similar model in which the rate is deterministic and system disturbances are bounded. This work has been later extended by Nair and Evans [10] to the more realistic case in which the rate is deterministic, but disturbances can have an unbounded support. Finally, Martins, Dahleh, and Elia [7] considered the case of a scalar system with random time-varying rate and bounded disturbances, and provided necessary and sufficient conditions for m-th moment stability. In our work, we allow both the system disturbances to have unbounded support *and* the rate to vary randomly. Furthermore, the encoder has access to output feedback rather than state feedback and we also consider the multi-dimensional case. This formulation is more practical and requires the use of an adaptive quantizer, as this must be capable of tracking the state when atypically large disturbances affect the system and must dynamically adapt to the rate that is instantaneously supported by the channel. Naturally, our results can recover the ones mentioned in the above papers.

We also want to spend a few words on a different approach that has been used in the literature to model control over wireless channels. This has a network-theoretic flavor rather than the information-theoretic one described above. In this case, instead of modeling the wireless channel using a time-varying bit rate, the channel uncertainty is modeled using random packet dropouts. Bits are grouped into packets that

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are considered as single entities that can be lost, each with some probability. Furthermore, channel state information is in this case modeled as packet acknowledgement at the transmitter. An extensive survey of different works following this approach appears in [5] and we refer the reader to this work for references. The network-theoretic equivalent of the data rate theorem is the proof of existence of a *critical dropout probability* above which the closed loop system cannot be stabilized, see for example [6] [13]. Our present paper reveals an important link between the two approaches. We can apply our results to an erasure channel, where the rate is R with probability $1-\epsilon$ and zero with probability ϵ. In the high data rate limit ($R \to \infty$), this channel can be seen as communicating real numbers with random i.i.d erasures, and in this case we obtain a necessary and sufficient condition for stabilization that is the same as the one in [6] and [13], obtained under the network theoretic model, with Bernoulli packet dropouts, acknowledgement of packet reception, and unbounded system disturbances.

Finally, we point out that while our formulation can be used to model erasures, by allowing the rate process to have value zero, we do not address more general channels with noise. In this case, data rate theorems are difficult to prove, as stabilization results generally depend on the given notion of stability and channel state information [8] [16] [17]. To overcome this drawback, Sahai and Mitter recently proposed an interesting alternative notion of information capacity, called *anytime capacity* [12]. Unlike Shannon's capacity, however, its main limitation is that it is often difficult to compute [11].

The rest of the paper is organized as follows. The main contributions are informally summarized and discussed next. Section III formally defines the problem. Section IV is devoted to the proof of the necessary and sufficient conditions for stabilizability in the scalar case. These are shown via the entropy-power inequality (necessary) and the construction of an adaptive, variable length encoder (sufficiency). Section V is devoted to the more complex multi-dimensional case, for which necessary and sufficient conditions are shown to be tight in some special cases.

II. Overview of the results

In the scalar case, we prove that a necessary and sufficient condition to stabilize a linear system of unstable mode $|\lambda| > 1$ in the second moment sense over a time-varying channel is,

$$\mathbb{E}\left[\left(\frac{|\lambda|^2}{2^{2R}}\right)^n\right] < 1, \tag{1}$$

where n is the length of the channel block during which the rate is constant.

The condition above is amenable to the following intuitive interpretation. If no information is sent over the link during a transmission block, the estimation error at the controller about the state of the system grows by $|\lambda|^{2n}$. The information sent by the encoder can reduce this error by at most 2^{2nR}, where nR is the total rate supported by the channel in a given block. However, if averaging over the fluctuation of the rate $|\lambda|^{2n}$ exceeds 2^{2nR}, then the information sent over the channel cannot compensate (on average) the dynamics of the system and it is not possible to stabilize the plant. Notice that if the variation of the rate is deterministic, then our condition reduces to the well known $R > \log_2 |\lambda|$. Similarly, it is also easy to see that when communicating over an erasure channel for which $R = \infty$ with probability $1-\epsilon$ and $R = 0$ with probability ϵ, then when $n = 1$ the necessary and sufficient condition for stabilization reduces to

$$\epsilon < \frac{1}{\lambda^2}, \tag{2}$$

which is the same critical loss probability derived in [6] [13] for systems with unbounded disturbances under the network-theoretic model.

The proof of the result is based on an information-theoretic argument based on the entropy-power inequality (necessary condition), and on an explicit construction of an adaptive quantizer (sufficient condition). In the latter case, the main challenge is to design a *successively refinable* quantizer that adapts dynamically to the rate process and can handle atypically large disturbances. The construction of the coder-decoder pair is similar to the one by Nair and Evans [10]. However, while in [10] the time variation of the transmission rate is part of the coder design, in the model under consideration the variation of $\{R_k\}_{k=1}^{\infty}$ is uniquely determined by a stochastic process. As the rate process is assumed constant over blocks of n channels uses, coder and decoder exploit the causal knowledge of the rate, transmitting one packet per block, and encoding each packet at the rate supported by the channel. Successive refinements of the state are transmitted to the decoder by coding across many channel blocks and exploiting the fact that the quantizer utilized is successively refinable.

The extension to multi-dimensional linear systems entails the difficulty of the rate allocation to the different unstable modes. In this case, we show that the stabilizability region has a special polymatroid structure. When the rate variation is deterministic this polymatroid reduces to

$$\tilde{H} := \sum_{|\eta_i| \geq 1} \log_2 |\eta_i| < R,$$

where $\eta_1, \ldots, \eta_n$ are the open loop eigenvalues (raised to their corresponding algebraic multiplicities). Again, this is in agreement with the well known data rate theorem for vector systems with deterministic rate ([10], [15]). Finally, as in the scalar case, in the high data rate limit over an erasure channel, we recover the necessary condition on the critical dropout probability of [13].

Finally, we provide a coder-decoder construction whose rate allocation can be applied to any rate distribution and show that this is optimal in some limiting cases.

III. Problem formulation

Consider the partially-observed, discrete-time state-space unstable stochastic linear system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{v}_k,\ \mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{w}_k,\ \forall k \in \mathbb{W}, \tag{3}$$

where $\mathbf{x}_k \in \mathbb{R}^f$ is the state process, $\mathbf{u}_k \in \mathbb{R}^m$ is the control input, $\mathbf{v}_k \in \mathbb{R}^f$ process disturbance, the measurement $\mathbf{y}_k$ and measurement noise $\mathbf{w}_k$ are random vectors in $\mathbb{R}^p$. For simplicity, suppose $\mathbf{A}$ is uniquely composed by unstable modes. No Gaussian assumptions are made on the initial condition $\mathbf{x}_0$ and the disturbances, but the following assumptions are supposed to hold

- **A0.** $(\mathbf{A}, \mathbf{B})$ is reachable and $(\mathbf{C}, \mathbf{A})$ observable.
- **A1.** $\mathbf{x}_0$,$\mathbf{v}_k$ and $\mathbf{w}_j$ are mutually independent for all $k, j \in \mathbb{W}$.
- **A2.** $\exists \epsilon > 0$ such that $\mathbf{x}_0$,$\mathbf{v}_k$ and $\mathbf{w}_j$ have uniformly bounded $(2+\epsilon)$th absolute moments over $k \in \mathbb{W}$.
- **A3.** $\inf_{k\in\mathbb{W}} h(\mathbf{v}_k) > -\infty$. Thus, $\exists \beta > 0$ such that $e^{\frac{2}{f}h(\mathbf{v}_k)} > \beta$ for all $k \in \mathbb{W}$ and $\mathbf{v}_k \in \mathbb{R}^f$.

Suppose that observer and controller are connected by a *time-varying* digital link. The transmission rate supported by the digital link is assumed constant over blocks of $n \in \mathbb{Z}_+$ channel uses but changes independently from block to block according to a given probability distribution. In each of the n uses of the digital link during the j-th block symbols coming from an alphabet of size 2^{R_j} are transmitted without error from encoder to controller. Let $S_k \in \{1, \dots, 2^{R_j}\}$ denote the symbol sent at time $k \in \{jn, \dots, (j+1)n - 1\}$. An underlying stochastic process determines the alphabet size 2^{R_j} in each block. Suppose that for all $j \in \mathbb{W}$ R_j are i.i.d random variables distributed as R, where R can only take integer values, such that

$$\Pr\{R = r\} = p_r, \quad r \in \mathcal{R} \subseteq \mathbb{W},$$

for $0 \le r_{min} \le r \le r_{max} < \infty$, $\sum_{r\in\mathcal{R}} p_r = 1$. At the beginning of each block j, coder and decoder are assumed to know R_j (and hence all $\{R_i\}_{i=0}^{j}$), while the realization of the rate process in future blocks, $\{R_i\}_{i=j+1}^{\infty}$, is unknown to them.

Each transmitted symbol can depend on all past and present measurements, the present channel state and the past symbols,

$$S_k = g_k(\mathbf{y}_0, \dots, \mathbf{y}_k, S_0, \dots, S_{k-1}, R_j),$$
$$k \in \{jn, \dots, (j+1)n - 1\}, \forall j \in \mathbb{W}, \tag{4}$$

where $g_k(\cdot)$ is the coder mapping at time k. The control sequence, on the other hand, can depend on all past and present channel symbols

$$\mathbf{u}_k = f_k(S_0, \dots, S_k), \quad \forall k \in \mathbb{W}, \tag{5}$$

where $f_k(\cdot)$ is the controller mapping at time k. From (5),(4) and (3) observe that $\mathbf{x}_{jn}$ does not depend on R_j and furthermore $\mathbf{x}_{jn} \to S_0^{j-1} \to R_j$ form a Markov chain.

The aim is to construct a coder-decoder which stabilizes the plant in the mean square sense

$$\sup_{k\in\mathbb{W}} \mathbb{E}\left[\| \mathbf{x}_k \|^2\right] < \infty, \tag{6}$$

using the finite data rate provided by the time-varying digital link.

IV. Scalar systems

In this section it is assumed that the plant in (3) is scalar and has a representation of the following type:

$$x_{k+1} = \lambda x_k + u_k + v_k,\ y_k = x_k + w_k,\ \forall k \in \mathbb{W}, \tag{7}$$

where $|\lambda| > 1$, so that the system is unstable. The result for the scalar case is now stated:

Theorem 4.1: Under assumptions A1.-A3. above, necessary and sufficient condition for stabilizability in the mean square sense is that

$$\mathbb{E}\left[\left(\frac{|\lambda|^2}{2^{2R}}\right)^n\right] < 1, \tag{8}$$

where n is the length of the channel block with the same rate.

The proof of the necessity is omitted here and can be found in the full version of the present paper [9].

A. Sufficiency

We first describe the adaptive quantizer that is based on the construction given in [10]. A fundamental property of this quantizer that we exploit is then stated as a lemma, whose proof appears in [10].

1) Quantizer: The quantizer partitions the real line into non-uniform regions, in such a way that most of the regions are concentrated in a symmetrical way around the origin. A parameter $\rho > 1$ determines the speed at which the quantizer range increases. The quantizer generates 2^ν, $\nu \ge 0$, quantization intervals labeled from left to right by $I_\nu(0), \dots, I_\nu(2^\nu - 1)$. Let $I_0(0) := (-\infty, \infty)$, $I_1(0) := (-\infty, 0]$ and $I_1(1) := (0, \infty)$. If $\nu \ge 2$ the quantization intervals are generated by

- partitioning the set $[-1, 1]$ into $2^{\nu-1}$ intervals of equal length,
- partitioning the sets $(\rho^{i-2}, \rho^{i-1}]$, $[-\rho^{i-2}, -\rho^{i-1})$ into $2^{\nu-1-i}$ intervals of equal length, $i \in [2, \dots, \nu - 1]$.

The two open sets $(-\infty, -\rho^{\nu-2}]$ and $(\rho^{\nu-2}, \infty)$ are respectively the leftmost and rightmost intervals of the quantizer. Let

- $\kappa_\nu(\omega)$ be half-length of interval $I(\omega)$ for $\omega \in [1, \dots, 2^{n-2}]$, be equal to $\rho^\nu - \rho^{\nu-1}$ when $\omega = 2^\nu - 1$ and equal to $-(\rho^\nu - \rho^{\nu-1})$ when $\omega = 0$.
- $q_\nu(x) := \bar{\omega}_\nu(\omega)$ be midpoint of interval $x \in I(\omega)$ for $\omega \in [1, \dots, 2^{\nu-2}]$, be equal to ρ^ν when $\omega = 2^\nu - 1$ and equal to $-\rho^\nu$ when $\omega = 0$.

A property of this construction is that $\forall \nu \ge 2$ the quantization intervals $I_\nu(\cdot)$ can be generated recursively starting from $q_2(\cdot)$. In fact, for any $2 \le m \in \mathbb{Z}_+$ the quantizer intervals for $q_{m+1}(\cdot)$ are formed by partitioning each bounded interval $I_m(\omega)$ $\omega \in [1, \ldots, 2^{m-2}]$ into two uniform subintervals, and partitioning the semi-infinite interval $I_m(0) = (-\infty, -\rho^{m-2}]$ into two intervals $I_{m+1}(0) = (-\infty, -\rho^{(m+1)-2}]$ and $I_{m+1}(1) = (-\rho^{(m+1)-2}, -\rho^{m-2}]$ and, similarly, partitioning the semi-infinite interval $I_m(2^m - 1) = (\rho^{m-2}, \infty)$ into two intervals $I_{m+1}(2^{m+1}-2) = (\rho^{m-2}, \rho^{(m+1)-2}]$ and $I_{m+1}(2^{m+1}-1) = (\rho^{(m+1)-2}, +\infty)$.

If a real number x is in $I(\omega)$ for some $\omega \in \{0, \ldots, 2^{\nu-1}\}$, then the quantizer approximates x with $\bar{\omega}_\nu(\omega)$. The quantization error is not uniform over $x \in \mathbb{R}$, but is bounded by $\kappa_\nu(\omega)$ for all $\omega \in \{1, \ldots, 2^{\nu-2}\}$. A fundamental property of the quantizer is that the average quantization error diminishes like the inverse square of the number of levels, $2^{-2\nu}$. More precisely, if the $(2+\epsilon)$-th moment of X is bounded for some $\epsilon > 0$, then an upper bound of the second moment of the estimation error decays as $2^{-2\nu}$. The higher moment of X is useful to bound the estimation error (using Chebyshev's inequality) when X lies in one of the two open intervals $(\rho^{\nu-1}, \infty)$ and $(-\infty, -\rho^{\nu-1}]$.

Let L be a strictly positive random variable, define the functional

$$M_\epsilon[X, L] \equiv \mathbb{E}[L^2 + |X|^{2+\epsilon} L^{-\epsilon}]. \tag{9}$$

The functional $M_\epsilon[X, L]$ is an upper bound to the second moment of X:

$$\mathbb{E}[|X|^2] = \mathbb{E}[|X|^2(1_{|X| \leq L} + 1_{|X|>L})] \leq M_\epsilon[X|L]. \tag{10}$$

Define the conditional version of $M_\epsilon[X, L]$ given a random variable R as $M_\epsilon[X, L|R] \equiv \mathbb{E}[L^2 + |X|^{2+\epsilon} L^{-\epsilon}|R]$. The fundamental property of the quantizer described above is given by the following result:

Lemma 4.2: [10, Lemma 5.2] Let X and $L > 0$ and $R \geq 0$ be random variables with $\mathbb{E}X^{2+\epsilon} < \infty$ for some $\epsilon > 0$, and $n \in \mathbb{W}$. If $\rho > 2^{2/\epsilon}$, then the quantization error $X - Lq_{nR}(X/L)$ satisfies

$$M_\epsilon[X - Lq_\nu(X/L), L\kappa_{nR}(\omega)|R] \leq \frac{\zeta}{2^{2nR}} M_\epsilon[X, L], \tag{11}$$

for some finite constant $\zeta > 2$ determined only by ϵ and ρ.

Next, the coder and controller are described.

2) Coder: The first stage of the encoding process consists of computing the linear minimum variance estimator of the plant state based on the previous measurements and control sequences. The filter process satisfies a recursive equation of the same form as (7), namely

$$\bar{x}_{k+1} \;=\; \lambda \bar{x}_k + u_k + z_k, \ \forall z \in \mathbb{W}. \tag{12}$$

where the $(2+\epsilon)$-th moment of the innovation z_k can be shown to be bounded, under assumption A2. From the orthogonality principle the stability of $\bar{x}$ is equivalent to that of x. The output $\bar{x}_k$ of the filter (or a function of it) must be transmitted using the finite number of bits supported on the digital channel. Coder and decoder share a state estimator $\hat{x}_k$ based uniquely on the symbols sent over the digital link. Since $\hat{x}_k$ is available both at the coder and decoder, while the minimum variance estimator is available at the coder only, the encoder utilizes a predictive quantizer to encode the error between $\bar{x}_k$ and $\hat{x}_k$. The error is scaled by an appropriate coefficient and then recursively encoded using the quantizer in section IV-A.1. An accurate approximation of the error is obtained by transmitting the quantization index across many channel blocks. The fact that the random rate available at future times is not known in advance is not a problem, as the quantizer is successively refinable and can dynamically adapt to the rate that is instantaneously supported by the channel. By transmitting for a large enough number of blocks, the error between the two estimator can be kept bounded.

Time $k \in \mathbb{W}$ is divided into *cycles* $[jn\tau, \ldots, (j+1)n\tau]$, $j \in \mathbb{W}$, of integer duration $n\tau \in \mathbb{Z}_+$. Notice that each cycle consists of τ channel blocks.

At time $k = jn\tau$, just before the start of the j-th cycle, the coder sets the quantization rate equal to nR_{jn}, i.e. the rate in the first channel block in the j-th cycle, and computes

$$\bar{\omega}_{nR_{jn}}(\omega_{jn}) = q_{nR_{jn}}(x_j), \quad x_j := \frac{\bar{x}_{jn\tau} - \hat{x}_{jn\tau}}{l_j}. \tag{13}$$

where l_j is a scaling factor updated at the beginning of each cycle. This factor is utilized to scale $\bar{x}_{jn\tau} - \hat{x}_{jn\tau}$ close to the origin, where the quantizer provides better estimates. The index $\omega_{jn\tau}$ of the quantization level is converted into a string of $nR_{jn\tau}$ bits and transmitted using the n channel uses of the $jn\tau$-th channel block. Denote with $I_{nR_{jn}}(\omega_{jn\tau})$ the quantization interval labeled by $\omega_{jn\tau}$. After the first n transmissions in the cycle, coder and decoder agree on the fact that $x_j \in I_{nR_{jn}}(\omega_{jn\tau})$. The remaining $(n-1)\tau$ transmissions in the cycle are devoted to reducing the size of the uncertainty interval $I_{nR_{jn}}(\omega_{jn\tau})$.

At time $k = jn\tau + n$, the rate $R_{jn\tau+n}$ supported during the next channel block becomes known at both coder and decoder. Thus, coder and decoder divide up $I_{nR_{jn}}(\omega_{jn\tau})$ into $2^{nR_{jn\tau+n}}$ sub-intervals in the manner described above (uniform partitions of bounded intervals and exponential partition of semi-infinite intervals), sequentially generating the partitions $I_{nR_{jn}+nR_{jn\tau+n}}(\cdot) \subseteq I_{nR_{jn}}(\omega_{jn\tau})$ of the quantizer $q_{nR_{jn}+nR_{jn\tau+n}}(x_j)$

Then, the coder sends to the decoder the index of the sub-interval containing x_j. At the end of the second channel block in the cycle, coder and decoder agree on the fact that $I_{nR_{jn}+nR_{jn\tau+n}}(\omega_{jn\tau+n})$.

Continue this process until the end of the τ-th channel block. After receiving the last sequence of bits, the decoder computes the final uncertainty interval $I_{v_j}(\omega_j)$, $\omega_j := \omega_{(j+1)n\tau-n}$, corresponding to the uncertainty set formed by the quantizer $q_{v_j}(x_j)$, where the random variable

$$\nu_j = nR_{jn\tau} + nR_{jn\tau+n} + ... + nR_{(j+1)n\tau-n} \tag{14}$$

indicates the cumulative number of bits sent in the j-th cycle.

Before the $(j+1)$-th cycle, the coder updates the state estimator as follows,

$$\begin{aligned}\hat{x}_{(j+1)n\tau} = \lambda^{n\tau}[\hat{x}_{jn\tau} + l_j q_{v_j}(x_j)] + \\ + \sum_{k=jn\tau}^{(j+1)n\tau-1} \lambda^{(j+1)n\tau-1-k} L\hat{x}_k,\end{aligned} \tag{15}$$

where

$$\hat{x}_{k+1} = (\lambda + L)\hat{x}_k, \quad \forall k \in [jn\tau, \dots, jn\tau + n - 2] \quad (16)$$

and $\hat{x}_0 = 0$. L is the certainty-equivalent control coefficient such that $\lambda + L < 1$. Finally the scaling coefficient l_j is updated as follows

$$l_{j+1} \;=\; \max\{\sigma, l_j|\lambda|^{n\tau}\kappa_{\nu_j}(\omega_{(j+1)n\tau-n})\}, \quad (17)$$

with $l_0 = \sigma$ and where $\sigma^{2+\epsilon}$ is a uniform bound in the $(2+\epsilon)$-moment of

$$g_j := \sum_{i=0}^{n\tau-1} \lambda^{n\tau-i} z_{jn\tau+i}, \quad j \in \mathbb{W}. \quad (18)$$

It is important to note, from (17) and the independence of the rate process, that the l_j does not depend on the rate process during the j-th cycle.

3) Controller: At time $k = jn\tau$ coder and controller are synchronized and have common knowledge of the state estimator $\hat{x}_{jn\tau}$. During times $jn\tau, \dots, (j+1)n\tau - 2$, the controller sends to the plant a certainty-equivalent control signal

$$u_k = L\hat{x}_k \quad \forall k \in [jn\tau, \dots, (j+1)n\tau - 2], \quad (19)$$

where $\hat{x}_k$ is updated as in (16). At the end of the each channel block in the j-th cycle, the decoder receives estimates of the states in the way described above.

At time $(j+1)n\tau - 1$, once received $q_{v_j}(x_j)$ the controller updates the estimator $\hat{x}_{(j+1)n\tau}$ using (15). Synchronism between coder and observer is ensured by the fact that the initial value $\hat{x}_0$ is set equal to zero at both coder and decoder, and by the fact that the digital link is noiseless.

4) Analysis: In this section it is shown that the coder/controller pair described above ensures that the second moment of $\bar{x}$ is bounded if (8) is satisfied. Define the coder error at time $k \in \mathbb{W}$ as $f_k = \hat{x}_k - \bar{x}_k$.

The following analysis is developed in three steps. First it is shown that f_k is bounded for all times $k = jn\tau$ $j \in \mathbb{W}$, i.e. the beginning of each cycle. Next, the analysis is extended to all $k \in \mathbb{W}$. Finally the stability of f_k for all $k \in \mathbb{W}$ is shown to imply that $\bar{x}$ (and so x) is bounded.

First we show that the coder error $f_k = \hat{x}_k - \bar{x}_k$ is bounded in mean square for all times $k = jn\tau$, $j \in \mathbb{W}$. Instead of looking at $\mathbb{E}[f_{jn\tau}^2]$, it is more convenient to consider the functional $M_\epsilon[X, L]$ defined in (9), with $X = f_{jn\tau}$ and $L = l_j$. Thus, let

$$\theta_j := M_\epsilon[f_{jn\tau}, l_j] \equiv \mathbb{E}[l_j^2 + |f_{jn\tau}|^{2+\epsilon} l_j^{-\epsilon}].$$

Equation (10) implies that $\mathbb{E}[f_{jn\tau}^2] < \theta_j$. Therefore, it suffices to show that $\sup_{j\in\mathbb{W}} \theta_j < \infty$.

Substituting (19) into (12), and iterating over a block duration,

$$\bar{x}_{(j+1)n\tau} = \lambda^{n\tau}\bar{x}_{jn\tau} + g_j + \sum_{k=jn\tau}^{(j+1)n\tau-1} \lambda^{(j+1)n\tau-1-k}(L\hat{x}_k), \quad (20)$$

where g_j is defined in (18). So subtracting (20) from (15),

$$\begin{aligned} f_{(j+1)n\tau} &= \hat{x}_{(j+1)n\tau} - \bar{x}_{(j+1)n\tau} \\ &= \lambda^{n\tau}[f_{jn\tau} - l_j q_{v_j}(x_j)] + g_j. \end{aligned} \quad (21)$$

Notice that, by assumption A3., the $(2+\epsilon)$-th moment of $f_{jn\tau}$ can be shown to be bounded for any finite $j \in \mathbb{W}$. Next, $f_{(j+1)n\tau}$ is used to derive an expression for θ_{j+1}. From the inequality $(|x| + |y|)^\alpha \le 2^{\alpha-1}(|x|^\alpha + |y|^\alpha)$ $\forall \alpha > 0$,

$$|f_{(j+1)n\tau}|^{2+\epsilon} \le \phi\left(|\lambda^{n\tau}|^{2+\epsilon}|f_{jn\tau} - l_j q_{\nu_j}(f_{jn\tau}/l_j)|^{2+\epsilon} + |g_j|^{2+\epsilon}\right),$$

with $\phi = 2^{1+\epsilon}$. Dividing by l_{j+1}^ϵ and taking expectations,

$$\begin{aligned} &\mathbb{E}[|f_{(j+1)n\tau}|^{2+\epsilon} l_{j+1}^{-\epsilon}] \\ &\le \phi\left(|\lambda^{n\tau}|^{2+\epsilon}\mathbb{E}\left[\frac{|f_{jn\tau} - l_j q_{\nu_j}(f_{jn\tau}/l_j)|^{2+\epsilon}}{l_{j+1}^\epsilon}\right] + \mathbb{E}\left[\frac{|g_j|^{2+\epsilon}}{l_{j+1}^\epsilon}\right]\right) \\ &\le \phi\left(|\lambda^{n\tau}|^{2+\epsilon}\mathbb{E}\left[\frac{|f_{jn\tau} - l_j q_{\nu_j}(f_{jn\tau}/l_j)|^{2+\epsilon}}{[l_j|\lambda^{n\tau}|\kappa_{\nu_j}(\omega_j)]^\epsilon}\right] + \mathbb{E}\left[\frac{|g_j|^{2+\epsilon}}{\sigma^\epsilon}\right]\right) \\ &= \phi\left(|\lambda^{n\tau}|^2\mathbb{E}\left[\frac{|f_{jn\tau} - l_j q_{\nu_j}(f_{jn\tau}/l_j)|^{2+\epsilon}}{[l_j\kappa_{\nu_j}(\omega_j)]^\epsilon}\right] + \mathbb{E}\left[\frac{|g_j|^{2+\epsilon}}{\sigma^\epsilon}\right]\right), \end{aligned} \quad (22)$$

where we used the fact that, from (17), $l_{j+1} = \max\{\sigma, l_j|\lambda^{n\tau}|\kappa_{\nu_j}(\omega_j)\}$. Observe that $\mathbb{E}[l_{j+1}^2] \le \sigma^2 + |\lambda^{n\tau}|^2\mathbb{E}\left[|l_j\kappa_{\nu_j}(\omega_j)|^2\right]$. Adding this to (22), using $\mathbb{E}[g_j^{2+\epsilon}] < \sigma^{2+\epsilon}$ and the definition of θ_j,

$$\begin{aligned} \theta_{j+1} &\le \phi\left(2\sigma^2 + |\lambda^{n\tau}|^2\mathbb{E}_{\nu_j}\left[\frac{\zeta}{2^{2\nu_j}} M_\epsilon\left[f_{jn\tau}|\, l_j\right]\right]\right) \\ &= \phi 2\sigma^2 + \phi\zeta\left(\mathbb{E}\left[\frac{|\lambda^2|^n}{2^{2nR}}\right]\right)^\tau \theta_j, \end{aligned}$$

where the inequality follows from Lemma (4.2), and the equality uses the fact that the rate process is i.i.d. and that f_{jn} and l_j are independent of $R_{jn\tau}, R_{jn\tau+n}, ..., R_{(j+1)n\tau-n}$ because of the causality constraint. Therefore, θ_j evolves according to the following recursive equation

$$\theta_{j+1} \le \phi 2\sigma^2 + \phi\zeta\left(\mathbb{E}\left[\left(\frac{|\lambda^2|}{2^{2R}}\right)^n\right]\right)^\tau \theta_j.$$

It follows that if $\mathbb{E}\left[\left(\frac{|\lambda^2|}{2^{2R}}\right)^n\right] < 1$, then by making τ sufficiently large we can ensure that the coefficient of θ_j is strictly less than 1. Thus we have established that θ_j remains bounded in the limit as j going to infinity and therefore $\sup_{j\in\mathbb{W}} \theta_j < \infty$. Hence, from (10) it follows that $\sup_{j\in\mathbb{W}} \mathbb{E}[f_{jn}]^2 < \infty$.

Next, for any $k \in \{0, \dots, n-1\}$ the triangle inequality implies $|f_{jn+k}| \le |\lambda|^k|f_{jn}| + \sum_{i=0}^{k-1}|\lambda^{k-1-i}L| \cdot |z_{jn+k}|$, so the error f_k is bounded for all $k \in \mathbb{W}$. Finally, by rewriting (12) as $\bar{x}_{k+1} = (\lambda + L)\bar{x}_k - Lf_k + z_k$,the fact that f_k and z_k are bounded and that $\lambda + L < 1$ ensures that $\mathbb{E}\bar{x}_k^2 < \infty$ for all $k \in \mathbb{W}$.

V. Systems of order higher than one.

In this section, we consider the case of multi-dimensional unstable linear systems. A necessary condition for stabilizability is derived using information theoretic techniques. It is proved that the stabilizability region is contained inside a polyhedron with a polymatroid structure. A sub-optimal coder-decoder construction is provided and its optimality is shown in some limiting regimes. The main difficulty in stabilizing a multi-dimensional system over time-varying channels consists of allocating optimally the bits to each unstable sub-system. The scheme proposed works for *any* rate distribution. For some specific rate distributions, however, it is possible to design more efficient schemes. We illustrate this point at the end of this section, studying the specific problem of stabilization over an erasure channel.

A. Real Jordan form

As usual, it is convenient to put $\mathbf{A}$ into real Jordan canonical form so as to decouple its dynamical modes. Denote the system matrix in real Jordan canonical form as $\mathbf{J}$. The matrices $\mathbf{J}$ and $\mathbf{A}$ are related via a similarity matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{J}\mathbf{T} = \mathbf{A}$. Let $\lambda_1, \ldots, \lambda_u \in \mathbb{C}$ be the distinct unstable eigenvalues (with conjugates excluded) of $\mathbf{A}$, and let m_i be the algebraic multiplicity of each λ_i. The real Jordan canonical form $\mathbf{J}$ then has the block diagonal structure $\mathbf{J} = \text{diag}(\mathbf{J}_1, \ldots, \mathbf{J}_u) \in \mathbb{R}^{f\times f}$, where the block $\mathbf{J}_i \in \mathbb{R}^{\mu_i\times\mu_i}$ and $\det \mathbf{J}_i = \lambda_i^{\mu_i}$, with

$$\mu_i = \begin{cases} m_i & \text{if } \lambda_i \in \mathbb{R} \\ 2m_i & \text{otherwise.} \end{cases} \tag{23}$$

As $\mathbf{A}$ is uniquely composed by unstable systems, we have that $\sum_{i=1}^{u} \mu_i = f$. Let $\mathcal{U} := [1, \ldots, u]$ denote the index set of unstable systems. Then, the dynamical system equation can be written as

$$\mathbf{x}_{k+1} = \mathbf{J}\mathbf{x}_k + \mathbf{T}\mathbf{B}\mathbf{u}_k + \mathbf{T}\mathbf{v}_k \ \in \mathbb{R}^f, \ \mathbf{y}_k = \mathbf{x}_k + \mathbf{w}_k \ \in \mathbb{R}^p, \tag{24}$$

with $\mathbf{x}_k = \left[\mathbf{x}_k^{(1)}, \ldots, \mathbf{x}_k^{(u)T}\right]^T \in \mathbb{R}^f$, and where each sub-system $\mathbf{x}_k^{(i)}$ evolves according to

$$\mathbf{x}_{k+1}^{(i)} = \mathbf{J}_i\mathbf{x}_k^{(i)} + (\mathbf{T}\mathbf{B}\mathbf{u}_k)^{(i)} + (\mathbf{T}\mathbf{v}_k)^{(i)} \ \in \mathbb{R}^{\mu_i} \ i \in \mathcal{U}. \tag{25}$$

As the states of (24) and (3) are related through the transformation matrix $\mathbf{T}$, in the following we will assume that the system evolves according to (24).

B. Necessity

Theorem 5.1 ([9]): Necessary condition for stabilizability in the mean square sense of the system in (3) is that

$$\sum_{i\in\mathcal{U}} a_i s_i \log_2|\lambda_i| < -\frac{\mu'(\mathbf{s})}{2n}\log_2 \mathbb{E}\left[2^{-\frac{2n}{\mu'(\mathbf{s})}R}\right] \tag{26}$$

for all $s_i \in \{0, \ldots, m_i\}$ and $i \in \mathcal{U}$, where $\mu'(\mathbf{s}) \equiv \sum_{i\in\mathcal{U}} a_i s_i$, and $a_i = 1$ if $\lambda_i \in \mathbb{R}$, and $a_i = 2$ otherwise.

The region determined in Theorem 5.1 has a special combinatorial structure.

Proposition 5.2 ([9]): The polyhedron defined by (26) is a polymatroid.

Remarks:

1. When the rate process is deterministic, the constraints in (26) reduce to the well known condition ([10],[15])
$$\sum_{|\eta_i|\geq 1} \log_2|\eta_i| \equiv \sum_{i\in\mathcal{U}} \mu_i \log_2|\lambda_i| < R, \tag{27}$$
and the stabilizability reduces to the region in the positive orthant strictly inside the hyperplane $\sum_{i\in\mathcal{U}} \mu_i \log_2|\lambda_i| = R$.
2. In the limit as n goes to infinity, (26) reduce to
$$\sum_{i\in\mathcal{U}} \mu_i \log_2|\lambda_i| < r_{min}, \tag{28}$$
and the stabilizability region is determined uniquely by r_{min}. As the digital link supports the same rate for an arbitrarily long time interval, the system has to guarantee stability under the worst possible rate. In the limit, stabilization is not possible for those channels where $r_{min} = 0$ (e.g. erasure channels).
3. In an erasure channel, for a fixed n, as r goes to infinity the stabilizability reduces to the n-dimensional cube described by
$$\log_2|\lambda_i| < \frac{1}{2n}\log_2\frac{1}{1-p} \quad \forall i.$$
In other words, the system in (24) cannot be stabilized if the erasure probability is such that
$$1-p \geq \frac{1}{\max_i|\lambda_i|^{\frac{2}{n}}},$$
In the case $n = 1$, this is the same condition derived in [13] in the context of the LQG problem with erasures.

C. Sufficiency

We now present a sufficient condition for second-moment stabilizability of a multi-dimensional system. The scheme is based on the adaptive quantizer introduced in section IV-A.1. We introduce a rate allocation vector which indicates what fraction of the available rate is allocated to each unstable sub-system.

Theorem 5.3 ([9]): Sufficient condition for stabilizability in the mean square sense of the system in (3) is that $(\log_2|\lambda_1|, \ldots, \log_2|\lambda_u|)$ are inside the convex hull of the region determined by

$$\log_2|\lambda_i| < -\frac{1}{2n}\log_2 \mathbb{E}\left[2^{-\frac{2n}{\mu_i}\alpha_i(R)R}\right], \forall i \in \mathcal{U}, \tag{29}$$

for some rate allocation vector $\boldsymbol{\alpha}(R) := [\alpha_1(R), \ldots, \alpha_u(R)]$ such that,

$$\begin{cases} \alpha_i(r) \in [0,1] \\ \frac{1}{\mu_i}\cdot\alpha_i(r)\cdot nr \in \mathbb{N} \\ \sum_{i=1}^{u}\alpha_i(r) \leq 1, \end{cases}, \quad \forall r \in \mathcal{R}\setminus\{0\}, i \in \mathcal{U}. \tag{30}$$

Suppose in a given channel block nr bits are available for transmission. The rate allocation vector indicates what is the fraction of these bits allocated to a particular sub-system. Each of the μ_i sub-modes the i-th sub-system is quantized with $\frac{\alpha_i(r)nr}{\mu_i}$ bits. Clearly, $\frac{\alpha_i(r)nr}{\mu_i}$ must be an integer number, and the total number of bits utilized in each block should not

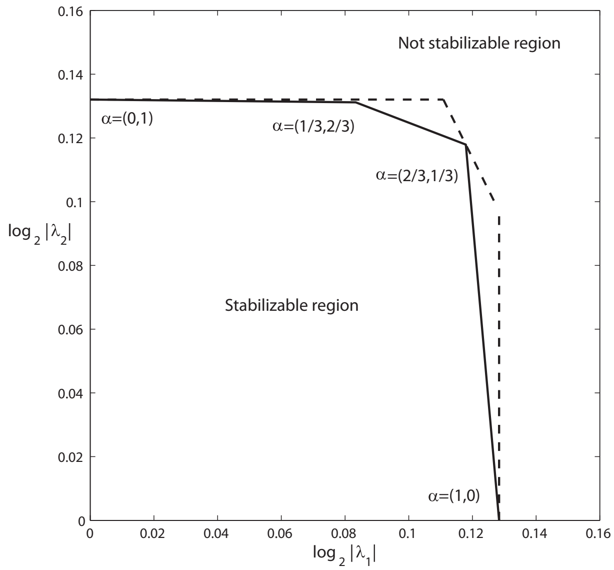

Fig. 2. Necessary conditions (dotted lines) and achievable region in Theorem 5.3 (solid lines) for stabilizability of a two-mode system communicating over a binary erasure channel with p=2/3, n=6. Mode λ_1 has dimensionality two, while λ_2 has dimensionality one.

exceed nr. Eqs. (30) state precisely these conditions. Before looking at the proof of the Theorem, consider the following Example:

Example 5.1: Consider a two-mode system with two distinct eigenvalues (λ_1, λ_2), where λ_1 has dimensionality $\mu_1 = 2$ while λ_2 has dimensionality $\mu_2 = 1$. Suppose that digital channel is a binary erasure channel with $n = 6$. In each channel block where $R = 1$, there are four allocations satisfying (30), namely $\alpha_1(1) = 1 - \alpha_2(1) = \frac{j}{6}$, for $j \in \{0, 2, 4, 6\}$. By time-sharing among these four policies, we can achieve all the points in the convex hull. Figure 2 shows the achievable stabilizability region when $p = \frac{2}{3}$.

Remarks:

1. If $\frac{rn}{f} \in \mathbb{N}$ for all $r \in \mathcal{R}$, then the rate allocation $\alpha^{(i)}(r) = \frac{\mu_i}{f}\ \forall r \in \mathcal{R} \setminus \{0\}$ is optimal when $\lambda := \lambda_1 = \ldots = \lambda_u$. In fact, from (29) sufficient condition for stabilizability is that
$$\log_2|\lambda| < -\frac{1}{2n}\log_2 \mathbb{E}\left[2^{-\frac{2n}{f}R}\right]. \qquad (31)$$
On the other hand, this condition is also necessary, as we can see from (26) when $s_i = m_i$ for all $i \in \mathcal{U}$. For example, in Example 5.1 we have that $nr = 6$ and $f = 3$, thus the rate allocation $\boldsymbol{\alpha} = (1/3, 2/3)$ is optimal (See Figure 2).
2. The scheme in Theorem 5.3 is optimal in the limit of n going to infinity, and the optimal coding scheme consists of time-sharing among the rate allocations $\boldsymbol{\alpha}^{(i)}(R) = \boldsymbol{e}_i$ for all $i \in \mathcal{U}$, where $\{\boldsymbol{e}_i\}_{i=1}^{f}$ are the canonical basis vectors of $\mathbb{R}^f$.
3. In an erasure channel, for a fixed n, as r goes to infinity the proposed achievable scheme is asymptotically optimal. Furthermore, the optimal coding scheme consists of time-sharing among the rate distributions $\boldsymbol{\alpha}^{(i)}(r) = \boldsymbol{e}_i$ for all $i \in \mathcal{U}$ and the allocation given in Remark 1., i.e. $\alpha^{(i)(u+1)}(r) = \frac{\mu_i}{f}$.
4. When the rate process changes deterministically, [10] showed that the necessary and sufficient conditions coincide. Once again, the optimal coding scheme consists of time-sharing among the rate distributions $\boldsymbol{\alpha}^{(i)}(R) = \boldsymbol{e}_i$ for all $i \in \mathcal{U}$.

VI. Conclusion

Motivated by control problems over wireless fading channels, we considered mean square stabilizability of a discrete-time, linear system with a noiseless time-varying digital communication link. Process and observation disturbances were allowed to occur over an unbounded support. Necessary conditions were derived employing information theoretic techniques, while a stabilization scheme based on an adaptive successively refinable quantizer was constructed. In the scalar case, this scheme was shown to be optimal. Furthermore, we have shown that in the vector case the necessary condition for stabilization has an interesting polymatroid structure, and have proposed a stabilization scheme that is optimal in some limiting regimes. An additional contribution is that we bridged the information-theoretic results of stabilization over rate limited channels, with the corresponding network-theoretic ones on critical dropout probabilities in systems with unbounded disturbances [13]. We have done so by recovering the latter results as a special case of our analysis.

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