Singularity-Free Joint Actuation in Omnidirectional Mobile Platforms With Powered Offset Caster Wheels

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This paper presents the analysis of singularity and motion capability of a mobile platform articulated by offset powered caster wheels. Specifically, it presents the analysis of the equations of motion resulting in the sufficient and necessary actuation condition to yield a workspace that is entirely free of singular configurations. This paper shows that powering both the steer and drive joints on two sets of offset caster wheels in a mobile platform guarantees a singularity-free condition throughout the entire workspace. Analysis and discussion on equations of motions that lead to this result are presented. [DOI: 10.1115/1.2885512]

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1 Introduction
Mobile platforms capable of omnidirectional planar motion form a significant part of mobile robots, especially in indoor applications. This is due to the space constraint in the indoor environment, such as in the narrow corridors, sharp turning corners, narrow door openings, and human traffic, where high degrees of maneuverability are required. An analysis of the degrees of kinematic mobility has been carried out in the past, such as in Refs. [1–4].

Omnidirectional motion on mobile platforms can be realized through the utilization of different types of wheel mechanisms, such as ball wheels, omnidirectional wheels, and offset caster wheels. A single ball wheel is kinematically capable of delivering omnidirectional motion to the mobile robot it is attached to [5,6]. However, in practice, it is difficult to actuate a ball wheel and to maintain its contact with the ground and with the actuation. Actuation is generally done through a direct rolling contact with the ball, which makes it sensitive to the dirt on the surface of the wheel. An omnidirectional wheel, often referred to as a Swedish wheel, mecanum wheel, or Ilnon wheel, is realized by placing roller bearings on the circumference of a conventional wheel, with the axes of the roller bearings placed nonparallel to the axis of the conventional wheel [7]. Three of these wheels are required to realize a planar 3DOF omnidirectional motion for a mobile robot. Motion using Swedish wheels are inherently nonsmooth due to the intermittent contact between rollers and the ground. There are, however, designs to minimize the nonsmoothness [8].

An offset caster wheel is not an omnidirectional wheel by itself. However, several of these wheels would provide omnidirectional motion to a mobile base. Each set of offset caster wheel is constructed out of a steer and a drive joint. The main feature that lends the ability to create an omnidirectional motion is the offset between the drive wheel axis and the steering joint axis [9].

The mobility of a mobile platform constructed out of a powered offset caster wheel is the subject of study in this paper. The mobility of the resulting mobile platform is often evaluated with different sets of actuated joint configurations. In many designs, there exists more than three actuated joints in the arrangement, making the robot an over-constrained closed-chain system. Several variations of methods have been presented to accommodate this over-constrained system. In Ref. [10], a set of independent joints was selected out of all the available actuated joints, while the rest of the actuated joints are defined as dependent joints. A given task space motion can produce an inverse kinematic solution on the joint space span by the three independent joints. A differential kinematic relationship between the independent and dependent joints was analytically determined and used to produce actuation reference to the dependent joints. In Ref. [4], velocities of the contact points between individual caster wheels with the ground were calculated and used as the reference for the joint velocities of each caster wheel. An efficient generalized inverse of the differential kinematic relationship was proposed to increase the sensitivity of the control strategy to the joint error in the over-constrained system. This approach of treating the mobile base as an over-constrained system in kinematics and as the sum of the contributions of individual wheel dynamics was derived from the augmented object model proposed in Ref. [11], which was implemented on a mobile manipulator with the powered caster omnidirectional platform Nomad XR4000 [12]. It should be noted that as the mobile platform and its wheels can be considered as an assembly of rigid links, any error (ill matching) in the task space motion generated by the set of caster wheels at the operational point would manifest in the slippage of the wheel-ground contact. Beyond generating the required joint velocities, the performance of the control algorithm in this over-constrained case can be improved by synchronization control [13], where the error of one set of caster wheels is “crossed over” to the control of other sets of caster wheels, i.e., used as a feedback to the control of other sets of caster wheels. This produces a result where the wheels would attempt to “synchronize” with one another.

The mobility and singularities of the mobile platform with powered caster wheels have been studied in the past, such as on the isotropicity of the design [1], on the effect of actuating different joints in a three-wheeled mobile base [2], and on kinematic modeling of the mobile base with caster wheels [10,14–16]. In this paper, a more general case of singularity-free condition for the entire workspace of a mobile robot with powered offset caster wheels is presented. Some references have been made in the past, such as in Ref. [14], to the examples of “admissible arrangements” for a singularity-free omnidirectional mobile robot. However, this condition was not formally studied and analytically demonstrated. In this paper, the mathematical proof of the sufficient and necessary condition of a singularity-free actuation strategy for an omnidirectional mobile robot is presented.

2 Kinematic Modeling
Many variations of kinematics models have been proposed for parallel-actuated mobile robots [10,17,18]. It is briefly defined in this paper so that it is possible to follow the symbolic proof that is presented. A mobile robot is defined as having N sets of powered caster wheels. Each of these N wheels could be completely powered (both its steer and drive joints) or selectively powered (only the steer or only the drive joint). (Note that a completely passive caster wheel, i.e., one whose steer and drive joints are not powered, is not included in the consideration of N). Figure 1 shows an offset caster wheel that is studied in this paper, with offset b and
radius $r$. It has a steer joint $\phi$ and drive joint $\rho$ that is modeled as a prismatic joint with displacement $r\rho$. In this paper, we follow the definition of an offset caster wheel; therefore, $b > 0$. Figure 2 shows the frame assignment and parameter definition of the mobile robot, treating the problem in two dimensions. Wheel $i$ has an offset of length $b_i$ and radius of $r_i$. The steer angle of wheel $i$ is defined as $\phi_i$, and the drive angle displacement as $\rho_i$. Frame $B$ is defined as the frame attached to the center of the mobile base and rotates with the base. The location of the origin of Frame $B$ is on the horizontal plane of the base. The location of the wheel steering joint attachment point to the horizontal base, with respect to Frame $B$ is defined by vector $h_i$, of length $h_i$, forming an angle $\beta_i$ with the $x$ axis of Frame $B$. The projection of the contact point between the wheel $i$ and the ground onto the horizontal base is defined as point $C_i$, and its position with respect to Frame $B$ is defined by vector $p_{C_i}$. A frame $C_i$ is defined with its origin at point $C_i$, with axis $y_{C_i}$ defined along the translational motion of the wheel due to $\rho_i$ and the axis $x_{C_i}$ perpendicular to $y_{C_i}$ so that it results in axis $z_{C_i}$ pointing vertically upward. All vectors $p_{C_i}$, $h_i$, $x_{C_i}$, $y_{C_i}$, $x_B$, and $y_B$ are on the same horizontal plane (i.e., base).

The kinematic relationship of a caster wheel $i$ with the operational point of the mobile robot can be derived by equating the velocities of the contact points $C_i$ as generated by the task space velocity at the center of the base ($\dot{x}=[v, \omega]^T$) and those generated by the joint space velocities ($[\dot{\phi}, \dot{\rho}]$). When expressed in Frame $B$, this relationship is described by the following equation:

$$Bv + b\omega \times p_{C_i} = b\phi \dot{h}_{C_i} + r\dot{\rho}$$

where $\dot{h}_{C_i}$ and $\dot{\rho}_{C_i}$ (expressed in Frame $B$) are

$$\begin{align*}
\dot{h}_{C_i} &= \begin{bmatrix} -\sin(\beta_i - \phi_i) \\
\cos(\beta_i - \phi_i) \end{bmatrix} \\
\dot{\rho}_{C_i} &= \begin{bmatrix} -\cos(\beta_i - \phi_i) \\
-\sin(\beta_i - \phi_i) \end{bmatrix}
\end{align*}$$

A vector $\dot{t}_{C_i}$ was obtained by rotating $p_{C_i}$ by $90$ deg anticlockwise and keeping the same magnitude. The result of the cross product $b\omega \times p_{C_i}$ is simplified as $\omega b\dot{t}_{C_i}$, where $\omega$ is the scalar value of the rotation of the base around the vertical $Z$ axis (see Fig. 2).

Equation (1) is rearranged into

$$\begin{bmatrix} I_{3 \times 2} & b_v \ \ & b_\omega \end{bmatrix} \begin{bmatrix} \dot{v} \\
\dot{\omega} \end{bmatrix} = \begin{bmatrix} b_\rho \dot{h}_{x_{C_i}} \\
\dot{r}_{y_{C_i}} \end{bmatrix} \begin{bmatrix} \dot{\phi}_i \\
\dot{\rho}_i \end{bmatrix}$$

(3)

For convenience, contact point velocities are expressed in individual Frame $C_i$ to reveal the individual contribution of the steer and drive joint motions of each wheel $i$. This is done by premultiplying both sides of Eq. (3) with rotation matrix $^B R_B \in \mathbb{R}^{3 \times 3}$, with $^B R_B = [x_{C_i}, y_{C_i}, z_{C_i}]^T$. The resulting equation of motion for the mobile robot, taking into account all the available joints, is expressed as

$$A \dot{x} = A \begin{bmatrix} b_v \\
\dot{\rho} \end{bmatrix} = Bq = B$$

(4)

where

$$A = \begin{bmatrix} R_{C_1} T_{C_1} & R_{C_1} T_{C_1} & \cdots & R_{C_1} T_{C_1} \\
R_{C_2} T_{C_2} & R_{C_2} T_{C_2} & \cdots & R_{C_2} T_{C_2} \\
\vdots & \vdots & \ddots & \vdots \\
R_{C_N} T_{C_N} & R_{C_N} T_{C_N} & \cdots & R_{C_N} T_{C_N} \\
\end{bmatrix}$$

(5)

$$B = \text{diag}[b_1, r_1, \ldots, b_N, r_N]$$

(6)

The odd and even rows of Eqs. (5) and (6) describe the velocity of contact point $C_i$ in the direction perpendicular ($x_{C_i}$) and inline ($y_{C_i}$) to wheel $i$, respectively (see Fig. 2), or, in other words, the velocity of $C_i$ as generated by the steer joint ($\phi_i$) and by the drive joint ($\rho_i$) of wheel $i$, respectively.

Equation (4) describes the kinematic relationship between the task space velocity of the mobile robot with all the available joints in the robot. Occasionally, it is desired to power only a subset of the available joints, with the assumption that the un-powered joints are fully back-drivable; otherwise it may not be practical as these joints might introduce resistance and unaccounted dynamics to the system.

Let the total number of actuated (active) joints be $n_a$. Let $k$ denote the active joint where $k=1, 2, \ldots, n_a$. In this case, matrices $A$ and $B$ define the relationship between task space velocities and the set of active joints $q_a$,

$$A^k \dot{x} = B q_a$$

(7)

where
\[
A = \begin{bmatrix}
    s_k^T (s_k^T \mathbf{t}_{C_i}) \\
    : \\
    : \\
    s_n^T (s_n^T \mathbf{t}_{C_i})
\end{bmatrix}
\]  
\tag{8}

and
\[
B = \text{diag}[d_1, d_2, \ldots, d_n]
\]  
\tag{9}

where \( s_k \) is \( B \mathbf{t}_{C_i} \) if the joint of interest is the steer joint (\( \phi_i \)) or \( B \mathbf{y}_{C_i} \) if the joint of interest is the drive joint (\( \rho_i \)). Matrix \( B \) is of size \( n_r \times n_r \), and \( d_1 = b_k \) if \( q_k \) is a steer joint and \( d_k = r_k \) if it is a drive joint.

Note that only the actuated joints are included in the equations. Passive joints are not included. In fact, if a wheel is completely passive, it is not included at all in the consideration. A completely passive wheel does not affect the mobility of the mobile base. It may be added for other purposes, such as to maintain balance as a third wheel when there are only two sets of actuated wheels or as an additional measure of odometry as a passive wheel is less likely to slip [19].

3 Singularity Analysis

Treating the mobile robots as closed-chained mechanisms, singular configurations occur when any one of the conditions are satisfied [20,21],

\begin{itemize}
    \item A is rank deficient
    \item B is rank deficient
\end{itemize}

where \( A \) and \( B \) are as defined by Eqs. (4)–(6) for complete actuation or (7)–(9) when selective actuation is involved. In this case, it is clear from Eqs. (6) and (9) that matrix \( B \) is always of full rank in both cases of complete and selective actuation. This is because \( r_i \) and \( b_i \) are always positive values greater than 0.

As defined earlier, \( n_{ra} \) is the number of actuated joints. It should be noted that matrix \( A \) is square only when \( n_{ra} = 3 \). That is also the minimum number of degrees of freedom that is required to produce a planar 3DOF motion. For cases where \( n_{ra} > 3 \), then 3DOF motion is possible when \( \det(A^T A) \) does not equate to zero.

In the following subsections, it is shown that complete actuation on two wheels are the necessary and sufficient condition for an omnidirectional mobile robot with caster wheels to be singularity-free throughout its workspace. The first subsection shows that a complete actuation on two wheels produces no singularity. The second shows that any additional actuation will not alter the singularity-free condition. The third subsection shows that anything less than a complete actuation of two caster wheels will still experience singular configurations, even if it possessed the same number of actuators (i.e., four actuated joints).

3.1 Complete Actuation on Two Wheels. When both the steer (\( \phi_i \)) and drive (\( \rho_i \)) joints are actuated for all wheels, \( A^T A \) simplifies to

\[
A^T A = \begin{bmatrix}
    N_1 & \frac{1}{L} \sum_{i=1}^{N} \mathbf{t}_{C_i} \\
    \frac{1}{L} \sum_{i=1}^{N} \mathbf{r}_{C_i} & \frac{1}{L} \sum_{i=1}^{N} \mathbf{t}_{C_i}^T \mathbf{r}_{C_i}
\end{bmatrix}
\]  
\tag{10}

where \( L \) is the characteristic length \( L \) obtained by

\[
L^2 = \frac{\sum_{i=1}^{N} \| \mathbf{t}_{C_i} \|^2}{N}
\]  
\tag{11}

It can be shown that in the case of a complete actuation and when \( N \geq 2 \), the determinant of \( A^T A \) for the mobile robot with a complete actuation on all its wheels can be expressed as

\[
\det(A^T A) = N^2 \sum_{i=1}^{N} \mathbf{t}_{C_i}^T \mathbf{r}_{C_i} - N \left( \sum_{i=1}^{N} \mathbf{t}_{C_i} \right) \left( \sum_{i=1}^{N} \mathbf{t}_{C_i}^T \right)
\]  
\tag{12}

Using the fact that

\[
\sum_{i=1}^{N} \| \mathbf{t}_{C_i} \| \geq \sum_{i=1}^{N} \mathbf{t}_{C_i}
\]  
\tag{13}

then

\[
\det(A^T A) = N^2 \sum_{i=1}^{N} \mathbf{t}_{C_i}^T \mathbf{r}_{C_i} - N \left( \sum_{i=1}^{N} \mathbf{t}_{C_i} \right) \left( \sum_{i=1}^{N} \mathbf{t}_{C_i}^T \right) > 0
\]  
\tag{14}

Equations (12)–(14) show that complete actuation guarantees a singularity-free condition. As a mobile platform requires a minimum of three actuated joints to display 3DOF planar motion; then, a minimum of two sets of completely actuated caster wheels is required for Eq. (14) to hold.

3.2 Effect of Additional Wheels. It can also be deduced that if two sets of wheels that are completely actuated provide enough degrees of mobility to guarantee the entire workspace to be singularity-free, then any additional wheels, even if they are selectively actuated, will not alter the singularity-free property of the workspace.

This can be also be verified by the application of the Binet–Cauchy identity,

\[
\det(A^T A) = \sum_{i=1}^{M} m_i^2
\]  
\tag{15}

where \( m_i \) are determinants of the \( 3 \times 3 \) minors of the \( A \) matrix. As matrix \( A^T \) is of size \( 3 \times n_r \), then there are \( M = n_r C_i \) minors from the combination of its columns.

If the first two wheels are completely actuated, then the first four columns of the matrix would form \( C_i \) minors \( 3 \times 3 \), which also make up a part of the \( M \) minors. From Eq. (14), it is guaranteed that the sum of the determinant of these first four minors would be positive. From Eq. (15), it is therefore shown that any additional actuation to the two completely actuated wheels would only add positive values to the determinant and will not cause any additional singularities. It is therefore shown that having two completely actuated sets of caster wheels is a sufficient condition of singularity-free workspace.

3.3 Necessary Condition. In this section, an analysis is performed to show that having two sets of completely actuated wheels is the necessary condition for a singularity-free workspace. To do that, it is necessary to show that singular configurations exist for a mobile platform with powered caster wheels if there exist less than two completely actuated wheels.

3.3.1 Three Actuated Joints. Firstly, the case of \( n_{ra} = 3 \) is analyzed. Without loss of generalities, it is defined that wheel 1 is completely actuated and that wheel 2 is only selectively actuated. Singular configurations can be identified by solving for \( \det(A) = 0 \).

Assuming that all wheels are of the same radius and offset length, and that \( h_j \) is also the same for all wheels, the expression for vector \( \mathbf{t}_{C_i} \) is

\[
\mathbf{t}_{C_i} = R_{d}(90 \text{ deg}) \mathbf{p}_{C_i}
\]  
\tag{16}

where for the 2D case, \( R_{d}(90 \text{ deg}) \) is defined as

\[
R_{d}(90 \text{ deg}) = \begin{bmatrix}
    \cos(90 \text{ deg}) & -\sin(90 \text{ deg}) \\
    \sin(90 \text{ deg}) & \cos(90 \text{ deg})
\end{bmatrix}
\]  
\tag{17}

and
For \( q_a = [\phi_1, \rho_1, \phi_2] \), the singular configurations are obtained as
\[
P_c = \begin{bmatrix} h \cos(\beta_1) + b \cos(\beta_2 - \phi_1) \\ h \sin(\beta_1) + b \sin(\beta_2 - \phi_1) \end{bmatrix}
\]
which leads to
\[
hS_{\phi_1} - hS_{\beta_1 - \beta_2 - \phi_2} - bS_{\beta_1 - \beta_2 - \phi_1 + \phi_2} = 0
\]
where \( S_q \) and \( C_q \) are the notations for \( \sin(q) \) and \( \cos(q) \), respectively. Given the joint displacement of wheel 1 (\( \phi_1 \)), the two steer angle displacements of joint 2 that would produce a singular configuration \( (\phi_{2S}) \) is expressed in a closed-form solution as
\[
\phi_{2S} = \tan^{-1} \left( \frac{-hS_{\beta_1 - \beta_2} - bS_{\beta_1 - \beta_2 - \phi_1}}{h - hC_{\beta_1 - \beta_2} - bC_{\beta_1 - \beta_2 - \phi_1}} \right) + k\pi
\]
where \( k = 0,1 \).

For the case of \( q_a = [\phi_1, \rho_1, \phi_2] \), the singular configurations are expressed as
\[
b + hC_{\phi_1} - hC_{\phi_1 - \phi_2 + \phi_2} - bC_{\phi_1 - \phi_2 + \phi_1 + \phi_2} = 0
\]
Equation (21) is rearranged to
\[
b + T_1C_{\phi_1} + T_2S_{\phi_2} = 0
\]
where
\[
T_1 = h - hC_{\beta_1 - \beta_2} - bC_{\beta_1 - \beta_2 - \phi_1}
\]
Utilizing the tangent half angle formulas, where
\[
\cos(\theta) = \frac{1 - t^2}{1 + t^2}
\]
the two closed-form solutions for \( \phi_2 \) that produce singular configurations with the given \( \phi_1 \) are obtained by substituting Eqs. (24) and (25) and solving the resulting quadratic equation in terms of \( t \).

Fig. 3 Example of a singular configuration for a case with three actuated joints. The actuated joints are \([\phi_k, \rho_k, \phi_3] \).

The singular configurations for the case of \( q_a = [\phi_2, \rho_2, \phi_3] \) and \( q_a = [\phi_1, \rho_1, \phi_2] \) are shown in Figs. 3 and 4, respectively. In the figures, the wheel that is completely actuated is labeled as wheel \( k \), and the wheel that is selectively actuated is labeled as wheel \( i \).

This shows that singular configurations exist for \( n_a = 3 \).

3.3.2 Four Actuated Joints. It is also necessary to show that singular configurations exist for \( n_a = 4 \) when the four actuated joints do not form two sets of completely actuated wheels. This would demonstrate that the criterion of having two sets of completely actuated wheels is a necessary condition for singularity-free omnidirectional motion.

Without loss of generality, the existence of singular configurations can be analyzed in the case of a mobile platform with three sets of caster wheels with the assumption that wheel 1 is completely actuated, while wheels 2 and 3 are selectively actuated. Essentially, a singular configuration can be thought of as two simultaneous occurrences of the three-actuated-joint cases between the completely actuated wheel \( k \) and the two selectively actuated wheels, as covered in Sec. 3.3.1. The effect of the singular configurations of both cases is the loss of degree of freedom to rotate about axis \( Z_k \), which in our case is the vertical axis at point \( C_1 \). It therefore follows that when wheels 1 and 2 form a singular configuration simultaneously as wheels 1 and 3 form a singular configuration (as defined in Sec. 3.3.1), the mobile platform of powered castor wheels with four actuated joints will be singular. Hence, it is demonstrated that without two completely actuated wheels, an omnidirectional mobile platform with offset powered caster wheels is not free of singular configurations.

To demonstrate the explanation above, a singular configuration is analyzed for the actuation scheme where wheel 1 is completely actuated, and wheel 2 is actuated on its steer joint and wheel 3 on its drive joint, i.e., the actuated joints are \( q_a = [\phi_1, \rho_1, \phi_2] \). As matrix \( A \) is now of \( 4 \times 3 \) dimension, singularity exists when det(\( A^T A \))=0. The Binet–Cauchy identity is utilized, and the determinants of the minors of the \( A \) matrix are expressed as
\[
m_1 = \det([a_{\phi_1} | a_{\rho_1} | a_{\phi_2}])
\]
\[
m_2 = \det([a_{\phi_1} | a_{\rho_1} | a_{\phi_2}])
\]
\[
m_3 = \det([a_{\phi_1} | a_{\rho_1} | a_{\phi_2}])
\]
where
\[
a_{\phi_1} = (S_{\phi_1} T_{\phi_1} C_{\phi_1})^T
\]
To show that these configurations are singular, it is necessary to show that \( m_1 = m_2 = m_3 = m_4 = 0 \).

It can be calculated that the explicit expressions of the minors are
\[
m_1 = b + hC_{\phi_2} - hC_{\phi_1 - \phi_2 + \phi_2} - bC_{\phi_1 - \phi_2 + \phi_1 + \phi_2}
\]
\[
m_2 = hS_{\phi_2} - hS_{\beta_1 - \phi_2 + \phi_1} - bS_{\beta_1 - \phi_2 + \phi_1 + \phi_3}
\]
\[
m_3 = -(b + hC_{\phi_2})C_{\phi_1 - \phi_2 + \phi_1 + \phi_3} + (b + hC - \phi_1)C_{\phi_1 - \phi_2 + \phi_1 + \phi_3}
\]
\[
-hS_{\beta_1 - \phi_2 + \phi_1 + \phi_3} - hS_{\phi_2} + bS_{\phi_2}
\]
The expressions \( m_1 \) and \( m_2 \) are shown to correspond directly to the cases of three actuated joints on two sets of wheels (21) and (19), as shown in Sec. 3.3.1. It can also be shown that when \( m_1 = 0 \) and \( m_2 = 0 \), then \( m_3 = 0 \) and \( m_4 = 0 \).

Therefore, it is demonstrated that singular configurations exist in mobile platforms with offset caster wheels with four actuated wheels. This concludes the proof that a complete actuation of two sets of offset caster wheels is the necessary and sufficient condition for a guaranteed singularity workspace of the resulting mobile platform.

4 Summary

In this paper, it is demonstrated that for an omnidirectional mobile platform with powered caster wheels, a singularity-free workspace can be achieved throughout when there is at least two sets of completely actuated wheels. This condition is shown as the minimum requirement, as any additional wheels, whether they are completely or selectively actuated, do not change this property. It is also shown as the necessary condition, as any cases with less number of completely actuated joints are shown in this paper to exhibit singular configurations within the workspace.

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References


