On non-local stability properties of extremum seeking control

Y. Tan, D. Nešić and I.M.Y.Mareels

The Department of Electrical & Electronics Engineering, The University of Melbourne, Melbourne, VIC 3010, Australia

Abstract

In this paper, we consider several extremum seeking schemes and show under appropriate conditions that these schemes achieve extremum seeking from an arbitrarily large domain of initial conditions if the parameters in the controller are appropriately adjusted. This *non-local* stability result is proved by showing semi-global practical stability of the closed-loop system with respect to the design parameters. We show that reducing the size of the parameters typically slows down the convergence rate of the extremum seeking controllers and enlarges the domain of the attraction. Our results provide guidelines on how to tune the controller parameters in order to achieve extremum seeking. Simulation examples illustrate our results.

1 Introduction

In many control applications the reference-to-output map has an extremum and the control objective is to regulate the output close to this extremum. For instance, consider the system:

$$\dot{\mathbf{x}} = f(\mathbf{x}, u), \qquad y = h(\mathbf{x}), \tag{1}$$

and suppose that there exists a unique $\mathbf{x}^*$ such that $y^* = h(\mathbf{x}^*)$ is the extremum of the map $h(\cdot)$. Due to uncertainty, it is often reasonable to assume that neither $\mathbf{x}^*$ nor $h(\cdot)$ are precisely known to the control designer. The main objective in extremum seeking control is to force the solutions of the closed loop system to eventually converge to $\mathbf{x}^*$ and to do so without any precise knowledge about $\mathbf{x}^*$ or $h(\cdot)$. Extremum seeking control is an old topic that was first investigated in the early 1950's under the names of "extremum regulation" [12,14] and "automatic optimization" [9–11]. Most results in the 1950's and 1960's focused on finding the optimal value of a static mapping and stability issues were largely ignored. It was not until the year 2000 that stability of an extremum seeking feedback scheme was proved rigorously in [6]. This sparked a new interest in the area and generated numerous new results and applications [4,3,7,6,16,17,27–29].

The analysis in [6] was based on classical singular perturbations and averaging results and a *local stability* result was demonstrated. Results in [6] do not characterize the achieved domain of attraction. However, in most engineering applications it is very useful to obtain an estimate of the domain of attraction or prove global or non-local stability in cases when this is possible.

The main purpose of this paper is to prove *non-local stability* properties of several extremum seeking controllers that are closely related to the ones considered in [6]. In other words, our results include explicit statements about the achieved domain of attraction for the closed loop. Moreover, we show under appropriate conditions that the considered extremum seeking controllers achieve *semi-global practical stability* of the closed loop system. In other words, given an arbitrarily large set of initial conditions B_Δ and an arbitrarily small neighborhood B_ν of the state $\mathbf{x}^*$ where the output achieves its extremum $y^* = h(\mathbf{x}^*)$, it is possible to adjust the controller parameters so that all solutions starting from the set B_Δ eventually converge to B_ν. At the same time we show that reducing the parameters in the controller reduces the speed of convergence of the algorithm and this poses a tradeoff that the designer needs to resolve when tuning the controller. To the best of our knowledge this is the first proof of non-local and semi-global practical stability properties of extremum seeking controllers in [6] with explicit bounds on convergence speed. Non-local stability results for a different class of extremum seeking controllers can be found in [27].

Several extremum seeking schemes are discussed. First, we investigate stability of a first order extremum seeking scheme, where the extremum seeking controller consists of an integrator and an appropriate excitation signal. This simplified extremum seeking scheme is of interest in its own right and we are not aware of whether it has already appeared in the literature. We show how to tune the controller parameters in order to achieve semi-global

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practical stability of the closed loop. This is done by showing that appropriate tuning of the controller leads to a desired separation of the two time scales in the closed loop system. The stability proof is simpler for this scheme and it lends itself to a novel interpretation of its operation via gradient optimization algorithms. Second, we consider several higher order variations of the first order extremum seeking controller, such as the scheme discussed in [6], by including various low-pass and high-pass filters. We show that all these schemes achieve semi-global practical stability in appropriately chosen parameters and provide guidelines for tuning the controller parameters. Our results are illustrated via two simulation examples.

It is worthwhile to note that our proof technique is novel (different from [6]) and is partly based on Lyapunov techniques and recent new developments in the theory of averaging [13,25,26] and singular perturbations [1,24] that are tailored for analysis of semi-global practical stability of systems that exhibit time scale separation. We do adopt, however, the same idea of time scale separations as in [6] that naturally leads to the use of singular perturbations and averaging techniques.

The paper is organized as follows. In Section 2 we present preliminaries, followed by the problem formulation in Section 3. The main results are stated in Section 4. Possible extensions are discussed in Section 5. An example with simulations is given in Section 6 and conclusions are presented in Section 7. Proofs as well as some auxiliary results are presented in the Appendices.

2 Preliminaries

The set of real numbers is denoted as R. The continuous function $\beta : R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is of class $\mathcal{KL}$ if it is nondecreasing in its first argument and converging to zero in its second argument. Given a measurable function x, we define its $\mathcal{L}_\infty$ norm $\| \cdot \| = \operatorname{ess\,sup}_{t \geq 0} |x(t)|$. A vector function $\mathbf{f}(\mathbf{x}, \epsilon) \in R^n$ is said to be $O(\epsilon)$ if for any compact set $\mathcal{D}$ if there exist positive constants k and ϵ^* such that $|\mathbf{f}(\mathbf{x}, \epsilon)| \leq k\epsilon$, for all $\epsilon \in (0, \epsilon^*]$, $\mathbf{x} \in \mathcal{D}$, where $| \cdot |$ is the Euclidean norm.

We will consider a parameterized family of systems:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \boldsymbol{\epsilon}) \ , \tag{2}$$

where $\mathbf{x} \in R^n$, $t \in R_{\geq 0}$ and $\boldsymbol{\epsilon} \in R^{\ell}_{>0}$ are respectively the state of the system, the time variable and the parameter vector. The stability of the system (2) can depend in an intricate way on the parameters. In order to illustrate this and to motivate our Definition 1, we consider next two typical examples.

Example 1 *Consider the linear parameterized system:*

$$\dot{x} = -\epsilon_1 \epsilon_2 (1 - \epsilon_1 \epsilon_2) x \tag{3}$$

where ϵ_1 and ϵ_2 are positive. It is easy to see that the system (3) is stable if and only if $0 < \epsilon_1\epsilon_2 < 1$ and the shaded region in Fig. 1A is the stability region for the system in the parameter space. Our proof techniques will be based on the results in averaging and singular perturbations that will require all parameters to be sufficiently small. Hence, our main concern will be how stability of the system depends on sufficiently small parameters. The system (3) is stable for all $\epsilon_1, \epsilon_2 \in (0, 1)$.

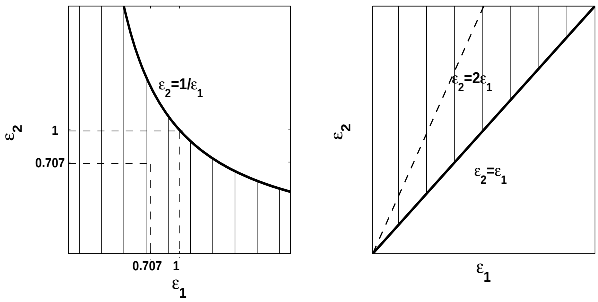

Fig. 1. A. Stability region in parameter space for the system (3); B. Stability region in parameter space for the system (5)

This is a favorable situation that we will refer to as[2] *"stability uniform in small (ϵ_1, ϵ_2)". Uniform stability means that we can fix a small "box" around the origin in the parameter space and the system is stable for all parameters in this box. Note that if we reduce the box, i.e. $\epsilon_1, \epsilon_2 \in (0, \frac{\sqrt{2}}{2})$, then the trajectories of the system satisfy the following transient bound:*

$$|x(t)| \leq e^{-\frac{\epsilon_1\epsilon_2}{2}t}|x(0)| \qquad \forall t \geq 0 \ . \tag{4}$$

Hence, the rate of convergence depends on the product of all parameters. Consider now a different example:

$$\dot{x} = -\epsilon_1(\epsilon_2 - \epsilon_1)x \ . \tag{5}$$

This example is stable whenever $\epsilon_2 > \epsilon_1$ and this stability region in parameter space in Fig. 1B is shaded. In this case, we can see that we can not fit a small "box" around the origin in the parameter space that will be completely contained in the stability region. To distinguish this case from the previous one we will just say that this system is "stable uniformly in small ϵ_1" (actually, we can also say that the system is "stable uniformly in small ϵ_2" but we will show later that the order in which parameters need to be tuned will be fixed in our proofs and this fact further motivates the terminology we are using). Moreover, if $\epsilon_2 > 2\epsilon_1$, the trajectories of (5) satisfy (4).

It is obvious that if there are 3 parameters, the situation is already much more complex since for any fixed value of one of the parameters, we can have different combinations of the above two cases for the remaining two

[2] In the sequel we will omit the word "small" from the definition of stability uniform in parameters.

parameters. The next definition captures situations in the above examples and applies to arbitrary nonlinear multi-parameter systems (2).

Definition 1 *The system (2) with parameter $\boldsymbol{\epsilon}$ is said to be semi-globally practically asymptotically (SPA) stable, uniformly in $(\epsilon_1, \ldots, \epsilon_j)$, $j \in \{1, \ldots, \ell\}$, if there exists $\beta \in \mathcal{KL}$ such that the following holds. For each pair of strictly positive real numbers (Δ, ν), there exist real numbers $\epsilon_k^* = \epsilon_k^*(\Delta, \nu) > 0, k = 1, 2, \ldots, j$ and for each fixed $\epsilon_k \in (0, \epsilon_k^*), k = 1, 2, \ldots, j$ there exist $\epsilon_i = \epsilon_i(\epsilon_1, \epsilon_2, \ldots, \epsilon_{i-1}, \Delta, \nu)$, with $i = j+1, j+2, \ldots, \ell$, such that the solutions of (2) with the so constructed parameters $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_\ell)$ satisfy:*

$$|\mathbf{x}(t)| \leq \beta(|\mathbf{x}_0|, (\epsilon_1 \cdot \epsilon_2 \cdot \ \cdots \ \cdot \epsilon_\ell)(t - t_0)) + \nu, \tag{6}$$

for all $t \geq t_0 \geq 0$, $\mathbf{x}(t_0) = \mathbf{x}_0$ with $|\mathbf{x}_0| \leq \Delta$. If we have that $j = \ell$, then we say that the system is SPA stable, uniformly in $\boldsymbol{\epsilon}$.

Note that in Definition 1 we can construct a small "box" around the origin for the parameters $\epsilon_k, k = 1, 2, \ldots, j$ so that the stability property holds uniformly for all parameters in this box, whereas at the same time we can not do so for the parameters $\epsilon_k, k = j+1, \ldots, l$. Sometimes we abuse terminology and refer to $(\epsilon_1 \cdots \epsilon_\ell)$ in the estimate (6) as the "convergence speed" (although the real convergence speed depends also on the function β).

Remark 1 *Definition 1 is tailored to families of systems that depend on several parameters. We will show that the closed loop system with an extremum seeking controller has the form (2) where $\epsilon_i, i = 1, 2, 3$ (i.e. $l = 3$) are design parameters that will be introduced and defined later. Typically, reducing the parameters ϵ_i will increase the size of the domain of attraction (larger Δ) and reduce the size of the set to which trajectories ultimately converge (smaller ν) but this will involve a penalty on the convergence rate: the smaller the ϵ_i, the slower the convergence rate is (see (6)). Although the relation between the size of the tuning parameters ϵ_i and the speed of convergence was mentioned in [7], the theoretical analysis of the relationship between the convergence speed and domain of attraction was not discussed in [6,7]. Definition 1 gives a clear picture of the relationship between the domain of attraction and the convergence rate of the closed loop system that all considered extremum seeking schemes will yield.*

Remark 2 *SPA stability in Definition 1 is more general from the one in the conference version of this paper [22, Definition 1]. Using the terminology of our Definition 1, we can say that [22, Definition 1] corresponds to SPA stability uniform in ϵ_1, which is a weak property. We prove in this paper several results under different assumptions that will yield SPA stability of the closed loop uniform in different parameters (Theorems 1, 2, 3, 4). In particular, our Theorems 1 and 2 contain stronger results than those in [22] since we prove uniformity of SPA stability with respect to all parameters $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_\ell)$.*

3 Problem Formulation

Consider the single-input-single-output (SISO) nonlinear model (1), where $\mathbf{f} : R^n \times R \to R^n$ and $h : R^n \to R$ are continuously differentiable [3]. $\mathbf{x}$ is the measurable state, u is the input and y is the output. Consider a family of control laws of the following form:

$$u = \alpha(\mathbf{x}, \theta), \tag{7}$$

where $\theta \in R$ is a scalar parameter. The closed-loop system (1) with (7) is then

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \alpha(\mathbf{x}, \theta)). \tag{8}$$

The requirement that θ is scalar and that (1), (7) is SISO is to simplify presentation. Multidimensional parameter situations with stronger Assumptions on the system (1) can be tackled and are left for future work. The following assumption is the same as [6, Assumption 2.1].

Assumption 1 *There exists a function $\mathbf{l} : R \to R^n$ such that*

$$f(\mathbf{x}, \alpha(\mathbf{x}, \theta)) = 0, \quad \text{if and only if} \quad \mathbf{x} = \mathbf{l}(\theta). \tag{9}$$

Assumption 2 is a natural extension of [6, Assumption 2.2] to prove non-local stability.

Assumption 2 *For each $\theta \in R$, the equilibrium $\mathbf{x} = \mathbf{l}(\theta)$ of the system (8) is globally asymptotically stable, uniformly in θ.*

Assumption 3 *Denoting $Q(\cdot) = h \circ \mathbf{l}(\cdot)$, there exists a unique θ^* maximizing $Q(\cdot)$ and, the following holds [4] :*

$$Q'(\theta^*) = 0 \qquad Q''(\theta^*) < 0 \tag{10}$$

$$Q'(\theta^* + \zeta)\zeta < 0 \qquad \forall \zeta \neq 0 \ . \tag{11}$$

Assumption 4 *We have that (10) holds and there exists $\alpha_Q \in \mathcal{K}_\infty$ such that:*

$$Q'(\theta^* + \zeta)\zeta \leq -\alpha_Q(|\zeta|) \qquad \forall \zeta \in R \tag{12}$$

Remark 3 *Assumption 3 is a stronger version of [6, Assumption 2.3], where it was assumed that there exists $\theta^* \in R$ such that (10) holds. Using conditions (10), only local stability properties of the extremum seeking scheme*

[3] In the sequel, all functions are assumed to be sufficiently smooth (all derivatives that we need are continuous).

[4] Without loss of generality we assume that the extremum is a maximum.

were analyzed in [6]. We use the stronger conditions (10) and (11) in Assumption 3 or (10) and (12) in Assumption 4 in order to obtain stronger (non-local) stability properties of the extremum seeking scheme. These assumptions appear natural in the context of non-local results. We will comment in the next section on how our proof techniques can be used to further relax Assumptions 3 and 4 and still achieve extremum seeking.

4 Main results

The main results are stated in this section. All proofs are found in Appendix A. This Section consists of two parts. In the first part we investigate stability of a first order extremum seeking scheme consisting of an integrator and an appropriate excitation signal. This simplified scheme appears to be novel and it lends itself to an interesting new interpretation of extremum seeking (via gradient descent optimization algorithms). In the second part we consider stability of several higher order extremum seeking schemes.

4.1 First order extremum seeking scheme

Consider the first order extremum seeking scheme, as shown in Fig.2, with the following dynamics

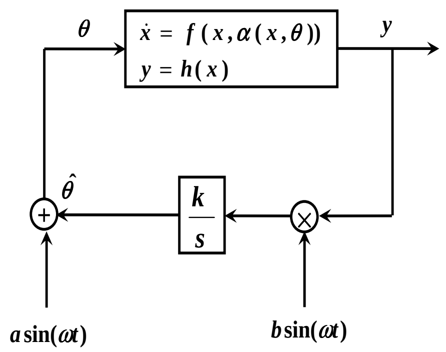

Fig. 2. The simplest peak seeking feedback scheme

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \alpha(\mathbf{x}, \hat{\theta} + a\sin(\omega t))) \\ \dot{\hat{\theta}} &= kh(\mathbf{x}) b \sin(\omega t), \end{aligned} \tag{13}$$

where (k, a, b, ω) are tuning parameters. Introduce the change of the coordinates, $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$, $\tilde{\theta} = \hat{\theta} - \theta^*$ and note that the point $(\mathbf{x}^*, \theta^*)$ is in general *not an equilibrium point* of the system (13). Nevertheless, we show that the system in new coordinates is SPA stable, which ensures extremum seeking. The system in new coordinates takes the form:

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}^*, \alpha(\tilde{\mathbf{x}} + \mathbf{x}^*, \tilde{\theta} + \theta^* + a\sin(\omega t))) \\ \dot{\tilde{\theta}} &= kh(\tilde{\mathbf{x}} + \mathbf{x}^*) b \sin(\omega t). \end{aligned} \tag{14}$$

We introduce [5] : $k \triangleq \omega \delta K$, $\sigma \triangleq \omega t$, where ω and δ are small parameters, $K > 0$ is fixed. The system equations expanded in time σ are:

$$\begin{aligned} \omega \frac{d\tilde{\mathbf{x}}}{d\sigma} &= \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}^*, \alpha(\tilde{\mathbf{x}} + \mathbf{x}^*, \tilde{\theta} + \theta^* + a\sin(\sigma))) \\ \frac{d\tilde{\theta}}{d\sigma} &= \delta K h(\tilde{\mathbf{x}} + \mathbf{x}^*) b \sin(\sigma). \end{aligned} \tag{15}$$

Note that the system (15) has the form (2) where the parameter vector is defined as $\boldsymbol{\epsilon} := [a\ b\ \delta\ \omega]^T$. For simplicity of presentation we let $b = a$ and [6]

$$\boldsymbol{\epsilon} := [a^2\ \delta\ \omega]^T . \tag{16}$$

The system (15) has a two-time-scale structure and our first main result is proved by applying the singular perturbations and averaging methods (see the Appendix A).

Theorem 1 *Suppose that Assumptions 1, 2 and 3 hold. Then, the system (14) (when $b = a$) with parameter $\boldsymbol{\epsilon}$ in (16) is SPA stable, uniformly in (a^2, δ).*

Remark 4 *Note that since $h(\cdot)$ is continuous, then for any $\nu > 0$, there exists $\nu_1 > 0$ such that*

$$|\tilde{\mathbf{x}}| \leq \nu_1 \implies |h(\tilde{\mathbf{x}} + \mathbf{x}^*) - y^*| \leq \nu . \tag{17}$$

Theorem 1 can be interpreted as follows. For any (Δ, ν) we can adjust $\boldsymbol{\epsilon}$ so that for all $|\mathbf{z}| \leq \Delta$, where $\mathbf{z} \triangleq |(\tilde{\mathbf{x}}, \tilde{\theta})|$, we have that $\limsup_{t\to\infty} |y(t) - y^*| \leq \nu$. In other words, the output of the system can be regulated arbitrarily close to the extremum value y^* from an arbitrarily large set of initial conditions by adjusting the parameters $\boldsymbol{\epsilon}$ in the controller. In particular, the parameters $\boldsymbol{\epsilon}$ are chosen so that Definition 1 holds with (Δ, ν_1) and ν_1 is defined in (17).

Remark 5 *Compared with the extremum seeking scheme in [6], the proposed extremum seeking scheme in Fig. 2 is simpler, containing only an integrator (without low-pass and high-pass filters that are used in [6]). Higher order schemes with low-pass and/or high-pass filters are considered in the next section.*

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[5] There is no loss of generality in assuming that $K = 1$. We considered the general case of $K > 0$ in order to make the connection to higher order schemes in the next section more apparent.

[6] Note that this is effectively the same as letting $\boldsymbol{\epsilon} = [a\ a\ \delta\ \omega]^T$ (see Definition 1).

Remark 6 *Theorem 1 is a stronger result than [6, Theorem 1] since we prove SPA stability, as opposed to local stability in [6]. However, our results are stated under stronger assumptions (Assumptions 1-3) than those in [6]. Assumptions 1-3 appear to be natural when nonlocal stability is investigated. Moreover, we note that it is not crucial in Assumptions 1 – 3 that all conditions hold globally. For instance, instead of requiring (11) in Assumption 3, we can assume:*

$$Q'(\theta^* + \zeta)\zeta < 0 \qquad \forall \zeta \in \mathcal{D}\ ,\ \zeta \neq 0\ , \tag{18}$$

where $\mathcal{D}$ is a bounded neighborhood of θ^*. We note that these conditions are not very restrictive, whereas their global version is (Assumptions 2 and 3). Indeed, if the maximum is isolated and all functions are sufficiently smooth, we can conclude that the condition (10) implies that there exists a set $\mathcal{D}$ satisfying (18). Similarly, we could assume only local stability in Assumption 2. If all of our assumptions were regional (as opposed to global) we could still state SPA stability with respect to the given bounded region.

Remark 7 *In the convergence speed analysis of the extremum seeking scheme, the "worst case" convergence speed is considered. That is, the convergence speed of the overall system depends on the convergence speed of the slowest sub-system. The first order extremum seeking controller (14), according to Theorem 1, yields the following stability bound:*

$$\begin{aligned} |\mathbf{z}(t)| &\leq \beta(|\mathbf{z}(t_0)|, (a^2\delta\omega)(t-t_0)) + \nu, \\ &= \beta\left(|\mathbf{z}(t_0)|, (a^2 k)\frac{(t-t_0)}{K}\right) + \nu, \end{aligned} \tag{19}$$

for all $t \geq t_0 \geq 0$ and $|\mathbf{z}(t_0)| \leq \Delta$, where $\mathbf{z} \stackrel{\triangle}{=} |(\tilde{\mathbf{x}}, \tilde{\theta})|$ and k, K were defined befor. Since $K > 0$ is fixed, the parameter $a^2 \cdot k$ affects convergence speed. The smaller $a^2 \cdot k$, the slower the convergence and the larger the domain of attraction.

Remark 8 *The proof of Theorem 1 in the Appendix A provides an interesting insight into the way the extremum seeking controller operates. The parameter ω is used to separate time scales between the plant (boundary layer) and the extremum seeking controller (reduced system), where the plant states are fast and they quickly die out*[7]*. Using the singular perturbation method, we obtain that the reduced system in the variable "θ_r" in time "$\sigma = \omega t$" is time varying and it has the form:*

$$\frac{d\theta_r}{d\sigma} = Ka\delta Q(\theta^* + \theta_r + a\sin(\sigma))\sin(\sigma)\ , \tag{20}$$

for which we introduce an "averaged" system:

$$\frac{d\theta_{av}}{d\sigma} = \frac{K}{2}a^2\delta Q'(\theta^* + \theta_{av})\ . \tag{21}$$

Hence, the averaged system (21) can be regarded as the "gradient system" whose globally asymptotically stable equilibrium θ^* corresponds to the global maximum[8] *of the unknown map Q. By combining the idea from [5, Section 10.4] and [18, Section 3.9], we introduce a change of coordinates*

$$\theta_r(\sigma) = w(\sigma) + K\delta a q(\sigma, w(\sigma), a), \tag{22}$$

where q will be defined in (47) in Appendix A. In the new coordinates we show

$$\frac{dw}{d\sigma} = \frac{Ka^2\delta}{2}Q'(\theta^* + w) + O(a^3\delta) + O(a^2\delta^2) + O(a^3\delta^2). \tag{23}$$

It is then obvious from (23) that reducing parameters a and δ reduces the mismatch between the averaged system (21) and the reduced system (20) and guarantees SPA stability of the reduced system.

Remark 9 *As indicated in Remark 8, first order extremum seeking scheme works on average as a "gradient search" method. Both the excitation signal and the integrator are necessary to achieve this. The excitation signal $a\sin(\omega t)$ is added to system (1) to get probing while the multiplication (modulation) of output and the excitation signal extracts the gradient of the unknown mapping $Q(\cdot)$. The role of the integrator is to get on average the steepest decent along the gradient of $Q(\cdot)$. Hence, the first order scheme is the simplest controller structure that achieves extremum seeking.*

Remark 10 *Note that we did not prove SPA stability, uniform in the whole vector $\boldsymbol{\epsilon}$ in Theorem 1 for system (14) with parameter $\boldsymbol{\epsilon}$. The stability properties is not uniform in ω. The result in Theorem 1 is still stronger than those in [22] where the SPA stability is only uniform in a^2.*

In the rest of this section we investigate two situations where SPA stability, uniform in $\boldsymbol{\epsilon}$, can be obtained under stronger conditions than those used in Theorem 1. First, we consider a static mapping without plant dynamics and then we consider the general case with plant dynamics but we use stronger assumptions.

Suppose that there is no plant dynamics and the reference to output map is static $y = h(u)$, as in [9–12,14]. In this case, the closed loop system in new coordinates $\tilde{\theta} = \theta - \theta^*$ becomes:

$$\dot{\tilde{\theta}} = K\delta h(\theta^* + \tilde{\theta} + a\sin(t))\ a\sin(t)\ . \tag{24}$$

[7] Because of Assumption 2.

[8] Because of our Assumption 3.

Note that in this case we do not need the parameter ω and we let $k = K\delta$ where $K > 0$ is arbitrary but fixed. Hence, in this case $\boldsymbol{\epsilon} := (a^2, \delta)$. Moreover, we do not need Assumptions 1 and 2 and we use Assumption 3 where Q is replaced by h. Then, from the proof of Theorem 1 in Appendix A we directly get the following:

Corollary 1 *Suppose that Assumption 3 holds with $Q(\cdot) = h(\cdot)$. Then, the system (24) with parameter $\boldsymbol{\epsilon} := (a^2, \delta)$ is SPA stable, uniformly in $\boldsymbol{\epsilon}$.*

Next we will prove SPA stability for general systems, uniform in $\boldsymbol{\epsilon}$ under stronger conditions. In particular, we will use Assumption 4 and Assumption 5 that will be stated below after some auxiliary results are presented. To this end, we first rewrite (15) to simplify the notation:

$$\begin{aligned} \omega\frac{d\tilde{\mathbf{x}}}{d\sigma} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, \tilde{\theta} + a\sin(\sigma)) \\ \frac{d\tilde{\theta}}{d\sigma} &= \delta K\tilde{h}(\tilde{\mathbf{x}})\ a\sin(\sigma)\ , \end{aligned} \tag{25}$$

where $\tilde{\mathbf{f}}(\mathbf{x}, \theta) := \mathbf{f}(\mathbf{x}+\mathbf{x}^*, \alpha(\mathbf{x}+\mathbf{x}^*, \theta^*+\theta))$ and $\tilde{h}(\mathbf{x}) := h(\mathbf{x}+\mathbf{x}^*)$. Then, we introduce "boundary layer" using $\bar{\mathbf{x}} := \tilde{\mathbf{x}} - \mathbf{l}(\theta^* + \tilde{\theta} + a\sin(\sigma)) =: \tilde{\mathbf{x}} - \tilde{\mathbf{l}}(\tilde{\theta} + a\sin(\sigma))$ and rewrite (25) back in original time scale "t" as follows:

$$\dot{\bar{\mathbf{x}}} = \tilde{\mathbf{f}}(\bar{\mathbf{x}} + \tilde{\mathbf{l}}(\tilde{\theta} + a\sin(\sigma)), \tilde{\theta} + a\sin(\sigma)) + \omega a\Delta_1 \tag{26}$$

$$\dot{\tilde{\theta}} = a\delta K\tilde{h}\circ\tilde{\mathbf{l}}(\tilde{\theta} + a\sin(\sigma))\sin(\sigma) + a\delta\Delta_2, \tag{27}$$

where

$$\begin{aligned} \Delta_1 &:= \tilde{\mathbf{l}}'(\tilde{\theta} + a\sin(\sigma))\left[\delta K\tilde{h}(\bar{\mathbf{x}} + \tilde{\mathbf{l}}(\tilde{\theta} + a\sin(\sigma)))\sin(\sigma)\right. \\ &\quad \left. + \cos(\sigma)\right], \\ \Delta_2 &:= K\left[\tilde{h}(\bar{\mathbf{x}} + \tilde{\mathbf{l}}(\tilde{\theta} + a\sin(\sigma)))\right. \\ &\quad \left. - \tilde{h}\circ\tilde{\mathbf{l}}(\tilde{\theta} + a\sin(\sigma))\right]\sin(\sigma). \end{aligned} \tag{28}$$

We consider the system (26), (27) as a feedback connection of two systems. We first show that under our assumptions, we can conclude that the two systems are input to state stable [9] (ISS) in an appropriate sense and their gains are parameter dependent. Then, we will state Assumption 5 that will allow us to prove our next main result (Theorem 2) using the small gain theorem (Lemma 2) stated in the Appendix B.

Proposition 1 *Suppose that Assumption 4 holds. Then, there exist $\beta_1 \in \mathcal{KL}$ and for any $\Delta_1 > \nu_1 > 0$ and $\omega^* > 0$ there exist $\gamma_1^\theta, \gamma_2^\theta \in \mathcal{K}_\infty$, $a^* > 0$ and $\delta^* > 0$ such that for all $a \in (0, a^*)$, $\delta \in (0, \delta^*)$, $\omega \in (0, \omega^*)$ and $\max\{|\tilde{\theta}_0|, ||\bar{\mathbf{x}}||\} \le \Delta_1$, with $\tilde{\theta}_0 := \tilde{\theta}(t_0)$ we have that the solutions of the subsystem (27) satisfy:*

$$|\tilde{\theta}(t)| \le \max\left\{\beta_1(|\tilde{\theta}_0|, (a^2\delta\omega)(t - t_0)), \gamma^1_{\boldsymbol{\epsilon}}(||\bar{\mathbf{x}}||), \nu_1\right\}, \tag{29}$$

for all $t \ge t_0 \ge 0$, where

$$\gamma^1_{\boldsymbol{\epsilon}}(s) := \gamma_1^\theta\left(\frac{1}{a}\gamma_2^\theta(s)\right)\ .$$

Proposition 2 *Suppose that Assumptions 1 and 2 hold. Then, there exist $\beta_2 \in \mathcal{KL}$ and for any positive Δ_2, ν_2, a^* and δ^* there exist $\gamma_1^z, \gamma_2^z \in \mathcal{K}_\infty$ and $\omega^* > 0$, such that for all $a \in (0, a^*)$, $\delta \in (0, \delta^*)$ $\omega \in (0, \omega^*)$ and $\max\{|\bar{\mathbf{x}}_0|, ||\tilde{\theta}||\} \le \Delta$, with $\bar{\mathbf{x}}_0 := \bar{\mathbf{x}}(t_0)$, we have that the solutions of the subsystem (26) satisfy:*

$$|\bar{\mathbf{x}}(t)| \le \max\left\{\beta_2(|\bar{\mathbf{x}}_0|, t - t_0), \gamma^2_{\boldsymbol{\epsilon}}(||\tilde{\theta}||), \nu_2\right\}, \tag{30}$$

for all $t \ge t_0 \ge 0$, where

$$\gamma^2_{\boldsymbol{\epsilon}}(s) := \gamma_1^z\left(\omega a\gamma_2^z(s)\right)\ .$$

Remark 11 *Note that the ISS gains $\gamma^1_{\boldsymbol{\epsilon}}$ and $\gamma^2_{\boldsymbol{\epsilon}}$ in Propositions 1 and 2 depend on a and ω. Moreover, the gain $\gamma^1_{\boldsymbol{\epsilon}}$ increases to infinity as a is reduced to zero. Typically, this behavior leads to lack of stability in the interconnection. However, in this case, it is possible to counteract this increase of $\gamma^1_{\boldsymbol{\epsilon}}$ through the decrease of $\gamma^2_{\boldsymbol{\epsilon}}$ as $\gamma^2_{\boldsymbol{\epsilon}}$ decreases to zero as a decreases. Moreover, it is sometimes possible to achieve this in a manner that will guarantee SPA stability, uniform in $\boldsymbol{\epsilon}$, of the system via Theorem 2 in the appendix. A condition that is needed for this to hold is summarized in the next assumption.*

Assumption 5 *Let the gains $\gamma^1_{\boldsymbol{\epsilon}}, \gamma^2_{\boldsymbol{\epsilon}}$ come from Propositions 2 and 1. Assume that there exists $\gamma \in \mathcal{K}_\infty$ such that for any $0 < s_1 < s_2$ there exist ω^* and a^* such that for all $\omega \in (0, \omega^*)$, $a \in (0, a^*)$ and $s \in [s_1, s_2]$ we have that the following small gain conditions hold:*

$$\gamma^1_{\boldsymbol{\epsilon}}\circ\gamma^2_{\boldsymbol{\epsilon}}(s) \le \gamma(s) < s, \qquad \gamma^2_{\boldsymbol{\epsilon}}\circ\gamma^1_{\boldsymbol{\epsilon}}(s) \le \gamma(s) < s. \tag{31}$$

Remark 12 *Note that the conditions (31) do not imply each other, as can be easily seen from the case when $\gamma_2^\theta\circ\gamma_1^z(s) = s^q$, $q > 1$ and $\gamma_2^z\circ\gamma_1^\theta(s) = s^p$, $p > 1$ in which case the conditions (31) become respectively $\gamma_1^\theta\left(\omega a^{q-1}\left(\gamma_2^z(s)\right)^q\right) < s$ and $\gamma_1^z\left(\omega a^{1-p}\left(\gamma_2^\theta(s)\right)^p\right) < s$. It is obvious, that in the first case we can choose ω and a independent of each other so that the first condition in (31) holds, whereas it is impossible to do so for the second condition in (31). This also illustrates that conditions of Assumption 5 may not hold for some gains.*

Remark 13 *We note that if all the gains $\gamma_1^\theta, \gamma_2^\theta, \gamma_1^z, \gamma_2^z$ are linear then Assumption 5 holds. In particular, the*

[9] For more details on ISS, refer to [19].

small gain conditions (31) become independent of a and can be achieved by reducing ω only.

Our second main result is stated next:

Theorem 2 *Suppose that Assumptions 1, 2, 4 and 5 hold. Then, the closed-loop system (14) (when $b = a$) with parameter $\boldsymbol{\epsilon} = (a^2\ \delta\ \omega)^T$ is SPA stable, uniformly in $\boldsymbol{\epsilon}$.*

Remark 14 *Note that the conditions in Assumption 5 and Remark 12 shed light on the result proved in Theorem 1. Indeed, Remark 12 illustrates that it is not possible to always satisfy the conditions of Assumption 5 uniformly in small a and ω. However, if we were interested in proving a weaker stability property, i.e. SPA stability uniform in (a^2, δ), then for any fixed a we can always find $\omega^* > 0$ such that for all $\omega \in (0, \omega^*)$ we have that the small gain conditions in Assumption 5 hold. Then, we can use the same steps as in proof of Theorem 2 to conclude that the system is SPA stable uniform in (a^2, δ) that we proved in Theorem 1 under weaker conditions.*

It is an open question whether there is a genuine gap between Theorems 1 and 2, that is whether there exists an example satisfying all conditions of Theorem 1 but not conditions of Theorem 2 that is not *SPA stable uniformly in $\boldsymbol{\epsilon}$.*

4.2 Higher order extremum seeking schemes

Higher order schemes have been considered in the literature where the first order controller from the previous section is augmented with low-pass and/or high-pass filters [6]. From the analysis in the previous subsection we can see that such filters are not needed to achieve extremum seeking. However, we show that the same techniques can be used to prove SPA stability of these higher order schemes. An outcome of this analysis is a result on SPA stability of an extremum seeking controller that was considered in [6]. We note that our analysis in the previous section relies on the results on second order averaging for periodic systems [18, Section 3.9] and the analysis in the previous section is greatly simplified for the scalar extremum seeking scheme. This analysis is more complicated for higher order systems of this section. Hence, for space reasons we state and prove in this section weaker results that are easier to prove (SPA stability uniform in a^2). Results in this section have appeared in the conference version of this paper [22].

Consider the extremum seeking controller given in Fig. 3, when $W_L(s) = \frac{\omega_l}{s+\omega_l}$ and $W_H(s) = 1$. The extremum seeking controller contains an integrator and a low pass filter. The low-pass filter is useful when we need to filter out high frequency measurement noise in the system. Moreover, tuning the filter parameter ω_l may lead to a possible improvement in the transient response. When

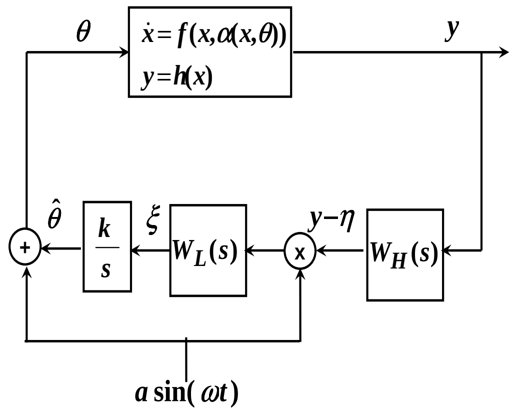

Fig. 3. A peak seeking feedback scheme with a low-pass filter (LF) and a high pass filter (HF)

$\omega_h = 0$, the following equations describe the closed loop system in Fig. 3 :

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \alpha(\mathbf{x}, \hat{\theta} + a\sin(\omega t))) \\ \dot{\hat{\theta}} &= k\ \xi \\ \dot{\xi} &= -\omega_l \xi + \omega_l h(\mathbf{x}) a \sin(\omega t). \end{aligned} \tag{32}$$

Introduce the change of the coordinates, $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$, $\tilde{\theta} = \hat{\theta} - \theta^*$, $\tilde{\xi} = \xi$. The system in new coordinates takes the following form:

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}^*, \alpha(\tilde{\mathbf{x}} + \mathbf{x}^*, \tilde{\theta} + \theta^* + a\sin(\omega t))) \\ \dot{\tilde{\theta}} &= k\ \tilde{\xi} \\ \dot{\tilde{\xi}} &= -\omega_l\ [\tilde{\xi} - h(\tilde{\mathbf{x}} + \mathbf{x}^*)\ a \sin(\omega t)]\ . \end{aligned} \tag{33}$$

Fix $\omega_L, K \in R_{>0}$ and define

$$\omega_l \triangleq \omega\ \delta\ \omega_L; \qquad k \triangleq \omega\ \delta\ K\ , \tag{34}$$

where δ and ω are small parameters. We introduce the new time "$\sigma = \omega\ t$" and obtain:

$$\begin{aligned} \omega \frac{d\tilde{\mathbf{x}}}{d\sigma} &= \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}^*, \alpha(\tilde{\mathbf{x}} + \mathbf{x}^*, \tilde{\theta} + \theta^* + a\ \sin(\sigma))) \\ \frac{d\tilde{\theta}}{d\sigma} &= \delta\ K\ \tilde{\xi} \\ \frac{d\tilde{\xi}}{d\sigma} &= -\delta\ \omega_L\ [\tilde{\xi} - h(\tilde{\mathbf{x}} + \mathbf{x}^*)\ a\ \sin(\sigma)]\ . \end{aligned} \tag{35}$$

Note that the system (35) has the form (2) where the parameter vector is defined as before $\boldsymbol{\epsilon} := [a^2\ \delta\ \omega]^T$. This system exhibits the same two time-scale structure as the system in the previous section. The main difference is that the slow system is in this case second order system instead of the scalar system in the previous section. Nevertheless, one can prove the same results that we proved for the scalar scheme. For space reasons, we only state counterparts of Theorem 1 for higher order schemes and sketch proofs of these results in the last section.

Theorem 3 *Suppose that Assumptions 1, 2 and 3 hold. Then, the closed-loop system (33) with parameter $\boldsymbol{\epsilon}$ is SPA stable, uniformly in a^2.*

We now turn to the extremum seeking scheme in Fig. 3 investigated in [6], where $W_L(s) = \frac{\omega_l}{s+\omega_l}$ and $W_H(s) = \frac{s}{s+\omega_h}$:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \alpha(\mathbf{x}, \hat{\theta} + a\sin(\omega\ t)) \\
\dot{\hat{\theta}} &= k\ \xi \\
\dot{\xi} &= -\omega_l \xi + \omega_l (y - \eta) a \sin(\omega t) \\
\dot{\eta} &= -\omega_h \eta + \omega_h y\ ,
\end{aligned} \tag{36}
$$

where besides (34), we also use $\omega_h \stackrel{\triangle}{=} \omega\ \delta\ \omega_H$ for some fixed $\omega_H > 0$. Note that the system (36) also has the form (2) where the parameter vector is defined as $\boldsymbol{\epsilon} := [a^2\ \delta\ \omega]^T$. We can prove the following result:

Theorem 4 *Suppose that Assumptions 1, 2 and 3 hold. Then, the closed-loop system (36) with parameter $\boldsymbol{\epsilon} = (a^2\ \delta\ \omega)^T$ is SPA stable, uniformly in a^2.*

Remark 15 *The key issue for higher order schemes is the parametrization of filter coefficients with ω and δ to get an appropriate time scale separation. With an appropriate parametrization of an arbitrary causal stable filter $\frac{N(s)}{D(s)}$ we can get a desired time scale separation that can be used to prove that the extremum seeking controller achieves SPA stability.*

Remark 16 *Note that we considered only stability and it is an interesting question how different schemes compare to each other from a performance point of view. However, this question is outside the scope of this paper.*

5 Possible Extensions

We briefly comment on possible extensions of our results and indicate how our techniques may be useful to analyze these important issues. First, we note that the stability results in [27] that were given for a different class of extremum seekers were derived under more general conditions than what we use: multi-parameter case was considered, non-smooth output to reference maps could be treated and results were applicable to infinite dimensional systems. Addressing stability for the class of extremum seekers considered in this paper under each of these relaxed conditions would be very important and is left for further research. We discuss in more detail two other important questions for which our techniques seem to be well suited: (i) passing through local extrema to achieve global optimization; (ii) excitation signal design.

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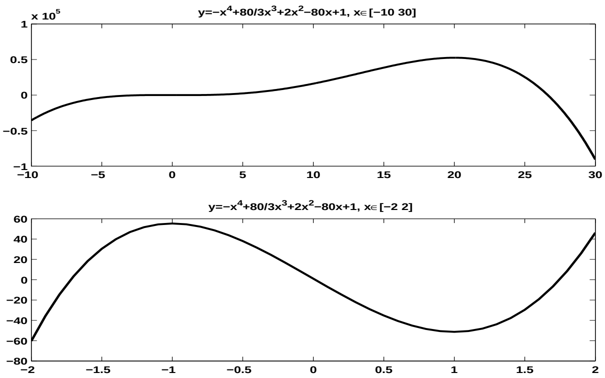

Fig. 4. Nonlinear mapping $h(x)$

5.1 Passing Through Local Extrema

It has been observed by different researchers and practitioners [10] that extremum seeking controllers sometimes have the ability to pass through a local extremum in order to converge to a global extremum. For example, sometimes if the amplitude a of the excitation signal is sufficiently large, then it is possible to pass through a local extremum. In other words, the extremum seeking controllers were observed to converge to the global extremum even when our Assumption 3 does not hold. The obvious question is whether one can provide general proofs of convergence when our Assumption 3 does not hold. While this phenomenon has been observed in simulations, no theoretical analysis was proposed to date. This issue is related to the technique of simulated annealing where global optimization is achieved through an appropriate design and tuning of the optimization algorithm. We show next by way of an example that our non-local proof techniques are suited for addressing this question. We will address this issue in more generality in our future work.

Consider the following static mapping (no dynamics)

$$y = h(x) = -x^4 + \frac{80}{3}x^3 + 2x^2 - 80x + 1,$$

with the first order extremum seeking controller to obtain the closed loop system (we use $K = 1$):

$$\dot{\theta} = \delta h(\theta + a\sin(t))\ a\sin(t).$$

For simplicity, we assume that $h(\cdot)$ is known in order to carry out the analysis. As seen from Fig. 4, the output mapping has one global maximum $x^* = 20$, one local maximum at $x = -1$ and one local minimum at $x = 1$.

[10] The authors would like to thank anonymous reviewers for pointing out this issue which motivated us to consider it more closely.

Denoting $\tilde{\theta} = \theta - 20$, in new coordinates, we have

$$\dot{\tilde{\theta}} = \delta h(\tilde{\theta} + 20 + a\sin(t))\, a\sin(t) \triangleq a\delta\tilde{h}(\tilde{\theta}, t), \tag{37}$$

Obviously, $h(\tilde{\theta} + 20)$ reaches its local maximum at $\tilde{\theta} = -21$ and global maximum at $\tilde{\theta} = 0$. Hence, our Assumption 3 does not hold. Next, we average the right hand side of (37) and introduce an averaged system [11]

$$\frac{d\theta_{av}}{dt} \triangleq \delta a \frac{1}{2\pi}\int_0^{2\pi} \tilde{h}(\theta_{av}, t)dt = \delta a^2 f_{av}(\theta_{av}, a). \tag{38}$$

where $f_{av}(\theta_{av}, a) = \frac{1}{2}[4\theta_{av}^3 + 160\theta_{av}^2 + 1596\theta_{av} + 3a^2\theta_{av} + 40a^2]$. Introduce a quadratic Lyapunov function $V(\theta_{av}) := \frac{1}{2}\theta_{av}^2$ and consider the sign of its derivative along solutions of the average system for different values of a.

If $a = 0.1$ we have $\dot{V} < 0$, for all $\theta_{av} \in S_{0.1}$, where $S_{0.1} = (-\infty, -20.998) \cup (-19.001, -0.0002) \cup (0, +\infty)$. A closer inspection reveals that all trajectories starting in the set $(-\infty, -19.001)$ converge to the point $\theta_{av} = -20.998$ that is in the vicinity of the local extremum. On the other hand, all trajectories starting in the set $(-19.001, -0.0002) \cup (0, +\infty)$ converge to the set $[-0.0002, 0]$ that is in the vicinity of the global extremum $\theta_{av} = 0$.

Suppose now that $a = 2.1$. Then, $\dot{V} < 0$ for all $\theta_{av} \in S_{2.1}$, where $S_{2.1} = (-\infty, -0.11) \cup (0, +\infty)$. Hence, we can conclude that all trajectories of the average system converge to the set $[-0.11, 0]$ that is in the vicinity of the global extremum. We note that by reducing δ we can make the actual system behave in approximately the same manner as the average system. Hence, by increasing a and reducing δ we could show analytically that we achieve global extremum seeking, i.e. if $\tilde{\theta}(0) < -21$ we have that $\tilde{\theta}(t) \to [-0.11, 0]$ - the extremum seeking controller passes through the local extremum. Simulations support these theoretical findings that were made possible using our non-local stability analysis.

Finally, we remark that if a time-varying amplitude a is employed so that $a(0)$ is sufficiently large and $a(t) \to 0$ sufficiently slowly, similar to the idea of simulated annealing, the global maximum value would be obtained by the extremum seeking system. This will complicate the analysis since it appears that we will need to introduce another (slowest) time scale for the evolution equation of $a(\cdot)$. Our techniques are suitable for addressing this issue in its full generality and this is a topic of future research.

[11] Note that this average system is different from the one used in the proofs of Theorems 1 and 2 where the a^4 terms were ignored.

5.2 *Excitation Signal Design*

We note that our proof techniques can be used to prove the same results when the sinusoidal excitation signal $v(t) = a\sin(\omega t)$ is replaced by an arbitrary symmetric periodic signal, such as sawtooth or square wave signals. A natural question arises: is it possible to show that some excitation signals are better in some sense than other excitation signals? In particular, is it possible to improve convergence or robustness of the closed loop by designing the excitation signal? The answer to both of these questions is affirmative and they can be treated using the same methodology that we have developed in this paper. We address these questions in a follow up paper [23] that is currently under preparation.

6 Simulation Example

We simulate several different extremum seeking schemes to illustrate SPA stability (e.g. reducing the tuning parameters would slow down the convergence speed of the system and increase the domain of attraction) for a system with a unique global maximum. Consider the system:

$$\dot{x} = -x + u^2 + 4u; \qquad y = -(x+4)^2 \,. \tag{39}$$

The initial condition is chosen as $x(0) = 2$. It is obvious that when $x = -4$, y reaches its global maximum $y^* = 0$. Let control input $u = \theta$, we have $\theta^* = -2$, $x^* = -4$ and $y^* = 0$. The initial value of $x(0)$ is far away from the desired one $x^* = -4.0$.

(a) The first order scheme. Let $\hat{\theta}(0) = 0$. By choosing $a = 0.3$, $\delta = 0.5(K = 4)$ and $\omega = 0.5$, the performance of the first order scheme is shown in Fig. 5 where $|\mathbf{z}| = |(\tilde{x}, \tilde{\theta})|$, where $\tilde{x}$ and $\tilde{\theta}$ are the same as in Equation (14). It can be seen that, the state $\mathbf{z}$ converges to the neighborhood of the origin. The output also converges to the vicinity of the extremum value. We increase a such that $a = 0.6$ while keeping $\delta = 0.5$ and $\omega = 0.5$. From the result of Theorem 1, increasing a will get a fast convergent speed, while the domain of the attraction would be smaller. It can be seen clearly from Fig. 5 that, though both $y(t)$ and $|\mathbf{z}|$ converge very fast, it converges to a much *larger* neighborhood of the optimal values.

Now, we fix $a = 0.3$ and $\omega = 0.5$, first let $\delta = 0.25$, as seen from Fig. 6, the state $\mathbf{z}$ converges to the neighborhood of the origin. The output also converges to the vicinity of the extremum value. Similarly, we increase δ to be 0.75, the performance of the extremum seeking scheme is shown in Fig. 6. The convergence speed of latter one is much faster. However, when we further increase δ, for example, any $\delta \geq 1.40$, unstable performance can be observed.

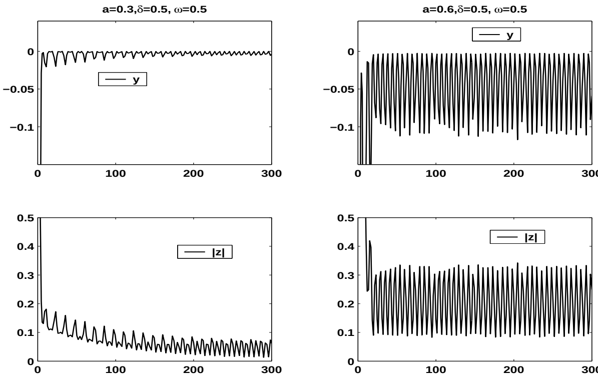

Fig. 5. The performance of the simplest extremum seeking scheme

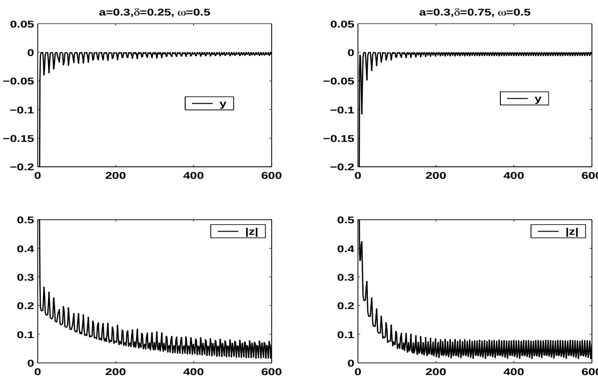

Fig. 6. The performance of the simplest extremum seeking scheme

By choosing $a = 0.3$, $\delta = 0.5$ and $\omega = 0.1$, the performance of the first order scheme is shown in Fig. 7. It can be seen that, the state $\mathbf{z}$ converges to the neighborhood of origin. The output also slowly converges to the vicinity of the extremum value. If we increase ω such that $\omega = 0.5$ while keeping $a = 0.5$ and $\delta = 0.5$, we can see from Fig. 6 that, though both $y(t)$ and $|\mathbf{z}|$ converge very fast, they converge to a *larger* neighborhood of the desired one compared with the performance when a smaller ω ($\omega = 0.1$) is employed.

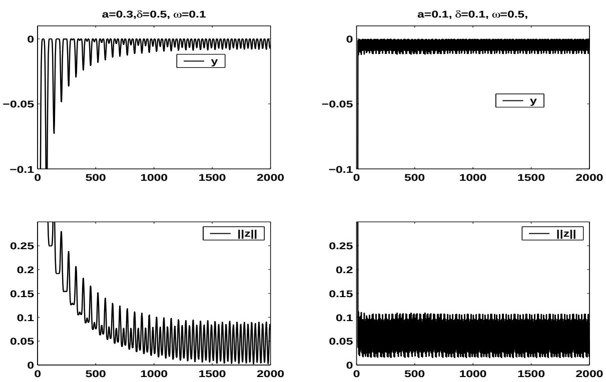

Fig. 7. The performance of the simplest extremum seeking scheme

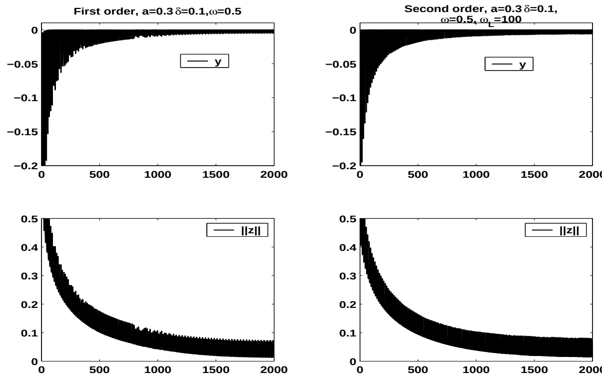

Fig. 8. The comparison between two extremum seeking schemes

(b) Comparison of the first and higher order schemes

In the simulation, two extremum seeking schemes (in Fig. 2 and Fig. 3) are compared with same dynamics (39). We fix the parameters $a = 0.3$, $k = 0.2$ ($\delta = 0.1, K = 4$) and $\omega = 0.5$ for both schemes.

In the extremum seeking scheme with a low-pass filter, let $\omega_L = 10$ such that $\omega_l = 0.5 = \omega$, $\xi(0) = 2$ while keeping other parameters the same as the first order scheme, the performances of two extremum seeking schemes are shown in Fig. 8, where $|\mathbf{z}| \stackrel{\triangle}{=} |(x - x^*, \theta - \theta^*, \xi)|$. It can be seen clearly that, the steady-state of the two schemes are comparable.

7 Conclusion

We presented non-local stability results for several extremum seeking controllers. We use a novel proof technique that is based on recent results in averaging and singular perturbations. Our definition of semi-global practical stability appears to be novel in this context and it shows a tradeoff between the domain of attraction and the speed of convergence: reducing the tuning parameters in the controller typically enlarges the domain of attraction, while it slows down the speed of convergence. Simulations illustrate that this semi-global practical stability is indeed achieved for the considered extremum seeking schemes.

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Appendix A

Proof of Theorem 1: The system (15) is in the standard singular perturbation form, where the singular perturbation parameter is ω. To obtain the fast and slow systems, we set $\omega = 0$ and "freeze" $\tilde{\mathbf{x}}$ at its "equilibrium", $\tilde{\mathbf{x}} = \mathbf{l}(\theta^* + \tilde{\theta} + a \cdot \sin(\sigma)) - \mathbf{x}^*$ to obtain the reduced system in variable θ_r in the time scale $\sigma = \omega t$:

$$\frac{d\theta_r}{d\sigma} = K\delta Q(\theta^* + \theta_r + a\sin(\sigma))a\sin(\sigma). \tag{40}$$

Applying the Taylor series expansion (c.f. [2]):

$$Q(\theta^* + \theta + a \cdot \sin(\sigma)) = \left[f_1(\sigma,\theta) + af_2(\sigma,\theta) + a^2R\right], \tag{41}$$

where [12]

$$f_1(\sigma,\theta) \stackrel{\triangle}{=} Q(\theta^* + \theta), \quad f_2(\sigma,\theta) \stackrel{\triangle}{=} Q'(\theta^* + \theta)\sin(\sigma), \tag{42}$$

and $R = R(\sigma,\theta)$ contains higher order terms in $\sin(\sigma)$. Next, we introduce the average system:

$$\frac{d\theta_{av}}{d\sigma} = Ka\delta f_{av}(\theta_{av}) = Ka^2\delta\frac{1}{2}Q'(\theta^* + \theta_{av}), \tag{43}$$

where, using (42) and $u(t,\theta) := \int_0^t f_1(\tau,\theta)d\tau - \frac{1}{2\pi}\int_0^{2\pi}\int_0^t f_1(\tau,\theta)d\tau dt$, we defined (see [18, Section 3.9]):

$$\begin{aligned} f_{av}(\theta,a) &:= \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial f_1}{\partial\theta}(\tau,\theta)u(t,\theta) + af_2(\tau,\theta)\right]d\tau \\ &= \frac{a}{2}Q'(\theta^* + \theta)\ . \end{aligned} \tag{44}$$

[12] We suppressed the dependance of all functions on θ^* since it is constant.

Note that Assumption 3 guarantees that the average system (43) is globally asymptotically stable. Indeed, we can use the Lyapunov function $V(\theta_{av}) = \frac{1}{2}\theta_{av}^2$ to see that since we have along solutions of (43) that:

$$\frac{dV}{d\sigma} = \theta_{av}\frac{d\theta_{av}}{d\sigma} = \frac{1}{2}Ka^2 Q'(\theta^* + \theta_{av})\theta_{av} < 0, \qquad (45)$$

where $Q'(\theta^* + \theta_{av})\theta_{av}$ is negative definite in θ_{av} because of (11) in Assumption 3.

Next, we show that the same Lyapunov function can be used to prove that the reduced system (40) is SPA stable, uniformly in (a^2, δ). The uniformity in (a^2, δ) can be shown in the new coordinates w, which is defined as

$$\theta_r(\sigma) = w(\sigma) + Ka\delta q(\sigma, w(\sigma), a), \qquad (46)$$

where, using (42) and (44), we define:

$$q(\sigma, w, a) := \int_0^\sigma [f_1(\tau, w) + af_2(\tau, w) - af_{av}(w)]d\tau. \qquad (47)$$

Since the integrand in (47) is periodic and has zero mean, the function q is periodic in σ. Hence, q is bounded for all σ and

$$\frac{\partial q}{\partial \sigma} = f_1(\sigma, w) + af_2(\sigma, w) - af_{av}(w) \qquad (48)$$
$$\frac{\partial q}{\partial w} = \int_0^\sigma \left[\frac{\partial f_1}{\partial w}(\tau, w) + a\frac{\partial f_2}{\partial w}(\tau, w) - a\frac{\partial f_{av}}{\partial w}(w)\right] d\tau,$$

are periodic in t and bounded. Differentiating (46) with respect to σ we obtain [13]:

$$\frac{d\theta_r}{d\sigma} = \frac{dw}{d\sigma} + Ka\delta\left[\frac{\partial q}{\partial \sigma} + \frac{\partial q}{\partial w}\cdot\frac{dw}{d\sigma}\right].$$

and by substituting this into (40) and then using (46) and (48) we obtain:

$$\begin{aligned}\frac{dw}{d\sigma}\left[1 + Ka\delta\frac{\partial q}{\partial w}\right] &= -Ka\delta\frac{\partial q}{\partial \sigma} + Ka\delta\,[f_1(\sigma, w + Ka\delta q) \\ &\quad + af_2(\sigma, w + Ka\delta q) + a^2 R] \\ &= Ka^2\delta f_{av}(w) + Ka^3\delta R \qquad (49) \\ &\quad + Ka\delta\,[f_1(\sigma, w + Ka\delta q) - f_1(\sigma, w)] \\ &\quad + Ka^2\delta\,[f_2(\sigma, w + Ka\delta q) - f_2(\sigma, w)]\,.\end{aligned}$$

Note that by adjusting a and δ we can make $\left[1 + Ka\delta\frac{\partial q}{\partial w}\right] = 1 + O(a\delta) > 0$ on arbitrarily large compact sets $|w| \le \Delta$ and in such cases we can divide both sides of (49) by this expression. Suppose that this is the case and note that using the Mean Value Theorem [15, page 125], we conclude that:

$$f_1(\sigma, w + Ka\delta q) - f_1(\sigma, w) = O(a\delta)$$
$$f_2(\sigma, w + Ka\delta q) - f_2(\sigma, w) = O(a\delta),$$

and we know that $R = R(\sigma, w + Ka\delta q) = O(1)$. Hence, for sufficiently small a and δ we can write (49) as follows:

$$\frac{dw}{d\sigma} = Ka^2\delta f_{av}(w) + O(a^3\delta) + O(a^2\delta^2) + O(a^3\delta^2). \qquad (50)$$

Using the Lyapunov function V from (45) as a candidate Lyapunov function for the system (50), we can easily conclude that the reduced system in w coordinates is SPA stable, uniformly in (a^2, δ). Then, using (46) it is immediate that the reduced system (40) in the original coordinates θ_r is SPA stable, uniformly in (a^2, δ).

Next we prove that the overall system (15) is SPA stable, uniformly in (a^2, δ). We do that by showing appropriate stability of the boundary layer system and then use Lemma 1 in Appendix B. Introducing $\bar{\mathbf{x}} \triangleq \tilde{\mathbf{x}} - \mathbf{l}(\theta^* + \tilde{\theta}(\sigma) + a\cdot\sin(\sigma)) - \mathbf{x}^*$, $t' = \frac{\sigma - \sigma_0}{\omega}$, where σ_0 is a fixed time instant in time scale σ, setting $\omega = 0$ and denoting $\theta_1 \triangleq \theta^* + \tilde{\theta}(\sigma_0) + a\cdot\sin(\sigma_0)$, the of boundary layer corresponding to the overall system (15) satisfies,

$$\frac{d\bar{\mathbf{x}}}{dt'} = \mathbf{f}(\bar{\mathbf{x}} + \mathbf{l}(\theta_1), \alpha(\bar{\mathbf{x}} + \mathbf{l}(\theta_1), \theta_1)). \qquad (51)$$

Assumption 2 guarantees that the above system is globally asymptotically stable, uniformly in θ_1. Hence, using Lemma 1 in the Appendix B, the system (14) (when $b = a$) with parameter $\boldsymbol{\epsilon}$ is SPA stable uniformly in (a^2, δ) (with the time scale t), which completes the proof. ■

Sketch of proof of Proposition 1: Introducing the change of coordinates $\tilde{\theta} := w + a\delta q(\sigma, w, a)$ like in (46) with (47), we can rewrite (27) in original time scale "t":

$$\dot{w} = a^2\delta\frac{K}{2}Q'(w+\theta^*) + (a^2\delta^2 + a^3\delta + a^3\delta^2)\Delta_3(w, \bar{\mathbf{x}}) + a\delta\tilde{\Delta}_2,$$

and $\tilde{\Delta}_2$ is obtained by substituting $w + a\delta q$ instead of $\tilde{\theta}$ in Δ_2 in (28). Moreover, it is not hard to show that Δ_3 is bounded and there exists $\tilde{\gamma} \in \mathcal{K}_\infty$ such that

$$\tilde{\Delta}_2(\bar{\mathbf{x}}, \theta) \le \gamma_2^\theta(|\bar{\mathbf{x}}|),$$

on compact sets, uniformly in small a and δ. Using the Lyapunov function $V(w) = \frac{1}{2}w^2$ and Assumption 4, we obtain that for any (Δ, ν) and ω^* there exist a^* and δ^* such that for any $a \in (0, a^*)$, $\delta \in (0, \delta^*)$,

[13] In the rest of the proof we suppress arguments of all functions whenever they are clear.

$\omega \in (0, \omega^*)$ and $\max\{|\bar{\mathbf{x}}|, |w|\} \leq \Delta$ we have that $|w| \geq \max\{\alpha_Q^{-1}\left(\frac{4}{a}\gamma_2^\theta(|\bar{\mathbf{x}}|)\right), \alpha_Q^{-1}\left(4(\delta+a)\Delta_3\right)\}$ implies

$$\frac{dV}{d\sigma} \leq -\frac{K}{4}a^2\delta\alpha_Q(|w|).$$

Standard comparison lemmas for ISS [5, Lemma 3.4] imply that an appropriate ISS bound holds for trajectories of system (27) in w coordinates. But then this immediately implies that the conclusion of Proposition 1 holds since $|\tilde{\theta}| \leq |w| + a\delta|q|$. ∎

Sketch of proof of Proposition 2: From Assumptions 1 and 2 we conclude using results from [8] that there exists a smooth Lyapunov function W such that for all $\bar{\mathbf{x}}, \tilde{\theta}$ we have:

$$\alpha_1(|\bar{\mathbf{x}}|) \;\leq\; W(\bar{\mathbf{x}}, \tilde{\theta}) \leq \alpha_2(|\bar{\mathbf{x}}|) \tag{52}$$

$$\frac{\partial W}{\partial \bar{\mathbf{x}}}(\bar{\mathbf{x}}, \tilde{\theta})\tilde{\mathbf{f}}(\bar{\mathbf{x}} + \tilde{\mathbf{l}}(\tilde{\theta}), \tilde{\theta}) \leq -\alpha_3(|\bar{\mathbf{x}}|), \tag{53}$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$. We use the Lyapunov function $U(\bar{\mathbf{x}}, \tilde{\theta}, \sigma) := W(\bar{\mathbf{x}}, \tilde{\theta} + a\sin(\sigma))$ as a candidate ISS Lyapunov function for the system (26). Taking the derivative of W along solutions of (26) in time scale "$\sigma = \omega t$", we obtain:

$$\begin{aligned}\frac{dU}{d\sigma} &= \frac{1}{\omega}\frac{\partial^T W}{\partial \bar{\mathbf{x}}}\tilde{\mathbf{f}}(\bar{\mathbf{x}} + \tilde{\mathbf{l}}(\tilde{\theta} + a\sin(\sigma)), \tilde{\theta} + a\sin(\sigma)) \\ &\quad + \frac{\partial^T W}{\partial \bar{\mathbf{x}}}\Delta_1 + \frac{\partial W}{\partial \tilde{\theta}}\left[\frac{d\tilde{\theta}}{d\sigma} + a\cos(\sigma)\right],\end{aligned} \tag{54}$$

and using the definition of Δ_1 in (27) and the condition in (53) we can see that for any a^*, δ^* and $\Delta > 0$ there exist $L, c > 0$ and $\gamma \in \mathcal{K}_\infty$ such that for all $\max\{|\bar{\mathbf{x}}|, |\tilde{\theta}|\} \leq \Delta$, $a \in (0, a^*)$ and $\delta \in (0, \delta^*)$ we have:

$$\frac{dU}{d\sigma} \leq -\frac{1}{\omega}\alpha_3(|\bar{\mathbf{x}}|) + a[c + L\alpha_3(|\bar{\mathbf{x}}|) + \gamma(|\tilde{\theta}|)].$$

Hence, we can see that on compact sets and for small ω we have that $|\bar{\mathbf{x}}| \geq \max\{\alpha_3^{-1}(4\omega a(\gamma(|\tilde{\theta}|))), \alpha_3^{-1}(4\omega ac)\}$ implies

$$\frac{dU}{d\sigma} \leq -\frac{1}{4\omega}\alpha_3(|\bar{\mathbf{x}}|),$$

from which the conclusion of the proposition follows immediately. ∎

Sketch of proof of Theorem 2: It follows directly from Propositions 1 and 2 and Theorem 2 in the Appendix B. ∎

Proof of Theorem 3: The system (35) has 2 time scales: fast dynamics $\tilde{\mathbf{x}}$ and slow dynamics $(\tilde{\theta}, \tilde{\xi})$ when ω is a small positive constant. We next use the singular perturbation method. To this end, we set $\omega = 0$ and "freeze" $\tilde{\mathbf{x}}$ at its "equilibrium", $\tilde{\mathbf{x}} = \mathbf{l}(\theta^* + \tilde{\theta} + a \cdot \sin(\sigma)) - \mathbf{x}^*$ to obtain the reduced system in variables (ξ_r, θ_r):

$$\begin{bmatrix} \frac{d\theta_r}{d\sigma} \\ \frac{d\xi_r}{d\sigma} \end{bmatrix} = \begin{bmatrix} K\delta\xi_r \\ -\delta\omega_L[\xi_r - Q(\theta^* + \theta_r + a\sin(\sigma))a\sin(\sigma)] \end{bmatrix}. \tag{55}$$

In the rest of the proof we use the same notation as in the proof of Theorem 1. In particular, using (41) and (44) we introduce the average system of (55):

$$\begin{bmatrix} \frac{d\theta_{av}}{d\sigma} \\ \frac{d\xi_{av}}{d\sigma} \end{bmatrix} = \begin{bmatrix} K\delta\xi_{av} \\ \delta\omega_L[-\xi_{av} + a^2 f_{av}(\theta_{av})] \end{bmatrix} =: F(\mathbf{z}, \delta, a), \tag{56}$$

where $\mathbf{z} := (\theta_{av}\ \xi_{av})^T$. First, we show that the following quadratic function:

$$V(\mathbf{z}) := \frac{1}{2}\mathbf{z}^T \begin{bmatrix} 1 & \frac{1}{c} \\ \frac{1}{c} & \frac{c^2+1}{c^2} \end{bmatrix} \mathbf{z} = \frac{1}{2}\mathbf{z}^T H \mathbf{z},$$

with $c \triangleq \dfrac{\omega_L}{K}$, is a Lyapunov function for the average system (56). It is obvious that V is positive definite and radially unbounded since there exist $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \cdot |\mathbf{z}|^2 \leq V(\mathbf{z}) \leq \alpha_2 \cdot |\mathbf{z}|^2 \qquad \forall \mathbf{z}. \tag{57}$$

Taking derivative of V along solutions of (56), we can write:

$$\begin{aligned}\frac{dV}{d\sigma} &= \frac{\partial V}{\partial \mathbf{z}}F(\mathbf{z}, a, \delta) = \mathbf{z}^T H F(\mathbf{z}, a, \delta) \\ &= K\delta\xi_{av}\left(\theta_{av} + \frac{1}{c}\xi_{av}\right) + \\ &\quad \omega_L\delta\left(\frac{1}{c}\theta_{av} + \frac{c^2+1}{c^2}\xi_{av}\right)(-\xi_{av} + a^2 f_{av}(\theta_{av})).\end{aligned} \tag{58}$$

Moreover, using $K = \frac{\omega_L}{c}$ and $\frac{\omega_L}{c^2} = \frac{K}{c}$, (58) becomes

$$\begin{aligned}\frac{\partial V}{\partial \mathbf{z}}F(\mathbf{z}, a, \delta) &= Ka^2\delta f_{av}(\theta_{av})\theta_{av} - \omega_L\delta\xi_{av}^2 \\ &\quad + \omega_L a^2\delta\left(1 + \frac{1}{c^2}\right)\xi_{av} f_{av}(\theta_{av}).\end{aligned} \tag{59}$$

Adding and subtracting $\frac{\delta\omega_L}{2} \cdot \left(a^2\left(1 + \frac{1}{c^2}\right)f_{av}(\theta_{av})\right)^2$ to (59) and using the completion of squares, we can write:

$$\begin{aligned}\frac{\partial V}{\partial \mathbf{z}}F(\mathbf{z}, a, \delta) &\leq Ka^2\delta f_{av}(\theta_{av})\theta_{av} - \frac{\omega_L}{2}\delta\xi_{av}^2 \\ &\quad -\frac{\omega_L\delta}{2}\left(\xi_{av} - a^2\left(1 + \frac{1}{c^2}\right)f_{av}(\theta_{av})\right)^2 + O(\delta a^4),\end{aligned} \tag{60}$$

Since $f_{av}(\theta_{av})\theta_{av} > 0$ is negative definite (see Assumption 3) and (57) holds, there exists a positive definite function $\alpha_3 : R_{\geq 0} \to R_{\geq 0}$ such that for all $a \in (0,1)$ we have:

$$\frac{\partial V}{\partial \mathbf{z}} \cdot F(\mathbf{z}, a, \delta) \leq -a^2\delta\alpha_3(V) + O(a^4\delta). \tag{61}$$

The rest of the proof follows using averaging results in [24] and Lemma 1 in the Appendix B (see [22] for more details). ∎

Sketch of proof of Theorem 4: In the new coordinate $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}^*$, $\tilde{\theta} = \hat{\theta} - \theta^*$, $\tilde{\xi} = \xi$ and $\tilde{\eta} = \eta - y^*$, the $\tilde{\eta}$-subsystem in the average does not affect the sub-system $(\tilde{\theta}, \tilde{\xi})$, that is, the average system of the sub-system $(\tilde{\theta}, \tilde{\xi})$ in (36) is exactly the same as the average of the sub-system $(\tilde{\theta}, \tilde{\xi})$ in (32). Hence, we have a cascade of the system $(\tilde{\theta}, \tilde{\xi})$ and the system $\tilde{\eta}$. In Theorem 3 we proved that the average of the sub-system $(\tilde{\theta}, \tilde{\xi})$ is SPA stable uniformly in (a^2, δ). Moreover, it is obvious that the $\tilde{\eta}$ subsystem is ISS in y. Using the stability results for cascade systems, we can conclude that the average system $(\tilde{\theta}, \tilde{\xi}, \tilde{\eta})$ is SPA stable, uniformly in (a^2, δ). The proof is then completed following exactly the same steps as in the proof of Theorem 3. ∎

Appendix B

In order to state Lemma 1, we consider the following nonlinear system

$$\begin{aligned} \dot{\mathbf{x}} &= \epsilon_\ell \mathbf{f}(\epsilon_\ell t, \mathbf{x}, \mathbf{z}, \epsilon_1, \cdots, \epsilon_{\ell-1}) \\ \dot{\mathbf{z}} &= \mathbf{g}(\epsilon_\ell t, \mathbf{x}, \mathbf{z}, \epsilon_1, \cdots, \epsilon_{\ell-1}, \epsilon_\ell), \end{aligned} \tag{62}$$

where $\mathbf{x} \in R^n$, $\mathbf{z} \in R^q$ and $[\epsilon_1, \epsilon_2, \cdots, \epsilon_{\ell-1}] \in R^{\ell-1}$. Let $\sigma = \epsilon_\ell t$, in the new time "σ",

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial \sigma} &= \mathbf{f}(\sigma, \mathbf{x}, \mathbf{z}, \epsilon_1, \cdots, \epsilon_{\ell-1}) \\ \epsilon_\ell \frac{\partial \mathbf{z}}{\partial \sigma} &= \mathbf{g}(\sigma, \mathbf{x}, \mathbf{z}, \epsilon_1, \cdots, \epsilon_{\ell-1}, \epsilon_\ell), \end{aligned} \tag{63}$$

Let $\epsilon_\ell = 0$, the state vector $\mathbf{z}$ becomes instantaneous and (63) takes the form

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial \sigma} &= \mathbf{f}(\sigma, \mathbf{x}, \mathbf{z}_s, \epsilon_1, \epsilon_2, \cdots, \epsilon_{\ell-1}) \\ 0 &= \mathbf{g}(\sigma, \mathbf{x}, \mathbf{z}_s, \epsilon_1, \cdots, \epsilon_{\ell-1}, 0), \end{aligned} \tag{64}$$

where $\mathbf{z}_s$ denotes a quasi-steady state for the fast state vector $\mathbf{z}$.

With $\mathbf{z}_s = \mathbf{h}(\sigma, \mathbf{x}, \epsilon_1, \cdots, \epsilon_{\ell-1})$, the following reduced system is obtained

$$\frac{\partial \mathbf{x}}{\partial \sigma} = \mathbf{f}(\sigma, \mathbf{x}, \mathbf{h}(\sigma, \mathbf{x}, \epsilon_1, \cdots, \epsilon_{\ell-1}), \epsilon_1, \cdots, \epsilon_{\ell-1}). \tag{65}$$

Introducing $\mathbf{y} = \mathbf{z} - \mathbf{h}(\sigma, \mathbf{x}, \epsilon_1, \cdots, \epsilon_{\ell-1})$, $\tau = \frac{\sigma - \sigma_0}{\epsilon_\ell}$ and setting $\epsilon_\ell = 0$, the boundary layer satisfies

$$\frac{d\mathbf{y}}{d\tau} = \mathbf{g}(\sigma_0, \mathbf{x}, \mathbf{h}(\sigma_0, \mathbf{x}, \epsilon_1, \cdots, \epsilon_{\ell-1}) + \mathbf{y}, \epsilon_1, \cdots, \epsilon_{\ell-1}, 0). \tag{66}$$

Lemma 1 Suppose the following conditions hold:

(1) The algebraic equation (64) possesses a unique root $\mathbf{z}_s = \mathbf{h}(\sigma, \mathbf{x}, \epsilon_1, \ldots, \epsilon_{\ell-1})$, where $\mathbf{h}$ and its partial derivatives $\frac{\partial \mathbf{h}}{\partial \mathbf{x}}$ are locally Lipschitz, uniformly in σ and small $(\epsilon_1, \ldots, \epsilon_{\ell-1})$.
(2) The reduced system (65) with parameter $(\epsilon_1, \cdots, \epsilon_{\ell-1})$, is SPA stable, uniformly in ϵ_1.
(3) The equilibrium $\mathbf{y} = \mathbf{0}$ of the boundary layer system in (66) is globally asymptotically stable, uniformly [14] in $\mathbf{x}$, σ_0 as well as $\epsilon_1, \epsilon_2, \cdots, \epsilon_{\ell-1}$.

Then, the system (62) with parameter $(\epsilon_1, \cdots, \epsilon_{\ell-1}, \epsilon_\ell)$ is SPA stable, uniformly in ϵ_1 (with the time scale t).

Consider a connection of two parameterized time varying systems (we refer to [20, Lemma A4][21, Theorem 5] for the techniques that can be used to prove this result):

$$\dot{x}_1 = f_1(t, x_1, x_2, \boldsymbol{\epsilon}) \tag{67}$$
$$\dot{x}_2 = f_2(t, x_2, x_1, \boldsymbol{\epsilon}), \tag{68}$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$ and $\boldsymbol{\epsilon} \in R^l_{>0}$ and we also denote $x := [x_1^T \ x_2^T]^T$. Then, the following small gain result holds:

Lemma 2 *Suppose that there exist $\beta_1, \beta_2 \in \mathcal{KL}$ such that for any strictly positive $\Delta_1, \Delta_2, \nu_1, \nu_2, s_1, s_2$ with $\Delta_i > \nu_i > 0, i = 1, 2$ and $0 < s_1 < s_2$ there exist $\boldsymbol{\epsilon}^* \in R^l_{>0}$ and for $\epsilon_i \in (0, \epsilon_i^*), i = 1, 2, \ldots, l$ there exist $\gamma^1_{\boldsymbol{\epsilon}}, \gamma^2_{\boldsymbol{\epsilon}} \in \mathcal{K}_\infty$ such that $\max\{|x_1(t_0)|, ||x_2||\} \leq \Delta_1$ implies that solutions of the subsystem (67) satisfy*

$$|x_1(t)| \leq \max\{\beta_1(|x_1(t_0)|, t - t_0), \gamma^1_{\boldsymbol{\epsilon}}(||x_2||), \nu_1\},$$

for all $t \geq t_0 \geq 0$ and $\max\{|x_2(t_0)|, ||x_1||\} \leq \Delta_2$ implies that solutions of the subsystem (68) satisfy

$$|x_2(t)| \leq \max\{\beta_2(|x_2(t_0)|, (\epsilon_1 \cdots \epsilon_l) \cdot (t - t_0)), \gamma^2_{\boldsymbol{\epsilon}}(||x_1||), \nu_2\},$$

for all $t \geq t_0 \geq 0$ and, moreover, for all $\epsilon_i \in (0, \epsilon_i^*), i = 1, 2, \ldots, l$ the small gain conditions hold for all $s \in [s_1, s_2]$:

$$\gamma^1_{\boldsymbol{\epsilon}} \circ \gamma^2_{\boldsymbol{\epsilon}}(s) \leq \gamma(s) < s \qquad \gamma^2_{\boldsymbol{\epsilon}} \circ \gamma^1_{\boldsymbol{\epsilon}}(s) \leq \gamma(s) < s, \tag{69}$$

where $\gamma \in \mathcal{K}_\infty$ is independent of $\boldsymbol{\epsilon}$. Then, the system (67), (68) is SPA stable uniformly in $\boldsymbol{\epsilon}$. In other words,

[14] Here, by uniform stability in $(\epsilon_1, \ldots, \epsilon_{\ell-1})$ we mean that the stability bounds are independent of all parameters, i.e. we do not use Definition 1.

there exists $\beta \in \mathcal{KL}$ *such that for any* (Δ, ν) *there exists* $\epsilon^* \in R^l_{>0}$ *such that for all* $|x(t_0)| \leq \Delta$ *and all* $\epsilon_i \in (0, \epsilon_i^*)$ *the solutions of the system (67), (68) satisfy:*

$$|x(t)| \leq \max\left\{\beta\left(|x(t_0)|, (\epsilon_1 \cdots \epsilon_l) \cdot (t - t_0)\right), \nu\right\},$$

for all $t \geq t_0 \geq 0$.