Input-to-state stability analysis via averaging for parameterized discrete-time systems

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Abstract

The paper studies semi-global practical input-to-state stability (SGP-ISS) of a parameterized family of discrete-time systems that may arise when an approximate discrete-time model of a sampled-data system with disturbances is used for controller design. It is shown under appropriate conditions that if the solutions of the time varying family of discrete-time systems with disturbances converge uniformly on compact time intervals to the solutions of the average family of discrete-time systems, then ISS of the average family of systems implies SGP-ISS of the original family of systems. A trajectory based approach is utilized to establish the main result.

1 Introduction

Sampled-data systems currently attract a lot of attention in the literature (see [1, 2, 4]). The presence of a sampler in the closed loop makes sampled-data systems time-varying even if the plant and controller are time invariant. This complicates the analysis of sampled-data systems, especially when the plant is nonlinear.

Recently, a prescriptive framework for stabilization of sampled-data nonlinear systems via their approximate discrete-time models was proposed in [9]. Within the above framework, one typically needs to verify uniform stability properties of a family of approximate discrete-time models that are parameterized with a sampling period. For instance, it was shown in [9] that if a family of approximate discrete-time models is uniformly globally asymptotically stable and a certain consistency condition holds, then the family of

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exact discrete-time models is semi-globally practically stable in the sampling period. Semi-global practical stability of the exact discrete-time models implies under weak conditions the same property for the actual sampled-data system, see [10]. Similar results were presented in [8] to provide conditions for semi-global practical input-to-state stability of sampled-data nonlinear systems.

The main goal of this paper is to present an averaging technique that can be used to conclude semi-global ISS of parameterized families of discrete time systems. Our results are fully consistent with the framework in [8] and they are a useful addition to the toolbox for analysis and design of sampled-data nonlinear systems.

Averaging methods for analysis of difference equations that depend on a small parameter have a long history. Most averaging results in the literature focus on stability properties of averaged and original systems, as presented in [3, 14]. Solo [14] proves that if the linearization of the averaged system is exponentially stable, then there exists a unique solution of the original system in a neighborhood of the equilibrium point of the averaged system. Moreover, if the equilibrium points of both systems are identical, then the original system is locally exponentially stable. We note that these results all focus on local exponential stability of non-parameterized discrete-time systems [5, 7, 16]. Such results are useful in situations when the exact discrete-time model of the sampled-data system is known. We are not aware of discrete-time averaging results for systems with disturbances, which is the main focus of our paper. Moreover, our results can be used together with [8] to analyze ISS of sampled-data nonlinear systems for which we can not compute the exact discrete-time model, and we need to use an approximate model for stability analysis or controller design.

Our results are closely related to the recent averaging techniques for continuous-time systems with disturbances given in [11] where ISS was investigated. In particular, we provide results similar to [11] for parameterized discrete-time systems. Using the notions of strong and weak averages that were introduced in [11], we present conditions under which ISS of the strong average implies SGP-ISS of the family of time-varying parameterized discrete-time systems. Using the notion of weak averages, we prove similar results where we conclude an ISS like property that requires derivatives of disturbances to be bounded.

The paper is organized as follows. Section 2 contains mathematical preliminaries. We present the main results in Sections 3 and 4, and an application of main results in Section 5. A summary is given in the last section. Technique lemmas are proved in the Appendix.

2 Preliminaries

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{G}$ if it is zero at zero, continuous and nondecreasing. A function $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{GG}$ if it is zero at zero, continuous and nondecreasing in both arguments. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class of $\mathcal{KL}$ if it is continuous and nondecreasing from zero in its first argument, and converging to zero in its second argument, and a continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class of $\mathcal{K}$ if it is of class-$\mathcal{G}$ and strictly increasing.

We often use sampled versions of a given continuous-time function. Given a function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and a positive sampling period $\tau > 0$, we define its sampled version as $w_\tau := \{w(k\tau) : k \in \mathbb{N}\}$ but we omit the subscript τ for notational simplicity. Then, we define its infinity norm as $\|w\|_\infty := \max_{k \geq 0} |w(k\tau)|$ and we write $w \in L_\infty$ if $\|w\|_\infty \leq r < \infty$.

Consider a family of time varying discrete-time systems parameterized by the sampling time interval $\tau > 0$:

$$\frac{\Delta x}{\Delta k} = F_\tau\left(k\tau, x, w\right) \qquad \Delta k = \tau, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $\Delta x := x(k\tau + \Delta k) - x(k\tau)$, $k \geq 0$. Suppose that the function $F_\tau(s, x, w)$ is locally Lipschitz in x and w uniformly in small τ and $s \geq 0$ and $F_\tau(s, 0, 0)$ is uniformly bounded in small τ and $s \geq 0$.

We also investigate the family of parameterized discrete-time systems that depends on a small parameter $\varepsilon > 0$:

$$\frac{\Delta x}{\Delta k} = F_\tau\left(\frac{k\tau}{\varepsilon}, x, w\right) \qquad \Delta k = \tau. \tag{2}$$

Remark 1 *Our results are strongly motivated by the results in [8, 9]. Indeed, our results can be used together with [8] to design controllers achieving ISS for nonlinear sampled-data systems for which the exact discrete-time model can not be analytically computed and we have to use an approximate discrete-time model for controller design and stability analysis. More precisely, consider a nonlinear sampled-data plant with disturbances*

$$\dot{x} = f(t, x, u, w) \ , \tag{3}$$

where we assume that $u(t) = u(k\tau), \forall t \in [k\tau, (k+1)\tau)$ *and for simplicity also* $w(t) = w(k\tau), \forall t \in [k\tau, (k+1)\tau)$. $\tau > 0$ *is the sampling period. If we want to carry out a controller design in discrete-time then we need to compute the (exact) discrete-time model of the plant:*

$$x(k+1) = F^e_\tau(k\tau, x(k), u(k), w(k)) \ , \tag{4}$$

which is obtained by integrating (3) over one sampling interval $[k\tau, (k+1)\tau]$ *from the initial time* $k\tau$ *and the initial state* $x(k) := x(k\tau)$ *with a constant* *control input $u(k) := u(k\tau)$ and disturbance input $w(k) := w(k\tau)$. However, since (3) is nonlinear, it is typically not possible to analytically compute the exact discrete-time model (4) for controller design. Instead, an approximate discrete-time model of the plant may still be obtained and used for controller design:*

$$x(k+1) = F^a_\tau(k\tau, x(k), u(k), w(k)) \; . \tag{5}$$

In this case, it is assumed that the sampling period τ is a design parameter which can be arbitrarily assigned. A natural question is if we design a family of controllers

$$u(k) = u_\tau(k\tau, x(k)) \; , \tag{6}$$

to stabilize in an appropriate sense the family of approximate discrete-time models (5), would the same family of controller stabilize (maybe in some weaker sense) the family of exact discrete-time models (4). It was shown in [8] this is not always true and in particular, it is required that[1]:

1. *The family F^a_τ is consistent in an appropriate sense with F^e_τ;*
2. *The family of control laws u_τ is bounded on compact sets uniformly in small τ.*
3. *The family (5), (6) of the approximate closed-loop models is ISS;*

The first condition is adapted from the numerical analysis literature and it holds for most commonly used approximations, such as Runge-Kutta methods. The second condition is easily checked once the control law (6) is obtained. The last condition is typically the hardest to check and it needs to be done on a case-by-case basis. This necessitates the development of various stability analysis tools for parameterized families of discrete-time systems (5), (6) that are useful in different situations. The main purpose of this paper is to develop several such stability analysis tools that are based on the averaging theory. In particular, our results are useful in this context whenever the parameterized family of discrete-time systems (5), (6) can be represented in the form (2) that is amenable to averaging analysis. □

Remark 2 *Define the function $k\tau = \varepsilon\zeta\tau$, using which we map the countable set $\{k_0\tau, k_1\tau, k_2\tau, \cdots\}$ into a set $\{\zeta_0\tau, \zeta_1\tau, \zeta_2\tau, \cdots\}$ with $k_0\tau = \varepsilon\zeta_0\tau$, $k_i\tau = k_0\tau + i\Delta k$ and $\zeta_i\tau = \zeta_0\tau + i\Delta\zeta$, for $i = 1, 2, \cdots$. Under the mapping, the corresponding family of systems (2) could be written as*

$$\frac{\Delta x}{\Delta \zeta} = \varepsilon F_\tau(\zeta\tau, x, w) \qquad \Delta\zeta = \frac{\tau}{\varepsilon}, \tag{7}$$

with the fixed initial time $k_0\tau = \varepsilon\zeta_0\tau$. Then, if (2) satisfies

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[1] Results in [8] are given for time invariant systems but these results can easily be extended to cover time-varying systems, see [6].

$$|x(k\tau)| \le \beta(|x_0|, (k-k_0)\tau), \tag{8}$$

where $x_0 = x(k_0\tau)$*, then the family of systems (7) satisfies*

$$|x(\zeta\tau)| \le \beta(|x_0|, \varepsilon(\zeta-\zeta_0)\tau) \tag{9}$$

with $x_0 = x(\varepsilon\zeta_0\tau)$*.*

□

Next we adapt the notions of strong and weak averages in [11] to families of discrete-time systems (2) so that we can obtain stability results that are fully consistent with [8, 9].

Definition 1 *(weak average) A locally Lipschitz function* F_τ^{wa} *is said to be the weak average of* F_τ *if there exists* $\beta_{wa} \in \mathcal{KL}$ *and* $T^* > 0$ *such that for all* $T > T^*$*, there exists* $\tau^* = \tau^*(T)$*, such that* $\forall \tau \in (0, \tau^*)$ *and* $N\tau \ge T$*, the following holds for all* $x \in \mathbb{R}^n$*,* $w \in \mathbb{R}^m$

$$\begin{aligned} &\left| F_\tau^{wa}(x,w) - \frac{1}{N\tau}\sum_{s=k}^{k+N} F_\tau(s\tau, x, w)\Delta s \right| \\ &\qquad \le \beta_{wa}(\max\{|x|, |w|, 1\}, N\tau). \end{aligned} \tag{10}$$

The weak average of the parameterized family of discrete-time systems (2) is then defined as

$$\frac{\Delta y}{\Delta k} = F_\tau^{wa}(y,w) \qquad \Delta k = \tau. \tag{11}$$

□

Remark 3 *Note the above definition of weak average depends on sampling intervals* τ*, the reason is as the approximation of continuous systems, the average functions of parameterized family of discrete-time systems could be rewritten as*

$$\begin{aligned} &\left| F_\tau^{wa}(x,w) - \frac{1}{N\tau}\sum_{s=k}^{k+N} F_\tau(s\tau, x, w)\Delta s \right| \\ &\le \left| F_\tau^{wa}(x,w) - \frac{1}{T}\int_t^{t+T} F_\tau(h,x,w)dh \right| \\ &\quad + \left| \frac{1}{T}\int_t^{t+T} F_\tau(h,x,w)dh - \frac{1}{N\tau}\sum_{s=k}^{k+N} F_\tau(s\tau, x, w)\Delta s \right| \\ &\le \tilde{\beta}_{wa}(\max\{|x|, |w|, 1\}, T) \\ &\quad + \frac{1}{T}\left| \int_t^{t+T} F_\tau(h,x,w)dh - \sum_{s=k}^{k+N} F_\tau(s\tau, x, w)\tau \right|, \end{aligned} \tag{12}$$

where $\tilde{\beta}(\cdot)$ is the definition of weak average for continuous systems (see [11]). When τ is sufficiently small such that we can bound the error between sum with integral with a function $\gamma(|x|,|w|)$ of class-$\mathcal{GG}$, and T is sufficiently large such that $\frac{1}{T}$ could be bounded by $\frac{2}{T+1}$, then the above inequality can be rewritten as

$$\begin{aligned} &\left| F_{av}(x,w) - \frac{1}{N\tau}\sum_{s=k}^{k+N} F_\tau(s\tau,x,w)\Delta s \right| \\ &\qquad \leq \tilde{\beta}_{wa}(\max\{|x|,|w|,1\},T) + \frac{2}{T+1}\gamma(|x|,|w|) \\ &\qquad := \beta_{wa}(\max\{|x|,|w|,1\},N\tau). \end{aligned} \tag{13}$$

For general periodic systems, sampling intervals τ is independent of T, but for the aim of generalization, $\tau^* = \tau^*(T)$ is used.

□

Definition 2 *(strong average) A locally Lipschitz function F_τ^{sa} is said to be the strong average of F_τ if there exists $\beta_{sa} \in \mathcal{KL}$ and and $T^* > 0$ such that for all $T > T^*$, there exists $\tau^* = \tau^*(T)$, such that $\forall \tau \in (0,\tau^*)$ and $N\tau \geq T$, the following holds for all $x \in \mathbb{R}^n$, $w \in L_\infty$*

$$\begin{aligned} &\left| \frac{1}{N\tau}\sum_{s=k}^{k+N} \{F_\tau^{sa}(x,w(s\tau)) - F_\tau(s\tau,x,w(s\tau))\}\Delta s \right| \\ &\qquad\qquad \leq \beta_{sa}(\max\{|x|,\|w\|_\infty,1\},N\tau). \end{aligned} \tag{14}$$

The strong average of the parameterized family of discrete-time systems (2) is then defined as

$$\frac{\Delta y}{\Delta k} = F_\tau^{sa}(y,w) \qquad \Delta k = \tau. \tag{15}$$

□

Note that the main difference between the weak and strong averages is that in the definition of weak average the disturbance is kept constant in (10) whereas in the definition of strong average the inequality (14) needs to hold for all disturbances $w \in L_\infty$. In case when $w \equiv 0$ both definitions of average coincide.

It was shown in [11] for continuous-time systems that strong averages exist for a smaller class of systems but using them we can state stronger stability results. On the other hand, weak averages exist for a larger class of systems but using them we can state weaker stability results. Nevertheless, weak averages are found useful in cases when disturbances are bounded and have bounded derivatives and one such situation arises when one deals with ISS of cascaded systems. Hence, both notions of the weak and strong averages are useful in different situations and we investigate both.

A complete characterization of strong averages for continuous-time periodic systems was given in [11]. It can be shown in a similar manner to [11] that any $F_\tau(s,x,w)$ that is periodic in s has a strong average if and only if the function F_τ has the structure as follows

$$F_\tau(k\tau,x,w) = F_\tau^1(k\tau,x) + F_\tau^2(x,w) \tag{16}$$

and there exists the average for $F_\tau^1(k\tau,x)$ according to either of our definitions (they coincide since F_τ^1 does not depend on the disturbance). Then, $F_\tau^{sa}(x,w) := F_{av}(x) + F_\tau^2(x,w)$ satisfies our definition of the strong average for F_τ.

The following example shows that for some systems, the weak average may exist whereas the strong average does not. The example is adapted from [11]. Consider the system

$$\frac{\Delta x}{\Delta k} = -0.5x^3 + \cos\left(\frac{k\tau}{\varepsilon}\right)x^3 w \tag{17}$$

where x,$w \in \mathbb{R}$. The weak average of $-0.5x^3 + \cos(k\tau)x^3w$ is

$$\frac{\Delta y}{\Delta k} = -0.5y^3 \ . \tag{18}$$

Indeed, setting $\tilde{s} = s\tau$ and $T := \tau N$ we can write for sufficiently small τ that

$$\begin{aligned}
\left|\frac{1}{N\tau}\sum_{s=k}^{k+N}\cos(s\tau)x^3w\cdot\tau\right| &= \left|\frac{1}{N\tau}\sum_{\tilde{s}=k\tau}^{k\tau+N\tau}\cos(\tilde{s})x^3w\cdot\Delta\tilde{s}\right| \\
&\le \left|\frac{x^3w}{T}\int_{k\tau}^{k\tau+T}\cos(\tilde{s})d\tilde{s}\right| \\
&\le \frac{|x^3w|\pi}{T} \le \frac{2(\max\{|x|,|w|,1\})^4\pi}{T+1}
\end{aligned} \tag{19}$$

where the last inequality holds when $T \geq 1$ and we can let $\beta_{wa}(s,t) := \frac{(2\max\{|x|,|w|,1\})^4}{T+1}$.

Now, we will show that there does not exist strong average for systems (17). Pick an arbitrary $\bar{x} \neq 0$ and note that, for any given function $F_\tau^{sa}(x,w)$, we have two possibilities

a. either $F_\tau^{sa}(\bar{x},w) + 0.5\bar{x}^3 = 0$, $\forall w$, or
b. $\exists \bar{w}$ such that $F_\tau^{sa}(\bar{x},\bar{w}) + 0.5\bar{x}^3 \neq 0$.

Suppose that $F^{sa}_\tau(x,w)$ is the strong average for $-0.5x^3+\cos(k\tau)x^3w$ and case **a** holds. Let $w(k\tau)=\cos(k\tau)$, $\tilde{s}=s\tau$, and $N_C\tau:=C\pi$, $C\in\mathbb{N}$, similarly like (19), we have

$$\begin{aligned}
\left|\frac{1}{N_C\tau}\sum_{s=k}^{k+N}\bar{x}^3\cos^2(s\tau)\Delta s\right| &= \left|\frac{1}{N_C\tau}\sum_{\tilde{s}=k\tau}^{k\tau+N_C\tau}\bar{x}^3\cos^2(\tilde{s})\Delta\tilde{s}\right| \\
&\le \left|\frac{1}{C\pi}\int_{k\tau}^{k\tau+C\pi}\bar{x}^3\cos^2(\tilde{s})d\tilde{s}\right| \\
&= \frac{1}{2}|\bar{x}^3|>0 \quad \forall C>0, \qquad (20)
\end{aligned}$$

which does not converge to zero as C approaches infinity ($N_c\tau\to\infty$). Suppose now that $F^{sa}_\tau(x,w)$ is the strong average for $-0.5x^3+\cos(k\tau)x^3w$ and case **b** holds. Pick $w(k\tau)=\bar{w}$, set $N_C\tau:=2C\pi$, one gets

$$\begin{aligned}
&\left|\frac{1}{N_C\tau}\sum_{s=k}^{k+N}(F^{sa}_\tau(\bar{x},\bar{w})+0.5\bar{x}^3-\bar{x}^3\bar{w}\cos(s\tau))\Delta s\right| \\
&= \left|\frac{1}{N_C\tau}\sum_{\tilde{s}=k\tau}^{k\tau+N_C\tau}(F^{sa}_\tau(\bar{x},\bar{w})+0.5\bar{x}^3-\bar{x}^3\bar{w}\cos(\tilde{s}))\Delta\tilde{s}\right| \\
&\le \left|\frac{1}{2C\pi}\int_{k\tau}^{k\tau+2C\pi}(F^{sa}_\tau(\bar{x},\bar{w})+0.5\bar{x}^3-\bar{x}^3\bar{w}\cos(\tilde{s}))d\tilde{s}\right| \\
&= |0.5\bar{x}^3+F^{sa}_\tau(\bar{x},\bar{w})|>0 \quad \forall C>0. \qquad (21)
\end{aligned}$$

The left hand side in the above expression is larger than zero for all $C\in\mathbb{N}$ and it does not converge to zero as C approaches infinity ($N_C\tau\to\infty$). Hence, there does not exist a strong average for systems (17). $\square$

3 Closeness of solutions

Results in this section are needed for the proofs of our main results that are given in the next section. Moreover, results on closeness of solutions on compact time intervals between the weak or strong average and the original system are of interest in their own right since they characterize precisely the approximating properties of the averages. Note also that results in this section are derived under weaker conditions than results in the next section and, in particular, we do not require the strong/weak average to be ISS. Instead, we use an appropriate notion of forward completeness that is much weaker than the ISS property. We need the following definitions of equi-boundedness, equi-uniformly Lipschitz and forward completeness for disturbance w.

Definition 3 *Let $\mathcal{W}$ be a set of locally bounded functions, the set $\mathcal{W}$ is equi-bounded if there exists a strictly positive real number r such that, for all $w \in \mathcal{W}$, $\|w\|_\infty \leq r$.* $\square$

Note that sampling a bounded continuous-time function $w(\cdot)$ at any sampling period yields its sampled version $w_\tau = w(k\tau)$ that is still bounded.

Definition 4 *Let $\mathcal{W}$ be a set of locally bounded functions, the set $\mathcal{W}$ is equi-uniformly Lipschitz if there exists a strictly positive real number ν and $\tau^* > 0$ such that, for all $\tau \in (0, \tau^*)$, $w \in \mathcal{W}$, $\left\|\frac{\Delta w}{\Delta k}\right\|_\infty \leq \nu < \infty$.* $\square$

Note that if we sample a continuous-time function $w(\cdot)$ then its sampled version $w_\tau = w(k\tau)$ will be equi-uniformly Lipschitz according to our definition if $\dot{w}(\cdot)$ is bounded.

Definition 5 *Let $\mathcal{W}$ be a set of locally bounded functions, the system*

$$\frac{\Delta y}{\Delta k} = F_\tau(y, w) \qquad y(k_0\tau) = y_0, \Delta k = \tau \tag{22}$$

is said to be $\mathcal{W}$-forward complete if for each $r > 0$ and $T > 0$ there exists $R \geq r$ and $\tau^* > 0$ such that, for all $\tau \in (0, \tau^*)$, $|y_0| \leq r$ and $w \in \mathcal{W}$, the solutions of (22) are contained in a closed ball of radius R for all $(k - k_0)\tau \in [0, T]$.

$\square$

The following Lemma 1 and Lemma 2 give conditions under which the solution of the family of systems (2) are close to the solutions of its weak or strong average on compact time intervals.

Lemma 1 *(Closeness to weak average) Suppose the family of parameterized functions $F_\tau(k\tau, x, w)$ is locally Lipschitz in (x, w) uniformly in $k\tau$, $F_\tau(k\tau, 0, 0)$ is bounded, and the set $\mathcal{W}$ is equi-bounded and equi-uniformly Lipschitz, the weak average of the family of discrete-time systems (2) exists and is $\mathcal{W}$-forward complete. Then, for each triple (r, δ, T) of strictly positive real numbers there exists a triple of $(\tau^*, \varepsilon^*, \mu)$ of strictly positive numbers such that, for each $\tau \in (0, \tau^*)$ and for all $\varepsilon \in (0, \varepsilon^*)$ so that, for all $k_0\tau \geq 0$, $|y_0| \leq r$, $w \in \mathcal{W}$, and for each x_0 such that $|x_0 - y_0| \leq \mu$, each solution $x(k\tau, k_0, x_0, w)$ of the family of systems (2) and the solution $y((k - k_0)\tau, y_0, w)$ of the weak average satisfy*

$$|x(k\tau, k_0\tau, x_0, w) - y((k - k_0)\tau, y_0, w)| \leq \delta \quad \forall k:\ (k - k_0)\tau \in [0, T]. \tag{23}$$

Proof. See Appendix. $\square$

Lemma 2 *(Closeness to strong average) Suppose the family of parameterized functions $F_\tau(k\tau, x, w)$ is locally lipschitz in (x, w) uniformly in $k\tau$, $F_\tau(k\tau, 0, 0)$ is bounded, and the set $\mathcal{W}$ is equi-bounded, the strong average of the family of discrete-time systems (2) exists and is $\mathcal{W}$-forward complete. Then, for each triple (r, δ, T) of the strictly positive real numbers there exists a triple $(\tau^*, \varepsilon^*, \mu)$ of the strictly positive numbers such that for each $\tau \in (0, \tau^*)$ and for all $\varepsilon \in (0, \varepsilon^*)$ so that, for all $k_0\tau \geq 0$, $|y_0| \leq r$, $w \in \mathcal{W}$ and for each x_0 such that $|x_0 - y_0| \leq \mu$, each solution $x(k\tau, k_0, x_0, w)$ of the family of systems (2) and the solution $y((k - k_0)\tau, y_0, w)$ of the strong average satisfies*

$$|x(k\tau, k_0\tau, x_0, w) - y((k - k_0)\tau, y_0, w)| \leq \delta \quad \forall k : \ (k - k_0)\tau \in [0, T]. \quad (24)$$

Proof. See Appendix. □

4 Input-to-State stability analysis

In this section, we present the main results of our paper (Theorems 2 and 3) where we present conditions under which ISS of weak or strong average implies an appropriate SGP-ISS property for the actual system. We also prove a preliminary result that shows for a given disturbance set $\mathcal{W}$, that the family of discrete-time systems are semi-globally practically ISS on the set $\mathcal{W}$ on finite time intervals, if and only if they are semi-globally practically ISS on the set $\mathcal{W}$. Precise definitions are given below:

Definition 6 *The parameterized family of discrete-time systems (2) is said to be semiglobally practically ISS on the set $\mathcal{W}$, if for each pair of (δ, r) with $r > \delta \geq 0$, there exist positive real numbers τ^* and ε^* such that for each $\tau \in (0, \tau^*)$ and for all $\varepsilon \in (0, \varepsilon^*)$, $k_0\tau \geq 0$, $w \in \mathcal{W}$ with $||w||_\infty \leq r$, and $x(k_0\tau) \in \mathbb{R}^n$ with $|x(k_0\tau)| \leq r$, we have*

$$|x(k\tau)| \leq \max\{\beta(|x(k_0\tau)|, (k - k_0)\tau), \gamma(\|w\|_\infty)\} + \delta \ \ \forall (k - k_0)\tau \geq 0. \quad (25)$$

□

Definition 7 *The parameterized family of discrete-time systems (2) is said to be semiglobally practically ISS on the set $\mathcal{W}$ on finite time intervals, if for each triple of (r, δ, T), with $r > \delta \geq 0$ and $T > 0$, there exist positive real numbers τ^* and ε^* such that for each $\tau \in (0, \tau^*)$ and for all $\varepsilon \in (0, \varepsilon^*)$, $k_0\tau \geq 0$, $w \in \mathcal{W}$ with $||w||_\infty \leq r$, and $x(k_0\tau) \in \mathbb{R}^n$ with $|x(k_0\tau)| \leq r$, we have*

$$|x(k\tau)| \leq \max\{\beta(|x(k_0\tau)|, (k - k_0)\tau), \gamma(\|w\|_\infty)\} + \delta \ \ \forall (k - k_0)\tau \in [0, T]. \quad (26)$$

□

A trajectory approach (also taken in [15]) is utilized to prove the following preliminary result.

Theorem 1 *The parameterized family of discrete-time systems (2) is semi-globally practically ISS on the set $\mathcal{W}$ on finite time intervals if and only if it is semi-globally practically ISS on the set $\mathcal{W}$.*

□

Proof. The sufficiency is straightforward. For considering the necessity, note $(k-k_0)\tau \in [0,T]$, take arbitrary $(\frac{\delta}{2}, r)$, and let $T>0$ be large enough such that for all $\beta(\max\{r,\gamma(r)+\delta\}, s\tau) \le \frac{\delta}{2}$, $\forall s\tau \in [T,\infty)$. From this, estimate the trajectory of the $x(k\tau)$ step by step and finish the proof.

A For all $x(k_0\tau) \in \mathbb{R}^n$ with $|x(k_0\tau)| \le \max\{r,\gamma(r)+\delta\}$, $w \in \mathcal{W}$ with $\|w\|_\infty \le r$ and $s\tau \in [T,\infty)$

$$\begin{aligned}
&\max\{\beta(|x(k_0\tau)|, s\tau), \gamma(\|w\|_\infty)\} + \frac{\delta}{2} \\
&\quad \le \max\{\beta(\max\{r,\gamma(r)+\delta\}, s\tau), \gamma(\|w\|_\infty)\} + \frac{\delta}{2} \\
&\quad \le \max\left\{\frac{\delta}{2}, \gamma(\|w\|_\infty)\right\} + \frac{\delta}{2} \\
&\quad \le \gamma(\|w\|_\infty) + \delta
\end{aligned} \tag{27}$$

B From the assumption that the family of systems (2) is semi-globally ISS on finite time intervals, for the particular values $2T$, $\frac{\delta}{2}$, $\max\{r,\gamma(r)+\delta\}>0$, we get a $\tau^*>0$ and a $\varepsilon^*>0$ such that for each $\tau \in (0,\tau^*)$ and for all $\varepsilon \in (0,\varepsilon^*)$ and $x(k_0\tau) \in \mathbb{R}^n$ with $|x(k_0\tau)| \le \max\{r,\gamma(r)+\delta\}$, $|x(k\tau)| \le \max\{\beta(|x(k_0\tau)|, (k-k_0)\tau), \gamma(\|w\|_\infty)\} + \frac{\delta}{2}$, $\forall (k-k_0)\tau \in [0,2T]$. Together with (27) it follows that $|x(k\tau)| \le \gamma(\|w\|_\infty)+\delta$, $\forall (k-k_0)\tau \in [T,2T]$, and in particular one gets that $|x(T)| \le \gamma(r)+\delta$.

C With initial value $\bar{x}(\bar{k}_0\tau) = x(T)$, repeated application of **A** and **B**, for $|\bar{x}(\bar{k}_0\tau)| \le \max\{r,\gamma(r)+\delta\}$ and $s\tau \in [\bar{k}_0\tau, \bar{k}\tau_0+T]$, we have $\max\{\beta(\|\bar{x}(\bar{k}_0\tau)|, s\tau), \gamma(\|w\|_\infty)\}+\frac{\delta}{2} \le \gamma(\|w\|_\infty)+\delta$, and $|\bar{x}(k\tau)| \le \max\{\beta(|\bar{x}(\bar{k}_0\tau)|, s\tau), \gamma(\|w\|_\infty)\}+\frac{\delta}{2}$, $\forall (k-\bar{k}_0)\tau \in [0,2T]$. It follows that $|x(k\tau)| \le \gamma(\|w\|_\infty)+\delta$, $\forall (k-k_0)\tau \in [T,3T]$ and repeating the process yields that, for all $k_0\tau \ge 0$ and $x(k_0\tau) \in \mathbb{R}^n$ with $|x(k_0\tau)| \le \max\{r,\gamma(r)+\delta\}$, $|x(k\tau)| \le \gamma(\|w\|_\infty)+\delta$ hold $\forall (k-k_0)\tau \in [T,\infty)$. □

Now, with the closeness of solutions on compact time interval we assume that the family of strong or weak averages is ISS, and we show the ISS properties for the actual parameterized discrete-time systems.

Theorem 2 *Suppose the parameterized family of discrete-time systems (2) has a family of weak average systems (11), if the family of weak average systems is globally ISS on the set $\mathcal{W}$, $\mathcal{W} \subset L_\infty$ is equi-bounded and equi-uniformly Lipschitz, then the family of discrete-time systems (2) is semi-globally practically ISS on the set $\mathcal{W}$.*

□

Proof. From Theorem 1, it is just necessary to show that the family of systems (2) is semiglobally practically ISS on finite time interval on the set $\mathcal{W}$. Taking arbitrary triple (r, δ, T), let $\tilde{\delta} > 0$ and $T > 0$ satisfy

$$\max_{d\in[0,r],(k-k_0)\tau\in[0,T]} \left[\beta(d+\tilde{\delta},(k-k_0)\tau) - \beta(d,(k-k_0)\tau)\right] + \tilde{\delta} \le \delta. \tag{28}$$

Using the result of Lemma 1, for some sufficiently small numbers $\tau^* > 0$ and $\varepsilon^* > 0$, for each $\tau \in (0, \tau^*)$ and for all $\varepsilon \in (0, \varepsilon^*)$, there exits $\tilde{\delta} \ge 0$ and $(k - k_0)\tau \in [0, T]$, such that the solution $x(k\tau, k_0\tau, x_0, w)$ of the family of systems (2) and the solution of the family of weak average systems satisfy

$$|x(k\tau, k_0\tau, x_0, w) - y((k-k_0)\tau, y_0, w)| \le \tilde{\delta}. \tag{29}$$

Using the simplified notation $x(k\tau)$ and $y(k\tau)$ to replace $x(k\tau, k_0\tau, x_0, w)$ and $y((k-k_0)\tau, y_0, w)$, the global ISS of the family of weak average systems on the set $\mathcal{W}$ guarantee that for any $y(k_0\tau) \in \mathbb{R}^n$ and $w \in \mathcal{W}$, we have

$$|y(k\tau)| \le \max\{\beta(|y(k_0\tau)|, (k-k_0)\tau), \gamma(\|w\|_\infty)\} \quad \forall (k-k_0)\tau \ge 0 \tag{30}$$

Note that for any $y(k\tau)$ and $x(k\tau)$ satisfy the inequality (29), $|x(k_0\tau) - y(k_0\tau)| \le \tilde{\delta}$ holds. Using (28), (29) and (30), one gets for all $(k-k_0)\tau \in [0, T]$,

$$\begin{aligned} |x(k\tau)| &\le |y(k\tau)| + |x(k\tau) - y(k\tau)| \\ &\le \max\{\beta(|y(k_0\tau)|, (k-k_0)\tau), \gamma(\|w\|_\infty)\} + \tilde{\delta} \\ &\le \max\{\beta(|x(k_0\tau)| + \tilde{\delta}, (k-k_0)\tau), \gamma(\|w\|_\infty)\} + \tilde{\delta} \\ &\le \max\{\beta(|x(k_0\tau)|, (k-k_0)\tau), \gamma(\|w\|_\infty)\} + \delta. \end{aligned} \tag{31}$$

The result then follows by applying Theorem 1. □

The corresponding results for strong average are:

Theorem 3 *Suppose the parameterized family of discrete-time systems (2) has a family of strong average systems (15), if the family of strong average systems is globally ISS on the set $\mathcal{W}$, $\mathcal{W} \subset L_\infty$ is equi-bounded, then the family of discrete-time systems (2) is semi-globally practically ISS on the set $\mathcal{W}$.* □

Proof. Same as the Theorem 2.

We emphasize that the conclusion of Theorem 2 that exploits weak averages holds only for sets of disturbances $\mathcal{W}$ that are equi-bounded and equi-uniformly Lipschitz. On the other hand, the conclusion of Theorem 3 that involves strong averages holds on larger sets of disturbances that are equi-bounded.

Our results can be directly applied to a disturbance free case:

$$\frac{\Delta x}{\Delta k} = F_\tau\left(\frac{k\tau}{\varepsilon}, x\right) \qquad \Delta k = \tau. \tag{32}$$

The following corollary is the special cases of Theorem 2 and 3 and the result is obvious. Note there exists the average for F_τ according to either of our definition of strong and weak average, as they coincide in the disturbance free case.

Corollary 1 *Suppose the parameterized family of discrete-time systems (32) has a family of average systems $\frac{\Delta y}{\Delta k} = F_\tau^{av}(y)$ where $\Delta k = \tau$, if the family of average systems is globally stable, then the family of discrete-time systems (32) is semi-globally practically stable.*

□

5 Application in an oscillator system with a periodically time-varying mass

To illustrate the applicability of our results, we address stabilization for the single-degree-freedom oscillator system with a periodically time-varying mass, which is an important model that arise in the application of biomechanics, robotics, conveyor systems, fluid structure interaction problems and many other situations [13]. We use the nonlinear model for Duffing oscillator with a periodically time-varying mass [12]:

$$y'' + kM(t)y + \gamma M(t)y^3 = u(t), \tag{33}$$

where $y(t)$ is the displacement of the center mass measured from its rest, $u(t)$ is the input, $k > 0$ and $\gamma \neq 0$ are stiffness coefficients of linear and cubic elastic restoring forces respectively. $M(t)$ is the total mass of the oscillator that is periodic in $\tilde{T}$ and satisfies

$$M(t) = \begin{cases} m & t \in [n\tilde{T}, n\tilde{T} + c) \\ 0 & t \in [n\tilde{T} + c, (n+1)\tilde{T}) \end{cases} \tag{34}$$

where $n = 0, 1, \cdots, m$ and c are positive constants. To illustrate our results, we assume that $M(t)$ is fast switching and its dynamics depends on a small parameter ε, u is implemented via a digital controller. Then, with $x_1 = y$ and $x_2 = y'$, we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kM\left(\frac{t}{\varepsilon}\right)x_1 - \gamma M\left(\frac{t}{\varepsilon}\right)x_1^3 + u(t), \end{aligned} \tag{35}$$

where $u(t) = u(k\tau) := u(k)$, $\forall t \in [k\tau, (k+1)\tau), k \in \mathbb{N}$, $\tau > 0$ is the sampling interval. Note that we can not compute the exact discrete time model of this sampled data system, we use the Euler approximation instead and get the family of approximate models parameterized by $\Delta k = \tau$:

$$\begin{aligned} \frac{\Delta x_1(k)}{\Delta k} &= x_2(k) \\ \frac{\Delta x_2(k)}{\Delta k} &= -kM\left(\frac{k\tau}{\varepsilon}\right)x_1(k) - \gamma M\left(\frac{k\tau}{\varepsilon}\right)x_1(k)^3 + u(k). \end{aligned} \tag{36}$$

Let $M_0 := \frac{cm}{T}$, and note that the definitions of the strong and the weak average coincide without disturbances, the average of the family of discrete time systems (36) is

$$\begin{aligned} \frac{\Delta x_1(k)}{\Delta k} &= x_2(k) \qquad (37)\\ \frac{\Delta x_2(k)}{\Delta k} &= -kM_0 x_1(k) - \gamma M_0 x_1^3(k) + u(k). \end{aligned}$$

Indeed, setting $\tilde{s} = s\tau$ and $T := \tau N$ we have for sufficiently small τ that

$$\begin{aligned} &\left|\frac{1}{N\tau}\sum_{s=k}^{k+N}(kM(s\tau)x_1 + \gamma M(s\tau)x_1^3 - kM_0x_1 - \gamma M_0 x_1^3)\tau\right| \\ &= \left|\frac{1}{N\tau}\sum_{\tilde{s}=k\tau}^{k\tau+N\tau}(kM(\tilde{s})x_1 + \gamma M(\tilde{s})x_1^3 - kM_0x_1 - \gamma M_0x_1^3)\Delta\tilde{s}\right| \\ &\le \left|\frac{kx_1 + \gamma x_1^3}{T}\int_{k\tau}^{k\tau+T}(M(\tilde{s}) - M_0)d\tilde{s}\right| \\ &\le \frac{2cm(k+\gamma)(\max\{|x|,1\})^3}{T+1}, \end{aligned} \tag{38}$$

where the last inequality holds when $T \geq 1$ and we can let $\beta_{sa}(s,t) = \beta_{wa}(s,t) := \frac{2M_0(k+\gamma)(\max\{|x|,1\})^3}{T+1}$ in this case. It is straight forward that under the control $u(k) = \gamma M_0 x_1^3 - 2\sqrt{kM_0}x_2$, the closed loop of the average system is a linear system whose real part of eigenvalues are all negative and then it is globally exponentially stable. With the result in Corollary 1, we conclude that the family of discrete time systems (36) is semi-globally practically stable. Moreover, with the condition that control law is bounded on compact sets uniformly in small τ, we recall the Remark 1 and conclude the semi-global practical stability of the sampled data system (35).

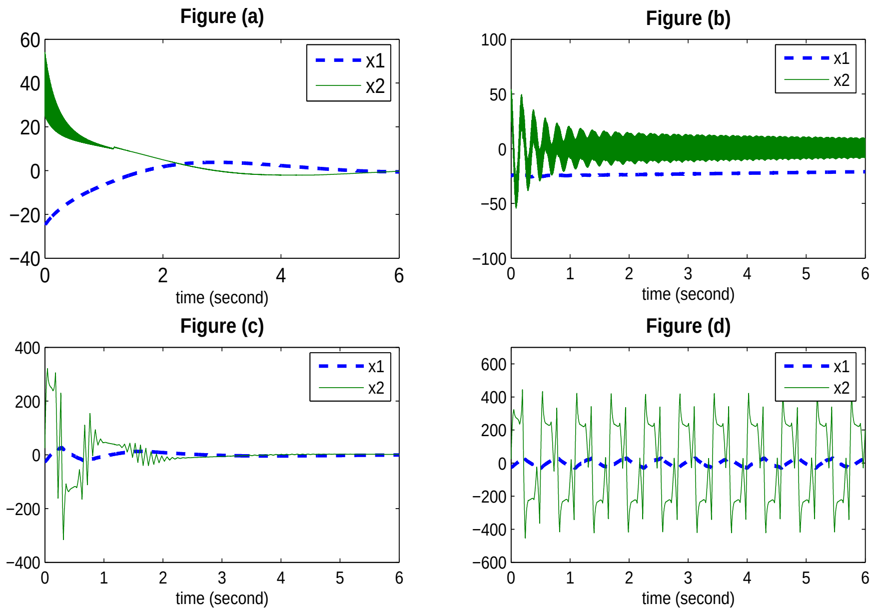

Figure 1: (a)$\tau = 0.005$ $\varepsilon = 0.005$ (b)$\tau = 0.1$ $\varepsilon = 0.005$ (c)$\tau = 0.01$ $\varepsilon = 0.09$ (d)$\tau = 0.01$ $\varepsilon = 0.097$

To check the semi-global practical stabilities of the sampled data system (35), we assume that $\tilde{T} = 1s$, and then simulate the trajectory of the solutions x_1 and x_2 under different values of parameter ε and sampling time interval τ. The simulation results, in Figure (a) and (c), show that when ε and τ are sufficiently small, the sampled data system (35) is asymptotically stable. While, when sampling time interval τ or parameter ε is chosen rather large, as $\tau = 0.1s$ in Figure (b) and $\varepsilon = 0.097$ in Figure (d), the asymptotical stability can not be guaranteed.

6 Conclusions

ISS of parameterized families of discrete-time systems was investigated via the averaging method. These results are useful when an approximate discrete-time model of a sampled-data system is used for stability analysis. We showed that under appropriate conditions, ISS of strong (or weak) average of the family of discrete-time systems implies SGP-ISS (or SGP-ISS like) properties for the actual family of systems. Via an example, we show that the results can be used with [8] to design controllers via approximate discrete-time models that achieve ISS of sampled-data nonlinear systems. Moreover, we presented general results on closeness of solutions between the weak or strong average and the actual system that only require the average system to be forward complete.

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Appendix

We now prove the technical lemma that were stated in Section 3. We first present a lemma that consider the closeness of points of some functions on finite time intervals which will be used in the following proof.

Lemma 3 *Let the set of functions $\tilde{w}_i(k\tau)(i = 1, \cdots, n)$, $\widetilde{\mathcal{W}}$, be a equi-bounded and equi-uniformly Lipschitz in compact sets, then for any $\tilde{\delta} > 0$ there exists $\rho^* > 0$, for all $\rho \in (0, \rho^*)$, $\tilde{w} \in \widetilde{\mathcal{W}}$ with $||\tilde{w}||_\infty \leq r$ and $||\frac{\Delta\tilde{w}}{\Delta k}||_\infty \leq \nu$, and for each $\tau < \rho$ such that*

$$|\tilde{w}_i(k\tau) - \tilde{w}_i(k_0\tau)| \leq \tilde{\delta} \quad \forall (k - k_0)\tau \in [0, \rho] \tag{39}$$

□

Proof. Let r and ν come from the equi-bounded and equi-uniformly Lipschitz definition, then define

$$\begin{aligned} \tilde{\delta} : &= (\exp(\nu\rho) - 1)r \\ \rho^* &= \frac{1}{\nu}\ln\left(1 + \frac{\tilde{\delta}}{r}\right), \end{aligned} \tag{40}$$

and

$$e_k := |\tilde{w}_i(k\tau) - \tilde{w}_i(k_0\tau)| \quad \forall (k-k_0)\tau \in [0,\rho]. \tag{41}$$

As $\widetilde{\mathcal{W}}$ is a equi-bounded and equi-uniformly Lipschitz compact set, then for each $\tilde{w}_i \in \widetilde{\mathcal{W}}$, we have

$$|\tilde{w}_i((k+1)\tau) - \tilde{w}_i(k\tau)| \leq \nu\tau|\tilde{w}(k\tau)|, \tag{42}$$

Assume for the purpose of induction that

$$e_m \leq (\exp(\nu m\tau) - 1)\,|\tilde{w}_i(k_0\tau)|, \quad m\tau \in [0,\rho]. \tag{43}$$

Note that this is trivially true for $m = 0$, and

$$\begin{aligned}
e_{m+1} &= |\tilde{w}_i((m+1)\tau) - \tilde{w}_i(k_0\tau)| \\
&\leq |\tilde{w}_i(m\tau) - \tilde{w}_i(k_0\tau)| + |\tilde{w}_i((m+1)\tau) - \tilde{w}_i(m\tau)| \\
&\leq |\tilde{w}_i(m\tau) - \tilde{w}_i(k_0\tau)| + \nu\tau|\tilde{w}_i(m\tau)| \\
&\leq |\tilde{w}_i(m\tau) - \tilde{w}_i(k_0\tau)| + \nu\tau(|\tilde{w}_i(m\tau) - \tilde{w}_i(k_0\tau)| + |\tilde{w}_i(k_0\tau)|) \\
&= (1+\nu\tau)e_m + \nu\tau|\tilde{w}_i(k_0\tau)| \\
&\leq (1+\nu\tau)(\exp(\nu m\tau) - 1)|\tilde{w}_i(k_0\tau)| + \nu\tau|\tilde{w}_i(k_0\tau)| \\
&= \{(1+\nu\tau)\exp(\nu m\tau) - 1\}|\tilde{w}_i(k_0\tau)| \\
&\leq (\exp(\nu(m+1)\tau) - 1)|\tilde{w}_i(k_0\tau)|, \quad m\tau \in [0,\rho]
\end{aligned} \tag{44}$$

we get that the inductive hypothesis (43) holds. Note that $\tilde{w}(k_0\tau) \in \tilde{\mathcal{W}}$ satisfying $|\tilde{w}(k_0\tau)| \leq r$, $m\tau \leq \rho$, and the definition of the $\tilde{\delta}$, and ρ^*, then for each $\tau < \rho$ and $\rho \in (0, \rho^*)$, it follows that (43) satisfies

$$|e_m| \leq \tilde{\delta}, \quad \forall m\tau \in [0,\rho]. \tag{45}$$

this complete the proof of lemma. □

A Proof of Lemma 1

Part 1. Definition of τ^*, ε^* and μ:

Given the triple (r, δ, T), without loss generality, assume $\delta < 1$. Suppose $R \geq r$ and τ_1^* comes from $\mathcal{W}$-forward completeness of the weak average, and let r and ν come from equi-boundedness and equi-uniformly Lipschitz of $\mathcal{W}$. Then, from the definition of weak average and Lipschitz condition of $F_\tau(k\tau, x, w)$, it follows that for all w satisfying $|w| \leq r$, there exist $L > 0$ such that, for all $\tau \in (0, \tau_1^*)$, $|y| \leq R+1$ and $|x| \leq R+1$

$$|F_\tau^{wa}(x,w) - F_\tau^{wa}(y,w)| \leq L|x-y|, \tag{46}$$

and a finite positive number B such that

$$B := \max_{k\tau \geq 0, |x| \leq R+1, |y| \leq R+1, |w| \leq r} \{|F_\tau(k\tau, x, w)|, |F_\tau^{wa}(y, w)|\}. \tag{47}$$

Then, define

$$\mu := \frac{\delta}{2 \exp \frac{2LT + L^2 T}{2}}, \tag{48}$$

and in preparation for defining ε^*, define

$$G\left(\frac{k\tau}{\varepsilon}, \tilde{w}\right) = \tilde{w}_1^T \left\{F_\tau\left(\frac{k\tau}{\varepsilon}, \tilde{w}_2, \tilde{w}_3\right) - F_\tau^{wa}(\tilde{w}_2, \tilde{w}_3)\right\}, \tag{49}$$

where the $\tilde{w}_i$ being components of appropriate dimension, of a vector $\tilde{w}$. Let $\widetilde{\mathcal{W}}$ be the set of functions

$$\tilde{w}(k\tau) := \begin{bmatrix} \tilde{w}_1(k\tau) \\ \tilde{w}_2(k\tau) \\ \tilde{w}_3(k\tau) \end{bmatrix}, \tag{50}$$

that are equi-bounded and equi-uniformly Lipschitz. Let τ_2^* comes from equi-uniformly Lipschitz of $\mathcal{W}$, and $\rho > 0$ be such that, for all $\tilde{w} \in \widetilde{\mathcal{W}}$, $k_i\tau \geq 0$, $(k - k_i)\tau \in [0, \rho]$ and $\tau \in (0, \tau_2^*)$, we have

$$\left|G\left(\frac{k\tau}{\varepsilon}, \tilde{w}(k\tau)\right) - G\left(\frac{k\tau}{\varepsilon}, \tilde{w}(k_i\tau)\right)\right| \leq \frac{\delta^2}{8T \exp(2LT + L^2 T)}, \tag{51}$$

where such ρ exists since G is Lipschitz uniformly in $\tilde{w}$, and from Lemma 3, $\tilde{w}(k\tau)$ and $\tilde{w}(k_i\tau)$ can be arbitrarily close for each $\tau \in (0, \tau_2^*)$ and all $(k - k_i)\tau \in [0, \rho]$, if ρ is sufficiently small. Moreover, the quantity being bounded in (51) is zero when $k\tau = k_i\tau$. Then, the left hand of (51) can be made arbitrarily small when ρ is small enough.

Let $\beta_{wa} \in \mathcal{KL}$ and $T^* > 0$ and τ_3^* come from the definition of weak average, and let $\tilde{T} > T^*$, $\tau_3^* = \tau_3^*(\tilde{T})$ and for all $\tau \in (0, \tau_3^*)$ and $\tilde{N}\tau \geq \tilde{T}$ satisfy

$$\beta_{wa}(\max\{(R + 1), r\}, \tilde{N}\tau) \leq \frac{\delta^2}{8T(1 + 3B) \exp(2LT + L^2 T)}, \tag{52}$$

then, for $\tau \in (0, \tau^*)$, where

$$\tau^* := \min\{\tau_1^*, \tau_2^*, \tau_3^*\}. \tag{53}$$

Without lose generality for the fast sampling system, we assume $\tau^* < 1$. Then, define

$$\varepsilon^* := \min\left\{\frac{\rho}{\tilde{N}\tau}, \frac{\delta^2}{16B\tilde{N}\tau(1+3B)\exp(2LT+L^2T)}\right\}. \tag{54}$$

Part 2. Error of solutions:

For any fixed $\tau \in (0,\tau^*)$, let $|y_0| \le r$, $k_0 > 0$, $(k-k_0)\tau \in [0,T]$, $\varepsilon \in (0,\varepsilon^*)$, and $w \in \mathcal{W}$, consider any x_0 such that $|x_0 - y_0| \le \mu$. Let

$$E(k\tau) := x(k\tau, k_0, x_0, w) - y((k-k_0)\tau, y_0, w), \tag{55}$$

and note that $|E(k_0\tau)| \le \mu \le \frac{\delta}{2} < 1$. If $|E(k\tau)| < 1$ for all $(k-k_0)\tau \in [0,T]$, then define $\bar{k}\tau = k_0\tau + T$. Otherwise, define

$$\bar{k}\tau := \max_{s\in[0,T]}\{s : |E(k\tau)| < 1 \quad \forall k\tau \in [0,s]\}. \tag{56}$$

Note that $\bar{k}\tau > k_0\tau$, $E(\cdot)$ and $x(\cdot, k_0\tau, x_0, w)$ are defined on $[k_0\tau, \bar{k}\tau]$. Let $\tilde{w}(k\tau) \in \widetilde{\mathcal{W}}$ be such that, for all $k\tau \in [k_0\tau, \bar{k}\tau]$,

$$\begin{bmatrix} \tilde{w}_1(k\tau) \\ \tilde{w}_2(k\tau) \\ \tilde{w}_3(k\tau) \end{bmatrix} = \begin{bmatrix} E^T(k\tau) + \tau\phi^T(k\tau) + \frac{\tau}{2}\psi^T(k\tau) \\ x(k\tau, k_0, x_0, w) \\ w(k\tau) \end{bmatrix}, \tag{57}$$

and in (57)

$$\begin{aligned} \psi(k\tau) &:= F_\tau\left(\frac{k\tau}{\varepsilon}, x, w\right) - F_\tau^{wa}(x,w), \\ \phi(k\tau) &:= F_\tau^{wa}(x,w) - F_\tau^{wa}(y,w). \end{aligned} \tag{58}$$

Such a $\tilde{w}(k\tau) \in \widetilde{\mathcal{W}}$ exist since $\tilde{w}_3 \in \mathcal{W}$, and for all $k\tau \in [k_0\tau, \bar{k}\tau]$, $|E(k\tau)| < 1$, and from (47), we know $|\phi| \le 2B$ and $|\psi| \le 2B$. Then, for each $\tau \in (0,\tau^*)$ and all $k\tau \in [k_0\tau, \bar{k}\tau]$, it follows that $||\tilde{w}_1||_\infty \le (1+3B)$, and $||\frac{\Delta \tilde{w}_1}{\Delta k}||_\infty \le 2B + 3L(B+\nu)$. Moreover, since $|y((k-k_0)\tau, y_0, w)| \le R$ for all $k\tau \in [0,T]$, it follows that $|x(k\tau, k_0, x_0, w)| \le R+1$ for all $k\tau \in [k_0\tau, \bar{k}\tau]$ and $||\frac{\Delta \tilde{w}_2}{\Delta k}||_\infty \le B$ from (47).

Using the simplified notation x and y to replace $x(k\tau, k_0\tau, x_0, w)$ and $y((k-k_0)\tau, y_0, w)$, and define the difference of $E(k\tau)$ as

$$\begin{aligned} H(k\tau) &:= \frac{\Delta E(k\tau)}{\Delta k} \\ &= \frac{E(k\tau + \Delta k) - E(k\tau)}{\Delta k} \\ &= \frac{x(k\tau+\Delta k) - x(k\tau)}{\Delta k} - \frac{y(k\tau+\Delta k) - y(k\tau)}{\Delta k} \\ &= F_\tau\left(\frac{k\tau}{\varepsilon}, x, w\right) - F_\tau^{wa}(y,w), \end{aligned} \tag{59}$$

where the last equality comes from (2) and (11). Noting (59),(58), one gets $H(k\tau) = \psi(k\tau) + \phi(k\tau)$. Moreover, for all $(k-k_0)\tau \in [0, \bar{k}\tau]$, for the scalar-valued function $V(k\tau) := \frac{1}{2}E^T(k\tau)E(k\tau)$, we have

$$
\begin{aligned}
\frac{\Delta V(k\tau)}{\Delta k} &= \frac{1}{2}\frac{E^T(k\tau+\Delta k)E(k\tau+\Delta k) - E^T(k\tau)E(k\tau)}{\Delta k} \\
&= \frac{1}{2}\frac{(E(k\tau+\Delta k)+E(k\tau))^T(E(k\tau+\Delta k)-E(k\tau))}{\Delta k} \\
&= \frac{1}{2}\left(2E(k\tau) + \Delta k\frac{\Delta E(k\tau)}{\Delta k}\right)^T \frac{\Delta E(k\tau)}{\Delta k} \\
&= E^T(k\tau)H(k\tau) + \frac{1}{2}\Delta k H^T(k\tau)H(k\tau),
\end{aligned}
\tag{60}
$$

Substituting $H(k\tau)$ with the expression of $\phi(k\tau)$ and $\psi(k\tau)$ in (60), noting (57) and the definition of $G(\cdot)$ in (49), and using the following inequality from the Lipschitz condition of weak average

$$
|\phi(k\tau)| \leq L|E(k\tau)|, \tag{61}
$$

then for $(k-k_0)\tau \in [0, \bar{k}\tau]$, it follows that

$$
\begin{aligned}
\frac{\Delta V(k\tau)}{\Delta k} &= E^T(k\tau)(\psi+\phi) + \frac{1}{2}(\psi+\phi)^T(\psi+\phi)\tau \\
&= E^T(k\tau)\phi + \frac{\tau}{2}\phi^T\phi + E^T(k\tau)\psi + \tau\phi^T\psi + \frac{\tau}{2}\psi^T\psi \\
&\leq V(k\tau)(2L+L^2\tau) + E^T(k\tau)\psi + \tau\phi^T\psi + \frac{\tau}{2}\psi^T\psi \\
&= V(k\tau)(2L+L^2\tau) + (E^T(k\tau) + \tau\phi^T + \frac{\tau}{2}\psi^T)\psi \\
&= V(k\tau)(2L+L^2\tau) + G\left(\frac{k\tau}{\varepsilon}, \tilde{w}\right).
\end{aligned}
\tag{62}
$$

By standard comparison theorems in [5], there exists $W(k\tau)$ with $W(k_0\tau) = \frac{1}{2}\mu^2$ such that $V(k\tau) \leq W(k\tau)$ and satisfy the equation

$$
W((k+1)\tau) = (2L\tau + L^2\tau^2 + 1)W(k\tau) + G\left(\frac{k\tau}{\varepsilon}, \tilde{w}\right)\Delta k. \tag{63}
$$

Noting $N\tau \leq T$ and the definition of μ in (48), one knows $V(k_0\tau) \leq \frac{1}{2}\mu^2 = \frac{\delta^2}{8\exp(2LT+L^2T)}$. Then, with the inequality

$$
\{1 + (2L\tau + L^2\tau^2)\}^N \leq \exp(2LN\tau + L^2N\tau^2), \tag{64}
$$

we have for any $\tau \in (0, \tau^*)$,

$$
\begin{aligned}
V(k\tau) &\le (2L\tau + L^2\tau^2 + 1)^{k-k_0} V(k_0\tau) \\
&\quad + \sum_{s=k_0}^{k-1} (2L\tau + L^2\tau^2 + 1)^{k-1-s} G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right) \Delta s \\
&\le \exp(2LN\tau + L^2N\tau^2) V(k_0\tau) \\
&\quad + \exp(2LN\tau + L^2N\tau^2) \sum_{s=k_0}^{k-1} G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right) \Delta s \\
&\le \exp(2LT + L^2T\tau) V(k_0\tau) \\
&\quad + \exp(2LT + L^2T\tau) \sum_{s=k_0}^{k-1} G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right) \Delta s \\
&\le \frac{\delta^2}{8} + \exp(2LT + L^2T) \sum_{s=k_0}^{k-1} G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right) \Delta s. \qquad (65)
\end{aligned}
$$

Fix $k\tau \in [k_0\tau, \bar{k}\tau]$ and set m to be the largest nonnegative integer such that $m \le \frac{(k-k_0-1)\tau}{\varepsilon\tilde{N}\tau}$ with $\varepsilon\tilde{N}\tau \le \rho$, where ρ is a positive real number that makes inequality (51) hold, for $i = 0, 1, \cdots, m$, define $(k_i - k_0)\tau = i\varepsilon\tilde{N}\tau$. Then, we have

$$
\begin{aligned}
V(k\tau) &\le \frac{\delta^2}{8} + \exp(2LT + L^2T) \sum_{s=k_m}^{k-1} G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right) \Delta s \\
&\quad + \exp(2LT + L^2T) \sum_{i=0}^{m-1} \sum_{s=k_i}^{k_{i+1}} G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right) \Delta s. \qquad (66)
\end{aligned}
$$

From the definition of m, we know for $k\tau \in [k_0\tau, \bar{k}\tau]$, $(k-k_m-1)\tau \le \varepsilon\tilde{N}\tau$, $(k_{i+1} - k_i)\tau = \varepsilon\tilde{N}\tau$. Noting (47) and $|E(k\tau)| < 1$ for all $k\tau \in [k_0\tau, \bar{k}\tau]$, then for $\tau \in (0, \tau^*)$, we have

$$
\left| G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right) \right| \le 2B|\tilde{w}_1(k\tau)| \le 2B(1+3B), \qquad (67)
$$

then when $\varepsilon \in (0, \varepsilon^*)$, it follows that

$$
\begin{aligned}
&\exp(2LT + L^2T) \sum_{s=k_m}^{k-1} G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right) \Delta s \\
&\qquad \le \varepsilon\tilde{N}\tau 2B(1+3B) \exp(2LT + L^2T) \le \frac{\delta^2}{8}. \qquad (68)
\end{aligned}
$$

Noting Remark 2, the function $k\tau = \varepsilon\zeta\tau$ maps the countable set $\{k_0\tau, k_1\tau, k_2\tau, \cdots\}$ into a set $\{\zeta_0\tau, \zeta_1\tau, \zeta_2\tau, \cdots\}$, with $k_0\tau = \varepsilon\zeta_0\tau$, $k_i\tau = k_0\tau + i\varepsilon\tilde{N}\tau$ and

$\zeta_i\tau = \zeta_0\tau + i\tilde{N}\tau$, for $i = 1, 2, \cdots$. Then, with the definition (52), it follows that

$$\begin{aligned}
&\left|\sum_{s=k_i}^{k_{i+1}} G(\frac{s\tau}{\varepsilon}, \tilde{w}(k_i\tau))\Delta s\right| \\
\leq\; & |\tilde{w}_1^T(k_i\tau)| \cdot \left|\varepsilon\tilde{N}\tau F_\tau^{wa}(\tilde{w}_2, \tilde{w}_3) - \sum_{k=k_i}^{k_{i+1}} F_\tau(\frac{s\tau}{\varepsilon}, \tilde{w}_2, \tilde{w}_3)\Delta s\right| \\
=\; & (1+3B) \cdot \left|\varepsilon\tilde{N}\tau F_\tau^{wa}(\tilde{w}_2, \tilde{w}_3) - \varepsilon \sum_{\zeta=\zeta_i}^{\zeta_{i+1}} F_\tau(\zeta\tau, \tilde{w}_2, \tilde{w}_3)\Delta\zeta\right| \\
\leq\; & \varepsilon\tilde{N}\tau(1+3B) \cdot \left|F_\tau^{wa}(x, w) - \frac{1}{\tilde{N}\tau} \sum_{\zeta=\zeta_i}^{\zeta_i+\tilde{N}} F_\tau(\zeta\tau, x, w)\Delta\zeta\right| \\
\leq\; & \varepsilon\tilde{N}\tau(1+3B)\beta_{wa}(\max\{(R+1), r\}, \tilde{N}\tau) \\
\leq\; & \varepsilon\tilde{N}\tau\frac{\delta^2}{8T\exp(2LT + L^2T)}. \qquad (69)
\end{aligned}$$

For the scalar function $V(k\tau)$, substituting (68) in (66), noting the fact $m\varepsilon\tilde{N}\tau \leq T$ and combining with the inequalities (69), (51), then we have

$$\begin{aligned}
V(k\tau) \;\leq\; & \frac{\delta^2}{4} + \exp(2LT + L^2T) \cdot \sum_{i=0}^{m-1} \sum_{s=k_i}^{k_{i+1}} \left\{G\left(\frac{s\tau}{\varepsilon}, x(k_i\tau), w(k_i\tau)\right)\right. \\
& \left. + \left|G\left(\frac{s\tau}{\varepsilon}, x(s\tau), w(s\tau)\right) - G\left(\frac{s\tau}{\varepsilon}, x(k_i\tau), w(k_i\tau)\right)\right|\right\} \Delta s \\
\leq\; & \frac{\delta^2}{4} + \exp(2LT + L^2T)m\varepsilon\tilde{N}\tau \\
& \cdot \left\{\frac{\delta^2}{8T\exp(2LT + L^2T)} + \frac{\delta^2}{8T\exp(2LT + L^2T)}\right\} \\
\leq\; & \frac{\delta^2}{2} \qquad (70)
\end{aligned}$$

As $V(k\tau) \leq \frac{\delta^2}{2}$ for all $k\tau \in [k_0\tau, \bar{k}\tau]$ and $V(k\tau) = \frac{1}{2}E^T(k\tau)E(k\tau)$, it follows that $|E(k\tau)| \leq \delta < 1$ for all $k\tau \in [k_0\tau, \bar{k}\tau]$. From the definition of $\bar{k}$, it follows that $(\bar{k} - k_0)\tau \leq T$ so that $|E(k\tau)| \leq \delta$ for all $(k - k_0)\tau \in [0, T]$. This establishes the result. $\square$

B Proof of Lemma 2

The proof of Lemma 2 follows exactly the same steps as the proof of Lemma 1 with following changes. With strong average definition, instead of (69) we use

$$
\begin{aligned}
&\left|\sum_{s=k_i}^{k_{i+1}} G(\frac{s\tau}{\varepsilon}, \tilde{w}_1, \tilde{w}_2, \tilde{w}(s\tau))\Delta s\right| \\
&\leq \quad \varepsilon\tilde{N}\tau|\tilde{w}_1^T|\frac{1}{\tilde{N}\tau}\cdot\left|\sum_{\zeta=\zeta_i}^{\zeta_{i+1}}\{F(\zeta\tau, \tilde{w}_2, \tilde{w}_3(\varepsilon\zeta\tau)) - F_{sa}(\tilde{w}_2, \tilde{w}_3(\varepsilon\zeta\tau))\}\Delta\zeta\right| \\
&\leq \quad \varepsilon\tilde{N}\tau(1+3B)\beta_{sa}(\max\{(R+1), r\}, \tilde{N}\tau) \\
&\leq \quad \varepsilon\tilde{N}\tau\frac{\delta^2}{8T\exp(2LT+L^2T)} \qquad (71)
\end{aligned}
$$

And in same way like (51), for each $\tau \in (0, \tau^*)$ there exists sufficiently small $\rho > 0$ such that, for all $w \in \mathcal{W}$ and $k_i \geq k_0$, $(k-k_i)\tau \in [0, \rho]$, such that

$$
\left|G\left(\frac{k\tau}{\varepsilon}, \tilde{w}(k\tau)\right) - G\left(\frac{k\tau}{\varepsilon}, \tilde{w}_1(k_i\tau), \tilde{w}_2(k_i\tau), \tilde{w}_3(k\tau)\right)\right| \leq \frac{\delta^2}{8T\exp(2LT+L^2T)} \qquad (72)
$$

This ρ exists since G is Lipschitz uniformly in $\tilde{w}$, note that we only need the closeness of functions $\tilde{w}_1(k\tau)$ and $\tilde{w}_2(k\tau)$, which have been proved in the proof of Theorem 2 to be included in function set $\widetilde{\mathcal{W}}$ that is equi-bounded and equi-uniformly Lipschitz. Note here $\tilde{w}_3 \in \mathcal{W}$ only need satisfies the equi-bounded condition. From Lemma 1, $\tilde{w}_1$ and $\tilde{w}_2$ can be arbitrarily close for each $\tau \in (0, \tau^*)$ if ρ is small enough. Moreover, for $k\tau = k_i\tau$ the quantity being bounded in (72) is zero, then the left hand of (72) can be made arbitrarily small if ρ is small enough.

Using the inequality (68), (71), (72), and the fact $m\varepsilon\tilde{N}\tau \leq T$, then for $k\tau \in [k_0\tau, \bar{k}\tau]$, it follows that the scalar function $V(k\tau)$ satisfies

$$
\begin{aligned}
V(k\tau) \quad \leq \quad & \frac{\delta^2}{4} + \exp(2LT+L^2T)\cdot\sum_{i=0}^{m-1}\sum_{s=k_i}^{k_{i+1}}\left\{G\left(\frac{s\tau}{\varepsilon}, \tilde{w}_1(k_i\tau), \tilde{w}_2(k_i\tau), \tilde{w}_3(s\tau)\right)\right. \\
& \left. + \left|G\left(\frac{s\tau}{\varepsilon}, \tilde{w}(s\tau)\right) - G\left(\frac{s\tau}{\varepsilon}, \tilde{w}_1(k_i\tau), \tilde{w}_2(k_i\tau), \tilde{w}_3(s\tau)\right)\right|\right\}\Delta s \\
& \leq \frac{\delta^2}{4} + \exp(2LT+L^2T)m\varepsilon\tilde{N}\tau \\
& \quad \cdot\left\{\frac{\delta^2}{8T\exp(2LT+L^2T)} + \frac{\delta^2}{8T\exp(2LT+L^2T)}\right\} \\
& \leq \frac{\delta^2}{2} \qquad (73)
\end{aligned}
$$

This establishes the result in the same way as Lemma 2. $\square$