Integral versions of ISS for sampled-data nonlinear systems via their approximate discrete-time models

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Abstract

Two integral versions of input to state stability are considered: integral input to state stability (iISS) and integral input to integral state stabilty (iIiSS). We present sufficient conditions that guarantee that if a controller achieves semiglobal practical iISS (respectively iIiSS) of an approximate discrete-time model of a nonlinear sampled-data system, then the same controller achieves semiglobal practical iISS (respectively iIiSS) of the exact discrete-time model by reducing the sampling period. Recent results on numerical methods for systems with measurable disturbances can be used to generate approximate models that we consider. Results are presented for arbitrary dynamic controllers that can be discontinuous in general.

1 Introduction

The main stumbling block in the controller design for nonlinear sampled-data systems appears to be the absence of a good model for controller design even in the cases when the continuous-time plant model is known. An approach for stabilization of sampled-data nonlinear system via their approximate discrete-time models has been proposed in [10]. These results were further extended in [8] to cover plants modeled as differential inclusions, dynamic controllers and stability with respect to arbitrary non-compact sets. These papers provide a framework for controller analysis but they do not present recipes for controller design. An example of control design within this framework can be found in [9] where backstepping controllers were developed based on the Euler approximate model of strict feedback systems. Simulation studies presented in [9] indicate that this approach may yield much better behaviour than the controller design based on the continuous-time model followed by a discretization of the controller.

Since plants with disturbances are prevalent in control theory, there is a strong motivation to extend the approach of [10, 8] to this class of plants. The first step in this direction was [7] where a framework for input to state stabilization (ISS) of sampled-data nonlinear systems via their approximate discrete-time models was presented. Input to state stability (see [12]) has found a widespread use in control theory but it is just one of the possible types of stability for systems with disturbances that may be of interest.

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For instance, the less restrictive notion of integral input to state stability (iISS) (see [3, 2, 1, 13]) is proving to be as useful as ISS.

It is the main purpose of this paper to present a framework for design of controllers based on approximate discrete-time models that achieve two integral versions of ISS: integral input to state stability (iISS) and integral input to integral state stability (iIiSS). Note that iISS was investigated in [1] in the case when the exact discrete-time model of the plant is known. Our results are different since we do not assume existence of the exact discrete-time model, which was a standing assumption in [1]. We consider dynamic control laws that can be discontinuous in general and present sufficient conditions that guarantee that if a controller achieves semiglobal practical iISS (respectively iIiSS) for an approximate discrete-time plant model, then the same controller achieves semiglobal-practical iISS (respectively iIiSS) of the exact discrete-time plant model. We emphasize that the semiglobal part of our iISS definition is different from the one used in [2], whereas the "practical" iISS that we consider appears to be new and we are not aware of related results. Our approach benefits from the results in numerical analysis literature [14] and in particular from the recent results in [4, 5].

The paper is organized as follows. In Section 2 we present preliminaries and definitions needed in the sequel. Section 3 contains main results and proofs are presented in Section 4.

2 Preliminaries

Sets of real and natural numbers are denoted respectively as $\mathbb{R}$ and $\mathbb{N}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and unbounded. A function λ is of class $\mathcal{L}$ if it is continuous and $\lambda(s)$ decreases to zero as $s \to +\infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s > 0$. For a given function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, we use the following notation: $w_T[k]$ is the restriction of the function $w(\cdot)$ to the interval $t \in [kT, (k+1)T]$ where $k \in \mathbb{N}$ and $T > 0$; and $w(k)$ is the value of the function $w(\cdot)$ at $t = kT, k \in \mathbb{N}$. We denote the norms $\|w_T[k]\|_\infty = \sup_{\tau \in [kT,(k+1)T]} |w(\tau)|$ and $\|w\|_\infty := \sup_{\tau \geq 0} |w(\tau)|$ and in the case when $w(\cdot)$ is a measurable function (in the Lebesgue sense) we use the essential supremum in the definitions. If there exists $r > 0$ such that $\|w\|_\infty \leq r$ or $\int_0^\infty \gamma(|w(s)|)ds \leq r$, with $\gamma \in \mathcal{K}_\infty$, then we write respectively $w \in \mathcal{L}_\infty(r)$ and $w \in \mathcal{L}_\gamma(r)$.

Consider a continuous-time nonlinear plant with disturbances:

$$\dot{x}(t) = f(x(t), u(t), w(t)) \ , \tag{1}$$

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^p$ are respectively the state, control input and exogenous disturbance. It is assumed that f is locally Lipschitz and $f(0,0,0) = 0$. The control is taken to be a piecewise constant signal $u(t) = u(kT) =: u(k)$, $\forall t \in [kT, (k+1)T)$, $k \in \mathbb{N}$, where $T > 0$ is the sampling period. Also, we assume that some combination (output) or all of the states ($x(k) := x(kT)$) are available at sampling instant $kT, k \in \mathbb{N}$. The exact discrete-time model for the plant (1), which describes the plant behavior at sampling instants kT, is obtained by integrating the initial value problem

$$\dot{x}(t) = f(x(t), u(k), w(t)) \ , \tag{2}$$

with given $w_T[k]$, $u(k)$ and $x_0 = x(k)$, over the sampling interval $[kT, (k+1)T]$. If we denote by $x(t)$ the solution of the initial value problem (2) at time t with given $x_0 = x(k)$, $u(k)$ and $w_T[k]$ and $t_k := kT$, then the exact discrete-time model of (1) can be written as:

$$\begin{aligned} x(k+1) &= x(k) + \int_{t_k}^{t_{k+1}} f(x(\tau), u(k), w(\tau)) d\tau \\ &=: F_T^e(x(k), u(k), w_T[k]) \ . \end{aligned} \tag{3}$$

We refer to (3) as a *functional difference equation* since it depends on $w_T[k]$. We emphasize that F_T^e is not known in most cases. Indeed, in order to compute F_T^e we have to solve the initial value problem (2) analytically and this is usually impossible since f in (1) is nonlinear. Hence, we will use an approximate discrete-time model of the plant to design a controller.

Different approximate discrete-time models can be obtained using different methods. Recently, numerical integration schemes for systems with measurable disturbances were considered in [4, 5]. Using these numerical integration techniques we can obtain an approximate discrete-time model

$$x(k+1) = F_T^a(x(k), u(k), w_T[k]) \ , \tag{4}$$

which is in general a functional difference equation. For instance, the simplest such approximate discrete-time model, which is analogous to Euler model, has the following form $x(k+1) = x(k) + \int_{kT}^{(k+1)T} f(x(k), u(k), w(s)) ds$ (see [5]). Since we will consider semiglobal stability properties (see Definition 4), we will think of F_T^e and F_T^a as being defined globally for all small T, even though the initial value problem (2) may exhibit finite escape times (see discussion on pg. 261 in [10]).

The sampling period T is assumed to be a design parameter which can be arbitrarily assigned. Since we are dealing with a family of approximate discrete-time models F_T^a, parameterized by T, in order to achieve a certain objective we need in general to obtain a family of controllers, parameterized by T. We consider a family of dynamic feedback controllers

$$\begin{aligned} z(k+1) &= G_T(x(k), z(k)) \\ u(k) &= u_T(x(k), z(k)) \ , \end{aligned} \tag{5}$$

where $z \in \mathbb{R}^{n_z}$. To shorten notation, we introduce $\tilde{x} := (x^T \ z^T)^T$, $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$, where $n_{\tilde{x}} := n_x + n_z$ and

$$\mathcal{F}_T^i(\tilde{x}, w_T) := \begin{pmatrix} F_T^i(x, u_T(x,z), w_T) \\ G_T(x,z) \end{pmatrix} . \tag{6}$$

The superscript i may be either e or a, where e stands for *exact* model, a for *approximate* model. We omit the superscript if we refer to a general model. We use the following:

Definition 1 *u_T is said to be locally uniformly bounded if for any $\Delta_{\tilde{x}} > 0$ there exist strictly positive numbers T^* and Δ_u such that for all $T \in (0, T^*)$, $|\tilde{x}| \leq \Delta_{\tilde{x}}$ we have $|u_T(\tilde{x})| \leq \Delta_u$.*

In order to prove our main results, we need to guarantee that the mismatch between F_T^e and F_T^a is small in some sense. We define a consistency property, which will be used to limit the mismatch. Similar definitions can be found in numerical analysis literature (see Definition 3.4.2 in [14]) and recently in the context of sampled-data systems with disturbances (for instance, see [7]). In the sequel we use the notation $x = x(k)$, $u = u(k)$, $w_T = w_T[k]$.

Definition 2 *The family F_T^a is said to be one-step consistent with F_T^e if given any strictly positive real numbers $(\Delta_x, \Delta_u, \Delta_w)$, there exist a function $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that, for all $T \in (0, T^*)$, all $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^m, w \in \mathcal{L}_\infty$ with $|x| \leq \Delta_x$, $|u| \leq \Delta_u$, $\|w_T\|_\infty \leq \Delta_w$, we have $|F_T^e - F_T^a| \leq T\rho(T)$.*

Sufficient checkable conditions for one-step consistency are given next (for the proof see [7]).

Lemma 1 *F_T^a is one-step consistent with F_T^e if the following conditions hold: 1. F_T^a is one-step consistent with $\tilde{F}_T^{Euler}(x, u, w_T) := x + \int_{kT}^{(k+1)T} f(x, u, w(s))ds$; 2. given any strictly positive real numbers $(\Delta_x, \Delta_u, \Delta_w)$, there exist $\rho_1 \in \mathcal{K}_\infty$, $T^* > 0$, such that, for all $T \in (0, T^*)$ and all $x_1, x_2 \in \mathbb{R}^{n_x}$ with $\max\{|x_1|, |x_2|\} \leq \Delta_x$, all $u \in \mathbb{R}^m$ with $|u| \leq \Delta_u$ and all $w \in \mathbb{R}^p$ with $|w| \leq \Delta_w$, the following holds $|f(x_1, u, w) - f(x_2, u, w)| \leq \rho_1(|x_1 - x_2|)$.*

The following lemma was proved in [3] and is needed in the sequel.

Lemma 2 *Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous positive definite function. Then there exist $\rho_1 \in \mathcal{K}_\infty$ and $\rho_2 \in \mathcal{L}$ such that $\rho(r) \geq \rho_1(r)\rho_2(r), \forall r \geq 0$.*

3 Main results

In this section we state and prove the main results of this paper. The first main result (Theorem 1) presents sufficient conditions on the continuous-time plant model, the controller and the approximate discrete-time plant model that guarantee that if the controller achieves semiglobal practical Lyapunov iISS for the approximate model (see Definition 3), then the same controller would yield a semiglobal practical iISS bound on the solutions of the exact discrete-time plant model (see Definition 4). The second main result (Theorem 2) presents similar conditions for integral input to integral state stability. We emphasize that examples in [10], where only stability of input-free systems was considered, show that our results are close in some sense to being necessary. Indeed, we can find systems satisfying *all but one (arbitrary) condition* of the above Theorems and the exact dicrete-time system is not semi-globally practically stable.

In order to state the following two definitions, we consider the family of systems:

$$\tilde{x}(k+1) = \mathcal{F}_T(\tilde{x}(k), w_T[k]) \ . \tag{7}$$

Definition 3 *Suppose that the following property holds: there exist continuous functions $\alpha_1, \alpha_2, \alpha_3$ and $\tilde{\gamma} \in \mathcal{K}$ and for any strictly positive real numbers $(\Delta_1, \Delta_2, \Delta_3, \delta_1)$ there exist strictly positive real numbers T^* and L such that for all $T \in (0, T^*)$ there exists a function $V_T : \mathbb{R}^{n_{\tilde{x}}} \to \mathbb{R}_{\geq 0}$ such that for all $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$ with $|\tilde{x}| \leq \Delta_1$ and all $w \in \mathcal{L}_\infty(\Delta_2) \cap \mathcal{L}_{\tilde{\gamma}}(\Delta_3)$ the following holds:*

$$\begin{array}{l} \alpha_1(|\tilde{x}|) \leq V_T(\tilde{x}) \leq \alpha_2(|\tilde{x}|) \\ \frac{\Delta V_T}{T} \leq -\alpha_3(|\tilde{x}|) + \frac{1}{T}\int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)ds + \delta_1 \ , \end{array} \tag{8}$$

where $\Delta V_T := V_T(\mathcal{F}_T(\tilde{x}, w_T)) - V_T(\tilde{x})$ and, moreover, for all x_1, x_2, z with $\left|(x_1^T \ z^T)^T\right|, \left|(x_2^T \ z^T)^T\right| \in [0, \Delta_1]$ and all $T \in (0, T^*)$, we have $|V_T(x_1, z) - V_T(x_2, z)| \leq L\,|x_1 - x_2|$.

1. If the above property holds with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite function α_3, then the family of systems (7) is Lyapunov semiglobally practically integral input to state stable (Lyapunov-SP-iISS). The function V_T is then called an iISS-Lyapunov function for the family $\mathcal{F}_T$.

2. If the above property holds with $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ then the family of systems (7) is Lyapunov semiglobally practically integral input to integral state stable (Lyapunov-SP-iIiSS). The function V_T is then called an iIiSS-Lyapunov function for the family $\mathcal{F}_T$.

We note that iIiSS is strictly stronger property than iISS, as shown in [13] for continuous-time systems.

Definition 4 *Suppose that there exist $\beta \in \mathcal{KL}$ and $\alpha, \gamma, \chi \in \mathcal{K}_\infty$ such that:*

1. for any strictly positive real numbers $(\Delta_{\tilde{x}}, \Delta_{w1}, \Delta_{w2}, \delta)$ there exists $T^* > 0$ such that for all $T \in$

$(0,T^*)$, $|\tilde{x}(0)| \le \Delta_{\tilde{x}}$ and $w \in \mathcal{L}_\infty(\Delta_{w1}) \cap \mathcal{L}_\gamma(\Delta_{w2})$, the solutions of the system exist and satisfy $\alpha(|\tilde{x}(k)|) \le \beta(|\tilde{x}(0)|, kT) + \int_0^{kT} \gamma(|w(s)|)ds + \delta, \forall k \in \mathbb{N}$. Then the family of systems (7) is said to be semiglobally practically integral input to state stable (SP-iISS).

2. for any strictly positive real numbers $(\Delta_{\tilde{x}}, \Delta_w, \delta)$ there exists $T^* > 0$ such that for all $T \in (0,T^*)$, $|\tilde{x}(0)| \le \Delta_{\tilde{x}}$ and $w \in \mathcal{L}_\infty(\Delta_w)$, the solutions of the system exist and satisfy $\sum_{i=0}^{k} \alpha(|\tilde{x}(i)|)T \le \chi(|\tilde{x}(0)|) + \int_0^{kT} \gamma(|w(s)|)ds + Tk\delta, \forall k \in \mathbb{N}$. Then the family of systems (7) is said to be semiglobally practically integral input to integral state stable (SP-iIiSS).

The following theorem contains the first main result of this paper. It gives checkable conditions on the approximate model, controller and the continuous-time plant model that guarantee that if a controller achieves Lyapunov-SP-iISS of the approximate discrete-time plant model, the same controller would achieve SP-iISS of the exact discrete-time plant model.

Theorem 1 *Suppose that: (i) The family of approximate discrete-time models $\mathcal{F}_T^a$ is Lyapunov-SP-iISS; (ii) F_T^a is one-step consistent with F_T^e; (iii) u_T is uniformly locally bounded. Then, the family of exact discrete-time models $\mathcal{F}_T^e$ is SP-iISS.*

The following theorem contains our second main result. It gives checkable conditions on the approximate model, controller and the continuous-time plant model which guarantee that if a controller achieves Lyapunov-SP-iIiSS of the approximate discrete-time plant model, the same controller would achieve SP-iIiSS of the exact discrete-time plant model.

Theorem 2 *Suppose that: (i) The family of approximate discrete-time models $\mathcal{F}_T^a$ is Lyapunov-SP-iIiSS; (ii) F_T^a is one-step consistent with F_T^e; (iii) u_T is uniformly locally bounded. Then, the family of exact discrete-time models $\mathcal{F}_T^e$ is SP-iIiSS.*

Remark 1 *We note that our main results allow the family of controllers to depend discontinuously on states. Moreover, under mild conditions (see for instance results in [11]) it is possible to overbound also inter-sample bahaviour and to conclude from Theorem 1 that: there exist $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$ such that for any strictly positive real numbers $(\Delta_{\tilde{x}}, \Delta_{w1}, \Delta_{w2}, \delta)$ there exists $T^* > 0$ such that for all $T \in (0,T^*)$, $|\tilde{x}(t_\circ)| \le \Delta_{\tilde{x}}$ and $w \in \mathcal{L}_\infty(\Delta_{w1}) \cap \mathcal{L}_\gamma(\Delta_{w2})$, the solutions of the system satisfy $\alpha(|\tilde{x}(t)|) \le \beta(|\tilde{x}(t_\circ)|, t - t_\circ) + \int_{t_\circ}^{t} \gamma(|w(s)|)ds + \delta$, $\forall t \ge t_\circ \ge 0$. A similar statement can be made for iIiSS, where we could prove that trajectories of the sampled-data system satisfy in a semiglobal practical sense the following bound $\int_{t_\circ}^{t} \alpha(|\tilde{x}(s)|)ds \le \chi(|\tilde{x}(t_\circ)|) + \int_{t_\circ}^{t} \gamma(|w(s)|)ds + \delta t$ $\forall t \ge t_\circ \ge 0$.*

Remark 2 *Similarly to results presented in [7], we may also start with an approximate discrete-time model of the plant for which we assumed that disturbances are constant during sampling intervals* $w(t) = w(kT) = const., \forall t \in [kT, (k+1)T), k \in \mathbb{N}$. *In this case, the approximate and exact models will depend on* $w(kT)$ *(not on* $w_T[k]$*) which means that they are difference equations (not functional difference equations). It was shown in [7] that a "weak" form of consistency property can be stated in this case and it can be used in a very similar manner to state a result similar to Theorem 1 or Theorem 2 except that the bounds in Definition 4 would hold for a smaller class of disturbances whose derivatives also need to be bounded. We did not pursue this direction for space reasons.*

4 Proofs of main results

The following lemmas are used in the proof of Theorem 1 (proofs are given at the end of this section).

Lemma 3 *Given any* $T > 0$ *and any continuous positive definite function* $\tilde{\rho} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, *there exists a* $\mathcal{KL}$ *function with the following property. Suppose that* $y : \mathbb{N} \to \mathbb{R}$ *and a nondecreasing function* $w : \mathbb{N} \to \mathbb{R}_{\geq 0}$ *satisfy for all* $k \in [0, k^*)$ *with* $0 < k^* \leq \infty$ *and all* $y_k \leq \Delta_y$ *where* $\Delta_y > c_1 + c_2$ *the following two conditions: (i)* $y_{k+1} \leq y_k + c_1$; *(ii)* $y_k \geq c_2 \Rightarrow y_{k+1} - y_k \leq -T\tilde{\rho}(\max\{y_k + w_k, 0\})$. *Then there exists* $\beta \in \mathcal{KL}$ *such that for all* $y_\circ \leq \Delta_y$ *and all* $k \in [0, k^*)$ *the following holds:*

$$y_k \leq \beta(y_\circ, kT) + w_k + c_1 + c_2 \ . \tag{9}$$

Lemma 4 *Suppose that* $T > 0$ *and* $y : \mathbb{N} \to \mathbb{R}_{\geq 0}$ *satisfy the following inequality for all* $k \in [0, k^*)$ $y_{k+1} - y_k \leq -T\rho_1(y_k) \cdot \rho_2(2y_k)$, *where* $k^* \in \mathbb{N} \cup \{\infty\}$, $\rho_1 \in \mathcal{K}_\infty$ *is locally Lipschitz and* $\rho_2 \in \mathcal{L}$. *Then, there exists* $\beta \in \mathcal{KL}$ *such that the following holds* $y_k \leq \beta(y_\circ, kT)$, $\forall k \in [0, k^*)$.

Proof of Theorem 1: Let α_3 come from item (i) of Theorem and let $\tilde{\rho}_1 \in \mathcal{K}_\infty$ and $\tilde{\rho}_2 \in \mathcal{L}$ be generated using Lemma 2 such that $\alpha_3(s) \geq \tilde{\rho}_1(s) \cdot \tilde{\rho}_2(s), \forall s \geq 0$. Let $\rho_1(s) := \tilde{\rho}_1 \circ \alpha_2^{-1}(s)$ and $\rho_2(s) := \tilde{\rho}_2 \circ \alpha_1^{-1}(s)$, $\rho_1'(s) := \frac{1}{2}\rho_1(s)$. Let β be generated via Lemma 4 using ρ_1' and ρ_2. Let $\gamma(s) := 2\tilde{\gamma}(s)$ and $\alpha(s) := \alpha_1(s)$.

Let $(\Delta_x, \Delta_{w1}, \Delta_{w2}, \delta)$ be given. Define $\Delta_1 := \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_{w2} + \delta) + 1$, $\Delta_2 := \Delta_{w1}$ and $\Delta_3 := \Delta_{w2}$. Let $\delta_1 := \frac{\rho_1(\delta/2) \cdot \rho_2(\Delta_1)}{4}$. Let $(\Delta_1, \Delta_2, \Delta_3, \delta_1)$ generate T_1^* and L via item (i) of Theorem, where without loss of generality we can assume that $L \geq 1$. Let Δ_1 generate Δ_u and T_2^* via item (iii) of Theorem. Let $(\Delta_1, \Delta_u, \Delta_2)$ generate ρ and T_3^* via item (ii) of Theorem. Let $T_4^* > 0$ be such that $LT_4^*\rho(T_4^*) \leq \min\left\{\frac{1}{2}, \frac{\delta}{4}\right\}$ and $L\rho(T_4^*) \leq \delta_1$. Let $\tilde{\delta} > 0$ be such that $\alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta + \tilde{\delta}) \leq \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta) + \frac{1}{2}$, and denote $T_5^* := \frac{\tilde{\delta}}{\tilde{\gamma}(\Delta_2) + \delta_1}$. Denote $T_6^* := \frac{\delta}{4(\tilde{\gamma}(\Delta_2) + \delta_1)}$. Finally, we introduce

$T^* := \min\{T_1^*, T_2^*, T_3^*, T_4^*, T_5^*, T_6^*\}$. To shorten notation we denote $V_k^e := V_T(\mathcal{F}_T^e(\tilde{x}(k), w[k]))$, $V_k^a := V_T(\mathcal{F}_T^a(\tilde{x}(k), w[k]))$ and $V_k := V(\tilde{x}(k))$.

Consider now an arbitrary $\tilde{x}_k$ such that $V_k \leq \alpha_2(\Delta_x) + \Delta_{w2} + \delta$ (this implies $|\tilde{x}_k| \leq \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_{w2} + \delta) < \Delta_1$), $w \in \mathcal{L}_\infty(\Delta_2) \cap \mathcal{L}_{\tilde{\gamma}}(\Delta_3)$ and $T \in (0, T^*)$. Using item (i) and our choice of T_1^*, we can write that:

$$V_k^e - V_k \leq -T\alpha_3(|\tilde{x}_k|) + \int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)ds + |V_k^e - V_k^a| + T\delta_1 \ . \tag{10}$$

From our choice of T_5^* we can write using item (i) of Theorem:

$$\alpha_1(|(F_T^a, G_T)|) \quad \leq \quad V_k^a \ \leq \ V_k + T\tilde{\gamma}(\Delta_2) + T\delta_1 \ \leq \ \alpha_2(\Delta_x) + \Delta_3 + \delta + \tilde{\delta},$$

which implies from the definition of $\tilde{\delta}$ that

$$|(F_T^a, G_T)| \quad \leq \quad \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta + \tilde{\delta}) \ \leq \ \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta) + 1/2 \ < \ \Delta_1$$

and from our choice of T_4^* and the fact that $L \geq 1$ we have:

$$\begin{aligned} |(F_T^e, G_T)| \quad &\leq \quad |(F_T^a, G_T)| + |(F_T^e, G_T) - (F_T^a, G_T)| \ \leq \ \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta) + 1/2 + |F_T^e - F_T^a| \\ &\leq \quad \alpha_1^{-1}(\alpha_2(\Delta_x) + \Delta_3 + \delta) + 1 = \Delta_1 \ . \end{aligned}$$

Hence, using local Lipschitz condition of V_T in item (i), item (ii) and our definition of T_2^*, T_3^* and T_4^* we can write that:

$$|V_k^e - V_k^a| \quad \leq \quad L|F_T^e - F_T^a| \ \leq LT\rho(T) \ \leq \ T\delta_1 \ . \tag{11}$$

From (10) and (11) and our definitions of ρ_1, ρ_2 we can write:

$$V_k^e - V_k \quad \leq \quad -T\rho_1(V_k)\rho_2(V_k) + \int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)ds + T2\delta_1 \ ,$$

and using the fact that $\rho_1(s)\rho_2(s) \geq 4\delta_1$ for all $s \in [\delta/2, \Delta_1]$, we can write:

$$V_k \geq \frac{\delta}{2} \quad \Rightarrow \quad V_k^e - V_k \leq -\frac{T}{2}\rho_1(V_k)\rho_2(V_k) + \int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)ds \ . \tag{12}$$

Moreover, using (10), (11) and the definitions of T_4^* and T_6^* we can write:

$$V_k^e \leq V_k^a + |V_k^e - V_k^a| \leq V_k + \frac{\delta}{4} + \frac{\delta}{4} \ . \tag{13}$$

Introduce $w_k := \int_0^{kT} \tilde{\gamma}(|w(s)|)ds$ and define $y_k := V_k - w_k$. Note that w_k is nondecreasing, $w_0 = 0$ and $y_\circ = V_0$. Then we have from (12) and (13), with $\tilde{\rho}(s) := \rho_1'(s)\rho_2(s)$ that

$$y_{k+1} \quad \leq \quad y_k + \frac{\delta}{2} \tag{14}$$

$$y_k \geq \frac{\delta}{2} \quad \Rightarrow \quad y_{k+1} - y_k \leq -T\tilde{\rho}(\max\{y_k + w_k, 0\}) \tag{15}$$

whenever $y_k \le \alpha_2(\Delta_x) + \Delta_3 + \delta - \Delta_3$. Note that since $V_k \ge 0$ and $w_k \le \Delta_{w2}$ for all $k \ge 0$, we have that $y_k \ge -\Delta_3, \forall k \ge 0$. Moreover, we show now by induction that $y_\circ \in [0, \alpha_2(\Delta_x) + \delta]$ implies that $y_k \le \alpha_2(\Delta_x) + \delta, \forall k \ge 0$. Suppose that $y_k \in [-\Delta_3, \alpha_2(\Delta_x) + \delta]$. Then we have that either $y_k \in [\delta/2, \alpha_2(\Delta_x) + \delta]$, in which case we have from (15) that $y_{k+1} \le y_k \le \alpha_2(\Delta_x) + \delta$ or we have that $y_k \in [-\Delta_3, \delta/2)$, in which case we have from (14) that $y_{k+1} \le y_k + \delta/2 < \delta < \alpha_2(\Delta_x) + \delta$. Hence, for any $y_\circ \in [0, \alpha_2(\Delta_x) + \delta]$ we have that $y_k \in [-\Delta_3, \alpha_2(\Delta_x) + \delta], \forall k \ge 0$ and therefore all conditions of Lemma 3 hold with $k^* = \infty$. We conclude from Lemma 3 with $\Delta_y = \alpha_2(\Delta_x) + \delta$, $c_1 = c_2 = \delta/2$ and $\tilde{\rho}(s) = \rho_1'(s)\rho_2(s)$ that $y_k \le \beta(y_\circ, kT) + w_k + \frac{\delta}{2} + \frac{\delta}{2}$, $\forall k \ge 0$, which implies (using the definition of y_k and the fact that $y_\circ = V_\circ$) that $V_k \le \beta(V_\circ, kT) + 2w_k + \delta$, $\forall k \ge 0$ and consequently

$$\alpha_1(|\tilde{x}(k)|) \quad \le \quad \beta(\alpha_2(|\tilde{x}_\circ|), kT) + 2\int_0^{kT} \tilde{\gamma}(|w(s)|)ds + \delta, \forall k \ge 0 \ ,$$

which completes the proof.

Proof of Theorem 2: We first need to show that a suitable dissipation inequality holds along trajectories of the exact discrete time model.

Claim: Given any triple $(\Delta_x, \Delta_w, \delta)$ of positive real numbers there exists $\hat{T}$ such that for any $T \in (0, \hat{T})$ there exists a family of functions $V_T(x)$ with $\alpha_1(|x|) \le V_T(x) \le \alpha_2(|x|)$ and

$$V_T(\mathcal{F}_T^e(\tilde{x}, w_T)) - V_T(\tilde{x}) \le -T\alpha_3(|\tilde{x}|) + \int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)\, ds + T\delta \tag{16}$$

for all $|\tilde{x}| \le \Delta_x$, all $w \in \mathcal{L}_\infty(\Delta_w)$.

To this end, consider a family of Lyapunov functions as in second part of Definition 3, which exists by virtue of assumption (i). The following obvious upper-bound holds:

$$V_k^e - V_k \le V_k^a - V_k + |V_k^e - V_k^a| \ . \tag{17}$$

Let us assume T_1^*, L and the family V_T as being generated by assumption (i) with the quadruple $\Delta_1 := \max\{\Delta_x, \alpha_1^{-1}(\alpha_2(\Delta_x) + 1) + 1\}$, $\Delta_2 := \Delta_w$, $\Delta_3 := \tilde{\gamma}(\Delta_w)$ and $\delta_1 := \delta/2$. Let $T_1^* \le 1$ without loss of generality. Then, since $\Delta_x \le \Delta_1$ and $\mathcal{L}_\infty(\Delta_w) = \mathcal{L}_\infty(\Delta_2) \cap \mathcal{L}_{\tilde{\gamma}}(\Delta_3)$ for signals defined over a time interval of less than 1 unit of time, we have that for all $\tilde{x}_k$s with $|\tilde{x}_k| \le \Delta_x$, for all $w \in \mathcal{L}_\infty(\Delta_w)$ and all $T \in (0, T_1^*)$

$$V_k^a - V_k \le -T\alpha_3(|\tilde{x}_k|) + \int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)\, ds + T\delta_1. \tag{18}$$

Consider now the difference $|V_k^e - V_k^a|$. In order to find a suitable bound we exploit the Lipschitzianity conditions which come from Lyapunov SP-iIiSS and from one-step consistency. Both of them can be applied provided that T is sufficiently small; in fact, for all $\tilde{x}_k$ and all w as in (18) we have

$$\alpha_1(|\mathcal{F}_T^a|) \quad \leq \quad V_k^a \leq V_k + T(\tilde{\gamma}(\Delta_2) + \delta_1) \ \leq \ \alpha_2(\Delta_x) + 1 < \alpha_1(\Delta_1) \tag{19}$$

where the last inequality only holds for all $T \in \big(0, 1/[\tilde{\gamma}(\Delta_2) + \delta_1]\big)$. Notice that (19) is satisfied with some margin, so that by exploiting (ii) and (iii) conditions of theorem we obtain:

$$|\mathcal{F}_T^e| \quad \leq \quad |\mathcal{F}_T^a| + |\mathcal{F}_T^e - \mathcal{F}_T^a| \ \leq \ \alpha_1^{-1}(\alpha_2(\Delta_x) + 1) + |F_T^e - F_T^a| \leq \Delta_1 \tag{20}$$

where the last inequality holds provided that we consider $T \in (0, \min\{T_2^*.T_3^*\})$, T_3^* being generated together with Δ_u from assumption (iii), and T_2^* such that $T_2^*\rho(T_2^*) \leq 1$ where ρ is generated by assumption (ii) with the triple $(\Delta_x, \Delta_u, \Delta_w)$. Combining (19) and (20) and exploiting the Lipschitz condition of V_T together with one-step consistency we are able to conclude

$$|V_k^e - V_k^a| \leq L\,|F_T^e - F_T^a| \leq LT\rho(T) \leq T\delta_1 \ , \tag{21}$$

for all $T \in (0, T_4^*)$ where T_4^* is such that $L\rho(T_4^*) \leq \delta_1$. We can now define $\hat{T} = \min\{T_1^*, T_2^*, T_3^*, T_4^*, 1/[\tilde{\gamma}(\Delta_2) + \delta_1]\}$, so that combining (17), (18) and (21) we get the desired dissipation inequality (16).

We show next how to derive SP-iIiSS by adding over time the inequality in (16). Let $(\tilde{\Delta}_x, \tilde{\Delta}_w, \tilde{\delta})$ be an arbitrary triple of positive reals. We generate the family of Lyapunov functions V_T in (16) by letting $\Delta_x = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}([\tilde{\gamma}(\tilde{\Delta}_w) + \tilde{\delta}]) + 1$, $\Delta_w = \max\{\tilde{\Delta}_w, \tilde{\gamma}^{-1} \circ \alpha_3(\tilde{\Delta}_x)\}$ and $\delta = \tilde{\delta}$. Notice that (16) implies:

$$V_T(\mathcal{F}_T^e(\tilde{x}_k, w_T)) - V_T(\tilde{x}_k) \quad \leq \quad -T\alpha_3 \circ \alpha_2^{-1}(V_T(\tilde{x}_k)) + \int_{kT}^{(k+1)T} \tilde{\gamma}(|w(s)|)\, ds + T\delta \tag{22}$$

for all $|\tilde{x}_k| \leq \Delta_x$, all $w \in \mathcal{L}_\infty(\Delta_w)$. Pick an arbitrary $\tilde{x}_0$ with $|\tilde{x}_0| \leq \tilde{\Delta}_x$. Then we have, by definition of Δ_w, $V_T(\tilde{x}_0) \leq \alpha_2(\tilde{\Delta}_x) \leq \alpha_2 \circ \alpha_3^{-1}(\gamma(\Delta_w) + \delta)$. It is by now a standard argument to show that $V_T(\tilde{x}_0) \leq \alpha_2 \circ \alpha_3^{-1}([\tilde{\gamma}(\Delta_w) + \delta])$ yields $V_T(\tilde{x}_k) \leq \alpha_2 \circ \alpha_3^{-1}([\tilde{\gamma}(\Delta_w) + \delta])$, for all $w \in \mathcal{L}_\infty(\Delta_w)$ and as long as (22) holds. This is a fortiori true also for all $w \in \mathcal{L}_\infty(\tilde{\Delta}_w) \subseteq \mathcal{L}_\infty(\Delta_w)$.

Therefore, since this in turn implies $|\tilde{x}_k| < \Delta_x$, we can show by contradiction that (22) actually holds for all $k > 0$. As a final step in the proof just take a sum of (22) from 0 to k in order to get:

$$V_T(\tilde{x}_k) - V_T(\tilde{x}_0) = \sum_{i=0}^{k-1} (V_T(\tilde{x}_{i+1}) - V_T(\tilde{x}_i)) \quad \leq \quad -T\sum_{i=0}^{k-1} \alpha_3 \circ \alpha_2^{-1}(V_T(\tilde{x}_i)) + \int_0^{kT} \tilde{\gamma}(|w(s)|)\, ds + Tk\delta.$$

Since $\alpha_1(|x|) \leq V_T(x) \leq \alpha_2(|x|)$, we can write:

$$T\sum_{i=0}^{k-1} \alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(|\tilde{x}_i|) \leq T\sum_{i=0}^{k-1} \alpha_3 \circ \alpha_2^{-1}(V_T(\tilde{x}_i)) \quad \leq \quad \int_0^{kT} \tilde{\gamma}(|w(s)|)\, ds + Tk\delta + \alpha_2(|\tilde{x}_0|). \tag{23}$$

Proof of Lemma 3: First we note that for all $y_k \geq c_2$ we have from (ii) that $y_{k+1} \leq y_k$ and if $y_k \leq c_2$ we have from (i) that $y_{k+1} \leq y_k + c_1 \leq c_1 + c_2$. Moreover, since $\Delta_y > c_1 + c_2$ we conclude that the set $\{y : y \le c_1 + c_2\}$ is forward invariant, that is, $y_\circ \le c_1 + c_2$ implies $y_k \le c_1 + c_2$ for all $k \in [0, k^*)$. Suppose now that $\Delta \ge y_\circ > c_1 + c_2 > 0$. Define $k_1 := \min\{k \ge 0 : y_k \le c_1 + c_2\}$ (with $k_1 = k^*$ if $y_k > c_1 + c_2$ for all $k \in [0, k^*)$). Hence, for all $k \ge k_1$ (if $k_1 < k^*$) we have that $y_k \le c_1 + c_2$ since the set $\{y : y \le c_1 + c_2\}$ is forward invariant and so (9) holds. Define now $k_0 := \min\{k \ge 0 : y_k \le w_k\}$ (with $k_0 = k_1$ if $y_k > w_k$ for all $k \in [0, k_1)$). Note that for all $k \in [0, k_1)$ we have from (ii) that y_k is non-increasing and also recall w_k is assumed to be a nondecreasing function of time. Hence, for all $k \in [k_0, k_1)$ (if $k_0 < k_1$) we have that $y_k \le w_k$ and so (9) holds. Finally, consider $k \in [0, k_0)$. Note that $y_k > w_k \ge w_i$ for all $i \in [0, k]$ and since y is non-increasing, we have that $y_i \ge y_k > w_i$ for all such i. Therefore, $0 \le y_i \le y_i + w_i \le 2y_i$ for all $i \in [0, k]$. From Lemma 2 and (ii) we can write that $y_{i+1} - y_i \le -T\rho_1(y_i)\rho_2(2y_i)$, $\forall i \in [0, k]$. From Lemma 4 we conclude that $y_i \le \beta(y_\circ, iT)$, $\forall i \in [0, k]$ and hence the bound (9) holds, which completes the proof.

Proof of Lemma 4: Consider an arbitrary $y_\circ$ and the corresponding sequence y_k. We introduce a new continuous and piecewise linear variable $\forall t \in [kT\rho_2(2y_k), (k+1)T\rho_2(2y_k)), k \in [0, k^* - 1)$: $\eta(t) = y_k + \left(\frac{t}{T\rho_2(2y_k)} - k\right)(y_{k+1} - y_k)$, and we let $\eta((k^* - 1)T\rho_2(2y_{k^*-1})) = y_{k^*-1}$ if $k^* \ne \infty$. Note that $\eta(kT\rho_2(2y_k)) = y_k$ for all $k \in [0, k^*)$. Denote $t^* := k^*T\rho_2(2y_{k^*})$. Since η is continuous and piecewise linear, it is differentiable for almost all $t \in [0, t^*)$. Hence, we can write that for all $t \in [kT\rho_2(2y_k), (k+1)T\rho_2(2y_k)), k \in [0, k^* - 1)$ we have:

$$\dot\eta(t) \quad = \quad \frac{y_{k+1} - y_k}{T\rho_2(2y_k)} \;\le\; -\rho_1(y_k) \tag{24}$$

Moreover, since $y_{k+1} \le y_k$ for all $k \in [0, k^*)$, we have $\eta(t) \le y_k$ for all $t \in [kT\rho_2(2y_k), (k+1)T\rho_2(2y_k)), k \in [0, k^* - 1)$. We can conclude from (24) that $\dot\eta(t) \le -\rho_1(\eta(t))$, $for\ a.a.\ t \in [0, t^*)$. Using the standard comparison principle (see Proposition 2.5 in [6]) and since ρ_1 is assumed locally Lipschitz, we conclude that there exists $\beta_1 \in \mathcal{KL}$ such that we have $\eta(t) \le \beta_1(\eta_\circ, t)$, $\forall t \in [0, t^*)$. We let $t = kT\rho_2(2y_k)$ to obtain $y_k \le \beta_1(y_\circ, kT\rho_2(y_k))$. Since $y_{k+1} \le y_k, k \in [0, k^* - 1)$ we conclude that $y_k \le y_\circ, \forall k \in [0, k^*)$ and we since $\rho_2 \in \mathcal{L}$, we can write $y_k \le \beta_1(y_\circ, kT\rho_2(y_k)) \le \beta_1(y_\circ, kT\rho_2(y_\circ)) =: \beta(y_\circ, kT), \forall k \in [0, k^*)$, where it is easy to see that $\beta(s, t) := \beta_1(s, t\rho_2(s)) \in \mathcal{KL}$.

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