Power characterizations of input-to-state stability and integral input-to-state stability

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Abstract

New notions of external stability for nonlinear systems are introduced, making use of average powers as signal norms and comparison functions as in the Input-to-State Stability (ISS) framework. Several new characterizations of ISS and integral ISS are presented in terms of the new notions. An example is discussed to illustrate differences and similarities of the newly introduced properties.

Keywords: Continuous-time, nonlinear, inputs, power, stability.

1 Introduction

The notion of input-to-state stability (ISS) has been now widely recognized and accepted as an important concept that is useful for a range of nonlinear control problems (see [4, 5, 6]). Since its conception in [11] a range of important equivalent characterizations of ISS have been proved in the literature (see [9, 10, 11, 12, 13]). These characterizations lead to a better understanding of the ISS property and provided the control engineers with a range of new tools that can be used in nonlinear control. ISS has been originally defined in $\mathcal{L}_\infty$ framework (see [11]) and it requires roughly speaking that "no matter what the initial state is, if the inputs are uniformly small, then the state must eventually be small". Results in [10] have shown that ISS systems also posses the property that bounded energy inputs imply bounded energy states. On one hand, this has shown that the ISS concept is more general than originally thought since it also covers L_2 stable systems. On the other hand, this research has lead to the introduction of a new property, the so called integral input-to-state stability property (iISS), which requires an ISS-like estimate for the solutions where the $\mathcal{L}_\infty$ norm of inputs is replaced by some energy function. The iISS property has been shown to be a natural generalization of ISS and it is anticipated that it will be at least as useful as ISS in the analysis of nonlinear control systems. A range of equivalent characterizations of iISS have been presented in [2, 3, 7, 10] and they serve to better understand the property itself and to provide new tools that may be useful in different situations.

The purpose of this paper is to provide new definitions of ISS-like and iISS-like properties. These are given in terms of powers of input and/or state signals and are novel compared to previous characterizations since they describe the system's behaviour for different classes of input signals. Since bounded power signals may fail to be uniformly bounded or may have unbounded energy, for such signals power estimates might turn out to be tighter than the ones provided by the original ISS and iISS definitions. On the other hand, bounds expressed in terms of averaged singals cannot be translated into hard bounds on pointwise signal norms without some careful handling. Nevertheless, it is somewhat surprising that both ISS and iISS properties do have equivalent "power" characterizations. For instance, under a mild assumption of local stability we show that ISS is equivalent to the property that "no matter what the initial state is, if the *power* of inputs is uniformly small, then the *power* of state must eventually be small".

The paper is organized as follows. In Section 2 we present preliminaries and definitions. Section 3 contains main results with proofs. In the last section we present an example which shows that our basic assumption of local stability is really necessary when characterizing the ISS or iISS properties in terms of powers of signals.

2 Preliminaries

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and zero at zero. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ if it is continuous and $\gamma(\cdot)$ is strictly decreasing to zero as $t \to +\infty$. The function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if for each fixed $s > 0$ $\beta(s, \cdot)$ is of class $\mathcal{L}$ and for each fixed $t \geq 0$ $\beta(\cdot, t)$ is of class $\mathcal{K}$. Similarly a function $\lambda : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{LL}$ if it is class $\mathcal{L}$ in both arguments. Given a measurable function w, we define its infinity norm $\|w\|_\infty := \operatorname{ess\,sup}_{t\geq 0} |w(t)|$. If we have $\|w\|_\infty < \infty$, then we write $w \in \mathcal{L}_\infty$. If $\int_0^{+\infty} \sigma(|w(s)|)ds < +\infty$ then we write $w \in \mathcal{L}_\sigma$. In the sequel we use the following notation

$$
\begin{aligned}
P_\sigma^t(w) &= \frac{\int_0^t \sigma(|w(s)|)ds}{t} \quad \text{for } t > 0 \\
P_\sigma^0(w) &= \sigma(|w(0)|)
\end{aligned}
$$

If for all $t \in [0, +\infty)$ we have that $P_\sigma^t(w) < +\infty$ then we write $w \in \mathcal{LP}_\sigma$.

Consider the nonlinear system:

$$
\dot{x} = f(x, u) \ , \tag{1}
$$

where f is locally Lipschitz and $f(0,0) = 0$. The system (1) is called forward complete if for any $\xi \in \mathbb{R}^n$ and any measurable locally essentially bounded function u the solution $x(t, \xi, u)$ of (1) exists for all $t \geq 0$. We first recall several properties from [2] and [12] that we will need in the sequel. The system (1):

1. is *Input-to-State Stable (ISS)* if there exist α and σ of class $\mathcal{K}_\infty$ and β of class $\mathcal{KL}$ such that for all $\xi \in \mathbb{R}^n$ and all $u \in \mathcal{L}_\infty$ we have:
$$\alpha(|x(t,\xi,u)|) \leq \beta(|\xi|,t) + \sigma(\left\|u_{[0,t]}\right\|_\infty) \qquad \forall t \geq 0 \ . \tag{2}$$
2. is *integral Input-to-State Stable (iISS)* if there exist α and σ of class $\mathcal{K}_\infty$ and β of class $\mathcal{KL}$ such that for all $\xi \in \mathbb{R}^n$ and all $u \in \mathcal{L}_\sigma$ we have:
$$\alpha(|x(t,\xi,u)|) \leq \beta(|\xi|,t) + \int_0^t \sigma(|u(s)|)ds \qquad \forall t \geq 0 \ . \tag{3}$$
3. is 0-*Locally Stable (*0*-LS)* if for each $\epsilon > 0$ there exists $\delta > 0$ such that $|\xi| \leq \delta$ and $u(\cdot) \equiv 0$ imply $|x(t,\xi,0)| \leq \epsilon, \forall t \geq 0$.
4. is 0-*globally asymptotically stable (*0*-GAS)* if there exist β of class $\mathcal{KL}$ such that for all $\xi \in \mathbb{R}^n$ and $u(\cdot) \equiv 0$ it holds
$$|x(t,\xi,0)| \leq \beta(|\xi|,t) \qquad \forall t \geq 0 \ . \tag{4}$$
5. has LIM property if there exists a class $\mathcal{K}$ function γ such that for all $\xi \in \mathbb{R}^n$ and all $u \in \mathcal{L}_\infty$ we have
$$\inf_{t\geq 0} |x(t,\xi,u)| \leq \gamma(\|u\|_\infty) \ . \tag{5}$$
6. is *Uniformly Bounded Energy Bounded State (UBEBS)* if for some α, γ and σ of class $\mathcal{K}_\infty$, $c \geq 0$ and all $\xi \in \mathbb{R}^n$ and $u \in \mathcal{L}_\sigma$ the following estimate holds:
$$\alpha(|x(t,\xi,u)|) \leq \gamma(|\xi|) + \int_0^t \sigma(|u(s)|)ds + c \qquad \forall t \geq 0 \ . \tag{6}$$

We need the following results that were respectively proved in [12] and [3]:

Proposition 2.1 The system (1) is ISS if and only if it is 0-LS and it has LIM property. ∎

Notice that, for autonomous systems, the result amounts to saying that GAS is equivalent to local stability plus a weaker notion of attractivity, namely $\inf_{t\geq 0} |x(t,\xi)| = 0$.

Proposition 2.2 The system (1) is iISS if and only if it is 0-GAS and UBEBS. ∎

We now introduce new properties that involve powers of input and/or state signals that we need in the sequel. We call the system (1)

1. *Power Input-to-State Stable (P-ISS)*, if it is forward complete and there exist α and σ of class $\mathcal{K}_\infty$ and β of class $\mathcal{KL}$ such that for all $\xi \in \mathbb{R}^n$ and all $u \in \mathcal{LP}_\sigma$ we have:
$$P_\alpha^t(x) \leq \beta(|\xi|,t) + P_\sigma^t(u) \qquad \forall t \geq 0 \ . \tag{7}$$

2. *Power integral Input-to-State Stable (P-iISS)*, if it is forward complete and there exist α, σ and γ of class $\mathcal{K}_\infty$ and β of class $\mathcal{KL}$ such that for all $\xi \in \mathbb{R}^n$ and $u \in \mathcal{L}_\sigma$ we have:

$$P_\alpha^t(x) \leq \beta(|\xi|, t) + \gamma\left(\int_0^t \sigma(|u(s)|)ds\right) \qquad \forall t \geq 0 \ . \tag{8}$$

3. *Moving Average Input-to-State Stable (MA-ISS)*, if it is forward complete and there exist α and σ of class $\mathcal{K}_\infty$, β of class $\mathcal{KL}$ and $T > 0$ such that for all $\xi \in \mathbb{R}^n$ and all $u \in \mathcal{L}_\infty$ we have:

$$\frac{1}{T}\int_t^{t+T} \alpha(|x(s, \xi, u)|)ds \leq \beta(|\xi|, t) + \sigma(\|u\|_\infty) \qquad \forall t \geq 0 \ . \tag{9}$$

4. *Moving Average integral Input-to-State Stable (MA-iISS)*, if it is forward complete and there exist α, σ and γ of class $\mathcal{K}_\infty$, β of class $\mathcal{KL}$ and $T > 0$ such that for all $\xi \in \mathbb{R}^n$ and $u \in \mathcal{L}_\sigma$ we have:

$$\frac{1}{T}\int_t^{t+T} \alpha(|x(s, \xi, u)|)ds \leq \beta(|\xi|, t) + \gamma\left(\int_0^{t+T} \sigma(|u(s)|)ds\right) \qquad \forall t \geq 0 \ . \tag{10}$$

3 Main results

Several different characterizations of ISS and iISS have appeared in the literature. Most of them use the infinity norms of disturbances or integral of disturbances and/or states, which can be interpreted as energies of signals. However, we are not aware of any characterizations of ISS and iISS that involve powers of inputs and/or states and these are interesting, for instance, when the input signal has unbounded energy but finite power. Our main results are stated in Theorems 1 and 2, which show respectively equivalence of ISS and iISS to certain power notions.

Theorem 1 *The following statements are equivalent for the system (1):*

1. *the system is ISS;*

2. *the system is P-ISS and* 0*-LS;*

3. *the system is MA-ISS and* 0*-LS.* ∎

Theorem 2 *The following statements are equivalent for the system (1):*

1. *the system is iISS;*

2. *the system is P-iISS and* 0*-LS;*

3. *the system is MA-iISS and* 0*-LS.* ∎

Now we present proofs of all main results.

ISS $\Rightarrow$ 0-LS + P-ISS: It is trivial that ISS implies 0-LS and forward completeness. It follows from the exponential ISS argument in [9] that ISS implies the existence of a smooth and proper V, such that along trajectories of (1):

$$\dot{V} \le -V + \sigma(|u|) \ , \tag{11}$$

for some σ of class $\mathcal{K}_\infty$. Then, taking the integral of (11) with $\xi := x(0)$, we obtain

$$\int_0^t V(x(s))ds \le V(\xi) - V(x(t)) + \int_0^t \sigma(|u(s)|)ds \le V(\xi) + \int_0^t \sigma(|u(s)|)ds \tag{12}$$

By a standard comparison principle applied to (11) we also have:

$$V(x(t)) \le V(\xi)e^{-t} + \int_0^t e^{-(t-s)}\sigma(|u(s)|)ds \tag{13}$$

and taking the integral of (13) gives:

$$\begin{aligned} \int_0^t V(x(s))ds &\le V(\xi)[1-e^{-t}] + \int_0^t \int_0^\tau e^{-(\tau-s)}\sigma(|u(s)|)\, ds d\tau \\ &= V(\xi)[1-e^{-t}] + \int_0^t \int_s^t e^{-(\tau-s)}\sigma(|u(s)|)\, d\tau ds \\ &\le V(\xi)[1-e^{-t}] + \int_0^t \sigma(|u(s)|)\, ds \ . \end{aligned} \tag{14}$$

Considering inequalities (12) and (14), respectively for $t > \log(2)$ and $t \le \log(2)$, yields:

$$\int_0^t V(x(s))ds \le \int_0^t \sigma(|u(s)|)ds + \left\{ \begin{array}{cc} V(\xi) & t > \log(2) \\ V(\xi)[1-e^{-t}] & t \le \log(2) \end{array} \right. \tag{15}$$

Dividing both sides by t, recalling that for all $x \in \mathbb{R}^n$ we have $\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$ where α_1, α_2 are functions of class $\mathcal{K}_\infty$ and defining $\beta \in \mathcal{KL}$ as

$$\beta(V,t) = \left\{ \begin{array}{cl} 2V[1-e^{-t}]/t & \text{for } t \le \log(2) \\ V/t & \text{for } t > \log(2) \end{array} \right. \tag{16}$$

we have for all $u \in \mathcal{LP}_\sigma$ that $P^t_{\alpha_1}(x) \le \beta(\alpha_2(|\xi|), t) + P^t_\sigma(u)$, $\forall t \ge 0$. ∎

P-ISS + 0-LS $\Rightarrow$ ISS: From Proposition 2.1, it is sufficient to show that P-ISS implies the LIM property. We claim that:

$$\inf_{t\ge 0} |x(t)| \le \liminf_{t\to+\infty} |x(t)| \le \alpha^{-1} \circ \sigma(\|u\|_\infty) \tag{17}$$

Assume by contradiction that

$$\exists\, \varepsilon > 0,\ \exists\, \tau \ge 0 :\ \forall t \ge \tau, |x(t)| > \alpha^{-1}(\sigma(\|u\|_\infty) + \varepsilon)$$

Without loss of generality take $\tau = 0$. Then, we have

$$\alpha(|x(t)|) > \alpha(\alpha^{-1}(\sigma(\|u\|_\infty) + \varepsilon)) = \sigma(\|u\|_\infty) + \varepsilon \tag{18}$$

Taking the integral of (18) we obtain

$$\int_0^t \alpha(|x(s)|)\, ds \geq \int_0^t [\sigma(\|u\|_\infty) + \varepsilon]\, ds \geq \int_0^t \sigma(|u(s)|)\, ds + \varepsilon t \tag{19}$$

Then, exploiting the definition of P-ISS, and inequality (19) we get to the following contradiction

$$t\beta(|\xi|, t) \geq \varepsilon t \qquad \forall t \geq 0\ .$$

Note also that if $u(\cdot) \equiv 0$, then the estimate $P_\alpha^t(x) \leq \beta(|\xi|, t), \forall t \geq 0$ and 0-LS imply 0-GAS. ∎

0-LS and MA-ISS $\Rightarrow$ ISS: From definition of MA-ISS in (9) and since for any $t \geq 0$ there exists $\tau \in [t, t+T]$ such that $\alpha(|x(\tau, \xi, u)|) \leq \frac{1}{T}\int_t^{t+T} \alpha(|x(s, \xi, u)|)ds$, we have that for any trajectory there exists a sequence of $\tau_k, k = 0, 1, \ldots$ such that $\tau_k \to +\infty$ as $k \to +\infty$ and $\alpha(|x(\tau_k, \xi, u)|) \leq \beta(|\xi|, \tau_k) + \sigma(\|u\|_\infty)$. This in turn implies $\inf_{t\geq 0} \alpha(|x(t)|) \leq \liminf_{k\to+\infty} \alpha(|x(\tau_k, \xi, u)|) \leq \sigma(\|u\|_\infty)$. Hence, the LIM property holds and the result follows from Proposition 2.1. ∎

ISS $\Rightarrow$ 0-LS and MA-ISS: ISS implies 0-LS and forward completeness. To see that ISS implies (9), for arbitrary $t \geq 0$ and any fixed $T > 0$ integrate the ISS $\mathcal{KL}$ estimate initialized at ξ on the interval $[t, t+T]$:

$$\begin{aligned} \int_t^{t+T} \alpha(|x(s, \xi, u)|)ds &\leq \int_t^{t+T} \beta(|\xi|, s)ds + \int_t^{t+T} \sigma(\|u\|_\infty)ds \\ &\leq T\beta(|\xi|, t) + T\sigma(\|u\|_\infty)\ , \end{aligned} \tag{20}$$

which proves the result. ∎

iISS $\Rightarrow$ 0-LS and P-iISS: 0-LS and forward completeness are direct consequences of iISS. We show now that iISS also implies inequality (8). It was shown in [2] that iISS is equivalent to any of the following properties.

Property 1: there exist functions $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ of class $\mathcal{K}_\infty$ such that the following estimate holds:

$$\int_0^t \gamma_1(|x(s)|)ds \leq \gamma_2(|\xi|) + \gamma_3\left(\int_0^t \gamma_4(|u(s)|)ds\right) \qquad \forall t \geq 0\ . \tag{21}$$

Property 2: there exist functions $\alpha_1, \alpha_2, \alpha_4$ of class $\mathcal{K}_\infty$, a positive definite function α_3 and a smooth $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for all x and u we have

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{22}$$

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_3(V) + \alpha_4(|u|)\ . \tag{23}$$

We split the analysis into two parts $t \in [0, 1]$ and $t > 1$. Consider first $t \in [0, 1]$. Using calculations similar to [2] one can show that (23) implies that there exists $\beta_1 \in \mathcal{KL}$ such that:

$$V(x(t)) \leq \beta_1(V(x(0)), t) + \int_0^t 2\alpha_4(|u(s)|)ds \qquad \forall t \geq 0\ . \tag{24}$$

Using Lemma 1 in [8] there is no loss of generality in assuming that $\beta_1(s,t) \le 2\beta_1(s,t+1), \forall s,t$. We integrate (24) to obtain:

$$\begin{aligned}
\int_0^t V(x(s))ds &\le \int_0^t \beta_1(V(x(0)),s)ds + \int_0^t\int_0^\tau 2\alpha_4(|u(s)|)dsd\tau \\
&\le \int_0^t 2\beta_1(V(x(0)),s+1)ds + t\int_0^t 2\alpha_4(|u(s)|)ds \\
&\le t2\beta_1(V(x(0)),1) + t\int_0^t 2\alpha_4(|u(s)|)ds \\
&\le t2\beta_1(V(x(0)),t) + t\int_0^t 2\alpha_4(|u(s)|)ds \ .
\end{aligned} \tag{25}$$

Finally, using (22) and dividing by t we can write that for all $\xi \in \mathbb{R}^n$ and $u \in \mathcal{L}_{2\alpha_4}$:

$$P^t_{\alpha_1}(x) \le 2\beta_1(\alpha_2(|\xi|),t) + \int_0^t 2\alpha_4(|u(s)|)ds \ , \ \forall t \in [0,1]. \tag{26}$$

Consider now $t > 1$. Dividing inequality (21) by t, we can write:

$$\begin{aligned}
P^t_{\gamma_1}(x) &\le \frac{\gamma_2(|\xi|)}{t} + \frac{1}{t}\gamma_3\left(\int_0^t \gamma_4(|u(s)|)ds\right) \\
&\le \frac{2\gamma_2(|\xi|)}{t+1} + \gamma_3\left(\int_0^t \gamma_4(|u(s)|)ds\right) \ , \ \forall t > 1.
\end{aligned} \tag{27}$$

Define $\alpha(s) := \min\{\alpha_1(s), \gamma_1(s)\}$, $\gamma(s) := \max\{\gamma_3(s), s\}$, $\sigma(s) := \max\{2\alpha_4(s), \gamma_4(s)\}$ and

$$\beta(s,t) = \max\left\{2\beta_1(\alpha_2(s),t), \frac{2\gamma_2(s)}{t+1}\right\} \ .$$

We see from (26) and (27) that (8) holds for all $t \ge 0$. ∎

0-LS + P-iISS $\Rightarrow$ iISS: As remarked earlier, P-iISS with $u(\cdot) \equiv 0$ implies $\inf_{t\ge 0}|x(t)| = 0$. By Proposition 2.1 (and related remark) this, together with 0-LS, yields 0-GAS. By Proposition 2.2 (the main theorem in [3]) we must show that a UBEBS estimate holds. We start by recalling a result in [1]. It is shown there that forward completeness of (1) implies (and is, in fact, equivalent to) there being an estimate of the following type along all trajectories:

$$|x(t,\xi,u)| \ \le \ \kappa_1(t) \ + \ \kappa_2(|\xi|) \ + \ \kappa_3\left(\int_0^t \tilde{\sigma}(|u(s)|)\,ds\right) \ + \ c \tag{28}$$

holding for some κ_1,κ_2,κ_3,$\tilde{\sigma}$ of class $\mathcal{K}_\infty$ and some $c \ge 0$. We choose functions like this, and let α, γ, σ and β be as in the estimate (8). We also introduce

$$\delta \ := \ \max\{\tilde{\sigma}, \sigma\} \, .$$

Let us define, for each $r \ge 0$:

$$m(r) \ := \ \sup\left\{|x(t,\xi,u)| \ : \ t \ge 0\,,\ |\xi| \le r\,,\ \int_0^{+\infty} \delta(|u(s)|)\,ds \le \gamma^{-1}(\alpha(r)/2)\right\} \ .$$

Note that m is a nondecreasing function. The main technical step is in showing that $m(r) < \infty$:

Lemma There exists a function θ so that for each $r > 0$,

$$m(r) \;\leq\; \kappa_1(\theta(r)) \;+\; \kappa_2(r) \;+\; \kappa_3(\gamma^{-1}(\alpha(r)/2)) \;+\; c\,. \tag{29}$$

Proof. Pick any $r > 0$ and denote for simplicity the right-hand side of (29) as $M(r)$. Pick any state ξ and input u so that $|\xi| \leq r$ and $\int_0^{+\infty} \delta(|u(s)|)\,ds \leq \gamma^{-1}(\alpha(r)/2)$. We need to show that, for all t, $|x(t,\xi,u)| \leq M(r)$. Assume that is not the case, so there is some $T > 0$ so that $|x(T,\xi,u)| > M(r)$. Let

$$\tau \;:=\; \sup\{t \leq T \;:\; |x(t,\xi,u)| \leq r\}\,,$$

so that $|x(t,\xi,u)| > r$ for all $t \in [\tau, T]$. Clearly, $|x(\tau)| = r$. It follows from (8), initialized at time τ that

$$\begin{aligned}
\alpha(r)(T-\tau) \;&\leq\; \int_\tau^T \alpha(|x(s,\xi,u)|)\,ds \\
&\leq\; (T-\tau)\beta(|x(\tau)|, T-\tau) + (T-\tau)\gamma\left(\int_\tau^T \sigma(|u(s)|)\,ds\right) \\
&\leq\; (T-\tau)\beta(|x(\tau)|, T-\tau) + (T-\tau)\gamma\left(\int_\tau^T \delta(|u(s)|)\,ds\right) \\
&\leq\; (T-\tau)\beta(r, T-\tau) + (T-\tau)\gamma(\gamma^{-1}(\alpha(r)/2)).
\end{aligned} \tag{30}$$

Hence, we have $\alpha(r) \leq 2\beta(r, T-\tau)$. By the $\mathcal{KL}$ function lemma in [10], there exists function θ_1 and θ_2 of class $\mathcal{K}_\infty$ such that $2\beta(r,s) \leq \theta_1(\theta_2(r)e^{-s})$. This allows to find an upper-bound for $T-\tau$ as follows

$$T - \tau \leq \log\left(\frac{\theta_2(r)}{\theta_1^{-1}(\alpha(r))}\right) =: \theta(r). \tag{31}$$

With the notation $\tilde{T} := T - \tau \leq \theta(r)$, $\tilde{\xi} := x(\tau,\xi,u)$, $\tilde{u}(\cdot) := u(\cdot - \tau)$, we have that

$$|x(T,\xi,u)| \;=\; |x(\tilde{T},\tilde{\xi},\tilde{u})| \;\leq\; \kappa_1(\tilde{T}) + \kappa_2(r) + \kappa_3(\gamma^{-1}(\alpha(r)/2)) + c \;=\; M(r)\,,$$

a contradiction. ∎

Now pick any ξ and u, and let

$$r \;:=\; \max\left\{|\xi|\,,\, \gamma^{-1} \circ \frac{\alpha}{2}\left(\int_0^{+\infty} \delta(|u(s)|)\,ds\right)\right\}\,.$$

By definition of m,

$$|x(t,\xi,u(\cdot))| \;\leq\; m(r) \;\leq\; \kappa(r) + c_1 \tag{32}$$

for all $t \geq 0$, where κ is any class-$\mathcal{K}_\infty$ function and c_1 is any constant such that $m(r) \leq \kappa(r) + c_1$ for all r (such κ and c_1 exist because m is nondecreasing and finite-valued). Thus, equation (32) gives us a UBEBS estimate, as wanted. ∎

0-LS and MA-iISS ⇒ iISS: The result will be proved using Proposition 2.2. Consider and arbitrary trajectory of the system (1). Let $\tau_0 = 0$ and consider the sequence of time intervals $[(2k-1)T, 2kT], k \geq 1$. Since for all $k \geq 1$ there exists $\tau_k \in [(2k-1)T, 2kT]$ such that $\alpha(|x(\tau_k)|) \leq \frac{1}{T}\int_{(2k-1)T}^{2kT} \alpha(|x(s)|)ds$, MA-iISS implies that there exists a sequence of times $\tau_k \in [(2k-1)T, 2kT]$ with:

$$\alpha(x(\tau_k, \xi, u)) \leq \beta(|\xi|, \tau_k) + \int_0^{2kT} \sigma(|u(s)|)ds \ , \ \forall k \geq 1. \tag{33}$$

For $u(\cdot) \equiv 0$ we have $|x(\tau_k, \xi, 0)| \to 0$, and together with 0-LS this is enough to conclude 0-GAS. Note that $T \leq \tau_k - \tau_{k-1} \leq 3T, \forall k \geq 1$. From forward completeness we have that (28) holds and for all $t \in [0, 2T]$ we can write:

$$|x(t, \xi, u)| \ \leq \ \kappa_1(2T) \ + \ \kappa_2(|\xi|) \ + \ \kappa_3\left(\int_0^t \tilde{\sigma}(|u(s)|)\, ds\right) \ + \ c \ , \tag{34}$$

Moreover, we can write that for all $t \in [2kT, 2(k+1)T], k \geq 1$ we have:

$$|x(t, \xi, u)| \ \leq \ \kappa_1(3T) \ + \ \kappa_2(|x(\tau_k)|) \ + \ \kappa_3\left(\int_{\tau_k}^t \tilde{\sigma}(|u(s)|)\, ds\right) \ + \ c \ , \tag{35}$$

which together with (33), which holds for $k \geq 1$, implies that UBEBS property holds. ∎

iISS ⇒ 0-LS and MA-iISS: iISS trivially implies 0-LS and forward completeness. To see that iISS implies MA-iISS, integrate the iISS $\mathcal{KL}$ estimate initialized at ξ on the interval $[t, t+T]$:

$$\begin{aligned} \int_t^{t+T} \alpha(|x(s, \xi, u)|)ds \ &\leq \ \int_t^{t+T} \beta(|\xi|, s)ds + \int_t^{t+T}\int_t^{\tau} \sigma(|u(s)|)dsd\tau \\ &\leq \ T\beta(|\xi|, t) + T\int_t^{t+T} \sigma(|u(s)|)ds \ , \end{aligned} \tag{36}$$

which proves the result. ∎

4 Why is local stability necessary

In general P-ISS and P-iISS do not imply 0-LS as will be shown in the last part of the section by means of an example. However, in some situations P-ISS and P-iISS may imply 0-LS. Indeed, if there exists a class $\mathcal{K}_\infty$ function κ such that the β function in the estimate (7) or (8) satisfies $t\beta(s,t) \leq \kappa(s), \forall s, t \geq 0$, then we have that for any $\xi \in \mathbb{R}^n$, with $u(\cdot) \equiv 0$, the solution of the system (1) satisfies $\int_0^t \alpha(|x(s, \xi, 0)|)ds \leq \kappa(|\xi|), \forall t \geq 0$ and by Theorem 1 in [10] the system (1) is 0-GAS, which implies 0-LS. For instance, this is true if $\beta(s,t) = \alpha(s)e^{-\lambda t}$ for some $\alpha \in \mathcal{K}_\infty$ and $\lambda > 0$, which includes exponentially stable systems.

We show next, with an example, that the 0-LS assumption in Theorem 1 is indeed necessary in general. We want to build an unstable input-free system, such that, on a subset of state space we have $P_\alpha^t(x) \leq$

$\beta(|\xi|, t), \forall t \geq 0$. Consider the following system evolving in $\mathbb{R}^2$:

$$\begin{aligned} \dot{x} &= x^2 - y^2 \\ \dot{y} &= 2xy. \end{aligned} \tag{37}$$

This is a well known system in control literature, [14]. Trajectories are circular homoclinic orbits. Identifying the complex plane with $\mathbb{R}^2$, through the standard change of variables $z = x + jy$ yields for system (37) the equation $\dot{z} = z^2$. It is therefore possible to integrate the system; the generic solution with initial state $z(0)$ is as follows:

$$z(t) = \frac{z(0)}{1 - z(0)t}, \tag{38}$$

which in terms of the original coordinates reads:

$$\begin{aligned} x(t) &= \frac{x(0) - [x(0)^2 + y(0)^2]t}{1 - 2x(0)t + t^2[x(0)^2 + y(0)^2]} \\ y(t) &= \frac{y(0)}{1 - 2x(0)t + t^2[x(0)^2 + y(0)^2]}. \end{aligned} \tag{39}$$

Equation (37) defines the vector field of interest inside the closed circle $\mathcal{C} := \{(x, y) : x^2 + (y-1)^2 \leq 1\}$. Outside consider the system that in polar coordinates with respect to the point $(x = 0, y = 1)$ reads:

$$\begin{aligned} \dot{r} &= (-r + 1)\,\omega(r, \theta) \\ \dot{\theta} &= \omega(r, \theta). \end{aligned} \tag{40}$$

If $\omega(r, \theta) > 0$, trajectories of (40) are spirals that converge exponentially to the circle $\mathcal{C}$. In order for the two vector fields to patch continuously we must take care of choosing $\omega(r, \theta)$ in a suitable way, so that the vector fields coincide over $\partial\mathcal{C}$. Letting $\omega(r, \theta) = x^2 + y^2$, in Cartesian coordinates, yields

$$\begin{aligned} \dot{x} &= [x^2 + y^2]\left[-x - y + 1 + \frac{x}{\sqrt{x^2 + (y-1)^2}}\right] \\ \dot{y} &= [x^2 + y^2]\left[-y + 1 + x + \frac{y-1}{\sqrt{x^2 + (y-1)^2}}\right]. \end{aligned} \tag{41}$$

It is straightforward to verify that equation (41) and (37) match over the boundary of $\mathcal{C}$. The patched vector field looks like in figure 4.

Exploiting (39) explicit computation of the energy is possible,

$$\int_0^t [x^2(s) + y^2(s)]\,ds = \frac{x(0)^2 + y(0)^2}{y(0)} \left[\operatorname{atn}\left(\frac{(x(0)^2 + y(0)^2)t - x(0)}{y(0)}\right) + \operatorname{atn}\left(\frac{x(0)}{y(0)}\right)\right] \tag{42}$$

for all $(x(0), y(0)) \in \mathcal{C}$. We show next that inside $\mathcal{C}$ the power can be bounded from above by a $\mathcal{KL}$ function. Consider the two cases $y(0)^2 \leq |x(0)|^3$ and $y(0)^2 > |x(0)|^3$ separately. For $|x(0)|^3 < y(0)^2$, and recalling that $x(0)^2 + y(0)^2 \leq 2y(0)$, equation (42) yields:

$$\begin{aligned} \frac{\int_0^t [x^2(s) + y^2(s)]\,ds}{t} &\leq \frac{y(0)^{4/3} + y(0)^2}{t\,y(0)} \left[\operatorname{atn}\left(\frac{(x(0)^2 + y(0)^2)t - x(0)}{y(0)}\right) + \operatorname{atn}\left(\frac{x(0)}{y(0)}\right)\right] \\ &\leq \frac{y(0)^{1/3} + y(0)}{t} \left[\operatorname{atn}\left(2t - \frac{x(0)}{y(0)}\right) - \operatorname{atn}\left(-\frac{x(0)}{y(0)}\right)\right] \end{aligned} \tag{43}$$

We let $\gamma \in \mathcal{K}_\infty$ be $\gamma(r) = r^{1/3} + r$. Consider the function $g : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined as:

$$g(t,r) = \frac{\text{atn}(2t-r) - \text{atn}(-r)}{t}. \tag{44}$$

Clearly $g(t,r) \leq [(2t-r) - (-r)]/t = 2$. Moreover,

$$\begin{array}{rcll} \lim_{r\to\pm\infty} g(t,r) & = & 0 & \forall \text{ fixed } t \\ \lim_{t\to+\infty} g(t,r) & = & 0 & \forall \text{ fixed } r \end{array} \tag{45}$$

Hence, there exists a function of class $\mathcal{LL}$, λ, such that $|g(t,r)| \leq \lambda(t,|r|)$. By virtue of (43) then

$$\frac{\int_0^t [x^2(s) + y^2(s)]\, ds}{t} \leq \gamma\left(\sqrt{x(0)^2 + y(0)^2}\right) \lambda\left(t, \frac{|x(0)|}{y(0)}\right) \leq \gamma\left(\sqrt{x(0)^2 + y(0)^2}\right) \lambda\left(t, 0\right) \tag{46}$$

If instead $y(0)^2 \leq |x(0)|^3$, equality (42) yields

$$\begin{aligned} \frac{\int_0^t x^2(s) + y^2(s)\, ds}{t} &\leq \frac{2}{t}\left[\text{atn}\left(\frac{(x(0)^2+y(0)^2)t - x(0)}{y(0)}\right) + \text{atn}\left(\frac{x(0)}{y(0)}\right)\right] \\ &\leq \frac{2}{t}\left[\text{atn}\left(2t - \frac{x(0)}{y(0)}\right) + \text{atn}\left(\frac{x(0)}{y(0)}\right)\right] \\ &\leq 2\lambda\left(t, \frac{|x(0)|}{y(0)}\right) \leq 2\lambda\left(t, \frac{1}{y(0)^{1/3}}\right) \leq 2\lambda\left(t, \frac{1}{\sqrt{x(0)^2+y(0)^2}^{1/3}}\right). \end{aligned} \tag{47}$$

Hence, if we define $\alpha(s) = s^2$ and the $\mathcal{KL}$ function $\beta(r,t) := \max\{\gamma(r)\lambda(t,0), 2\lambda(t, 1/r^{1/3})\}$, then we can write that $\xi \in \mathcal{C}$ implies $P_\alpha^t(x) \leq \beta(|\xi|, t)$, $\forall t \geq 0$. It is hard to show that a similar estimate holds for all $\xi \in \mathbb{R}^2$ since we do not have an explicit expression for energy of trajectories outside the circle $\mathcal{C}$. However, numerical simulations of (41), augmented with the equation $\dot{e} = x^2 + y^2$, which was used in order to compute energy, clearly showed that a $\mathcal{KL}$ bound is enforced also for $\xi \notin \mathcal{C}$ (Fig. 4 shows power as a function of time for initial conditions in a small neighborhood of the origin).

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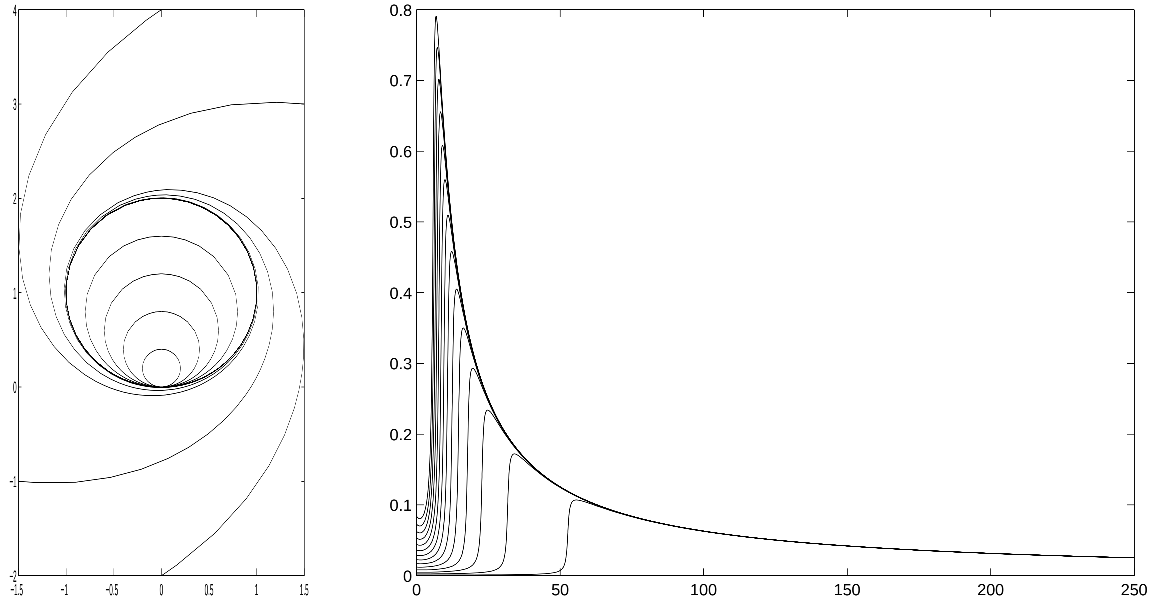

4
3
2
1
0
-1
-2
-1.5
-1
-0.5
0
0.5
1
1.5
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
50
100
150
200
250

Caption page:

Figure 1: An example of unstable vector field with $\mathcal{KL}$ bounded power.