Changing supply functions in input to state stable systems: the discrete-time case

D.Nešić*
Department of Electrical and Electronic Engineering,
The University of Melbourne, Parkville, 3052, Victoria, Australia

A.R.Teel†
CCEC, Electrical and Computer Engineering Department,
University of California, Santa Barbara, CA, 93106-9560, USA

February 2, 2001

Abstract

We characterize possible supply rates for input-to-state stable discrete-time systems and provide results that allow some freedom in modifying the supply rates. In particular, we show that the results reported in [7] for continuous-time systems are achievable for discrete-time systems.

Keywords: discrete-time, input-to-state stability.

1 Introduction

The notion of input-to-state stability (ISS) provides a very useful framework for $\mathcal{L}_\infty$ stability analysis of nonlinear systems as illustrated by its numerous applications to important nonlinear control problems (see, for instance, the paper where ISS was introduced [6], Sections 10.4, 10.5, 10.6 and 11.4 in [4] and Sections 2.4, 5.2, 5.3, 6.1, 6.7, 9.1 and 9.8 in [5]).

A very useful application of ISS is in investigation of stability of cascaded systems [4, 5] where it can be shown that a series connection of two ISS systems is ISS. Cascaded systems often arise in recursive controller designs, such as backstepping [4, 5]. In this context, it is often important to determine a Lyapunov function for a composite system from Lyapunov functions for its subsystems. A particularly easy choice is when one is able to use the sum of the Lyapunov functions for subsystems as a Lyapunov function for the overall system. While this is not possible to do always, it may be possible to change the supply rates (by modifying the Lyapunov functions for subsystems) in order to achieve this. Results on changing supply rates for continuous-time systems were first considered

*Research supported by the Australian Research Council under the Large Grants Scheme.
†Research supported in part by the AFOSR under grant F49620-00-1-0106.

in [7]. These results are very useful since they provide a construction which shows how Lyapunov functions can be modified to achieve any feasible supply pair (see also Section 10.5 of [4]). Another case when changing supply rates may be useful is when one wants to use an input-output dissipation inequality with a detectability condition to conclude about ISS (see, for example [1]).

This paper is concerned with proving the discrete-time counterpart of the results in [7]. In particular, we show that the results reported in [7] for continuous-time systems are achievable for discrete-time systems. These results have the same important utility as the continuous-time results. The proof technique is quite similar to that used in the continuous-time case with the notable exception being in the proof of Lemma 1 where a judicious use of the mean value theorem is required.

2 Main results

$\mathcal{SN}$ denotes the class of all smooth nondecreasing functions $q : [0,\infty) \to [0,\infty)$, which satisfy $q(t) > 0$ for $t > 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}_\infty$ ($\gamma \in \mathcal{K}_\infty$) if it is continuous, strictly increasing, zero at zero and unbounded. Functions of class-$\mathcal{K}_\infty$ are invertible. Given any two scalar functions α_1, α_2, we use the notation $\alpha_1 \circ \alpha_2(s)$ and $\alpha_1(s) \cdot \alpha_2(s)$ to denote, respectively, the composition (when it makes sense) and product of the functions.

We consider a parameterized family of discrete-time systems:

$$x_{k+1} = F_T(x_k, u_k) \; , \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ are, respectively, the state and input of the system and $T > 0$ is a parameter (perhaps the sampling period). We are motivated to consider the parameterized systems (1) as they arise when an approximate discrete-time model is used to design a digital controller for a nonlinear sampled-data system [3]. Our results apply directly to non-parameterized systems $x_{k+1} = F(x_k, u_k)$ by fixing $T > 0$ (without loss of generality setting $T = 1$) in (1) and in (2) and (3) to follow.

Following the lead of [6] and further motivated by results in [3] and [2], the system (1) is said to be ISS if there exist $\alpha_1, \alpha_2, \alpha, \gamma \in \mathcal{K}_\infty$, $T^* > 0$ and for all $T \in (0, T^*)$ there exists a smooth function $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that:

$$\alpha_1(|x|) \;\leq\; V_T(x) \;\leq\; \alpha_2(|x|) \tag{2}$$

$$V_T(F_T(x,u)) - V_T(x) \;\leq\; T\gamma(|u|) - T\alpha(|x|) \; , \tag{3}$$

for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$.

Definition 1 *A pair of class-$\mathcal{K}_\infty$ functions (γ, α) is a supply pair for system (1) if there is some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $T^* > 0$ and for all $T \in (0, T^*)$ a smooth function V_T so that (2) and (3) hold.* ■

As shown in [7], it is of interest to characterize all possible supply pairs. The following theorems specify the constraints under which we can achieve certain supply pairs for the system (1). The proofs in the next section contain the Lyapunov function modifications to achieve the results stated in the theorems.

Theorem 1 *Assume that (γ, α) is a supply pair. Suppose that $\tilde{\gamma}$ is a $\mathcal{K}_\infty$ function such that $\gamma(r) = O[\tilde{\gamma}(r)]$ as $r \to \infty$. Then, there exists $\tilde{\alpha} \in \mathcal{K}_\infty$ such that $(\tilde{\gamma}, \tilde{\alpha})$ is a supply pair.* ■

Theorem 2 *Assume that (γ, α) is a supply pair. Suppose that $\tilde{\alpha}$ is a $\mathcal{K}_\infty$ function such that $\alpha(s) = O[\tilde{\alpha}(s)]$ as $s \to 0^+$. Then, there exists $\tilde{\gamma} \in \mathcal{K}_\infty$ such that $(\tilde{\gamma}, \tilde{\alpha})$ is a supply pair.* ■

A direct consequence of Theorems 1 and 2 is the following result whose proof is identical to the proof of Corollary 1 in [7].

Corollary 1 *Let two ISS systems be given. Then there are $\mathcal{K}_\infty$ functions $\tilde{\gamma}, \tilde{\alpha}_1$ and $\tilde{\alpha}_2$ such that $[\frac{1}{2}\tilde{\alpha}_2, \tilde{\alpha}_1]$ is a supply pair for the first system and $[\tilde{\gamma}, \tilde{\alpha}_2]$ is a supply pair for the second system.* ■

Corollary 1 is now used to give a Lyapunov proof of ISS for cascade discrete-time ISS systems:

$$x_{k+1} = f_T(x_k, z_k) \tag{4}$$

$$z_{k+1} = g_T(z_k, u_k)\ . \tag{5}$$

Using Corollary 1, we conclude that there exist Lyapunov functions $U_T(x)$ and $W_T(z)$ so that $U_T(f_T(x,z)) - U_T(x) \le -T\tilde{\alpha}_1(|x|) + \frac{T}{2}\tilde{\alpha}_2(|z|)$ and $W_T(g_T(z,u)) - W_T(z) \le -T\tilde{\alpha}_2(|z|) + T\tilde{\gamma}(|u|)$. Hence, in this case the Lyapunov function for the overall system can be taken to be $V_T(x,z) = U_T(x) + W_T(z)$ and its first difference satisfies $V_T(f_T(x,z), g_T(z,u)) - V_T(x,z) \le -T\tilde{\alpha}_1(|x|) - \frac{T}{2}\tilde{\alpha}_2(|z|) + T\tilde{\gamma}(|u|)$ and the overall system is ISS since $(\tilde{\gamma}, \tilde{\alpha})$ is a supply pair for the overall system, where $\tilde{\alpha} := \min\{\tilde{\alpha}_1, \frac{1}{2}\tilde{\alpha}_2\}$.

3 Proofs

In this section we provide proofs of all main results. The proofs are similar to the proofs presented in [7]. The Lyapunov function modification that we use is the same as the one used in [7]. Given an arbitrary $q \in \mathcal{SN}$, we introduce:

$$\rho(s) := \int_0^s q(\tau) d\tau \ , \tag{6}$$

where it is easy to see that $\rho \in \mathcal{K}_\infty$ and ρ is smooth. Then we consider a new Lyapunov function $W_T(x) := \rho(V_T(x))$. The next lemma is the discrete-time counterpart of results in [7]. The proof of this result is the main location where the discrete-time result requires techniques different than those used in the continuous-time case.

Lemma 1 *Let α_1, α_2, α and γ all belong to class-$\mathcal{K}_\infty$, let $T^* > 0$ and suppose that for all x and u and all $T \in (0, T^*)$ we have that (2) and (3) hold. Let $\rho \in \mathcal{K}_\infty$ be such that $q(s) := \frac{d\rho}{ds}(s)$ is well-defined and nondecreasing. Then we have*

$$\rho(V_T(F_T(x,u))) - \rho(V_T(x)) \leq \frac{T}{2}\left[2 \cdot q \circ \theta_T(|u|) \cdot \gamma(|u|) - q \circ \frac{1}{2}\alpha_1(|x|) \cdot \alpha(|x|)\right] \tag{7}$$

for all x and u and all $T \in (0, T^*)$, where

$$\theta_T(s) := \alpha_2 \circ \alpha^{-1} \circ 2\gamma(s) + T\gamma(s) \ . \tag{8}$$

■

Proof. In what follows we will use the shorthand notation $V_T(F_T) := V_T(F_T(x,u))$ and $V_T := V_T(x)$. We also will rely on the mean value theorem and the fact that $\frac{d\rho}{ds}(\cdot)$, i.e., $q(\cdot)$, is nondecreasing. These two items yield

$$\rho(a) - \rho(b) \leq q(a)[a - b] \qquad \forall a \geq 0 \ , b \geq 0 \ . \tag{9}$$

Finally, we will use

$$\begin{aligned} V_T &\geq \max\{\alpha_1(|x|), T\alpha(|x|)\} && (10) \\ V_T(F_T) &\leq \alpha_2(|x|) + T\gamma(|u|) && (11) \end{aligned}$$

which both come from the combination of (2) and (3).

Let $T \in (0, T^*)$. We cover all possible values of x and u with three cases:

<u>Case 1:</u> $V_T(F_T) \leq \frac{1}{2}V_T$.

Using, in succession, $\rho \in \mathcal{K}_\infty$, inequality (9), $q(\cdot)$ nondecreasing, and inequality (10), we have

$$\begin{aligned} \rho(V_T(F_T)) - \rho(V_T) &\leq \rho\left(\tfrac{1}{2}V_T\right) - \rho(V_T) \\ &\leq q\left(\tfrac{1}{2}V_T\right)\left[-\tfrac{1}{2}V_T\right] \\ &\leq -\tfrac{T}{2} q \circ \tfrac{1}{2}\alpha_1(|x|) \cdot \alpha(|x|) \ . \end{aligned} \tag{12}$$

So the bound (7) holds in this case.

Case 2: $V_T(F_T) \geq \frac{1}{2}V_T$ and $\gamma(|u|) \leq \frac{1}{2}\alpha(|x|)$,

Using, in succession, inequalities (9) and (3), the condition $\gamma(|u|) \leq \frac{1}{2}\alpha(|x|)$, $q(\cdot)$ nondecreasing, and (10), we have

$$\begin{aligned} \rho(V_T(F_T)) - \rho(V_T) &\leq q(V_T(F_T))T\left[\gamma(|u|) - \alpha(|x|)\right] \\ &\leq q(V_T(F_T))\left[-\tfrac{T}{2}\alpha(|x|)\right] \\ &\leq q\left(\tfrac{1}{2}V_T\right)\left[-\tfrac{T}{2}\alpha(|x|)\right] \\ &\leq -\tfrac{T}{2}q \circ \tfrac{1}{2}\alpha_1(|x|) \cdot \alpha(|x|) \ . \end{aligned} \tag{13}$$

So the bound (7) holds in this case.

Case 3: $V_T(F_T) \geq \frac{1}{2}V_T$ and $\gamma(|u|) \geq \frac{1}{2}\alpha(|x|)$.

Using, in succession, inequalities (9) and (3), $q(\cdot)$ nondecreasing, and inequalities (10) and (11), we have

$$\begin{aligned} \rho(V_T(F_T)) - \rho(V_T) &\leq q(V_T(F_T))T\left[\gamma(|u|) - \alpha(|x|)\right] \\ &\leq T \cdot q \circ \theta_T(|u|) \cdot \gamma(|u|) - Tq\left(\tfrac{1}{2}V_T\right) \cdot \alpha(|x|) \\ &\leq T \cdot q \circ \theta_T(|u|) \cdot \gamma(|u|) - \tfrac{T}{2}q \circ \tfrac{1}{2}\alpha_1(|x|) \cdot \alpha(|x|) \ . \end{aligned} \tag{14}$$

So the bound (7) holds in this case. ■

In order to prove Theorems 1 and 2 we need the following two results which follow directly from the results proved in [7]:

Lemma 2 *Assume that the functions $\gamma, \tilde{\gamma}, \theta \in \mathcal{K}_\infty$ are given and $\gamma, \tilde{\gamma}$ are such that $\gamma(r) = O[\tilde{\gamma}(r)]$ as $r \to +\infty$. Then there exists a function $q \in \mathcal{SN}$ so that $q \circ \theta(r) \cdot \gamma(r) \leq \tilde{\gamma}(r) \quad \forall r \geq 0$.* ■

Lemma 3 *Assume that the functions $\alpha, \alpha_1, \tilde{\alpha}, \in \mathcal{K}_\infty$ are given and $\alpha, \tilde{\alpha}$ are such that $\alpha(s) = O[\tilde{\alpha}(s)]$ as $s \to 0^+$. Then there exists a function $q \in \mathcal{SN}$ so that $\frac{1}{2}q \circ \frac{1}{2}\alpha_1(s) \cdot \alpha(s) \geq \tilde{\alpha}(s) \quad \forall s \geq 0$.* ■

Proof of Theorem 1: Let the supply pair (α, γ) and $\tilde{\gamma}$ be given. Using Lemma 1 we can see that we need to show that there exists $\tilde{\alpha}$ and $\rho \in \mathcal{K}_\infty$, with $q = \frac{d\rho}{ds}(s) \in \mathcal{SN}$, so that:

$$Tq \circ \theta_T(r) \cdot \gamma(r) - T\frac{1}{2}q \circ \frac{1}{2}\alpha_1(s) \cdot \alpha(s) \leq T\tilde{\gamma}(r) - T\tilde{\alpha}(s) \qquad \forall r, s \geq 0 \ T \in (0, T^*). \tag{15}$$

Define $\theta = \theta_{T^*}$ and note that $q \circ \theta_T(r) \cdot \gamma(r) \leq q \circ \theta(r) \cdot \gamma(r)$. Using Lemma 2 we conclude that we can always find ρ with the desired properties so that (15) holds with $\tilde{\alpha}(s) := \frac{1}{2}q \circ \frac{1}{2}\alpha_1(s) \cdot \alpha(s)$ and $\tilde{\alpha} \in \mathcal{K}_\infty$ since $q \in \mathcal{SN}$ and $\alpha, \alpha_1 \in \mathcal{K}_\infty$. ■

Proof of Theorem 2: Let the supply pair (α, γ) and $\tilde{\alpha}$ be given. Using Lemma 1 we can see that we need to show that there exists $\tilde{\gamma}$ and $\rho \in \mathcal{K}_\infty$, with $q = \frac{d\rho}{ds}(s) \in \mathcal{SN}$, so that (15) holds. Using Lemma 3 we conclude that we can always find ρ with the desired properties so that (15) holds with $\tilde{\gamma}(r) := q \circ \theta_{T^*}(r) \cdot \gamma(r)$ and $\tilde{\gamma} \in \mathcal{K}_\infty$ since $q \in \mathcal{SN}$ and $\theta_{T^*}, \gamma \in \mathcal{K}_\infty$. ■

References

[1] D. Angeli, "Input-to-state stability of PD controlled robotic systems", *Automatica*, vol. 35, pp. 1285-1290, 1999.

[2] Z. P. Jiang, E. D. Sontag and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems", in *Proc. 14th IFAC World Congress*, Beijing, China, 1999, Vol. E, pp. 277-282.

[3] D. Nešić, A.R. Teel and P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations", *Sys. Contr. Lett.*, vol. 38, pp. 259-270, 1999.

[4] A. Isidori, *Nonlinear control systems II.* London: Springer Verlag, 1999.

[5] M. Krstić, I. Kanellakopoulos and P. V. Kokotović, *Nonlinear and adaptive control design.* New York: J. Wiley & Sons, 1995.

[6] E. D. Sontag, "Smooth stabilization implies coprime factorization", *IEEE Trans. Automat. Contr.*, vol. 34, pp. 435-443, 1989.

[7] E. D. Sontag and A. R. Teel, "Changing supply functions in input/state stable systems", *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1476-1478, 1995.