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Minimum Phase Properties for Input Nonaffine Nonlinear Systems

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***Abstract*—For input nonaffine nonlinear control systems, the minimum phase property of the system in general depends on the control law. Switching or discontinuous controllers may offer advantages in this context. In particular, there may not exist a continuous control law which would keep the output identically equal to zero and for which the zero output constrained dynamics are locally stable, whereas a discontinuous controller which achieves this exists. For single-input/single-output input nonaffine nonlinear systems we give sufficient conditions for existence and present a method for the design of discontinuous switching controllers which yield locally stable zero dynamics.**

***Index Terms*— Input nonaffine systems, minimum phase, nonlinear, switching control.**

I. Introduction

The notion of minimum phase (MP) is of great importance for a number of nonlinear control theoretic questions. Loosely speaking, a nonlinear system is termed MP if it has locally stable zero output constrained dynamics (zero dynamics), which are obtained when the output of the system is kept identically equal to zero [9]. The notion of MP has found applications in a number of nonlinear problems, such as input–output linearization [9], output dead-beat control, stabilizability, output tracking [1], [4], [9], etc.

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An important difference between affine and nonaffine nonlinear systems is that in the latter case the zero dynamics are not always well defined. Indeed, the algorithm of Byrnes and Ho in [1] typically produces a finite number of continuous control laws, all of which have the property of ensuring that the output remains identically at zero. It is obvious that different choices of the continuous control law may yield either stable or unstable zero dynamics.

Moreover, by partitioning the state space and applying a (different) continuous control law at each subset, we may design an infinite number of discontinuous controllers which keep the output of the system at zero. In a conference version of this paper [8] we presented an example of a nonaffine system for which there does not exist a continuous control law which yields stable zero dynamics but a discontinuous control law which achieves stability of zero dynamics does exist. The example motivates the consideration of discontinuous (or switched) controllers in the investigation of MP property for nonaffine nonlinear systems. In this paper we use ideas from [11] where switched linear controllers were investigated to construct locally stable zero dynamics. By considering discontinuous controllers, we enlarge the class of nonaffine nonlinear systems that can be termed MP.

This paper is organized as follows. In Section II we present mathematical preliminaries, define the class of systems and the problem, and provide some motivation. In Section III we introduce the notion of switched controllers. In Section IV we propose an algorithm to test for MP nonaffine nonlinear systems. Finally, we summarize our results in the last section.

II. Preliminaries

In this paper, we consider the following class of nonaffine nonlinear systems:

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ and $f, h \in C^\infty$. A good discussion on the motivation for considering the zero dynamics of (1) can be found in [1]. One way of investigating (1) is to introduce an integrator at the plant input [9], which transforms it into a control affine system. However, the new augmented system may have some undesirable properties as outlined in [1].

The Euclidean norm of a vector x is denoted as $|x|$ and the corresponding induced norm of a matrix P as $\|P\|$. We denote an open ball with radius d and centered at $x^* \in \mathbb{R}^n$ as $B_d(x^*) \subset \mathbb{R}^n$.

Definition 1: A state $x^* \in \mathbb{R}^n$ is termed an equilibrium for the system (1) if there exists at least one $u^* \in \mathbb{R}$ such that

$$f(x^*, u^*) = 0, \qquad h(x^*) = 0. \tag{2}$$

Without loss of generality it can be assumed that the origin, $x^* = 0$, is an equilibrium.

Definition 2 [1]: A closed set S, $S \subset \mathbb{R}^n$, is said to be a viable set of the system (1), if there exists a (continuous[1]) feedback control law $u = u(x)$ defined on S such that for any $x_0 \in S$ there exist a number $T > 0$ (it may be that $T = \infty$) and a unique solution $x(t, x_0, u(\cdot))$ of the system $\dot{x} = f(x, u(x))$ that satisfies

$$x(t, x_0, u(\cdot)) \in S, \qquad \forall t: 0 \le t < T. \tag{3}$$

[1] Definitions 2 and 4 are taken from [1], where continuity of the output zeroing (OZ) controllers is required. We do not use this assumption in the sequel. However, in the cases when we use results of [1] the continuity of control laws in the definitions is assumed.

Definition 3: Suppose that $S_O \subset h^{-1}(0)$, $S_O \neq \emptyset$ is a viable set. Any control law, $u = u(x)$, which achieves (3) for S_O is called an *output zeroing (OZ) controller*.

Definition 4 [1]: Consider the following sets:

$$\begin{aligned} L_0 &= h^{-1}(0) \\ S &= \{S_\alpha \subset L_0\colon S_\alpha \text{ viable}\} \\ M^* &= \cup_{S_\alpha \in S} S_\alpha. \end{aligned} \tag{4}$$

Consider an equilibrium x^*. If $M^* \neq \emptyset$ and $x^* \in M^*$, then there exist zero dynamics for the system (1).

Definition 5: Suppose that there exist zero dynamics for (1) according to Definition 4. Then the zero dynamics are stabilizable at an equilibrium $x^* \in M^*$ if there exists an OZ control law $u = u(x)$ with the following properties.

1) $\forall \varepsilon > 0, \exists \delta > 0$ such that if $x_0 \in M^* \cap B_\delta(x^*)$ then $\forall t$ it follows that $x(t, x_0, u(x)) \in M^* \cap B_\varepsilon(x^*)$.
2) $\exists \Delta > 0$ such that if $x_0 \in M^* \cap B_\Delta(x^*)$ then $\lim_{t\to\infty} |x(t, x_0, u(x)) - x^*| = 0$ and $\forall t$ we have that $x(t, x_0, u(x)) \in M^*$.

Any control law which satisfies the above given conditions is referred to as an *minimum phase (MP) controller*.

If in Definition 5 we let $M^* = \mathbb{R}^n$, we obtain the usual (unconstrained) $\varepsilon - \delta$ notions of stability and asymptotic stability which we also use in the sequel.

Definition 6: System (1) is termed MP at x^* if its zero dynamics are stabilizable at x^*.

Assumption 1: There exists zero dynamics and an *a priori* known equilibrium point $x^* \in M^*$ at which we wish to investigate the MP property.

Our definition of MP differs from the usual definitions found in [4] and [9] since we do not require continuity of OZ controllers.

In order to analyze the MP property it is useful to transform (1) into a normal form [9]. Suppose that the system (1) has a relative degree $r < n$ at an equilibrium x^* [9, p. 417]. Then, there exists a locally invertible coordinate transformation $(\eta^T \xi^T)^T = \Phi(x)$ such that the system (1) is transformed into the following form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \cdots \\ \dot{z}_r &= g(\eta, \xi, u) \\ \dot{\xi} &= F(\eta, \xi, u) \\ y &= z_1 \end{aligned} \tag{5}$$

where $\eta = (z_1 \cdots z_r)^T$ and $\xi \in \mathbb{R}^{n-r}$.

Example 1: To motivate our work, consider the nonaffine system in normal form

$$\begin{aligned} \dot{x}_1 &= (x_1 + x_2^2 - u)(x_1^3 - x_2^2 - u) \\ \dot{x}_2 &= u \\ y &= x_1. \end{aligned} \tag{6}$$

It is easy to see that the following controllers (with $x_1(0) = 0$) $u_1(x) = x_2^2$; $u_2(x) = -x_2^2$ are both OZ controllers for the system (6). However, neither of them is an MP controller. Notice, however that if one uses the following switching strategy:

$$u^*(x) = \begin{cases} x_2^2, & x_2 \le 0 \\ -x_2^2, & x_2 > 0 \end{cases}$$

this globally stabilizes the zero dynamics $\dot{x}_2 = u^*(x)$. The phenomenon we described is at the heart of the present paper. Indeed, we propose an MP test based on a switching strategy which can be used to locally stabilize zero dynamics of nonaffine nonlinear systems (1). In this case, the switching MP controller is continuous in x. However, for higher dimensional zero dynamics one usually obtains discontinuous in x controllers if the same technique is used (see [8]). Hence, the structure of the system (1) may be such that *it is necessary* to use hybrid control ideas (design of discontinuous controllers) in order to fully understand MP properties of the system.

We note that this paper is concerned with the zero dynamics only and we do not consider the relationship of MP and the stabilization problem.

III. Switched Controllers Design

In order to tackle the problem of stabilizability of the zero dynamics of system (1), we use an approach from the design of hybrid (switched) controllers. We start our investigation by considering linear switched controllers. Consider the system

$$\dot{x}(t) = Ax + Bu, \qquad B \neq 0 \tag{7}$$

where $x(t) \in \mathbb{R}^n$ is the *state* and $u(t) \in \mathbb{R}$ is the *control input.*

1) Controlled Switching: Suppose we have a collection of N linear controllers

$$u_1(t) = K_1 x(t),\ u_2(t) = K_2 x(t), \cdots, u_k(t) = K_N x(t) \tag{8}$$

where $\mathcal{L} = \{K_1, \cdots, K_N\}$ is the set of gain matrices, with $K_j \neq K_i, \forall i, j$. The controllers in (8) are called basic controllers. We will consider the following class of state feedback controllers, introduced in [11]. Let $I(x(\cdot))$ be a function which maps from the set of state measurements $\{x(\cdot)\}$ to the set $\{1, 2, \cdots, N\}$. We consider the following state feedback controller:

$$u(t) = K_i x(t) \qquad \forall t \in [0, \infty) \text{ where } i \triangleq I(x(\cdot)). \tag{9}$$

Hence our control strategy is a rule for switching from one basic controller to another. $I(x(\cdot))$ is called a switching function.

Definition 7: The plant in (7) is stabilizable via controlled switching with the basic controllers (8) if there exists a state feedback controller of the form (9) such that the origin of the closed loop system (7)–(9) is asymptotically stable.

We use the usual $\varepsilon - \delta$ stability definition—see Definition 5 and the remark below the definition.

Definition 8: Let $Z_1 = Z_1^T, Z_2 = Z_2^T, \cdots, Z_N = Z_N^T$ be given real matrices. The collection $\{Z_1, Z_2, \cdots, Z_N\}$ is said to be complete if for any $x \in \mathbb{R}^n$ there exists at least one $i \in \{1, 2, \cdots, N\}$ such that $x^T Z_i x \le 0$. A collection $\{Z_1, Z_2, \cdots, Z_N\}$ is said to be strictly complete if for any $x \in \mathbb{R}^n/\{0\}$, there exists at least one $i \in \{1, 2, \cdots, N\}$ such that $x^T Z_i x < 0$.

2) Remark: It can be shown that if there exist constants $\tau_1 \ge 0, \tau_2 \ge 0, \cdots, \tau_N \ge 0$, not all zero and such that

$$\tau_1 Z_1 + \tau_2 Z_2 + \cdots + \tau_N Z_N \le 0 \qquad (<0) \tag{10}$$

then the collection $\{Z_1, Z_2, \cdots, Z_N\}$ is complete (strictly complete). This condition is necessary and sufficient for $N \le 2$ and only sufficient for $N > 2$ (see [11] and references therein). Consider the plant (7) as a switched linear system where controller switching occurs when the plant reaches some specified point in the state space. The following theorem was presented in [11].

Theorem 1: Consider the system (7) with the basic controllers (8). If there exists a real matrix $P = P^T > 0$ such that the set of matrices

$$Z_i \triangleq (A + BK_i)^T P + P(A + BK_i), \qquad i = \{1, \cdots, N\} \tag{11}$$

is strictly complete, then the system (7) is stabilizable using controlled switching with basic controllers (8).

Below we present an alternative proof to that presented in [11]. The proof is constructive in that it specifies the switching controller which achieves stability. A controller of the following form is used in the proof:

$$\begin{aligned} I^*(x) &= \arg\min_{i\in\{1,2,\cdots,N\}} x^T[(A+BK_i)^TP \\ &\qquad + P(A+BK_i)]x \\ u^*(x) &= K_{I^*(x)}x. \end{aligned} \tag{12}$$

Notice that the above controller specifies a switching function $I^*(x)$. We say that the switching function $I^*(x)$ is single-valued for a state x if $I^*(x)$ is a singleton. It is multivalued at x if $I^*(x) = \{k_1, \cdots, k_l\}$, $k_i \in \{1, 2, \cdots, N\}$, $l \le N$. In order to precisely specify the properties of the closed loop system (7) and (12), the following lemma is useful.

Lemma 1: Let $I^*(x)$ be the switching function defined by (12). $I^*(x)$ is multivalued on a set $S_w \subset \mathbb{R}^n$, which is a nondense subset of the state space.

Proof: The switching function $I^*(x)$ is multivalued only for those x for which there exist $j, k \in \{1, \cdots, N\}$, $j \ne k$ such that

$$\begin{aligned} &x^T[(A+BK_j)^TP + P(A+BK_j)]x \\ &\quad = x^T[(A+BK_k)^TP + P(A+BK_k)]x. \end{aligned}$$

This implies that the set S_w is the union of sets of the form

$$2x^TPB(K_j-K_k)x = 0, \qquad j, k \in \{1, 2, \cdots, N\},\ j \ne k.$$

The set of points for which this relationship holds belongs to a real variety defined by the above given equation. This variety is dense in $\mathbb{R}^n$ if and only if the polynomial which describes it is identically equal to zero [2]. Since by definition P is positive definite and $K_j \ne K_k$, $j \ne k$, this can happen only if $B = 0$ (the zero vector), which contradicts the definition of the system (7). Q.E.D.

Therefore, we can conclude that Lemma 1 tells us that in general the closed-loop system (7) and (12) has a discontinuous right-hand side and the theory developed in [3] can be used to analyze its stability properties. We state several definitions and results from [3] which are used in the sequel. Consider the system

$$\dot{x} = f(x) \tag{13}$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ is measurable and essentially locally bounded.

Definition 9 (Filippov): A vector function $x(\cdot)$ is called a solution of (13) on $[t_0, t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$

$$\dot{x} \in C[f](x) \tag{14}$$

where

$$C[f](x) \equiv \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{co} f(B(x,\delta) - N) \tag{15}$$

where $\overline{co} f(x)$ denotes the convex closure of f at x. Also, $\bigcap_{\mu N=0}$ denotes the intersection over all sets N of Lebesgue measure zero.

Definition 10: The upper derivative of V "along" the differential inclusion (14) is defined as

$$\dot{V}^*(x) = \sup_{y\in C[f](x)} \frac{\partial V(x)}{\partial x} y.$$

Notice that the differential inclusion (14) defines a regular o.d.e. on the region $\mathbb{R}^n - S_w$ and the upper derivative becomes $\dot{V}^* = \dot{V}_i = x^T(A_i^TP + PA_i)x$. The above given definition is used to prove the so-called strong stability properties in the sense of [3], which coincides with our definition of stability.

We now state a theorem which easily follows from [3], [10] for differentiable Lyapunov functions.

Theorem 2: Consider the system (7) with $0 \in C[f](0)$. Suppose that there exists a positive definite and radially unbounded function $V(x)$ and a class $\mathcal{K}$ function $\omega(\cdot)$ with the property

$$\dot{V}^*(x) = \sup_{y\in C[f](x)} \frac{\partial V(x)}{\partial x} y \le -\omega(|x|)$$

then the origin of the system (13) is asymptotically stable. Notice that in [10] discontinuous Lyapunov functions were considered and consequently Dini derivatives were needed. However, with minor modifications and with the Filippov definition of solution [3], one can arrive at the above given statement.

Proof of Theorem 1: The Lyapunov function $V(x) = x^TPx$ that we are using in the proof is positive definite, differentiable, and radially unbounded. System (7) with the controller (12) has solutions for which

$$\dot{x} \in C[Ax + Bu^*(x)].$$

We use the notation

$$\dot{V}_i = 2x^T((A+BK_i)^TP + P(A+BK_i))x. \tag{16}$$

The upper derivative of the function $V(x)$ is

$$\dot{V}^*(x) = \sup_{y\in C[Ax+Bu^*(x)])} \frac{\partial V(x)}{\partial x} y.$$

Consider now the following function:

$$\omega_1(|x|) = -\sup_{y,\,|y|\ge|x|} \dot{V}^*(y).$$

Since $\dot{V}_i(0) = 0, \forall i$, we have that $\omega_1(0) = 0$. The function $\omega_1(|x|)$ is continuous since its discontinuity would imply that some of the functions $\dot{V}_i$ (16) is discontinuous in x, which is not true since $\dot{V}_i$ are quadratic polynomial functions in x. Because of the strict completeness condition and the choice of the controller (12), the function $\omega_1(|x|)$ is strictly positive for any $x \ne 0$. Indeed, if we assume that there exists $\hat{x} \ne 0$ such that $\omega_1(|\hat{x}|) = 0$, we necessarily have that $\dot{V}_i(\hat{x}) = 0, \forall i$, which contradicts the strict completeness condition. $\omega_1(|x|)$ is monotonically nondecreasing, positive and zero only at zero. It is now easy to construct a strictly increasing and continuous (class $\mathcal{K}$ function) $\omega(|x|)$ such that $\omega_1(|x|) \ge \omega(|x|), \forall x$. From the construction of $\omega(|x|)$ it follows that $\dot{V}^*(x) \le -\omega_1(|x|) \le -\omega(|x|)$ and hence the origin of the closed loop system is asymptotically stable. Q.E.D.

We now extend the results of [11] to nonlinear systems.

Theorem 3: Consider a nonlinear system (1) with the basic controllers

$$u_i = K_i(x),\ K_i(0) = 0,\ \forall i = \{1, \cdots, N\} \tag{17}$$

and $f(0, K_i(0)) = 0, \forall i$. Suppose that the vector functions $f(x, K_i(x))$ are at least once differentiable and that

$$\dot{x} = \left.\frac{\partial f(x, K_i(x))}{\partial x}\right|_{x=0} x + g_i(x) = F_ix + g_i(x)$$

where $g_i(x)$ denotes higher order terms satisfying

$$\lim_{|x|\to 0} \frac{|g_i(x)|}{|x|} = 0, \qquad \forall i = \{1, 2, \cdots, N\}. \tag{18}$$

System (1) is locally stabilizable by switching with the basic controllers (17) if there exists a matrix $P = P^T > 0$ and positive numbers $\tau_i \geq 0$, $\sum_i \tau_i > 0$ such that

$$\sum_i \tau_i (F_i^T P + PF_i) = -Q, \qquad Q = Q^T > 0. \tag{19}$$

Proof: The proof is very similar to the standard linearization result [5]. Suppose the conditions of theorem are satisfied. We use the following notation: $\dot{V}_i = 2x^T Pf(x, K_i(x))$. Using ideas in the proof of Theorem 1, it is not difficult to see that if $\forall x \in B_r(0)$, $r > 0$, $\exists i \in \{1, 2, \cdots, N\} \Rightarrow \dot{V}_i < 0$, then the nonlinear system is stabilizable by switching with the basic controllers. A sufficient condition for this to hold is that there exists a set of numbers $t_i \geq 0$, $\sum_i t_i > 0$ such that the following holds: $\mathcal{V}(x) = \sum_{i=1}^N 2t_i x^T Pf(x, K_i(x)) < 0$, $\forall |x| < r$. In other words, the function $\mathcal{V}(x)$ is negative definite. The linearizations F_i produce a set of matrices satisfying condition (19). Consider the matrix P and the values τ_i for which the linearizations F_i yield the condition (19). Consider now the function $\mathcal{V}(x)$

$$\begin{aligned}\mathcal{V}(x) &= \sum_{i=1}^N \tau_i \Big[x^T Pf(x, K_i(x)) + f^T(x, K(x))Px\Big] \\ &= \sum_{i=1}^N \tau_i \Big[x^T (F_i^T P + PF_i)x + 2x^T Pg_i(x)\Big]. \end{aligned} \tag{20}$$

Since (18) holds, we have that for any $\gamma_i > 0$, there exists $r_i > 0$ such that $|g_i(x)| < \gamma_i |x|$, $\forall i$, $\forall x \in B_{r_i}(0)$. We denote $r = \min r_i$. Due to (19), we have $\sum_i \tau_i (F_i^T P + PF_i) = -Q$, $Q = Q^T > 0$. We can write (20) as follows:

$$\mathcal{V}(x) = -x^T Qx + 2\sum_{i=1}^N \tau_i \gamma_i \|P\| \, |x|^2, \qquad \forall x \in B_r(0). \tag{21}$$

Notice that $x^T Qx \geq \lambda_{\min}(Q)|x|^2$, where $\lambda_{\min}(Q)$ is the minimum eigenvalue of Q, which a real positive number. Therefore, we can write $\mathcal{V}(x) < -[\lambda_{\min}(Q) - 2(\sum_i \tau_i \gamma_i) \, \|P\|]|x|^2$, $\forall x \in B_r(0)$. By choosing γ_i so that $2\sum_i \tau_i \gamma_i < (\lambda_{\min}(Q))/\|P\|$, we obtain that the function $\mathcal{V}(x)$ is negative $\forall x \in B_r(0)$, which completes the proof. Q.E.D.

IV. The Minimum Phase Test

In order to analyze the MP property it is useful to transform the system into the normal form (5). Consider a system in the normal form (5). If want to keep the output identically equal to zero, we necessarily have that $\eta(0) = 0$. The set of controllers u must satisfy

$$g(0, \xi, u) = 0. \tag{22}$$

All of these controllers are OZ controllers. Assume that there are N real solutions to the above given equation:

$$\mathcal{U} = \{u_1(\xi), u_2(\xi), \cdots, u_N(\xi)\} \tag{23}$$

which are defined on a neighborhood of the equilibrium $x^* = \Phi^{-1}(\eta^*, \xi^*)$. In order to use the results that we have presented so far to test for MP at x^*, we proceed as follows.

Step 1: Find the set (23). Assume that the set of controllers $\mathcal{U}$ (23) is not empty.

Step 2: Consider the following set of systems obtained from (5) with (23):

$$\dot{\xi} = F(0, \xi, u_i(\xi)), \qquad i = 1, 2, \cdots, N. \tag{24}$$

Step 3: Find all linearizations for zero dynamics which are obtained with admissible candidate controllers:

$$F_i = \frac{\partial F(0, \xi, u_i(\xi))}{\partial \xi}\bigg|_{\xi^*}. \tag{25}$$

Step 4: Check whether the conditions of Theorem 3 are satisfied, that is whether there exist a set of positive numbers $\tau_i \geq 0$, $\sum_i \tau_i > 0$ and matrix $P = P^T > 0$ such that the collection of matrices

$$F_i^T P + PF_i, \qquad i = \{1, 2, \cdots, N\}$$

satisfies the condition (19). If so, (5) is MP at x^*.

Comment 1: We emphasize that we are not advocating the use of discontinuous switching controllers in the cases when a continuous MP controller exists. However, if no continuous controller exists, there may exist a discontinuous switching controller which can stabilize the zero dynamics (see for instance [8]). Hence, our results enlarge the class of systems that can be termed MP. On the other hand, note that even if a continuous MP controller exists but we have more than one OZ controller, by switching we may still obtain faster convergence.

Comment 2: The above given test is a sufficient condition used to check local asymptotic stability of the origin of the zero dynamics. However, for nonlinear systems for which there exists a nonempty subset of the set of OZ controllers $\mathcal{U}_L \subset \mathcal{U}$ such that $u_i(\xi)$ are linear in ξ and also $\dot{\xi} = F(0, \xi, u) = F\xi + gu$, the test can be used to check global asymptotic stability of zero dynamics, if they are well defined. We can see that a real challenge is to consider stabilizability of switched nonlinear systems since this could lead to global results (global asymptotic stability of the zero dynamics). Also, an interesting question is to show a relationship between MP in our sense and stabilizability of the system (5).

Comment 3: The procedure for testing the MP property as outlined in the test verifies the existence of an MP controller based on the linearization. Suppose that such a controller exists and we used the quadratic Lyapunov function $V(\xi) = \xi^T P\xi$ to prove the local stability result based on the linearizations. Denote $\dot{V}_i(\xi) = 2\xi^T PF(0, \xi, u_i(\xi))$ [see (24)]. The controller of the form $u^*(\xi) = u_{I^*(\xi)}(\xi)$ where $I^*(\xi) = arg\ \min_{i \in \{1, 2, \cdots, N\}} \dot{V}_i(\xi)$ may achieve a much larger basin of attraction than that obtained by applying the linear MP controller computed for the linearizations.

Comment 4: Our method can be in principle used for testing MP of multi-input/multi-output (MIMO) systems. However, the strict completeness condition is more difficult to check in this case since we may have that the linearizations of zero dynamics are of the form $F_i x + g_i v_i$, $i = 1, 2, \cdots, N$, where v_i is arbitrary. Even if we choose $v_i = L_i x$, we need to check the strict completeness condition of the form: does there exist L_i and P such that $(F_i + g_i L_i)^T P + P(F_i + g_i L_i)$ is strictly complete, which is much harder to check, since L_i introduce new parameters. Hence, testing MP for MIMO nonaffine systems motivates the investigation of switched controlled systems, which is an interesting hybrid control problem: given a set of controlled systems $\dot{x} = f_i(x, v_i)$, $i = 1, 2, \cdots, N$, what are the conditions for the existence of controllers $v_i = v_i(x)$ so that $\dot{x} = f_i(x, v_i(x))$, $i = 1, 2, \cdots, N$ can be stabilized by switching?

V. Summary

We considered the local property of MP at a specific equilibrium for the class of nonaffine nonlinear systems. We investigated both continuous and discontinuous switched controllers as the class of admissible control laws. We have also presented a MP test and a design method for switched controllers which can be used to stabilize zero dynamics. Our results enlarge the class of nonaffine nonlinear systems that can be termed MP. Moreover, our results open several avenues for further research, such as MIMO cases and relations between MP and stabilizability for nonaffine systems.

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Single Sample Path-Based Sensitivity Analysis of Markov Processes Using Uniformization

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***Abstract*—Using the notion of perturbation realization factor, Cao and Chen [2] provide sensitivity formulas of discrete-time Markov chains and uniformizable Markov processes. In this paper, the estimators of the realization factors of a uniformizable Markov process are provided by its uniformized Markov chain. It is proved that estimators given by the uniformized Markov chain have smaller variances than those provided by the original Markov process, which are increasing functions of the uniformization parameter and thus have minimum.**

***Index Terms*—Markov process, sensitivity analysis, uniformization.**

I. Introduction

Markov processes have been widely used to analyze the performance and reliability of a variety of systems. Sensitivity (derivative) with respect to model parameters plays an important role in model optimization, in studying the effect of uncertainties in parameter values, and in reduction of model complexity by aiding further abstraction of the model if it is relatively insensitive to certain parameters. Sensitivity analysis of discrete-events dynamic system (DEDS) based on a single sample path is one of the most active research fields at present. Several sensitivity analysis techniques have been developed, among which infinitesimal perturbation analysis (IPA), when applicable, is the most efficient one available. However, it fails for a broad class of parameters, e.g., the transition probabilities of Markov chain. Various extensions (cf., [2], [6], and references cited therein) of IPA have been developed. Infinitesimal perturbation of the transition probability of a Markov chain yields finite change in the sample path. Due to this discontinuity, the sensitivity analysis of the Markov process is challenging for a long period. Cao *et al.* [1], Dai [4], [5], and Fu and Hu [7] study the single sample path-based sensitivity analysis of Markov chains. Recently, Cao and Chen [2] have developed a new technique to evaluate the performance sensitivity of a denumerable Markov process with respect to linear perturbation of its infinitesimal generator.

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Uniformization is a modification of an idea introduced by Jensen [12] and used by Heidelberger and Goyal [10], Hordijk *et al.* [11], Keilson [13], Shanthikumar [14], Sonderman [15], and a lot of other authors. The basic idea of the uniformization of a continuous time stochastic process $Z = \{Z(t), t \in [0, \infty)\}$ is that under certain conditions Z can be represented as the composition of a discrete time stochastic process $Z' = \{Z'_n, n \geq 0\}$ and a Poisson process $\{N(t), t \in [0, \infty)\}$, i.e., $Z(t) = Z'_{N(t)}$. Sonderman [15] illustrates how a semi-Markov process could be uniformized. Shanthikumar [14] uses uniformization to develop hybrid simulation/analytical models of renewal process and shows that the estimators for the number of renewals in a time interval $[0, t)$ obtained from the hybrid models have lower variances than those from the traditional simulation models. When Z is a uniformizable Markov process, then Z' itself is a discrete-time Markov chain which is independent of Poisson process $\{N(t)\}$ and has the same stationary distribution as Z (see [11], [14], and [15]). Hordijk *et al.* [11] use the Markov property of Z' in the simulation of Z. In particular, they simulate Z' and use the estimator for its stationary distribution as the estimators for the stationary distribution of Z. In this paper, we simulate Z' and use the estimators for the sensitivity of the performance measure of Z' to evaluate that of Z. It is proved that estimators obtained from Z' have smaller variances than those from the original Markov process Z.

II. Main Results

Let $\{X_t, t \in T\}$ be an irreducible and aperiodic Markov chain $(T = \{0, 1, \cdots, n, \cdots\})$ or irreducible Markov process $(T = [0, \infty))$ with state space $S = \{1, 2, \cdots\}$ and transition probability matrix $P = (p_{ij}), i, j \in S$ or infinitesimal generator $A = (a_{ij}), i, j \in S$, where $a_{ij} \geq 0, i \neq j, a_{ii} < 0$, satisfying $\sup\{-a_{ii}, i \in S\} < \infty$, and $\Sigma_{j=1}^{\infty} a_{ij} = 0$, for all $i \in S$. Let $f(x)$: $S \to R$ be a real function and $\pi = (\pi_1, \pi_2, \cdots)$ be the stationary distribution of $\{X_t, t \in T\}$. Then, under the assumption $E_\pi(|f|) = \Sigma_{i=1}^{\infty} \pi_i |f(i)| < \infty$, the steady-state performance measure is defined by

$$\eta = E_\pi(f) = \sum_{i=1}^{\infty} \pi_i f(i). \tag{1}$$

Let $\delta > 0$ be a small enough real number and $Q = (q_{ij}), i, j \in S$ be an infinite matrix with $Qe = 0$, where $e = (1, 1, \cdots)^T$ and ^{T} denotes the transpose operator of a matrix. Then the sensitivity of η, with respect to $P(T = \{0, 1, \cdots\})$ or $A(T = [0, \infty))$, in the direction of Q is defined by

$$\frac{\partial \eta}{\partial Q_P} \left(\text{or } \frac{\partial \eta}{\partial Q_A} \right) = \lim_{\delta \to 0} \frac{\eta' - \eta}{\delta} \tag{2}$$