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Controllability for a class of simple Wiener–Hammerstein systems

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Abstract

Controllability for a class of simple Wiener–Hammerstein systems is considered. Necessary and sufficient conditions for dead-beat and complete controllability for these systems are presented. The controllability tests consist of two easy-to-check tests for the subsystems.

Keywords: Controllability; Discrete time; Wiener–Hammerstein systems; Dead-beat; Nonlinear

1. Introduction

Controllability is one of the fundamental notions in control theory because of its strong connection to a number of important properties of control systems, such as stabilizability. It is a well-known fact that testing controllability of non-linear systems is a computationally hard problem and few non-linear systems allow for an easy-to-check global controllability test. Hence, it is very important to characterize classes of nonlinear systems which allow for simple controllability tests.

One such class of models are of Wiener–Hammerstein type, whose detailed classification is given in [7, 8]. These models arise in black-box identification of non-linear systems and they consist of series and/or parallel connections of linear dynamical blocks (L) and static polynomial non-linearities (N). For example, simple Hammerstein models consist of a linear block which is fed through a non-linearity, that is we have the N-L configuration. If several simple Hammerstein systems of the form N_i-L_i, $i = 1, 2, \ldots, r$, are connected in parallel, we obtain an URYSON model, etc. Some examples of practical applications of Wiener–Hammerstein models can be found in [1, 10].

Wiener–Hammerstein systems that we consider can be regarded as a subclass of a larger class of discrete-time polynomial systems. Several controllability results for classes of polynomial systems can be found in [4–6, 13, 14]. Results of this paper are closely related to [11] where controllability properties for a class of generalized Hammerstein systems (a parallel connection of a linear and a simple Hammerstein system) was investigated and [12] where we considered controllability for a class of URYSON models (a multiple parallel connection of simple Hammerstein systems). In [11, 12] we showed how the *parallel connections* of these special classes of polynomial systems can be used to simplify the controllability tests. In this paper we show that a *series connection*, which normally arises in Wiener–Hammerstein systems, can also be used to simplify the controllability tests. Together with results in [11, 12], results of this paper settle the controllability question for

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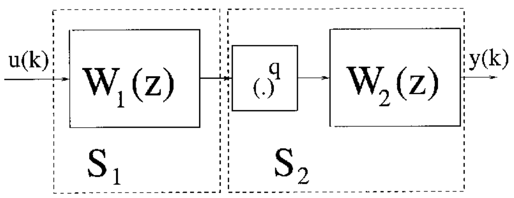

Fig. 1. Simple Wiener–Hammerstein model.

basic parallel and series connections which normally arise in Wiener–Hammerstein models. Using our method, however, generalizations to more complicated Wiener–Hammerstein models are possible.

The class of systems that we consider consists of two linear blocks between which we have a static nonlinearity of the form $N(\cdot)=(\cdot)^q, q\in\mathbb{N}, q>1$ (see Fig. 1). The standard structure of simple Wiener–Hammerstein systems is therefore *L-N-L*. If we denote the first linear subsystem as S_1 and the series connection of nonlinearity and the second linear subsystems as S_2 (see Fig. 1), the main result of this paper states that the overall system is dead-beat controllable if and only if both S_1 and S_2 are dead-beat controllable. Notice that if $q=1$, that is the purely linear case, dead-beat controllability of subsystems *does not always imply* dead-beat controllability of the series connection. We identify the phenomenon, which shows the difference between the linear ($q=1$) and non-linear ($q>1$) cases. For complete controllability we need an extra requirement which is similar to the one arising in purely linear series connections [9].

The paper is organized as follows. We first present preliminary results and notation. Main results and two examples are presented in Section 3. Summary is given in the last section.

2. Preliminaries

Sets of real, natural and complex numbers are respectively denoted as $\mathbb{R}, \mathbb{N}$ and $\mathbb{C}$. We consider simple Wiener–Hammerstein discrete-time systems of the following form:

$$S_1: x_1(k+1)=Ax_1(k)+bu(k),$$
$$S_2: x_2(k+1)=Fx_2(k)+g(cx_1(k))^q,$$
$$y(k)=hx_2(k), \tag{1}$$

where $x_i\in\mathbb{R}^{n_i}$, $i=1,2$, $n_1+n_2=n$, $u\in\mathbb{R}$, $q\in\mathbb{N}$, $q>1$ and matrices $A\in\mathbb{R}^{n_1\times n_1}, F\in\mathbb{R}^{n_2\times n_2}, b\in\mathbb{R}^{n_1\times 1}, g\in\mathbb{R}^{n_2\times 1}, c\in\mathbb{R}^{1\times n_1}, h\in\mathbb{R}^{1\times n_2}$.

This class of systems can be represented by the block diagram given in Fig. 1. The system consists of a series connection of two linear dynamical blocks

$$W_1(z)=\frac{y_1(z)}{u_1(z)}=c(zI-A)^{-1}b=\frac{b_1(z)}{a_1(z)},$$
$$W_2(z)=\frac{y_2(z)}{u_2(z)}=h(zI-F)^{-1}g=\frac{b_2(z)}{a_2(z)} \tag{2}$$

interconnected via the static nonlinearity $u_2(k)=(y_1(k))^q$. We also have that the input of the overall system $u(k)=u_1(k)$ and output of the overall system $y(k)=y_2(k)$. Roots of polynomials $a_i(z)$ and $b_i(z)$ are respectively poles and zeros of $W_i(z)$. For simplicity, we assume without loss of generality that there are no feed-through terms for transfer functions $W_i(z)$.

In the sequel, we often refer to the following decomposition of the simple Wiener–Hammerstein system. We say that the linear dynamical block $W_1(z)$ is the subsystem S_1 and that the series connection of the static nonlinearity $(\cdot)^q$ and the linear block $W_2(z)$ is the system S_2 (see Fig. 1).

We denote a sequence of controls $\{u(0),u(1),\ldots\}$ as U where $u(i)\in\mathbb{R}$ and its truncation of length N, that is $\{u(0),\ldots,u(N-1)\}$, as U_N. The state of the system (1) at time step N, which is obtained when a sequence U_N is applied to the system and which emanates from the initial state $x(0)$, is denoted as $x(N,x(0),U_N)$. We give below the definitions that are used in the sequel:

Definition 1. The system (1) is dead-beat controllable if for any initial state $x(0)\in\mathbb{R}^n$ there exists an integer N and a control sequence U_N such that $x(N,x(0),U_N)=0$.

Definition 2. If in Definition 1 there exists a fixed integer $\bar{N}\in\mathbb{N}$ such that $\forall x(0)\in\mathbb{R}^n$ it holds that $N(x(0))<\bar{N}$, then we say that there exists a uniform bound on the dead-beat time.

Definition 3. The system (1) is completely controllable if $\forall x(0)$, $x^*\in\mathbb{R}^n$ there exists a positive integer $N=N(x(0),x^*)$ and U_N such that $x(N,x(0),U_N)=x^*$.

The characteristic polynomial of a matrix F is denoted as $P_F(\lambda)=\det(\lambda I-F)$. Given a polynomial

$$P(\lambda)=\lambda^t+a_{t-1}\lambda^{t-1}+\cdots+a_1\lambda+a_0,$$

we introduce a new polynomial $P^q(\lambda)$ which is obtained from $P(\lambda)$ when all the coefficients a_i are taken with a power $q>0$, $q\in\mathbb{N}$, that is we write

$$P^q(\lambda)=\lambda^t+a^q_{t-1}\lambda^{t-1}+\cdots+a^q_1\lambda+a^q_0.$$

Also, if we are given a polynomial $H=\lambda^s+h_{s-1}\lambda^{s-1}+\cdots+h_1\lambda+h_0$, we use the notation

$$(P\cdot H)^q(\lambda)=\lambda^{t+s}+(h_{s-1}+a_{t-1})^q\lambda^{t+s-1}+\ldots+(h_1a_0+h_0a_1)^q\lambda+(a_0h_0)^q.$$

Since $q\in\mathbb{N}$, we can use repeatedly the binomial formula $(a+b)^q=\sum_{i=0}^q\binom{q}{j}a^{q-j}b^j$, where $\binom{q}{j}=q!/j!(q-j)!$, to find the coefficients of $(P\cdot H)^q(\lambda)$. Hence, the polynomial $(P\cdot H)^q(\lambda)$ is obtained when we first multiply the polynomials P and H and then take qth powers of all the coefficients of the product polynomial. Notice that $(P\cdot H)^q(\lambda)=(H\cdot P)^q(\lambda)$.

Definition 4 (Cox et al. [2]). A greatest common divisor of polynomials p_1, p_2 is a polynomial G such that: G divides p_1 and p_2; if p is another polynomial which divides p_1 and p_2, then p divides G. When G has these properties we write $G=\mathrm{GCD}(p_1,p_2)$.

Definition 5. Given a monic polynomial p_1 with roots $\{\sigma_1,\ldots,\sigma_{n_1}\}$ and monic polynomial p_2 with roots $\{\zeta_1,\ldots,\zeta_{n_2}\}$, their resultant is denoted as $\mathrm{Res}(p_1,p_2)$ and is defined as

$$\mathrm{Res}(p_1,p_2)=\prod_{i,j}(\sigma_i-\zeta_j).$$

Two polynomials have a common root if and only if their resultant is equal to zero. Resultant of two polynomials can be obtained as a function of coefficient of the polynomials by using the Sylvester matrix [2].

Given a set of vectors $B_i\in\mathbb{R}^{n\times 1}$, $i=1,2,\ldots,f$, the cone generated by the vectors is denoted as

$$C(B_1,\ldots,B_f)=\left\{x:x=\sum_{i=1}^f B_i\alpha_i,\ \alpha_i\geqslant 0\right\}.$$

Negative cone C^- of a cone $C=(B_f,\ldots,B_1)$ is given by $C^-=C(-B_f,\ldots,-B_1)$. The span of vectors B_i is denoted as $\mathrm{sp}\{B_1,\ldots,B_f\}$.

The following theorem is needed in the sequel:

Theorem 1 (Evans and Murthy [4]; Nešić [11]). *Consider the linear system with positive controls* $u\geqslant 0$:

$$x(k+1)=Ax(k)+bu(k). \tag{3}$$

The system is completely (dead-beat) controllable if and only if

1. $\mathrm{rank}[\lambda I-A:b]=n,\forall\lambda\in\mathbb{C}$ $(\mathrm{rank}[\lambda I-A:b]=n,\forall\lambda\in\mathbb{C}-\{0\})$.
2. *A has no real positive or zero eigenvalues (A has no real strictly positive eigenvalues).*

Notice that conditions for complete controllability are stronger than that for dead-beat controllability.

Theorem 2 (Evans and Murthy [4]; Evans [3]). *Suppose that a matrix A has no zero or positive real eigenvalues. Then there exists a polynomial $c(\lambda)=\sum_{i=0}^N c_i\lambda^i$, $c_i>0$ such that $c(A)=0$.*

Theorem 3 (Evans and Murthy [4]). *Consider a set of vectors A^ib, $i=0,\ldots,r$. If there exist $\alpha_i>0$ such that the following condition is satisfied*:

$$\sum_{i=0}^r\alpha_iA^ib=0 \tag{4}$$

then $C(A^rb,\ldots,Ab,b)=\mathrm{sp}\{A^rb,\ldots,Ab,b\}$.

Comment 1. A special form of Theorem 3 can be interpreted in a geometric way, which is more suitable for our purposes. In Theorem 3 suppose that $r>n$ and that the first n vectors $A^ib\in\mathbb{R}^n$, $i=0,1,\ldots,n-1$ are linearly independent. Then obviously a positive linear combination of the $r-n$ remaining vectors $A^ib, i=n,\ldots,r$ must be in the interior of the negative cone $C^-=C(-A^{n-1}b,\ldots,-b)$ in order for the condition (4) to hold.

This holds in general and we can state the following: Consider n linearly independent vectors $A^{k_{n-1}}b,\ldots,A^{k_1}b,A^{k_0}b\in\mathbb{R}^n$ with $k_{i+1}>k_i$. Consider the cone $C=C(A^{k_{n-1}}b,\ldots,A^{k_1}b,A^{k_0}b)$. The cone C has a non-empty interior in $\mathbb{R}^n$. If there exist vectors $A^{k_n}b,\ldots,A^{k_{N-1}}b$ and positive numbers α_i such that the vector $\sum_{i=n}^{N-1}\alpha_iA^{k_i}b$ belongs to the interior of the negative cone $\mathring{C}^-$, then we have that $C(A^{k_{N-1}}b,A^{k_{N-2}}b,\ldots,A^{k_1}b,A^{k_0}b)=\mathbb{R}^n$. This implies that there exist positive numbers α_i such that

$$\sum_{i=0}^{N-1}\alpha_iA^{k_i}b=0,\quad k_{i+1}>k_i,\ k_i\in\mathbb{N}. \tag{5}$$

3. Main results

In this section we present and prove the main result of the paper. The proof is based on similar ideas to [11, 12] where parallel connections of simple Hammerstein systems were considered. For simplicity, we assume below that (A, b, c) and (F, g, h) are in controllability canonical form.

The main result is summarized below.

Theorem 4. *Consider the system* (1), *with* $q > 1$. *The system is dead-beat controllable if and only if both subsystems* S_1 *and* S_2 *are dead-beat controllable.*

Hence, we have the following controllability test for systems (1). Consider the conditions:
1. $\text{rank}[\lambda I - A : b] = n_1, \forall \lambda \in \mathbb{C} - \{0\}$.
2. $\text{rank}[\lambda I - F : g] = n_2, \forall \lambda \in \mathbb{C} - \{0\}$.
3. F has no real strictly positive eigenvalues.

The system (1) dead-beat controllable if and only if:
1. (for odd $q > 1$) Conditions 1 and 2 hold,
2. (for even $q > 1$) Conditions 1–3 hold.

In order to prove the main result we need several lemmas.

Lemma 1. *Suppose that a pair of matrices* $A \in \mathbb{R}^{n \times n}$ *and* $b \in \mathbb{R}^{n \times 1}$ *is controllable and* A *is non-singular. Then, given any positive integer* $T \in \mathbb{N}$, *there exist a sequence of positive integers of the form*

$$k_0 = 0,$$

$$k_{i+1} \geqslant k_i + T, \quad i = 0, 1, \ldots, n-1, \tag{6}$$

such that

$$\text{sp}[A^{k_{n-1}}b : A^{k_{n-2}}b : \ldots : A^{k_1}b : b] = \mathbb{R}^n. \tag{7}$$

In other words, there exist integers of the form (6) such that $\text{rank}[A^{k_{n-1}}b : A^{k_{n-2}}b : \ldots : A^{k_1}b : b] = n$.

Proof of Lemma 1. Because of controllability of (A, b) and non-singularity of A we can write

$$\text{rank}\, A^k[A^{n-1}b : \ldots : b] = n, \ \forall k \in \mathbb{N}. \tag{8}$$

Pick an integer $s_1 \geqslant T$. If $A^{s_1}b$ and b are linearly independent, let $k_1 = s_1$. Suppose that the vectors $A^{s_1}b$ and b are linearly dependent. Hence, there exists $e_1 \in \mathbb{R}$ such that $A^{s_1}b = e_1 b$. Consider now the vector $A^{s_1+1}b$ and b and suppose that they are linearly dependent. That implies that $A^{s_1}b$ and $A^{s_1+1}b$ are also linearly dependent, which contradicts Eq. (8). Hence, we can let $k_1 = s_1 + 1$. The construction of the remaining k_i is carried out in the same manner by considering the linearly independent vectors $A^{k_{i-1}}b, \ldots, b$ and a new vector $A^{s_i}b$ with $s_i \geqslant k_{i-1} + T$, which proves Lemma 1. □

Lemma 2. *Suppose that a pair of matrices* (A, b) *is controllable*, A *is non-singular and* A *has no real strictly positive eigenvalues. Then, given any positive integer* $T \in \mathbb{N}$, *there exist an integer* P *and a sequence of positive integers of the form*

$$m_0 \geqslant T,$$

$$m_{i+1} \geqslant m_i + T, \quad i = 0, 1, \ldots, P-1, \tag{9}$$

such that

$$C(A^{m_P}b, A^{m_{P-2}}b, \ldots, A^{m_1}b, A^{m_0}b) = \mathbb{R}^n. \tag{10}$$

Proof. The lemma is proved by contradiction.

Since A has no positive real eigenvalues, it satisfies a polynomial equation with positive coefficients (see Theorem 2)

$$A^N + \alpha_{N-1}A^{N-1} + \cdots + \alpha_1 A + \alpha_0 I = 0, \quad \alpha_i > 0$$

and according to Theorem 3 the vectors $A^i b$, $i = 0, 1, \ldots, N$, span the whole state space since $A^i b$, $i = 0, 1, \ldots, n-1$, are linearly independent. Also, since A is non-singular, it follows that given any $k \in \mathbb{N}$ there does not exist a separating hyper-plane $H := \{x : hx = 0\}$ for the vectors $A^{k+N}b, \ldots, A^k b$, such that $A^{k+i}b \in H_+ := \{x : hx \geqslant 0\}, \forall i = 0, 1, 2, \ldots, N$.

From Lemma 1 we construct a sequence $k_2, \ldots, k_{n+1}$, satisfying Eq. (9) and $k_2 > 3(T+N)$, such that $A^{k_i}b$, $i = 2, \ldots, n+1$, are linearly independent. Hence, $C(A^{k_{n+1}}b, \ldots, A^{k_2}b)$ has a non-empty interior in $\mathbb{R}^n$. Given any T, we can find $k_{n+2} \in \{k_{n+1} + T, \ldots, k_{n+1} + N + T\}$ such that $A^{k_{n+2}}b \notin C(A^{k_{n+1}}b, \ldots, A^{k_2}b, A^{k_1}b)$, where $k_1 \in \{2(T+N), \ldots, 3(T+N)\}$. To shorten notation we write

$$C_i = C(A^{k_i}b, \ldots, A^{k_2}b, A^{k_1}b).$$

If there is $i^* \in \{0, 1, \ldots, N\}$ such that $A^{k_{n+1}+i^*+T}b \in C^-_{n+1}$, then we let $k_{n+2} = k_{n+1} + i^* + T$ and have $C_{n+2} = \mathbb{R}^n$ If this is not true, choose k_{n+2} for which $C_{n+1} \subset C_{n+2}$. By this construction we can see that at each step we enlarge the cone generated by the vectors $A^{k_i}b$, $i = 1, 2, 3 \ldots$. We can show by contradiction that we have

$$\exists H_+, H_+ \subseteq \lim_{i \to \infty} C(A^{k_i}b, \ldots, A^{k_1}b)\ . \tag{11}$$

Indeed, suppose that using the construction that we described above we have that Eq. (11) is not satisfied. This implies that there exists a half-space H_+ such that $\lim_{i\to\infty} C(A^{k_i}b,\dots,A^{k_1}b)\subset H_+$. This also implies that $\lim_{i\to\infty} C(A^{k_i}b,\dots,A^{k_2}b)\subset H_+$. However, under the conditions of Theorem, given any H_+ there exists $k_1\in\{2(T+N),\dots,3(T+N)\}$ such that $A^{k_1}b\notin H_+$, a contradiction.

If we have that H_+ is a proper subset of $\lim_{i\to\infty} C(A^{k_i}b,\dots,A^{k_1}b)$ then we actually have that $\lim_{i\to\infty} C(A^{k_i}b,\dots,A^{k_1}b)=\mathbb{R}^n$ and this would suffice for the proof.

Otherwise, we have $H_+=\lim_{i\to\infty} C(A^{k_i}b,\dots,A^{k_1}b)$ and we proceed as follows. Suppose that there does not exist $m_0\in\{T,T+1,\dots,T+N\}$ and vectors $A^{m_i}b,\ m_i\in\{k_1,k_2,\dots\},\ i=1,2,\dots,P$ such that $C(A^{m_P}b,\dots,A^{m_1}b,A^{m_0}b)=\mathbb{R}^n$. This contradicts the fact that, H_+ is generated by $A^{k_i}b,\ i=1,2,\dots$. Indeed, since A has no positive real eigenvalues, there exists $m_0\in\{T,T+1,\dots,T+N\}$ such that $A^{m_0}b\notin H_+$. If there are no $A^{m_i}b, m_i\in\{k_1,k_2,\dots,\},\ i=1,2,\dots,P$, such that $A^{m_0}b$ is in the negative cone generated by $A^{k_i}b$, then H_+ is not generated by $A^{k_i}b$. It follows that there exist $A^{m_i}b,\ m_i\in\{k_1,k_2,\dots,\},\ i=1,2,\dots,P$ such that $A^{m_0}b$ is in the (non-empty) interior of $-C(A^{m_P}b,\dots,A^{m_1}b)$. Therefore, we can always choose m_i so that $C(A^{m_P}b,\dots,A^{m_1}b,A^{m_0}b)=\mathbb{R}^n$. □

Lemma 3. *Consider polynomials*

$$\begin{aligned} &p_1(\lambda)=\lambda^{n_1}+b_{n_1-1}\lambda^{n_1-1}+\cdots+b_1\lambda+b_0,\\ &\quad b_i\in\mathbb{R},\quad b_0\neq 0,\\ &p_2(\lambda)=\lambda^{n_2}+a_{n_2-1}\lambda^{n_2-1}+\cdots+a_1\lambda+a_0,\\ &\quad a_i\in\mathbb{R},\quad a_0\neq 0. \end{aligned}\tag{12}$$

Suppose that $p_1(\lambda)$ and the polynomial $p_2^q(\lambda)$, $q\in\mathbb{N}$, $1\leqslant q$ have common roots. There exists a polynomial $H(\lambda)$ with real coefficients of the degree at most n_2 such that the polynomials $p_1(\lambda)$ and $(p_2\cdot H)^q(\lambda)$ have no common roots if and only if $q>1$.

Proof. *Necessity*: Suppose that $q=1$. Obviously, $p_2^q(\lambda)=p_2(\lambda)$. Then, if $\mathrm{GCD}(p_1,p_2)\neq 1$ we have for any polynomial H that $\mathrm{GCD}(p_1,Hp_2)\neq 1$, which proves necessity.

Sufficiency: Suppose that $q>1$. Denote the set of roots of the polynomial $p_1(\lambda)$ as $\Sigma=\{\sigma_1,\sigma_2,\dots,\sigma_{n_1}\}$.

Notice that the resultant of p_1 and p_2^q can be written as

$$\mathrm{Res}(p_1,p_2^q)=\prod_{i=1}^{n_1}p_2^q(\sigma_i).\tag{13}$$

The polynomials p_1 and p_2^q have a common root if and only if the resultant (13) vanishes. The lemma is proved if we find a polynomial H with real coefficients such that $\forall\sigma_i\in\Sigma$, the polynomials $(H\cdot p_2)^q(\sigma_i)$ is a nontrivial polynomial in the coefficients of H. Indeed, then we can choose coefficients of H so that $\mathrm{Res}(p_1,(H\cdot p_2)^q)$ is not equal to zero and the lemma holds.

Assume that $p_2^q(\sigma_j)=0$ for some σ_j, a root of $p_1(\lambda)$. Notice that if $\sigma_j=0$, then $p_2^q(\sigma_j)\neq 0$ because $a_0\neq 0$. Introduce the polynomial $H=\lambda^{n_1}+h$ and consider

$$\begin{aligned}(p_2\cdot H)^q(\lambda)=\lambda^{2n_1}&+a^q_{n_1-1}\lambda^{2n_1-1}+\cdots+a_1^q\lambda^{n_1+1}\\ &+(a_0+h)^q\lambda^{n_1}+h^q(a^q_{n_1-1}\lambda^{n_1-1}+\cdots+a_0^q).\end{aligned}\tag{14}$$

If we consider now the roots σ_i of $p_1(\lambda)$ for which $p_2^q(\sigma_i)\neq 0$ we obtain

$$(p_2\cdot H)^q(\sigma_i)=p_2^q(\sigma_i)h^q+\cdots$$

which is a non-trivial polynomial in h. If, on the other hand, we consider a root σ_j of $p_1(\lambda)$ for which $p_2^q(\sigma_j)=0$, we obtain

$$(p_2\cdot H)^q(\sigma_j)=\left(\sum_{i=1}^{q-1}\binom{q}{j}a_0^{q-j}h^j\right)\sigma_j^{n_1}.\tag{15}$$

Since $a_0\neq 0$ and $b_0\neq 0$ implies $\sigma_j\neq 0$, it follows that the polynomial (15) is a non-trivial polynomial in h *if only if* $q>1$. Indeed, the sum on the right-hand side of Eq. (15) is defined only if $q-1>0$. On the other hand, if $q-1>0$, then the polynomial is non-trivial in h. Hence, the resultant

$$\mathrm{Res}(p_1,(p_2\cdot H)^q)=\prod_{i=1}^{n_1}(p_2\cdot H)^q(\sigma_i)$$

is a non-trivial polynomial in h and we can always chose $h\in\mathbb{R}$ so that the resultant does not vanish. □

Proof of Theorem 4. Necessity part of the proof is obvious. Suppose now that conditions of the theorem are satisfied.

We first notice that there is no loss of generality if we concentrate on the case when the matrix F is non-singular. Indeed, if F has some zero eigenvalues, supposing that $x_1(0)=0$ (because system S_1 is dead-beat controllable), we can see that by simply applying zero input sequence the zero modes of S_2 converge to the origin in finite time. The same applies to zero modes of S_1.

Hence, we suppose that A and F are non-singular. For simplicity, suppose also that (A,b,c) and (F,g,h) are in controllability canonical form. There is no loss of generality if we introduce a non-singular feedback transformation $u(k)=Kx_1(k)+v(k)$, where $Kx_1(k)$ is a minimum time dead-beat controller for the subsystem S_1 and $v(k)$ is a new control variable. With the transformation, the system (1) becomes

$$x_1(k+1)=Jx_1(k)+bv(k),$$
$$x_2(k+1)=Fx_2(k)+g(cx_1(k))^q, \tag{16}$$

where

$$J=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \vdots & & \vdots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad b=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$$c=\begin{pmatrix} b_0^1 & b_1^1 & b_2^1 & \dots & b_{n_1-1}^1 \end{pmatrix}$$

and $b_i^1 \in \mathbb{R}$ are coefficients of the polynomial $b_1(z)$ in (2). Notice that $cJ^ib=b_{n_1-1-i}^1$, $\forall i=0,1,\dots,n_1-1$. By applying the following control sequence:

$$\begin{aligned}
&v(i_1)=0, \quad 0\leqslant i_1\leqslant k_0-1, k_0>n_1,\\
&v(k_0)=w(0),\\
&v(i_2)=0, \quad k_0+1\leqslant i_2\leqslant k_1-1,\ k_1>k_0+n_1,\\
&v(k_1)=w(1),\\
&\vdots \quad \vdots\\
&v(k_{T-1})=w(T-1),\\
&v(i_T)=0, \quad k_{T-1}+1\leqslant i_T\leqslant P-1,\\
&\qquad\qquad P>k_{T-1}+n_1+1,
\end{aligned} \tag{17}$$

we obtain that the state of the system at time step P is

$$\begin{aligned}
&x_1(P)=0, \qquad (18)\\
&x_2(P)=f(x_1(0),x_2(0))\\
&\qquad +b_1^q(F)[F^{m_0}g:F^{m_1}g:\dots:F^{m_{T-2}}g:F^{m_{T-1}}g]\\
&\qquad [w^q(0)\ w^q(1)\ \dots w^q(T-1)]^{\mathrm{T}},
\end{aligned}$$

where $m_i=P-1-k_i-n_1$, $i=0,1,\dots,T$ and $f(x_1(0),x_2(0))$ is a (non-linear) function of initial states of the system. Suppose that q is odd. We let $T=n_2$. If $b_1^q(F)$ and $[F^{m_0}g:\dots:F^{m_{n_2-1}}g]$ are non-singular matrices, the system is dead-beat controllable. From Lemma 1 it follows that we can always chose m_i so that $[F^{m_0}g:\dots:F^{m_{n_2-1}}g]$ is non-singular.

Similarly, if q is even, using Lemma 2, we can always find m_i so that $C(F^{m_0}g:\dots:F^{m_{T-2}}g:F^{m_{T-1}}g)=\mathbb{R}^n$. If $b_1^q(F)$ is singular, from Lemma 3 it follows we can find a transfer function

$$W^*(z)=\frac{z^{l_1}+h}{z^{l_1}+d_{l_1-1}z^{l_1-1}+\dots+d_0}, \quad l_1\leqslant n_1,$$

such that the numerator $\bar{b}_1(z)=(z^{l_1}+h)b_1(z)$ of the augmented transfer function $\bar{W}=W^*(z)W_1(z)$ satisfies

$$\operatorname{Res}(\bar{b}_1^q(z), p_F(z))\neq 0.$$

Moreover, it is easy to see that a possible pole - zero cancelations can easily be avoided in the transfer function $\bar{W}(z)$ with appropriate choices of the coefficients of $W^*(z)$. In other words, we can always choose $W^*(z)$ so that the augmented subsystem $\bar{W}(z)$ is controllable and at the same time $\bar{b}_1^q(F)$ is non-singular. Now we can apply a similar control sequence to Eq. (19) to the system (1) which is augmented with $W^*(z)$ to complete the proof. □

Comment 2. The control sequence (17) may be modified in many ways and we may be able to prove the controllability result. Indeed, it is easy to see that if the degree of the polynomial $b_1(z)$ is $h_1\leqslant n_1$, then instead of having $k_{i+1}>n_1+k_i$, we can have $k_{i+1}>k_i+h_1$ and we can still use the same arguments. This situation is illustrated in Example 2. However, the sequence (17) works always since the degree of $b_1(z)$ is at most n_1.

If the transfer function $W_1(z)$ has no zeros, that is, $b_1(z)=\text{const.}$, the matrix $b_1^q(F)$ in Eq. (19) is equal to an identity matrix for any F. From the proof of Theorem 4 it follows that there exists a uniform bound on the dead-beat time if the system (1) is dead-beat controllable.

A very similar statement can be derived for complete controllability of Wiener–Hammerstein systems, which is given below without proof. In this case, the only critical common roots of $b_1^q(z)$ and $a_2(z)$, which are not allowed, are equal to zero (at the origin). We introduce the sets $\mathscr{Z}_1 := \{z \in \mathbb{C} : b_1(z) = 0\}$, $\mathscr{P}_2 := \{z \in \mathbb{C} : a_2(z) = 0\}$.

Theorem 5. *The system* (1) *is completely controllable if and only if both subsystems* S_1 *and* S_2 *are completely controllable and* $0 \notin \mathscr{Z}_1 \cap \mathscr{P}_2$.

We do not prove Theorem 5 since the proof is carried out along the same lines as that of Theorem 4. However, we show why $b_1(0) = a_1(0) = 0$ leads to loss of complete controllability. Consider the state equations (16). We assume, without loss of generality that F has all eigenvalues equal to zero. Denote the entries of vectors $x_1(k)$ and $x_2(k)$, respectively, as $\xi_j(k)$ and $\eta_i(k)$. Since $b_1(0) = 0$ we have that the matrix $c = (0\ 0\ \dots\ b_l^1\ b_{l+1}^1\ \dots\ b_{n_1-1}^1)^{\mathrm{T}}$. It is not difficult to calculate that

$$\eta_{n_2}(k) = (b_l^1 \xi_l(k) + b_{l+1}^1 \xi_{l+1}(k) + \cdots + b_{n_1-1} \xi_{n_1-1}(k))^q, \quad \forall k \in \mathbb{N}.$$

Hence, the algebraic variety,

$$V = \{x : \eta_{n_2} = (b_l^1 \xi_l + b_{l+1}^1 \xi_{l+1} + \cdots + b_{n_1-1} \xi_{n_1-1})^q\}$$

is invariant under control. The variety is non-empty in $\mathbb{R}^n$ and hence no initial state in the variety can be mapped outside of it by means of controls. The system is not completely controllable.

Theorems 4 and 5 differ from the linear series connection case [9]. Indeed, when $q = 1$ in (1), controllability of the series connection is (dead-beat) controllable if and only if $b_1(\lambda)$ and $a_2(\lambda)$ have no common (non-zero) roots. Lemma 3 can be used to see how the non-linear polynomial interconnection breaks this relationship. In a certain sense, we can say that by non-linearizing the interconnection we can recover certain controllability properties, which is exactly the same interpretation that we obtained for the parallel interconnections in [11, 12].

We illustrate our approach by two examples.

Example 1. To illustrate the claims of Theorems 4 and 5 consider the following simple Wiener–Hammerstein system:

$$\begin{aligned} x_1(k+1) &= x_1(k) + u(k), \\ x_2(k+1) &= (x_1(k) + u(k))^q, \end{aligned} \tag{19}$$

with q an odd integer. In this case, the subsystem S_1 is given by its transfer function $W_1(z) = z/(z+1)$ and the subsystem S_2 consists of the static non-linearity $(\cdot)^q$ and the linear block $W_2(z) = 1/z$. Notice that there is a feed-through term for W_1, but our results still apply. Both subsystems are completely and therefore dead-beat controllable.

Notice that W_1 has a zero and W_2 has a pole at the origin. According to Theorems 4 and 5, the overall system is dead-beat controllable but it is not completely controllable. Indeed, by applying $u(0) = -x_1(0)$, we can see that $x_1(1) = 0, x_2(0) = 0$, and the system is one-step dead-beat controllable.

On the other hand, it is easily checked that the variety $V = \{x : x_2 - x_1^q = 0\}$ is control invariant, that is if an initial state is in the variety $x_0 \in V$, then we have that $\forall N, \forall U_N, x(N, x_0, U_N) \in V$ and the system is not completely controllable. Notice that the control invariant variety V has similar meaning for the uncontrollable system (19) as control invariant linear subspaces of linear uncontrollable systems [9].

We illustrate in the next example our method and in particular the use of Lemma 3 in constructing a control sequence which is used in the proof of the main result.

Example 2. Consider the system (1) with $q = 3$, for which

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}; \qquad b = g = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}; \qquad c = \begin{pmatrix} -1 & 1 \end{pmatrix}.$$

It is easily verified that the above matrices with (1) are a state space realization of the Wiener-Hammerstein system with the transfer functions $W_1(z) = (z-1)/(z^2+z+1), W_2(z) = (1)/(z^2-2z+1)$.

Both subsystems are dead-beat controllable, according to Theorem 4, since $q = 3$ is odd and (F, g) and (A, b) are controllable pairs.

Suppose that we apply a non-singular input transformation of the form $u(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} x_1(k) + v(k)$. One sequence that can be used to prove dead-beat controllability is

$$v(0) = 0; \; v(1) = w(0); \; v(2) = 0;$$

$$v(3) = w(1); \; v(4) = v(5) = 0.$$

The state of the system at time step 6 is

$$x_1(6) = J^6 x_1(0) + \sum_{i=0}^{5} J^{5-i} b v(i) = 0,$$

$$x_2(6) = F^6 x_2(0) + F^5 g(cx_1(0))^3 + F^4 g(cJx_1(0))^3 + \underbrace{(F - I_2)}_{b_1^3(F)}[F^2 g : g][w^3(0)\ w^3(1)]^{\mathrm{T}}. \quad (20)$$

Since the matrix $b_1^3(F) = F - I_2$ is singular, we can not use $w(0)$, $w(1)$ to prove dead-beat controllability. Notice, however, that in the first subsystem the state $x(6) = 0$ irrespective of the applied controls $w(0)$, $w(1)$.

Let us use Lemma 3 so that we augment the first subsystem and then prove controllability for the augmented system. Introduce polynomial $H(z) = z + h$ and consider the polynomial

$$(b_1 \cdot H)^3(z) = z^2 + (h-1)^3 z + h^3 = z(z-1) + (-3h^2 + 3h)z + h^3(z-1).$$

Since both eigenvalues of F are equal to 1, we have that

$$\mathrm{Res}(p_F(z), (b_1 \cdot H)^3(z)) = (3h^2 - 3h)^2 \neq 0.$$

Hence, if we chose for instance $h = 2$, we have that $\mathrm{Res}(p_F(z), (b_1 \cdot H)^3(z)) = 36 \neq 0$. We augment the first subsystem with a linear block so that we can use our method for the augmented system. For example the transfer function $W^*(z) = (z+2)/(z+1)$ is of desired form since there are no pole-zero cancelations in the transfer function $\bar{W}(z) = W^*(z)W_1(z)$. Let us denote the numerator of the transfer function $\bar{W}(z)$ as $\bar{b}_1(z)$.

F and g matrices are the same for the second subsystem after the augmentation. However, for the first subsystem we have

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & -2 \end{pmatrix}; \quad \bar{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad \bar{c} = \begin{pmatrix} -2 & 1 & 1 \end{pmatrix}$$

and $\bar{J}$ and $\bar{x}_1$ are, respectively, the Jordan matrix of the same dimension as the augmented system and the state of the augmented first subsystem. We again introduce the input transformation $u(k) = (-1\ 2\ 2)^{\mathrm{T}}\bar{x}_1(k) + v(k)$ and apply, for example, the following control sequence:

$$v(0) = v(1) = v(2) = 0;\ v(3) = w(0);\ v(4) = v(5) = 0;$$

$$v(6) = w(1);\ v(7) = v(8) = v(9) = v(10) = 0.$$

The state of the system at time step 12 with the given control sequence is

$$\bar{x}_1(11) = 0,$$
$$x_2(11) = F^{11}x_2(0) + F^{10}g(\bar{c}\bar{x}_1(0))^3 + F^9 g(\bar{c}\bar{J}\,\bar{b}\bar{x}_1(0))^3 + F^8 g(\bar{c}\bar{J}^2\,\bar{b}\bar{x}_1(0))^3 + \underbrace{(F^2 + F - 8I_2)}_{\bar{b}_1^3(F)\text{-non-singular}}\underbrace{[F^4 g : Fg]}_{\text{non-singular}}[w^3(0)w^3(1)]^{\mathrm{T}} \quad (21)$$

Since both matrices $\bar{b}_1^3(F)$ and $[F^4 g : Fg]$ are non-singular, the state of the second subsystem can be arbitrarily assigned while keeping the state of the augmented subsystem at zero. Hence, the system is dead-beat controllable.

4. Conclusion

We presented necessary and sufficient conditions for dead-beat and complete controllability of simple Wiener–Hammerstein systems. Together with controllability results in [11, 12], the results of this paper provide controllability results for basic system structures (blocks) arising in identification of block oriented models. We believe that similar statements can be obtained for more general polynomial interconnected systems.

References

[1] W. Bamberger, R. Isermann, Adaptive on-line steady-state optimization of slow dynamic processes, Automatica 14 (1978) 223–230.
[2] D. Cox, J. Little, D. O'Shea, Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, Springer, New York, 1992.
[3] M.E. Evans, The convex controller: controllability in finite time, Int. J. System Sci. 16 (1985) 31–47.
[4] M.E. Evans, D.N.P. Murthy, Controllability of a class of discrete-time bilinear systems, IEEE Trans. Automat. Control 22 (1977) 78–83.
[5] M.E. Evans, D.N.P. Murthy, Controllability of discrete-time systems with positive controls, IEEE Trans. Automat. Control 22 (1977) 942–945.
[6] M.E. Evans, D.N.P. Murthy, Controllability of discrete-time inhomogeneous bilinear systems, Automatica 14 (1978) 147–151.
[7] R. Haber, L. Keviczky, Identification of nonlinear dynamic systems, in: N.S. Rajbman (Ed.), Identification and System Parameter Estimation, North-Holland, Amsterdam, 1978, pp. 79–127.

[8] R. Haber, H. Unbehauen, Structure identification of nonlinear dynamic systems – a survey of input/output approaches, Automatica 26 (1990) 651–677.
[9] T. Kailath, Linear Systems, Prentice-Hall, Englewood Cliffs, NJ, 1980.
[10] L. Keviczky, Nonlinear dynamic identification of a cement mill to be optimized, in: N.S. Rajbman (Ed.), Identification and System Parameter Estimation, North-Holland, Armsterdam, 1978, pp. 1185–1195.
[11] D. Nešić, A note on dead-beat controllability of generalized Hammerstein models, Systems Control Lett. 29 (1997) 223–231.
[12] D. Nešić, Controllability for a class of parallelly connected polynomial systems, submitted for publication.
[13] D. Nešić, I.M.Y. Mareels, Dead-beat control of simple Hammerstein models, IEEE Trans. Autoat. Control 43 (8) (1998) 1184–1189.
[14] D. Nešić, I.M.Y. Mareels, G. Bastin, R. Mahony, Output dead-beat control for a class of planar polynomial systems, SIAM J. Control Optim. 36 (1998) 253–272.