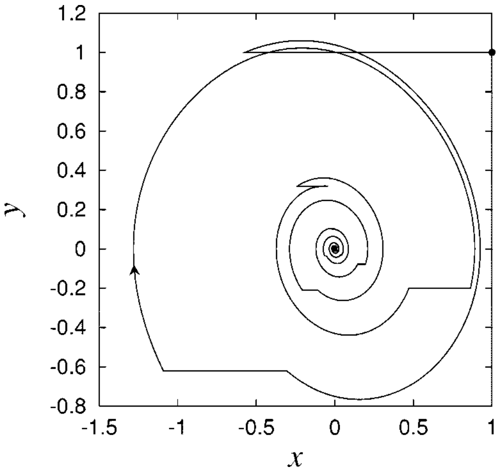

Fig. 2. Chaotic orbit of the periodically forced pendulum system is driven to zero equilibrium by the impulsive control.

for $i = i_0,\ i_0+1,\ i_0+2,\ \cdots$, where i_0 is a sufficiently large integer, then the conclusions of Theorem 1 hold.

Proof: For (4) and for the given constant q, chose a constant β such that $1 < \beta < q\exp[2b\tau/3 + b^2]$. Then, define $\mu \triangleq (1/\tau)\ln(\beta/q)$, which satisfies $\beta/e^{\mu\tau} = q < 1$ and $0 < \mu < (2b/3 + b^2)$. Since

$$\frac{a_{i+1}^2 e^{(i+1)\mu\tau}}{\beta^{(i+1)}}\,\frac{\beta^i}{a_i^2 e^{i\mu\tau}} = \frac{a_{i+1}^2 e^{\mu\tau}}{a_i^2 \beta^i} \le q^2\,\frac{e^{\mu\tau}}{\beta^i} = q < 1$$

the infinite series $= \sum_{i=1}^{\infty}(a_i^2 e^{i\mu\tau}/\beta^i)$ converges. This shows that the conditions of Theorem 1 are satisfied, so that the conclusions of Theorem 1 hold.

Remark 2: It is clear that, for the controlled system (3), if the control gains $\{g_k\}$ satisfy the conditions of Theorem 1 or Corollary 1, then the chaotic state of the pendulum system can be driven to its zero equilibrium. Moreover, the controlled system (3) is eventually exponentially asymptotically stable.

Example 1: For (3), if we take the control $u(t)$ with $g_k = (1 - (1/2^k))h$, namely

$$u(t) = \frac{\pi}{2}\sum_{k=1}^{\infty}\left(1 - \frac{1}{2^k}\right)h\delta(t - k\tau)$$

then we have $a_i = (\pi/2)(h - g_i) = (\pi h/2^{i+1})$. Obviously, $|(a_{i+1}/a_i)| = 1/2 < 1$. It then follows from Corollary 1 that the chaotic state of the pendulum system is driven to its zero equilibrium, and the controlled system (3) is eventually exponentially asymptotically stable. This control process is visualized by Fig. 2.

V. Conclusions

In this paper, we have developed a new impulsive control method for chaos suppression of a periodically forced pendulum system. Some simple and easily verified sufficient conditions for driving the chaotic state to the zero equilibrium have been presented, and some criteria for eventually exponentially asymptotical stability of the controlled system have been established. This work provides a rigorous theoretical analysis to support some early experimental observations on impulsive chaos control of the periodically forced pendulum system.

It would be interesting to compare the effects of this impulsive control with continuous feedback, which is a topic under further investigation. Besides, how to take advantage of certain special features of impulsive control for chaos synchronization and chaotification (making an originally nonchaotic system chaotic [8]) for some nonconventional applications is an even more interesting subject for future research.

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Output Feedback Stabilization of a Class of Wiener Systems

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***Abstract*—A globally stabilizing output feedback controller is designed for a class of continuous-time Wiener systems. The Wiener systems we consider consist of a linear dynamical block and an output polynomial nonlinearity connected in series. The (hybrid) controller consists of three modes of operation which are periodically applied to the system. The controller achieves a dead-beat response of the closed-loop system.**

***Index Terms*—Dead-beat, hybrid, output feedback, stabilization, Wiener systems.**

I. Introduction

This work is motivated by the need to further understand stabilization using partial state information, which is an issue that is at the core of current control theory research. While this question is well-understood for linear systems, it is very difficult to deal with in a general nonlinear situation. For instance, in contrast to linear systems, for general nonlinear systems it is in general not true that controllability and observability suffice for the existence of a (dynamic) output stabilizer. In [16], necessary and sufficient (but, except for certain special cases, cf. [17], hard to check) conditions for dynamic output regulation were obtained. Some recent references on the problem of stabilization using partial state feedback are [1]–[3], [9], and [18]. Since the problem is too difficult to deal with in general, it appears to be reasonable to try

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to understand it in particular situations of practical importance, such as the class of Wiener models considered here.

Wiener models consist of a linear dynamical block connected in series with a static (output) nonlinearity. The general continuous-time (SISO) Wiener model has the following form:

$$\dot{x} = Ax + bu \tag{1}$$

$$z = f(y) = f(cx) \tag{2}$$

where $x \in \mathbb{R}^n$, $u, y, z \in \mathbb{R}$ are, respectively, state of the system, control input, output of the linear subsystem, and output of the nonlinear system and $f(y)$ is a static nonlinearity. A discrete-time version of the model (1), (2) is also often used in the literature.

Wiener models have a long and rich history in the control systems literature and we review briefly some of the related results. Wiener models often arise from the so called "black-box identification" of nonlinear systems, which has been described in detail in the survey papers [4] and [5]. Observability of Wiener models with the output nonlinearities $f(y) = \text{sign}(y)$ and $f(y) = \text{sat}(y)$ was addressed, respectively, in [6] and [7]. Our work is most closely related to the results of [8] and [10]. In [8] a stabilizer was designed for Wiener models with output saturating nonlinearity, that is $f(y) = \text{sat}(y)$. Reference [10] deals with design of an input-to-state stabilizer for Wiener models with positive outputs and measurement disturbances d [in other words model (1), (2) is considered where $z = |y+d|$]. Finally, some related work on analysis of controllability and observability of several classes of Wiener–Hammerstein models can be found in [11]–[14].

The output feedback controller presented in this paper stabilizes under appropriate conditions Wiener systems of the form (1), (2) with polynomial output nonlinearity ($f(y) = \sum_{i=1}^{L} a_i y^i$, $a_i \in \mathbb{R}$). Due to different output nonlinearities, the design in [8] and the one presented here are notably different. On the other hand, results presented in [10] (for the case without disturbances $d(t) \equiv 0$) are more closely related to our work. In fact, the stabilizers for the class of systems considered in [10] can be easily modified to apply to Wiener systems whose output nonlinearity is of the form $f(y) = y^{2m}$, $m \in \mathbb{N}$, which is a particular class of systems considered here. Therefore, the results of the present paper can be regarded as an alternative approach for stabilization of the class of systems considered in [10]. More importantly, the stabilizing controller presented in this paper is applicable to a more general class of Wiener models with arbitrary polynomial output nonlinearity of the form $f(y) = \sum_{i=1}^{L} a_i y^i$, $a_i \in \mathbb{R}$. However, our controller may not perform well when the output is corrupted with measurement disturbances (whereas the controller in [10] is designed to deal with disturbances) and this may be an interesting topic for further research.

The paper is organized as follows. In Section II we present and comment on definitions and assumptions. In Section III we present the main result with the proof. A summary is given in the last section and several technical lemmas are stated with proofs in the Appendix.

II. Preliminaries

Consider the system

$$\begin{aligned} \dot{x} &= Ax + bu \\ z &= f(y) = f(cx) \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$, $u, z, y \in \mathbb{R}$ are, respectively, the state, input, output of Wiener system, and output of the linear subsystem and $f(s) = \sum_{i=1}^{L} a_i s^i$, $a_i \in \mathbb{R}$. It is assumed that the polynomial $f(s)$ is nontrivial, that is there exists $i \in \{1, \cdots, L\}$ such that $a_i \neq 0$. $|\cdot|$ denotes the absolute value of a number or the norm of a vector. $\|\cdot\|$ denotes the norm of a matrix. We denote controllability and observability matrices for the triple (A, b, c), respectively, as $\mathbf{C}(A, b)$ and $\mathbf{O}(A, c)$. Denote the trajectory of (3) at time t, starting from the initial state x and under the action of input $u_{[0,t]}$ as $\phi(t, x, u_{[0,t]})$. When the control input $u_{[0,t]}$ is clear from the context, we use the shorter notation $\phi(t, x)$. If constant input $u(t) \equiv c_1$, $t \geq 0$ is applied to (3), the solution that emanates from x is denoted as $\phi(t, x, c_1)$. We also use notation $z(t, x, c_1)$ and $y(t, x, c_1)$, which are, respectively, the output of the nonlinear system and output of the linear subsystem at time t that emanate from initial state x and with the control input $u(t) \equiv c_1$. Given arbitrary real numbers c_1 and c_2, we introduce the following set of states $\mathcal{X}[c_1 : c_2] := \{x \in \mathbb{R}^n: y(t, x, c_1) \equiv c_2, \forall t \geq 0\}$.

Definition 1: A controller stabilizes (3) if the following hold:

1) The origin is an equilibrium of the closed-loop system, that is $\phi(t, 0) = 0$, $\forall t \geq 0$.
2) For each initial state $x(0)$, the closed-loop state satisfies $\lim_{t\to\infty} |\phi(t, x(0))| = 0$.
3) For each $\varepsilon > 0$ there is some $\delta > 0$ such that, if $|x(0)| \leq \delta$, then the closed-loop state satisfies that $|\phi(t, x(0))| \leq \varepsilon$, $\forall t \geq 0$. □

In order to keep the formalism as simple as possible, we follow the presentations in [8] and [10] (see also [10, Remark 1]) and we do not define precisely the general meaning of "controller" and "closed-loop behavior." It will be clear from our constructions how one could represent our controller as a dynamic time-periodic "sampled-data like" system which operates on the continuous-time system (3). The following definitions are needed in the sequel.

Definition 2: The system (3) is 0-state detectable if for all x such that $z(t, x, 0) \equiv 0$ we have that the following hold:

1) $\lim_{t\to\infty} |\phi(t, x, 0)| = 0$;
2) for any $\epsilon > 0$ there exists $\delta > 0$ such that $|x| \leq \delta$ implies $|\phi(t, x, 0)| \leq \epsilon$, $\forall t \geq 0$. □

Definition 3: The system (3) is 0-state observable if $z(t, x, 0) = 0$, $\forall t \geq 0 \Longleftrightarrow x = 0$. □

We use the following assumptions and lemmas in the sequel.

A1) z is the only measured variable.
A2) (A, b, c) is a minimal triple.
A3′) The system (3) is 0-state detectable.
A3″) The system (3) is 0-state observable.
A3) Either A is nonsingular or 0 is the unique real root of $f(\cdot) = 0$.

Lemma 1 (A2 and A3′ $\Longrightarrow$ A3″): If (A, c) *is observable and the system (3) is 0-state detectable then (3) is 0-state observable.* □

Proof: Since the solutions of (3) are continuous and the polynomial $f(s)$ is a continuous function that has isolated real roots $\ell_i \in \mathbb{R}$, $i \in \{1, 2, \cdots, N\}$, $N \leq L$, we have that $z(t, x, 0) \equiv 0 \Longleftrightarrow x \in \bigcup_{i=1}^{N} \mathcal{X}[0 : \ell_i]$. Since A3′ holds, we can state a stronger claim.

If A3′ holds then $z(t, x, 0) \equiv 0 \Longleftrightarrow x \in \mathcal{X}[0 : 0]$.

($\Longleftarrow$) is trivial and we concentrate only on ($\Longrightarrow$). We use contradiction to prove this claim. Suppose that there exists $\ell \neq 0$ and $x \in \mathbb{R}^n$ such that $z(t, x, 0) \equiv 0 \Rightarrow x \in \mathcal{X}[0 : \ell]$ and the system is 0-state detectable. Then the following implications hold:

$$\begin{aligned} z(t, x, 0) \equiv 0 &\Longrightarrow c\phi(t, x, 0) = \ell, \qquad \forall t \geq 0 \\ &\Longrightarrow |\phi(t, x, 0)| \geq \frac{|\ell|}{|c|} > 0, \qquad \forall t \geq 0. \end{aligned}$$

Hence, $\lim_{t\to\infty} |\phi(t, x, 0)| \geq |\ell/c| > 0$, but this contradicts assumption A3′ and proves the claim. Using the claim we can write

$$z(t, x, 0) = 0, \forall t \geq 0 \Longleftrightarrow y(t, x, 0) = 0, \forall t \geq 0. \tag{4}$$

We again use contradiction to prove that the systems is 0-state observable. Suppose that the system is not 0-state observable and A2 holds. Then, there exists $x^* \neq 0$ such that $z(t, x^*, 0) = 0$, $\forall t \geq 0$. This implies [using (4)] that $y(t, x^*, 0) = ce^{At}x^* = 0$, $\forall t \geq 0$. However, this contradicts observability of (c, A) since $x^* \neq 0$. Q.E.D.

Lemma 2 (if A2, then A3″ ⇔ A3): Suppose that (A, c) is observable. Then, the system (3) is 0-state observable if and only if either 0 is the unique real root of $f(\cdot) = 0$ or A is nonsingular. □

Proof: If 0 is the unique real root of $f(\cdot) = 0$ it is obvious that observability of (A, c) is equivalent to 0-state observability of (3). Suppose now that $f(\cdot) = 0$ has at least one real root z^* other than zero. We need to show that nonsingularity of A is equivalent to 0-state observability of (3).

Necessity: Suppose A is nonsingular but (3) is not 0-state observable. This implies that there exists x^* such that $cx(t) = ce^{At}x^* \equiv z^* \neq 0$. Differentiate both sides with respect to t and we obtain $ce^{At}Ax^* \equiv 0$, $t \geq 0$, which from observability implies $Ax^* = 0$ and since A is nonsingular we have $x^* = 0$, which is a contradiction.

Sufficiency: Suppose A is singular and $cx(t) = ce^{At}x^* \equiv z^* \neq 0$. Then there exists $v^* \neq 0$ such that $cv^* = z^*$ and $Av^* = 0$, which contradicts 0-state observability since $x(0) = v^*$ yields $cx(t) \equiv 0$. We prove existence of v^* by contradiction. Suppose that there does not exists $v^* \neq 0$ such that $cv^* = z^*$ and $Av^* = 0$. Then the subspace $\{v\colon Av = 0\}$ must be confined to a plane H_1 parallel to the plane $\{v\colon cv = z^*\}$. Since $0 \in \{v\colon Av = 0\}$, we have $0 \in H_1$ and hence $H_1 = \{v\colon cv = 0\}$. However, if $\{v\colon Av = 0\} \subseteq H_1$, then the eigenvectors corresponding to 0 eigenvalues are in the null space of c contradicting observability of (A, c). Q.E.D.

Remark 1: In the sequel we design a controller that stabilizes (3) under Assumptions A1, A2, and A3$'$ and we briefly comment on each of the assumptions. Assumption A1 indicates that we deal with stabilization of the system (3) using only the output measurement z. Assumption A2 could be relaxed and our controller modified to require only that (A, b) is stabilizable and (c, A) detectable, which are obviously necessary conditions for the stabilization of (3) using output feedback (this can be seen by considering $f(s) = s$ and using the corresponding linear results). We use Assumption A2 only to simplify the exposition. Assumption A3$'$ is a necessary condition for stabilization using output feedback, as it was shown in [16]. Therefore, the stabilizer that we design is universal in a sense since it can be modified to stabilize any Wiener system (3) that is possible to stabilize using output feedback.

Lemma 1 is used to simplify the presentation since it shows that there is no loss of generality if instead of Assumption A3$'$ we use Assumption A3$''$. Moreover, Lemma 2 shows the equivalence of Assumptions A3$''$ and A3. Hence, we assume in the sequel that Assumptions A1, A2, and A3 hold. □

III. MAIN RESULT

The controller we design is a periodically time-varying "sampled-data like" scheme, which produces a dead-beat response of the closed-loop system. In this sense it is similar to controllers designed in [8] and [10]. The controller acts by cycling through three basic steps or "modes," each of the same duration nT. T is a strictly positive real number which can be thought of as a sampling period and n is the order of the plant. We first describe the purpose and role of each of the modes and then present the details. For simplicity, we describe only the first cycle on the time interval $[0, 3nT]$ (Mode 1–Mode 2–Mode 3) but the actual controller is periodic since after Mode 3 we switch back to Mode 1 on the time interval $[3nT, 4nT]$, then to Mode 2 on the interval $[4nT, 5nT]$, and so on. We note that the periodicity of the controller is not necessary to prove stability since we can apply another scheme which applies Modes 1–3 only once over the interval $[0, 3nT]$ and then $u \equiv 0$, $t \geq 3nT$. However, we use the periodic controller since it rejects any disturbance that acts over a finite time interval (nonpersistent disturbance) whereas the other scheme does not. An interesting open problem is to redesign the presented controller to achieve good performance under a larger class of disturbances, such as input-to-state stability of the closed loop for bounded (and perhaps persistent) measurement disturbances (see [10]).

1) In Mode 1, one applies a zero control and measures the output $z(t) := z(t, x(0), 0)$, $t \in [0, nT]$. The measured output is used to:
   a) detect if the initial state $x(0) = 0$;
   b) to compute a finite set, denoted as $\mathcal{X}$, of n dimensional vectors such that if $x(0) \neq 0$, then $x(0) \in \mathcal{X}$.
2) In Mode 2, we apply a piecewise constant control input $u(kT + t) = u(k) = const.$, $k = n, \cdots, 2n - 1$, $t \in [0, T)$ which is computed using $z(t)$, $t \in [0, nT]$ and the model of the system. The control input is such that:
   a) if $x(0) = 0$ then $u(k) = 0$, $k = n, \cdots, 2n - 1$;
   b) if $x(0) \neq 0$ then we can use the measured outputs $z(kT)$, $k = n, \cdots, 2n - 1$ in a test designed to select one vector η from the set $\mathcal{X}$ such that $x(0) = \eta$.
3) Mode 3 of the controller acts as a linear dead-beat controller that steers the state of the system from the (initial) state at time $2nT$ to the origin in over the time interval $[2nT, 3nT]$, that is $x(3nT) = 0$.

Remark 2: Modes 1 and 2 act as a dead-beat observer that computes the initial state $x(0)$ of the system from the measurements in finite time (in $2nT$ s). Note that this is not a pure sampled-data controller since we use $z(t)$, $t \in [0, nT]$ and not only $z(kT)$, $k = 0, \cdots, n - 1$ to reconstruct the initial state. □

A. Controller Description

We now present the details. The output $z(t)$, $t \geq 0$ is measured and used to compute the control signal which is a piecewise constant function of time $u(t) = u(kT) =: u(k) = const.$, $t \in [kT, (k + 1)T)$. The discrete-time model of the plant that describes the systems at sampling instants kT is

$$\begin{aligned} x(k+1) &= Fx(k) + gu(k) \\ z(k) &= f(cx(k)) \end{aligned} \tag{5}$$

where $F := e^{AT}$; $g := \int_0^T e^{As}b\,ds$. We use notation $x(k) := x(kT)$, $y(k) := y(kT)$, $z(k) := z(kT)$. Since assumption A2 holds, without loss of generality we can assume that T is such that (F, g, c) is a minimal triple and $cg \neq 0$. To simplify exposition we describe only the first cycle of the periodic controller.

Mode 1 ($t \in [0, nT]$): Apply $u(k) = 0$, $k = 0, 1, \cdots, n - 1$ and measure $z(t) = z(t, x(0), 0)$, $t \in [0, nT]$. The measured output $z(t)$ is used to compute a variable σ_0 (that has value either 0 or 1) and a finite set of n dimensional vectors $\mathcal{X}$ in the following manner:

Computation of σ_0

$$\sigma_0 := \begin{cases} 0, & \text{if } \displaystyle\int_0^{nT} |z(s)|\,ds = 0; \\ 1, & \text{if } \displaystyle\int_0^{nT} |z(s)|\,ds \neq 0. \end{cases} \tag{6}$$

Computation of $\mathcal{X}$: For each $z(k)$, $k \in \{0, 1, \cdots, n - 1\}$ we form the following set of real numbers

$$\{\xi_k^{i_k} \in \mathbb{R}\colon f(\xi_k^{i_k}) = z(k),\ i_k = 1, \cdots, l_k\}$$

(note that $l_k \leq L$). Form the column vectors $\eta_j := (\xi_0^{i_0}\ \xi_1^{i_1} \cdots \xi_{n-1}^{i_{n-1}})^T$, for all possible combinations of $i_k \in \{1, 2, \cdots, l_k\}$, $k = 0, \cdots, n - 1$ and note that there is at most $\hat{L} \leq L^n$ such vectors (L is the degree of the polynomial and n is the order of the system). Compute the vectors

$$\hat{x}_j := \mathbf{O}^{-1}(F, c)\eta_j, \qquad j = 1, 2, \cdots, \hat{L}$$

where $\mathbf{O}(F, c)$ is the (nonsingular) observability matrix for the pair (F, c). Finally, we introduce the set

$$\mathcal{X} := \{\hat{x}_j\colon \hat{x}_j \neq 0, \qquad j = 1, 2, \cdots, \hat{L}\}. \tag{7}$$

Remark 3: The system is assumed 0-state observable (Assumption A3) and we have that $\sigma_0 = 0$ if and only if $x(0) = 0$. Hence, the variable σ_0 acts as a "0-state detector." Also, if $x(0) \neq 0$, then $x(0) \in$

$\mathcal{X}$ and hence $\mathcal{X}$ is a nonempty set. Note that the zero vector is never included in the set $\mathcal{X}$. Consequently, $\mathcal{X}$ may be an empty set when $\hat{L} = 1$ and $\hat{x}_1 = x(0) = 0$! Note also that $\sigma_0 = 0$ if $\mathcal{X} = \emptyset$. □

Mode 2 ($t \in [nT, 2nT]$): In Mode 2 we apply the sequence of controls $u(k) = \alpha_k v$, $k = n, n+1, \cdots, 2n-1$ and measure the sequence of outputs $z(k)$, $k = n+1, \cdots, 2n$ which is used in a test to single out a vector $\hat{x}_j^*$ in the set $\mathcal{X}$ such that $x(0) = \hat{x}_j^*$. The choice of α_k and v, as well as the test are discussed below.

Choice of α_k: α_k are chosen so that

$$z(k) = f(cF^k x(0) + v), \qquad \forall k \in \{n+1, \cdots, 2n\}. \tag{8}$$

Lemma 3 in the Appendix guarantees that $\forall v \in \mathbb{R}$ there exist $\alpha_k \in \mathbb{R}$, $k = n, \cdots, 2n-1$ that satisfy (8).

Choice of v: To choose appropriate v we first introduce

$$\sigma_1 := \begin{cases} \min\limits_{\eta_j \in \mathcal{X}} |\eta_j|, & \text{if } \sigma_0 = 1 \\ 1, & \text{if } \sigma_0 = 0. \end{cases}$$

Suppose $\mathcal{X}$ is nonempty and form the set

$$\begin{aligned} \mathcal{V} := \{v \in \mathbb{R}\colon & f(cF^k \eta_i + v) = f(cF^k \eta_j + v), \\ & cF^k \eta_j \neq cF^k \eta_i,\ \eta_j, \eta_i \in \mathcal{X}, \\ & i, j \in \{1, \cdots, \hat{L}\},\ k \in \{n+1, \cdots, 2n\}\}. \end{aligned}$$

The set $\mathcal{V}$ is finite (this follows from Lemma 4 in the Appendix and the definition of the set). We introduce the variable

$$\sigma_2 := \begin{cases} \min\limits_{v \in \mathcal{V},\, v \neq 0} |v|, & \text{if } \sigma_0 = 1 \\ 1, & \text{if } \sigma_0 = 0. \end{cases}$$

From Lemma 5 in the Appendix it follows that if $x(0) \neq 0$ then for any $v \notin \mathcal{V}$ we have that there exists a unique $\hat{x}_j^* \in \mathcal{X}$ such that $z(k) = f(cF^k \hat{x}_j^* + v)$, $\forall k = n+1, \cdots, 2n$, and moreover $x(0) = \hat{x}_j^*$. It is also proved in Lemma 5 that an appropriate choice of v is

$$v = 0.5 \min\{\sigma_0, \sigma_1, \sigma_2\}. \tag{9}$$

The sequence of controls that we apply in Mode 2 is

$$u(k) = 0.5\alpha_k \min\{\sigma_0, \sigma_1, \sigma_2\}, \qquad k = n, \cdots, 2n-1. \tag{10}$$

Test: If $\sigma_0 = 0$, then $x(0) = 0$. If $\sigma_0 = 1$, then for every $\hat{x}_j \in \mathcal{X}$ compute

$$\zeta_j(k) := f(cF^k \hat{x}_j + 0.5 \min\{1, \sigma_1, \sigma_2\}), \qquad k = n+1, \cdots, 2n$$

and pick $\hat{x}_j^* \in \mathcal{X}$ such that the measured outputs $z(k)$ and computed $\zeta_j(k)$ satisfy the test

$$z(k) = \zeta_j(k), \qquad \forall k = n+1, \cdots, 2n. \tag{11}$$

Lemma 5 in the Appendix and our choice of v guarantee that there is only one $\hat{x}_j^* \in \mathcal{X}$ satisfying (11) and we let $x(0) = \hat{x}_j^*$.

Remark 4: From Lemma 5 in the Appendix, we can see that any sequence of controls $u(k) = \alpha_k v$ such that $v \notin \mathcal{V}$ can be used in the test above to pick the correct initial state $x(0)$ from the set $\mathcal{X}$. However, in order to prove stability of the origin of the closed loop system we needed to introduce also σ_0 and σ_1 in our definition of v. The properties of v defined by (9) that are needed in the stability proof are as follows. If $x(0) = 0$ then $\sigma_0 = 0$ and this implies $v = 0$. As a result, we have $u(k) = 0$, $k = n, \cdots, 2n-1$. Moreover, since for arbitrary $x(0) \neq 0$ we have $v \le 0.5\sigma_1 \le 0.5|x(0)|$ then there exists $\overline{K}_0 > 0$ such that for any $x(0) \in \mathbb{R}^n$ we have that $|u(k)| \le \overline{K}_0 |x(0)|$, $\forall k = 0, 1, \cdots, 2n-1$. □

Mode 3 ($t \in [2nT, 3nT]$): Using the initial state $x(0)$ that we obtained in Mode 2 and the known sequence of controls $u(k)$, $k = 0, 1, \cdots, 2n-1$ we compute $x(2n)$. The sequence of controls that is applied in Mode 3 ($u(k)$, $k = 2n, \cdots, 3n-1$) is computed as follows:

$$(u(2n) \cdots u(3n-1))^T = -\mathbf{C}^{-1}(F, g) F^n x(2n)$$

where $\mathbf{C}(F, g)$ is the nonsingular controllability matrix for the pair (F, g). Note that F is also nonsingular. This control sequence transfers the state $x(2n)$ to the origin in nT seconds, that is $x(3n) = 0$.

Theorem 1: If Assumptions A1, A2, and A3 hold then the periodic controller consisting of Modes 1, 2, and 3 stabilizes (3). □

Proof: First, we show that the origin is an equilibrium for the closed-loop system. If $x(0) = 0$, then $\phi(t, 0) = 0$, $\forall t \in [0, nT]$. Moreover, we have that $\sigma_0 = 0$ and the sequence (10) is a zero sequence, which implies $\phi(t, 0) = 0$, $t \in [nT, 2nT]$. This implies that $u(k) = 0$, $k = 2n, \cdots, 3n-1$ and $\phi(t, 0) = 0$, $t \in [2nT, 3nT]$. By induction we obtain that $\phi(t, 0) = 0$, $\forall t \ge 0$, and the first condition of Definition 1 holds.

Since for any initial state $x(0)$ we have $x(3n) = 0$, one can conclude (using an argument similar to the one above) that the origin of the closed loop system is globally attractive in finite time, that is

$$\phi(t, x(0)) = 0, \qquad \forall t \ge 3nT,\ \forall x(0) \in \mathbb{R}^n. \tag{12}$$

Hence, the second condition of Definition 1 holds.

Since $u(k) \le \overline{K}_0 |x(0)|$, $\forall k = 0, 1, \cdots, 2n-1$ (from Remark 4), there exists a positive constant $\overline{K}_1$ such that $|x(2n)| \le \overline{K}_1 |x(0)|$. Moreover, since (F, g) is controllable and F nonsingular there exists a positive constant $\overline{K}_2$ such that we have

$$\begin{aligned} |u(k)| &\le \left\|\mathbf{C}^{-1}(F, g)\right\| \|F^n\| |x(2n)| \\ &\le \left\|\mathbf{C}^{-1}(F, g)\right\| \|F^n\| \overline{K}_1 |x(0)| =: \overline{K}_2 |x(0)| \end{aligned} \tag{13}$$

for $k = 2n, \cdots, 3n-1$. Using the inequality above and Remark 4, we conclude that there exists a positive number K_1 such that $|u(k)| \le K_1 |x(0)|$, $\forall k = 0, 1, \cdots, 3n-1$ (just define $K_1 := \max\{\overline{K}_0, \overline{K}_2\}$). This further implies that there is a positive real number K_2 such that

$$|\phi(t, x(0))| \le K_2 |x(0)|, \qquad \forall t \in [0, 3nT]. \tag{14}$$

By combining (12) and (14) we have that $|\phi(t, x(0))| \le K_2 |x(0)|$, $\forall t \ge 0$. Finally, the third condition of Definition 1 holds since $\forall \epsilon > 0$ there exists $\delta > 0$ such that $|x(0)| < \delta$ implies $|\phi(t, x(0))| < \epsilon$, $\forall t \ge 0$ (just take $\delta = \epsilon / K_2$). Hence, the controller is stabilizing for (3). Q.E.D.

IV. Summary

A globally stabilizing output feedback controller is designed for a class of Wiener systems that satisfy the following conditions: the linear subsystem is controllable and observable; and the Wiener model is 0-state detectable. The controller is periodically time-varying "sampled-data like" scheme which achieves a dead-beat response of the closed loop-system.

Appendix

Lemma 3: Suppose that $u(k) = 0$, $\forall k = 0, 1, \cdots, n-1$. Given any $v \in \mathbb{R}$ there exist $\alpha_k \in \mathbb{R}$, $k = n, \cdots, 2n-1$ such that the sequence of controls $u(k) = \alpha_k v$, $k = n, \cdots, 2n-1$ yields

$$z(k) = f(cF^k x(0) + v), \qquad \forall k \in \{n+1, \cdots, 2n\}. \quad \square$$

Proof: In order to have the desired sequence of controls notice that the following matrix equation has to be solvable in α_i:

$$\begin{pmatrix} cg & 0 & 0 & \cdots & 0 \\ cFg & cg & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots \\ cF^{n-1}g & cF^{n-2}g & cF^{n-3}g & \cdots & cg \end{pmatrix} \begin{pmatrix} \alpha_n \\ \alpha_{n+1} \\ \vdots \\ \alpha_{2n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

and this is satisfied since the Hankel matrix is invertible (because the sampling period T is chosen so that $cg \neq 0$). Q.E.D.

Lemma 4: Consider an arbitrary nontrivial polynomial $f(s) = \sum_{i=1}^{L} a_i s^i$. Let $c_1, c_2 \in \mathbb{R}$ be fixed. Then we have that $f(c_1 + u) = f(c_2 + u)$, $\forall u \in \mathbb{R}$ if and only if $c_1 = c_2$. □

Proof: If $c_1 = c_2 = c$, then it is obvious that $f(c + u) = f(c + u)$, $\forall u \in \mathbb{R}$. Suppose now that there exist $c_1 \neq c_2$ and a nontrivial polynomial $f(s) = \sum_{i=1}^{L} a_i s^i$ such that $f(c_1 + u) = f(c_2 + u)$, $\forall u$. Hence, all coefficients of the polynomials $f(c_1 + u)$ and $f(c_2 + u)$ must be identical. Consider the coefficients that multiply u^{L-1} in both polynomials. Then we obtain that $La_L c_1 + a_{L-1} = La_L c_2 + a_{L-1}$, which implies $c_1 = c_2$, a contradiction. Q.E.D.

Lemma 5: Suppose that $x(0) \neq 0$, we apply the control sequence as in Lemma 3 and $v \notin \mathcal{V}$. Then there exists a unique $\hat{x}_j^* \in \mathcal{X}$ such that

$$f(cF^k \hat{x}_j^* + v) = z(k), \qquad \forall\, k = n+1, \cdots, 2n. \tag{15}$$

In particular, if $v = 0.5 \min\{\sigma_0, \sigma_1, \sigma_2\}$, then $v \notin \mathcal{V}$ and there exists a unique $\hat{x}_j^* \in \mathcal{X}$ that satisfies (15). Moreover, we have that $\hat{x}_j^* = x(0)$.

Proof: Since $x(0) \neq 0$, then $\mathcal{X}$ is a nonempty set and $x(0) \in \mathcal{X}$. Hence, there exists at least one $\hat{x}_j^* \in \mathcal{X}$ such that $f(cF^k \hat{x}_j^* + v) = z(k),\ \forall\, k = n+1, \cdots, 2n$. Suppose now that there exist $\hat{x}_j^*, \hat{x}_i^* \in \mathcal{X}$, $\hat{x}_j^* \neq \hat{x}_i^*$ such that $f(cF^k \hat{x}_j^* + v) = f(cF^k \hat{x}_i^* + v) = z(k),\ \forall\, k = n+1, \cdots, 2n$. From the choice of the control sequence (10) and Lemma 4 we have that this can happen if and only if $cF^k \hat{x}_j^* = cF^k \hat{x}_i^*,\ \forall\, k = n, \cdots, 2n-1$. Since F is nonsingular and (F, c) is observable, this implies that $\hat{x}_j^* = \hat{x}_i^*$, a contradiction. If $x(0) \neq 0$ then $\mathcal{V}$ is a finite nonempty set and we have $\sigma_0 = 1$ (see Remark 3), $\sigma_1 > 0$ and $\sigma_2 > 0$ (by definition). From definition of σ_2 and the set $\mathcal{V}$ it follows that $(0, \sigma_2) \cap \mathcal{V} = \emptyset$. Hence, if we choose $v = 0.5 \min\{\sigma_0, \sigma_1, \sigma_2\}$ then $v \in (0, 0.5\sigma_2] \subset (0, \sigma_2)$ and hence $v \notin \mathcal{V}$. Now it is obvious that (the unique) $\hat{x}_j^* \in \mathcal{X}$ that satisfies (15) is such that $x(0) = \hat{x}_j^*$. Q.E.D.

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Rational Suboptimal Continuous-Time Controller Design

Mark E. Halpern

***Abstract*—We obtain rational suboptimal continuous-time solutions to some optimal control problems specified via time domain performance criteria. These include $\mathcal{L}_1$ and $\mathcal{L}_\infty$ norms and peak overshoot. The approach uses linear semi-infinite programming to compute weights for a given finite set of rational basis functions and makes the best possible use of the basis. The closed-loop transfer functions obtained satisfy appropriate interpolation constraints for internal stability and the formulation allows additional linear constraints to be incorporated.**

***Index Terms*—Feedback systems, l_1 optimization, interpolation, linear semi-infinite program.**

I. Introduction

A great deal of recent work on controller design has developed from the l_1 optimal design method, proposed for the optimal rejection of persistent bounded disturbances [1]–[3]. The continuous-time version of the problem, that of designing a stabilizing continuous-time controller to minimize the $\mathcal{L}_1$ norm of the error impulse response for continuous-time plants, was posed in [2] and completely solved in [4]. It was found in [4] that for a rational plant, the optimal closed-loop transfer function is irrational and thus an irrational controller is required. This is in contrast to the discrete-time case [1], [3] where the optimal closed-loop transfer function is obtained with a rational controller.

The difficulty of implementing irrational controllers for the continuous-time problem motivated three approaches, [5]–[7] and [9], for obtaining rational suboptimal compensators. One [5] involves approximating the true optimal solution with a finite-dimensional rational solution. The second, [6], [7], is based around Euler Approximating Systems (EAS) and involves minimizing a computationally tractable upper bound on the actual $\mathcal{L}_1$ norm of the corresponding continuous-time transfer function. The same kind of upper bound has subsequently been minimized to obtain l_1-suboptimal pole placement [8]. The third approach [9] minimizes a different upper bound on the $\mathcal{L}_1$ norm and can give a smaller $\mathcal{L}_1$ norm than [6], [7] with lower order controllers. Mul-

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