Stability of Wireless and Wireline Networked Control Systems

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***Abstract*— This paper provides a general framework for analyzing the stability of general nonlinear networked control systems (NCS) with disturbances in the setting of $\mathcal{L}_p$ stability. We build on the presentation in [1] to provide sharper results for both $\mathcal{L}_p$ gain and maximum allowable transfer interval (MATI) and detail the property of uniformly persistently exciting scheduling protocols. This class of protocols were shown to lead to $\mathcal{L}_p$ stability for high enough transmission rates and were a natural property to demand, especially in the design of wireless scheduling protocols. The property is used directly in a novel proof technique based on the notions of vector comparison and (quasi)monotone systems. We explore these results through analytical comparisons to those in the literature as well as through simulations and numerical comparisons that verify that the uniform persistence of excitation property of protocols is, in some sense, the "finest" property that can be extracted from wireless scheduling protocols.**

***Index Terms*— networked control, stabilization, wireless networks, scheduling.**

I. Introduction

CENTRAL to the study of networked control systems (NCS) is the analysis and design of scheduling protocols. NCS depart from the use of dedicated point-to-point links for connectivity amongst sensors, controllers and actuators (NCS nodes), replacing some or all links with a shared network channel. As in traditional data networks, the problem of arbitrating multiple access on the channel becomes an issue, motivating the discussion of the *scheduling* of nodes and the design and analysis of scheduling protocols suitable for NCS applications. By scheduling, we mean the transmission of information across a link in the form of a discrete packet or frame.

Canonical NCS examples include so-called by-wire systems: drive-by-wire and fly-by-wire with analogues in industrial applications. Here, the *network* in NCS is thought of as in the sense of a traditional data network. More abstractly, an NCS can be thought of as any collection of nodes and a control law where the exchange of information (sensor and controller values) is governed by a scheduling protocol. Examples of "abstract" NCS include:

Example 1.1 (Traditional control systems): The scheduling protocol is trivial: nodes communicate via dedicated point-to-point connections and there are no constraints on information exchange to consider.

Example 1.2 (Embedded digital control systems): Transmission of controller and sensor values to and from the device executing the control law is governed by protocols of electrical bus e.g., a PCI bus, and, typically, the scheduler of an operating system. Even if the underlying control system employs point-to-point connections from nodes to the controller, communication within the controller and its constituent components are subject to the communication constraints of various electrical buses and the operating system.

Example 1.2 is one of the strongest motivations for studying NCS. It is perhaps taken for granted that the digital control systems designed and deployed in industry will continue to behave like their idealized continuous-time (resp., discrete-time) counterparts, save for the effects of sampling and quantization. As control systems increase in size and complexity and the levels of component integration increase, the flow of data between elements of the system is subject to constraints similar to that of a "real" network. Indeed, components of systems based around the PCI-Express architecture communicate via a switched serial network. Regardless of how controllers and sensors are connected, at least internally, every non-trival digital control system can be thought of as an NCS.

From designs based around traditional wireless and wireline networks to the growing internal complexity of "un-networked" control systems, an increasing number of practical NCS implementations and their respective traffic scheduling protocols now exist. Standards-based component connectivity offers lower implementation costs, greater interoperability and a wide range of choices in developing control systems. The price paid for these advantages is the added complexity in the initial design and analysis of NCS. As alluded to earlier, part of this complexity comes in the form of issues of arbitration of network access amongst links, or *scheduling*, which is of fundamental importance, but above and beyond scheduling, NCS also presents the designer with the limitations of

1) finite bandwidth of communication channels;
2) finite precision of encoding and decoding schemes for transmitted information;
3) pure (propagation) delays of channels;
4) and data dropouts from unreliable channels.

These limitations are not mutually exclusive, however, as transmission rates increase and with frame and packet sizes well in excess of machine (CPU) precision, effects of quantization and pure delay play an increasingly diminishing role in the analysis of most NCS and we forego their treatment in this paper.

A survey of scheduling and scheduling protocols is provided in [2] and stability and performance results of wireline NCS have been examined in [3], [2], [4], [5], [6] and [7]. Although

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round-robin (RR) scheduling is used almost exclusively in practice, the aforementioned works present various alternative protocols that demonstrate a performance gain over RR in simulations and, in special cases, demonstrate the superiority of the alternative protocols analytically. The NCS design approach adopted in [6], [8] and [4], [7], [3], [1], and this paper consists of the following steps:

1) design of a stabilizing controller ignoring the network;
2) and analysis of robustness of stability with respect to effects that scheduling within a network introduce.

The principal advantage of this approach is its simplicity – the designer of the NCS can exploit familiar tools for controller design and select an appropriate scheduling protocol and transmission rate such that the desired properties of the network-free system are preserved.

Let $\hat{y}(t)$ denote the most recently transmitted vector of output measurements and $\hat{u}(t)$ the most recently transmitted vector of control values. For a (multiple-input multiple output) MIMO NCS, the transmission of an output measurement from node i will result in $\hat{y}_i(t^+) = y_i(t)$ and the remaining components of $\hat{y}$ will be "stale", that is, $\hat{y}_j(t^+) = \hat{y}_j(t)$ for all $j \neq i$. The transmission of a control value to actuator i is analogous. We can now define the error term $e = (\hat{y} - y, \hat{u} - u)$.

The first analytic proof of stability (uniform global exponential stability) for linear NCS designed with this approach was provided in [6] that also introduced the try-once-discard (TOD) scheduling protocol. The idea behind TOD is to associate the error term e_i with each link i. At each transmission instant, the scheduler will select the node with the largest magnitude of error for transmission.

Implicit in this setup is that $\hat{y}$ and $\hat{u}$ are generally kept fixed between transmission instants and no estimation or decoding and encoding of transmissions take place. The NCS model we introduce does not preclude the use of such schemes, however, we only analyze their effects in terms of scheduling and their influence on the scheduling protocol.

Denoting the combined plant and controller state as x, [6], [8] and [4] model the dynamics of x and e *between network transmission instants with*:

$$\dot{x}(t) = f(t, x, e) \tag{1}$$

$$\dot{e}(t) = g(t, x, e) \tag{2}$$

and describe the node scheduling protocol *at transmission instants* via its effects on a particular component of e. The performance bounds and stability results in [6], [8] and [4] treat e as a perturbation to the nominal system $\dot{x}(t) = f(t, x, 0)$ which is assumed to be a priori stable.

The notion of a scheduling protocol was made precise in [7] and [3] which present a jump-continuous (hybrid system) model of general nonlinear NCS with disturbances of the form:

$$\dot{x}(t) = f(t, x, e, w) \quad \forall t \in [t_{i-1}, t_i] \tag{3}$$

$$\dot{e}(t) = g(t, x, e, w) \quad \forall t \in [t_{i-1}, t_i] \tag{4}$$

$$e(t_i^+) = h(i, e(t_i)), \tag{5}$$

where $\{t_n\}_{n \in \mathbb{N}}$ is a sequence of transmission instants and h is the effect of the scheduling protocol on the error e at the ith transmission instant. Ignoring the dynamics introduced by (4), we can regard (5) as discrete-time system that captures the behavior of the scheduling protocol. This notion of describing the protocol in this fashion allows one to speak of uniformly globally asymptotically and exponentially stable (UGAS and UGES) scheduling protocols whenever the associated discrete-time system is UGAS or UGES. Beyond taxonomy, the notion of UGES and UGAS protocols and the construction of smooth Lyapunov functions for the associated UGAS and UGES discrete-time systems is central to the stability analysis approach developed in [7] and [3]. Inferring stability of the NCS employing arbitrary UGES or UGAS scheduling protocols is tantamount to constructing an appropriate Lyapunov function, which may be difficult in general. Compared to [6], [3] provides better MATI bounds for both TOD and RR; requires less conservative conditions on the NCS to conclude stability results (linear-gain IOS) using a novel small-gain technique and the use of appropriate Lyapunov function constructions for auxiliary UGES and UGAS systems.

Intuition suggests that schemes such as TOD should perform better than RR, as the node with the greatest error is transmitted at each transmission instant. Indeed, it has been shown analytically in [3] that TOD yields better performance than RR, *when the scheduler has access to the entire error vector e at scheduling instants*. TOD is certainly implementable in variants of CAN[1] as the error can be encoded into an arbitration field in a frame but no such arbitration is possible for wireless channels and, indeed, many wireline channels and, hence, it is often unreasonable to assume knowledge of the entire error vector.

Several variants of TOD were introduced in [8] that "estimate" the error vector and were shown to outperform RR in simulations. Stability results are also provided for linear systems that lead to conservative estimates on performance bounds. Casting these variants of TOD in the framework of [3] requires a more general model than (3)-(5). One model of NCS that accommodates these variants was proposed in [1]

The variants of TOD presented in [8] as well as the RR scheduling protocol satisfy the following property: *there is a fixed number of transmissions T such that all nodes of the NCS have transmitted within T transmissions.* This T is related to the notion of a node's "silent-time" in [8]. This property is the point of departure of this paper and, for reasons that will become apparent, we call protocols that satisfy this property *uniformly persistently exciting scheduling protocols*, or simply, PE protocols. Whenever T is known, we say that the protocol is PE_T. We prove that all PE protocols lead to $\mathcal{L}_p$ stable NCS for high enough transmission rates when the network-free system is $\mathcal{L}_p$ stable and provide sharper performance bounds than [3] for NCS employing RR scheduling, the foremost example of a PE protocol and a protocol that is widely used in industry.

Our analysis framework analyzes the input-output $\mathcal{L}_p$ stability (IOS) of NCS, the essence of which is that outputs (or state) of an NCS verify a robustness property with respect to exogenous disturbances. This notion of stability is the cornerstone of modern robust and optimal control (see [9],

[1]Control Area Network.

for instance) and closely related to the notion of input-to-state stability (ISS). In fact, if the network-free system is $\mathcal{L}_p$ stable, we show that the NCS remains so with any scheduling protocol that always visits its links within every $T < \infty$ consecutive transmissions, whenever transmissions occur faster than once every τ^* secs, where

$$\tau^* = \frac{\ln(z)}{|A|T}, \tag{6}$$

and satisfies

$$\gamma \exp(-1) T z^{1+1/T} + z(|A| - \gamma \exp(-1)T) - 2|A| = 0,$$

with $z \in [1, 2]$. The parameter γ captures robustness properties of the network-free system and $|A|$ captures the rate at which the NCS departs from it nominal network-free behavior in the absence of exogenous disturbances. In particular, when $\gamma = 0$, $\tau^* = \ln(2)/(|A|T)$. Under additional technical conditions, $\mathcal{L}_p$ stability specializes to uniform global exponential stability (UGES) in the absence of exogenous disturbances.

The condition that a scheduling protocol visits all its links is particularly important in our analysis and, for reasons that will become apparent, we say that a scheduling protocol is uniformly persistently exciting in time T, or just PE_T, if *there is a fixed number $T < \infty$ such that all links of the NCS have transmitted within every T consecutive transmissions.* The technological limitations of communications channels, especially wireless channels, leave us with few guarantees about what information is available to the scheduler for making its scheduling decisions or it may be the case that the communication channel is using a collision protocol, in which case we have very few guarantees at all. Save for RR, the protocols described and analyzed in the literature (see [3], [2] and [10], for instance) require more information (NCS state information) than is typically available, particularly in a wireless setting. The results presented in [8] partially address the issue, concluding UGES for linear systems on wireless networks when using the protocols described therein. The results in this paper address the issue more precisely and in greater generality through PE and $\mathcal{L}_p$ stability.

PE in the sense we have described is verified by many network technologies. Ethernet and 802.11 are examples of CSMA/CD protocols where it is known (see [11], for instance) that for a finite number of users (links), the expected waiting time for a link is finite. Although our results are cast in a deterministic setting, we mention collision protocols to motivate how natural the notion of PE is in the context of real networks and to suggest that similar results maybe pursued in a stochastic setting. and many protocols described in the aforementioned NCS literature. Building on the informal definition of PE, the primary contributions of this paper are:

1) that we characterize the fundamental notion of PE scheduling protocols and demonstrate the importance of PE in stability analysis;
2) we generalize the NCS model presented in [3] to handle a wider class of scheduling protocols with a focus on PE and protocols implementable on wireless channels – in particular, RR, every protocol presented in [8] and the hybrid-TOD protocol that we introduce are all PE and can be analyzed with our framework;
3) under general conditions we show that every PE scheduling protocol leads to the $\mathcal{L}_p$ stability of general nonlinear NCS with disturbances whenever the network-free system is $\mathcal{L}_p$ stable with sharper $\mathcal{L}_p$ gain and MATI bounds than [1];
4) and we compare our results to those presented in [3] and [8] and show that we achieve MATI bounds that are analytically asymptotically larger for linear systems for many protocols, including RR. To that end, two case studies are presented and our results are also compared to simulation-based MATI bounds that place our theoretically-obtained MATI bound within an order of a magnitude of upperbounds for the "true" MATI.

A. Paper Outline

The paper is divided into eight additional sections: Section II presents preliminary definitions and results. Section III presents our model for NCS and outlines the limitations inherent in the design of scheduling protocols. The notion of PE is described in detail and characterized in Section IV, along with examples of NCS scheduling protocols. Section V contains our main results on the $\mathcal{L}_p$ stability of the error dynamics and $\mathcal{L}_p$ stability of the NCS as a whole together with several remarks on their use. Simulations, analytic comparisons and case studies are explored in Section VI and the proofs of the main results are gathered in Section VII. Concluding remarks are provided in Section VIII and several technical lemmas and results are developed in the Appendix.

II. Preliminaries

A. Notation

$\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$ denote, respectively, the sets of real, nonnegative real and natural numbers with the convention that $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}^+ = \{1, 2, \dots\}$. Let $\mathcal{K}$ denote the class of continuous functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that satisfy $f(0) = 0$ and $f(t_1) < f(t_2)$ for any $0 \leq t_1 < t_2$. We say that $f \in \mathcal{K}$ is of class $\mathcal{K}_\infty$ if it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if for each $s \geq 0$ the function $\beta(s, \cdot)$ is decreasing to zero in the second argument and for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$. A function $g : \mathbb{R} \to \mathbb{R}$ is said to be of order ϕ if there exists a constant K such $|g(t)| < K\phi(t)$ for $t \geq 0$ and we write $g = O(\phi)$. Given $t \in \mathbb{R}$ and a piecewise continuous function $f : \mathbb{R} \to \mathbb{R}^n$, we use the notation $f(t^+) = \lim_{s \to t, s > t} f(s)$. All vector (Euclidean) norms are denoted by $|\cdot|$, as is the induced matrix 2-norm. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a (Lebesgue) measurable function and define

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(s)|^p ds \right)^{1/p}$$

for $1 \leq p < \infty$ and define

$$\|f\|_\infty = \inf_{\text{vol}(S)=0} \sup_{t \in \mathbb{R} \setminus S} |f(t)|.$$

We say that $f \in \mathcal{L}_p$ for $p \in [1,\infty]$ whenever $\|f\|_p < \infty$. Let $f : \mathbb{R} \to \mathbb{R}^n$ and let $[a,b] \subset \mathbb{R}$. We use the notation

$$\|f[a,b]\|_p := \left(\int_{[a,b]} |f(s)|^p ds \right)^{1/p}$$

to denote the $\mathcal{L}_p$ norm of f when restricted to the interval $[a,b]$. Let $\mathbf{0}$ denote the zero matrix. We define a collection of sequences of $n \times n$ matrices, $\mathscr{S}_n(T)$, such that

$$\{A_i\}_{i\in\mathbb{N}} \in \mathscr{S}_n(T) \iff \prod_{i=s}^{s+T-1} A_i = \mathbf{0}$$

for all $s \in \mathbb{N}$.

Let $\mathcal{A}_n$ denote the set of all $n \times n$ matrices and let $\mathcal{A}_n^+$ denote the subset of all matrices that are positive semi-definite, symmetric and have positive entries and let $\mathbb{R}_+^n$ denote the nonnegative orthant. We will need to use the notion of a partial order on a vector space. Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$. The partial order $\preceq$ is given by:

$$x \preceq y \iff (\forall i \in \{1,\dots,n\})\, x_i \le y_i. \tag{7}$$

It is also possible to define an analogous partial order on elements of $\mathcal{A}_n^+$ in the natural way:

$$A \preceq B \iff B - A \in \mathcal{A}_n^+$$

A thorough discussion of the properties and consequences of the partial order $\preceq$ and the induced partial order on elements of $\mathcal{A}_n$ is given in Appendix I-B. We will often consider vectors of the form $\overline{x}$, where $x \in \mathbb{R}^n$ and $\overline{x} = (|x_1|, \dots, |x_n|)^T$. That is, $\overline{x}$ is the vector that results from taking the absolute value of each scalar component of x. It is important to note that the $|\overline{x}| = |x|$. For a function $f : \mathbb{R} \to \mathbb{R}^n$, $\overline{f}$ denotes the function that takes components of the image of f to their absolute value. That is,

$$t \overset{\overline{f}}{\mapsto} \overline{f(t)}.$$

Finally, let $Df(t)$ denote the left-handed derivative of $f : \mathbb{R} \to \mathbb{R}^n$:

$$Df(t) = \lim_{h\to 0, h<0} \frac{f(t+h) - f(t)}{h}.$$

B. Underlying Stability Theory

Let $\{t_i\}_{i=0}^{\infty}$ be a sequence of increasing time instants such that $0 < t_{i+1} - t_i < \tau < \infty$ for all $i \in \mathbb{N}$. Consider the hybrid system

$$\Sigma_z : \dot{z} = f(t,z,w) \quad t \in [t_i, t_{i+1}] \tag{8}$$

with "jump" equation

$$z(t_i^+) = h(i, z(t_i)), \tag{9}$$

initialized at (t_0, z_0) with input w and a prescribed output

$$y = g(t,z).$$

We assume enough regularity on f and h to guarantee existence of the solution $z(\cdot, t_0, z_0, w)$ on the interval of interest. By a solution we mean a (not necessarily unique) function $z(\cdot)$ such that $\frac{d}{dt} z(t,t_0,z_0,w) = f(t,z,w)$ for $t \in [t_i, t_{i+1}]$, and satisfying (9). A solution $z(t,t_0,z_0,w)$, $t \in [t_k, t_{k+1})$ can be constructed inductively by integrating (8) from the initial condition $(t_i, h(i, z(t_i)))$. This construction forgoes the discussion of the maximum interval of definition of each integral and the maximum interval of definition of the entire solution for which we refer the reader to [3, Section II-B].

Definition 2.1: Let $p \in [1,\infty]$ and $\gamma \ge 0$ be given. We say that Σ_z is $\mathcal{L}_p$ stable from w to y with gain γ if

$$\exists\, K \ge 0 : \|y[t_0,t]\|_p \le K|z_0| + \gamma \|w[t_0,t]\|_p.$$

Definition 2.2: Let $p, q \in [1,\infty]$ and $\gamma \ge 0$ be given. The state z of Σ_z is said to be $\mathcal{L}_p$ to $\mathcal{L}_q$ detectable from output y with gain γ if

$$\exists\, K \ge 0 : \|z[t_0,t]\|_q \le K|z_0| + \gamma\|y[t_0,t]\|_p + \gamma\|w[t_0,t]\|_p.$$

An exposition of these ideas as they pertain to NCS can be found in [3, Section II-B].

Consider the feedback interconnection of two systems of the same form as Σ_z:

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2, w) \quad t \in [t_i, t_{i+1}] \\ y_1 &= H_1(t, x_1, y_2, w) \\ x_1(t_i^+) &= h_1(i, x_1(t_i)) \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{x}_2 &= f_2(t, x_1, x_2, w) \quad t \in [t_i, t_{i+1}] \\ y_2 &= H_2(t, y_1, x_2, w) \\ x_2(t_i^+) &= h_2(i, x_2(t_i)). \end{aligned} \tag{11}$$

This interconnection admits a small-gain theorem presented in [3, Section II-B] in the same vein as that of systems without jumps:

Theorem 2.3: Suppose that $p \in [1,\infty]$, we have the following:

1) System (10) is $\mathcal{L}_p$ stable from (y_2, w) to y_1 with gain γ_1;
2) x_1 in (10) is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from (y_1, w);
3) system (11) is $\mathcal{L}_p$ stable from (y_1, w) to y_2 with gain γ_2;
4) x_2 in (11) is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from (y_2, w); and
5) the small-gain condition $\gamma_1\gamma_2 < 1$ holds.

Then, the system (10), (11) is $\mathcal{L}_p$ stable from w to (x_1, x_2).

We will often be interested in analyzing stability properties of the interconnected system (10), (11) in the absence of exogenous perturbations. To that end, we use the standard notion of uniform global exponential stability (UGES):

Definition 2.4: Consider system Σ_z and suppose that $w \equiv 0$. We say that Σ_z is uniformly globally exponentially stable if for every $z_0 \in \mathbb{R}^{n_z}$,

$$\exists\, K, L \ge 0 : |z(t,t_0,z_0)| \le K \exp(L(t_0 - t))|z_0| \quad \forall t \ge t_0$$

In particular, the system (10), (11) with $w \equiv 0$ is UGES whenever the origin of the system with $w \equiv 0$ is uniformly globally fixed time interval stable (UGFTIS)[2] with linear gain and is $\mathcal{L}_p$ stable. We will use the following sufficient condition in this paper to simplify the presentation:

[2] See [3, Section II-B] for a definition of the UGFTIS property.

Theorem 2.5: Suppose that in addition to satisfying the hypotheses of Theorem 2.3, systems (10) and (11) satisfy

$$\begin{aligned}
\exists\, L_1 \geq 0: &\qquad |f_1(t,x_1,x_2,0)| \leq L_1(|x_1| + |x_2|)\\
\exists\, L_2 \geq 0: &\qquad |f_2(t,x_1,x_2,0)| \leq L_2(|x_1| + |x_2|)\\
\exists\, L_3 \geq 0: &\qquad |h_1(i,x_1)| \leq L_3|x_1|\\
\exists\, L_4 \geq 0: &\qquad |h_2(i,x_2)| \leq L_4|x_2|,
\end{aligned}$$

for all $x_1 \in \mathbb{R}^{n_{x_1}}$, $x_2 \in \mathbb{R}^{n_{x_2}}$, all $t \geq t_0$ and all $i \in \mathbb{N}$. Then the system (10) and (11) with $w \equiv 0$ is UGES.

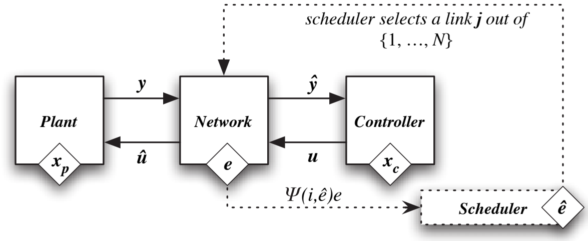

Fig. 1. Conceptual diagram of scheduled NCS.

III. WIRELESS AND WIRELINE NETWORKED CONTROL SYSTEMS

Computer networks and communications systems present rich and sophisticated models of varying degrees of complexity, within stochastic and deterministic settings, and of various underlying physical communication media. The network model presented in this paper aims to capture the essential aspects of control over networks in a deterministic setting. In that vein, we assume:

Assumption 1: All NCS links are connected via a shared, serial bus on which such links either have exclusive access to the network medium or no access at all. ◁

Assumption 2: Transmitted frames are received error-free and network throughput is high enough to ignore delays due to transmission time.This is especially true for wireline channels but some wireless channels may exhibit notable delays. As the focus of this paper is scheduling, we defer the treatment of delays and appeal to the robustness properties verified by the class of systems considered to assert that the results in this paper remain true for sufficiently small delays. ◁

A. A Model for NCS

We consider general nonlinear NCS with disturbances conceptually depicted in Figure 1, where x_P and x_C are, respectively, states of the plant and controller; y is the plant output and u is the controller output; $\hat{y}$ and $\hat{u}$ are the vectors of the most recently transmitted plant and controller output values via the network and e is the network-induced error defined as

$$e(t) := \begin{pmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{pmatrix}.$$

The key difference between this model and that presented in [3] is the addition of the *scheduler* and its corresponding decision-vector $\hat{e}$ that are represented by "dashed" elements in Figure 1. Special cases of $\hat{e}$-based scheduling were first considered in [8]. The model we introduced in [1] and used here formalizes the $\hat{e}$-based scheduling that was considered in [8] and it generalizes the NCS models considered in [3]. Motivation for considering the class of NCS depicted in Figure 1 is given later in Assumption 3 and the ensuing discussion.

We model the NCS in Figure 1 as a so-called jump-continuous (hybrid) system. Node data (controller and sensor values) are transmitted at transmission instants at times $\{t_0, t_1, \ldots, t_i\}, i \in \mathbb{N}$ satisfying $\epsilon < t_{j+1} - t_j \leq \tau$ for all $j \geq 0$ where $\tau > 0$ and $\epsilon > 0$.[3] The constant τ is the *maximum allowable transmission interval* (MATI). The system is initialized at time t_s with $t_s \in [0, t_0], t_0 - t_s < \tau$.

[3]This ensures that Zeno solutions cannot occur.

Our NCS model is prescribed by the following dynamical and jump equations. In particular, for all $t \in [t_{i-1}, t_i]$:

$$\dot{x}_P = f_P(t, x_P, \hat{u}, w) \tag{12}$$
$$\dot{x}_C = f_C(t, x_C, \hat{y}, w) \tag{13}$$
$$u = g_C(t, x_C) \quad y = g_P(t, x_P) \tag{14}$$
$$\dot{\hat{y}} = 0 \quad \dot{\hat{u}} = 0 \quad \dot{\hat{e}} = 0, \tag{15}$$

and at each transmission instant t_i,

$$e(t_i^+) = (I - \Psi(i, \hat{e}(t_i)))e(t_i) \tag{16}$$
$$\hat{e}(t_i^+) = \Lambda(i, \Psi(i, \hat{e}(t_i))e(t_i), \hat{e}(t_i)) \tag{17}$$

and we refer to Ψ as the scheduling function and Λ as the decision-update function. The effect of the protocol on the error is such that if the mth to nth nodes are scheduled. at transmission instant t_i the corresponding components of error, $e_n, \ldots, e_m$, experience a "jump". It may be the case that a single logical node (a "link") consists of several sensors or several actuators or both with the scheduling of that node having the effect of setting multiple components of e to zero. It may also be the case that the network allows the scheduling of more than one node at each transmission and our model allows for this extra degree of freedom. For transmission of nodes mth to nth nodes, though it is not necessary, we will always assume that $e_n(t_i^+), \ldots, e_m(t_i^+) = 0$ and, hence, $\Psi(i, \hat{e}(t_i))$ is a diagonal matrix consisting of entries $[a_{ij}]$, where $a_{ii} = 1$ for $n \leq i \leq m$ and 0 elsewhere. We group the nodes that are scheduled together into logical links, associating a partition of size s_i, denoted by $\mathbf{e}_i = (e_{i1}, e_{i2}, \ldots, e_{is_i})$, of the error vector e such that we can write $e = (\mathbf{e}_1, \ldots, \mathbf{e}_N)$. We say that the NCS has N links and $\sum_{i=1}^N s_i$ nodes. Note that this is purely a notational convenience and simplifies the description of scheduling protocols and the NCS itself. We combine the controller and plant states into a vector $x = (x_P, x_C)$ and similarly to [3, pp. 1653], assuming g_P, g_C are a.e. C^1, for example, we can rewrite (12)-(15):

$$\dot{x} = f(t, x, e, w) \quad t \in [t_{i-1}, t_i] \tag{18}$$
$$\dot{e} = g(t, x, e, w) \quad t \in [t_{i-1}, t_i] \tag{19}$$
$$\dot{\hat{e}} = 0 \quad t \in [t_{i-1}, t_i] \tag{20}$$
$$e(t_i^+) = (I - \Psi(i, \hat{e}(t_i)))e(t_i) \tag{21}$$
$$\hat{e}(t_i^+) = \Lambda(i, \Psi(i, \hat{e}(t_i))e(t_i), \hat{e}(t_i)) \tag{22}$$

where $x \in \mathbb{R}^{n_x}, e \in \mathbb{R}^{n_e}, w \in \mathbb{R}^{n_w}, \hat{e} \in \mathbb{R}^{n_{\hat{e}}}$. Note that implicit to our model is the following assumption:

Assumption 3: Only the transmitted value $\Psi(i, \hat{e}(t_i))e(t_i)$ is accessible by the scheduler – the full vector $e(t_i)$ is unavailable. ◁

IV. NCS SCHEDULING PROTOCOLS

Assumption 3 is quite realistic in a range of wireline NCS and archetypical in wireless NCS. It was the main motivation for introducing $\hat{e}$-based scheduling in [8]. Moreover, Assumption 3 is not satisfied by the protocols considered in [3], excepting RR, that otherwise assume that $e(t_i)$ is always accessible by the scheduler, where (21)-(22) are replaced with the single e jump equation:

$$e(t_i^+) = (I - \Psi(i, e(t_i)))e(t_i) \ . \tag{23}$$

The maximum-error-first try-once-discard (MEF-TOD or just TOD) protocol presented in [6] provides part of the rationale for Assumption 3 and operates as follows:

Example 4.1 (MEF-TOD): At a transmission instant t_j, the scheduler transmits link i with the largest error $|\mathbf{e}_i(t_j)|$ so that the instantaneous error is minimized after jumps. TOD can be written in the form (23), where $\Psi(i, e) = \Psi(e) = \mathrm{diag}\{d_j(e)I_{s_j}\}$, $j = [1, \ldots, N]$, I_{s_j} are $s_j \times s_j$ identity matrices and

$$d_j(\hat{e}) = \begin{cases} 1 & \text{if } j = \min\left(\arg\max_{1 \le k \le N} |\mathbf{e}_k|\right) \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

It was shown that TOD preserves stability properties of the network-free system in (linear systems) [4] and (nonlinear systems with disturbance) [3] for sufficiently small MATI.

Clearly, TOD does not satisfy Assumption 3 since it requires access to the full error vector e. It is easy to see that this is equivalent to the scheduler having access to, for example, each *remote* sensor's measurements *prior to transmission* which is quite unreasonable for most NCS. TOD is implementable through binary countdown in CAN-like medium access protocols and generally not implementable over wireless channels – see [12] and [2] for details. ◁

Example 4.2 (Round-robin): Round-robin scheduling is employed in the Token Ring and Token Bus network protocols as well as (once) being the ubiquitous scheduling protocol of time-sharing operating systems. Each link of the network is assigned a unique index and links are "visited" in order of index. In terms of NCS scheduling, the discrete-time system is a linear time-varying system with no dependence on state:

$$e^+ = (I - \Delta(i))e, \tag{25}$$

where $\Delta(i) = \mathrm{diag}\{\delta_1(i)I_{s_1}, \ldots, \delta_N(i)I_{s_N}\}$, and

$$\delta_k(i) = \begin{cases} 1 & \text{if } k - 1 = i \bmod N \\ 0 & \text{otherwise.} \end{cases}$$

It has been established (in [3] and [7], for instance) that RR preserves stability properties of the network-free system for high enough transmission rates. As the protocol does not depend on NCS state it makes RR easily implementable, and is an example of protocol satisfying Assumption 3.

For the majority of communications channels, we are resigned to employing scheduling protocols that are state-independent or rely upon a decision-vector whose evolution is only (partially) coupled to plant and controller states and only at transmission instants. Ignoring the potential presence of (continuous) decision-vector dynamics, the only way the discrete-time system (31) is coupled to the plant and controller states is through the term $\Psi(i, \hat{e})e$, which we can regard as an output of the combined error-plant-controller system that we can design or choose as the protocol designers. Examples of this style of protocol can be found in [8] and we describe how two protocols, constant-penalty TOD and hybrid time-scheduling, can be expressed in the context of our NCS model.

Loosely, if we could design $\hat{e}$ dynamics and the associated jump equation such that $\hat{e}$ was a good estimate of e, and the following protocol was used:

$$e^+ = (I - \Phi(\hat{e}))e, \tag{26}$$

where Φ comes from Example 4.1, the TOD scheduling function, the expectation is that, the NCS should be qualitatively similar to an NCS using the unmodified TOD protocol. The issue then becomes designing $\hat{e}$ so it is a good enough estimate for this to be true but there are several fundamental obstacles that make this difficult:

For unmodified TOD, it is easy to see that the scheduler may never visit a link because, for example, the link in question is sensing a set of outputs that remain constant or are evolving very slowly compared to other outputs. If this status quo ever changes, that is, if the outputs of the unvisited link suddenly "speed up", the magnitude of error for that link will eventually increase to a level where it will cause the scheduler to visit (transmit) the link.

TOD's reluctance to visit links until a link's error is sufficiently large is a major issue when we are using an estimate of error rather than the error itself: in light of the aforementioned scenario, if the magnitude of error $|\mathbf{e}_j|$ of a link j is initially low, the magnitude of the estimate $|\hat{\mathbf{e}}_j|$ should be low as well and TOD, using the estimate to make the scheduling decision $\Phi(\hat{e})$, will (correctly) not visit the link. Without $\hat{e}$ dynamics, when the link's error potentially grows to a stage where unmodified TOD would have visited it, $\hat{\mathbf{e}}_j$ is left unchanged because the scheduler has not yet visited the link. Paradoxically, the scheduler needed to have visited the link to see the growth in error in order to have updated $\hat{\mathbf{e}}_j$ appropriately and consequently to have known to visit link j.

An ideal solution would be to design $\hat{e}$ dynamics so that, by running a copy of x and e dynamics in the scheduling device, $\hat{e}$ is a copy of e.[4] Of course, in the presence of arbitrary integrable disturbances, this is not possible to do exactly. The protocols described in [8] are a first attempt at estimation in this loose sense but these protocols are forced to visit each link regularly through an additional mechanism described below, essentially acknowledging that the estimates can potentially be arbitrarily bad.

Finite silent-time ensures that links are visited within a fixed-length finite window (T) of transmission instants but it is not necessarily the case that the silent-time thresholds are

[4] Running both x and e dynamics would mean that the dimension of $\hat{e}$ is at least $n_e + n_x$ so by *copy of* e, we mean $\hat{e} = (e, x)$ and when scheduling, only make use of the e component of $\hat{e}$.

exceeded by links in the same order every T transmissions or it may be the case that the silent-time threshold is not exceeded by some links that were nevertheless visited.

An alternative approach to ensure that all links are visited within any contiguous T transmissions is to enforce interleaved RR scheduling every M transmission instants. Labelling the links $\{1, ..., N\}$ and transmission instants $\{t_1, t_2, \ldots,\}$, at transmission instants $t_1, t_{M+1}, \ldots, t_{(M-1)N+1}$, links $1, 2, \ldots, N$ would be transmitted, respectively. At other transmission instants, any other scheduling protocol can be used including the "estimated" TOD variant employed in CP-TOD.

This hybrid protocol, described in Example 4.5, RR scheduling (Example 4.2) as well as the protocols employing the mechanism of silent-time are particular examples of protocols verifying a general protocol robustness property that we refer to as *uniform persistency of excitation* and described in the proceeding section.

A. PET Scheduling Protocols

Motivated by the results in [3], we consider the following auxiliary discrete-time system induced by the protocol:

$$e^+ = (I - \Psi(i, \hat{e}))e \tag{27}$$

$$\hat{e}^+ = \Lambda(i, \Psi(i, \hat{e})e, \hat{e}) \tag{28}$$

Henceforth, we will "disconnect" the protocol from the NCS and refer to (27)-(28) as the protocol. It was shown in [3] that stability properties of the auxiliary system $e^+ = (I - \Psi(i, e))e$ induced by (23) are instrumental in proving $\mathcal{L}_p$ stability properties for a class of NCS for which Assumption 3 does not hold. For the class of NCS we consider, partial stability (only in e) of the system (27)-(28) is essential for stability. To motivate our PE definition, let $V(i, e, \hat{e}) = |e|$ in (27)-(28), we have $\Delta V = -|\Psi(i, \hat{e})e|$ and if there exists a $T < \infty$ such that, for every $k \in \mathbb{N}$, $\hat{e}(k) = \hat{e} \in \mathbb{R}^{n_{\hat{e}}}$, $e(k) = e \in \mathbb{R}^{n_e}$:

$$\sum_{i=k}^{k+T-1} |\Psi(i, \phi_{\hat{e}}(i))e| \geq |e| \ , \tag{29}$$

where $\phi_{\hat{e}}(i) := \phi_{\hat{e}}(i, e, \hat{e})$, we can conclude partial stability (in e) of the system (27)-(28) using persistency of excitation (PE) arguments (see [13], for instance). We can rephrase (29) by saying that each link is visited within T transmissions. It is tempting to conjecture that (29) is enough to prove the stability of (18)-(22) but, notwithstanding the results presented in [3], this is not true in general.

Indeed, it can be shown that if we integrate the equations (19)-(20) on the interval $[t_{i-1}^+, t_i]$ and then use (21), (22) at t_i, the NCS induces the following discrete-time system:

$$e^+ = (I - \Psi(i, \hat{e}))(e + d) \tag{30}$$

$$\hat{e}^+ = \Lambda(i, \Psi(i, \hat{e})(e + d), \hat{e}) \ , \tag{31}$$

where d captures the inter-sample behavior of $e(\cdot)$. For specific initializations $(k, e(k), \hat{e}(k))$ and specific (bounded) values of $d(j), j \geq k$ the solution of the system (30)-(31) coincides with that of (18)-(22) at time instants $t_j^+, j \geq k$. In contrast, while the system (27)-(28) may satisfy the PE condition (29) and, hence, is partially stable in e, there may exist a bounded disturbance d for its perturbed counterpart (30)-(31) that destroys the PE property along with partial stability in e. Hence, our formal definition of PE needs to possess an appropriate robustness property and will be stated using (30)-(31):

Definition 4.3: The protocol (27)-(28) is said to be (robustly) persistently exciting in T or PE_T if there exists $T \in (0, \infty)$ such that (29) holds for all $k \in \mathbb{N}$, $\hat{e}(k) = \hat{e} \in \mathbb{R}^{n_{\hat{e}}}$, $e(k) = e \in \mathbb{R}^{n_e}$ and all $d \in \ell_\infty$, where $\phi_{\hat{e}}(i) := \phi_{\hat{e}}(i, e, \hat{e}, d_{[k,i]})$ is the $\hat{e}$ component of the solution of the system (30)-(31). ◁

Definition 4.4: Uniform persistence of excitation in time T admits an equivalent definition to (29) in terms of the action of the scheduling protocol on the error e at each transmission instant:

$$\prod_{k=i}^{i+T-1} [I - \Psi(i, \phi_{\hat{e}}(k))] = \mathbf{0}, \tag{32}$$

for every $k \in \mathbb{N}$ and any initial condition $e(i), \hat{e}(i)$ where we have written $\phi_{\hat{e}}(k)$ in place of $\phi_{\hat{e}}(k, i, e(i), \hat{e}(i), d_{[k,i]})$.
More intuitively, a protocol is PE if it *regularly visits every NCS node within a fixed period of time*.

The protocols below are typical of what has been proposed in NCS literature and what is used in practice.

We will always assume an N*-link* NCS with the ith linking consisting of N_i nodes and an error vector $\mathbf{e}_i$.

Two PE_T protocols that satisfy Assumption 3 are presented next though we note that the simplest example of a PE protocol is RR (Example 4.2).

Example 4.5 (Hybrid RR-TOD Scheduling Protocol): The hybrid RR-TOD scheduling protocol enforces PE in a time-periodic manner. For a prescribed $M \in \mathbb{N}$, the protocol takes the form:

$$e^+ = (I - \Omega(i, \hat{e}))(e + d) \tag{33}$$

$$\hat{e}^+ = (I - \Omega(i, \hat{e}))\hat{e} + \Omega(i, \hat{e})(e + d) \ , \tag{34}$$

$$\Omega(i, \hat{e}) := \begin{cases} \text{diag}\{p_1(i)I_{s_1}, \ldots, p_N(i)I_{s_N}\}, \ \text{mod}(i, M) = 0 \\ \text{diag}\{d_1(\hat{e})I_{s_1}, \ldots, d_N(\hat{e})I_{s_N}\}, \ \text{otherwise,} \end{cases}$$

where, $p_n(i) = 1$ when $\text{mod}(i/M, N) = n - 1$ and $p_n(i) = 0$ otherwise with d_j defined in (24). The hybrid RR-TOD protocol is PE_T with $T := MN$. In particular, when $M = 1$, we obtain the simplest PE_T protocol: "classical" RR.

Example 4.6 (Constant-Penalty TOD): Constant-penalty TOD (CP-TOD) [8] uses the mechanism of "silent-time" to ensure that every link is eventually visited within a finite window of time: each link j has a counter r_j that is incremented at every transmission instant that it is *not* scheduled and reset to zero when it *is* scheduled. Irrespective of the underlying scheduling protocol, when a link's counter reaches a predetermined threshold, say M, it will be scheduled. This ensures that every link is scheduled within

$N + M$ transmission instants[5]. The protocols in [8] use the mechanism of "silent-time" to enforce PE: each link j has a counter r_j that is incremented at every transmission instant that it is *not* scheduled and reset to zero when it *is* scheduled. Irrespective of the underlying scheduling protocol, when a link's counter reaches a predetermined threshold, say M, it will be transmitted. This ensures that every link is scheduled within $M + N - 1$ transmission instants, hence, $T = M + N - 1$. The underlying scheduler in this example is TOD and corresponds to the constant-penalty TOD scheme in [8] with a penalty (vector) of Θ:

$$e^+ = (I - \Phi(r,\zeta))(e + d) \tag{35}$$
$$\zeta^+ = (I - \Phi(r,\zeta))(\zeta + \Theta) + \Phi(r,\zeta)(e + d) \tag{36}$$
$$r^+ = (I - \Phi(r,\zeta))(r + \mathbf{1}), \tag{37}$$

where $\mathbf{1} = [1\ \ldots\ 1]^T$, the scheduling function Φ is given by $\Phi(r,\zeta) = \mathrm{diag}\{\varphi_j(r,\zeta)I_{s_j}\}, j \in [1,\ldots,N]$ and

$$\varphi_n(r,\zeta) = \begin{cases} 1 & \text{if } [n = \min\{m : r_m \geq M\}) \\ & \vee \left(n = \min\left(\arg\max_{1\leq j\leq N} |\boldsymbol{\zeta}_j|\right)\right. \\ & \wedge (\forall\, m \in \{1,\ldots,N\})(r_m < M)] \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

The role of estimating e is played by ζ and through the term $\Phi(r,\zeta)e$, ζ is updated with $\mathbf{e}_j$ whenever the jth link is transmitted. For those links that are not transmitted, the estimated error is incremented by a fixed penalty Θ that might capture the worst-case growth of error (in the absence of disturbance) for a given MATI. In addition to performing this ad hoc estimation, the scheduling protocol counts the number of transmission instants that a link has not been visited for, the link's silent time, and schedules links that have exceeded a predetermined threshold for silent-time. In this way, if ζ is degenerating into an arbitrarily bad estimate of e, all links will continue to be visited in within a fixed-length, finite window of transmission instants through the mechanism of forcing a finite silent-time for each link. In a loose sense, the protocol's behavior will "often" be qualitatively similar to that of RR , a protocol that has been shown to lead to $\mathcal{L}_p$ stability of the NCS with appropriate conditions. A proof that this indeed holds true for CP-TOD and other silent-time protocols is provided in Section VII-A.

Theorem 5.1 and Theorem 5.2 confirm that PE in the sense of (29) is the only property needed to assert $\mathcal{L}_p$ stability of an NCS for high enough transmission rates when the network-free system is $\mathcal{L}_p$ stable. PE is clearly not a necessary condition for $\mathcal{L}_p$ stability as [3] shows that TOD leads to $\mathcal{L}_p$ stability under appropriate conditions. It may also be the case that the open-loop system in question is $\mathcal{L}_p$ stable. These two examples can be considered to be extremes as the former requires the free flow of error information from nodes to the scheduler and the latter is atypical in practice and theoretically uninteresting. For all other situations, PE seems to be a key property for stabilization of the NCS.

[5]The silent-time protocols described in [8] have the links measure *continuous time* as opposed to counting the number of transmission instants elapsed (discrete-time) and set the silent-time threshold in terms of an integer multiple of MATI, say $M\tau$. Since, for all $i \in \mathbb{N}$, $M\tau \geq M(t_{s_{i+1}} - t_{s_{i+1}})$, our silent-time threshold will be smaller for the same M but the protocol will behave in precisely the same manner as when using the verbatim definition of silent-time given in [8].

V. $\mathcal{L}_p$ Stability of NCS

A. $\mathcal{L}_p$ Stability Properties of Error Dynamics with PET Scheduling Protocols

The following theorem is the main component of our $\mathcal{L}_p$ stability result for the NCS as a whole. In essence, we show that for sufficiently small MATI, PE protocols lead to the finite $\mathcal{L}_p$ stability of the e-subsystem. Note that in the both this section and the proceeding section, we only consider stability of e and x. The decision-vector, if used in the protocol being analyzed, may fail to verify any stability properties but as $\hat{e}$ has no physical significance as a state vector whose evolution is governed by the protocol, this is generally not an issue.

Theorem 5.1: Suppose that the NCS scheduling protocol (18)-(22) is uniformly persistently exciting in time T and there exists $A \in \mathcal{A}^+_{n_e}$ and a continuous $\tilde{y} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_e}_+$ so that the error dynamics (19) satisfy

$$\overline{g}(t,x,e,w) \preceq A\overline{e} + \tilde{y}(x,w) \tag{39}$$

for all $(x,e,w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_w}$, all $t \in (t_i, t_{i+1})$, for all $i \in \mathbb{N}$. Further suppose that MATI satisfies $\tau \in (\kappa, \tau^*), \kappa \in (0, \tau^*)$ where

$$\tau^* = \frac{\ln(2)}{|A|T}. \tag{40}$$

Then, the NCS error subsystem (19)-(22) is $\mathcal{L}_p$ stable from $\tilde{y}$ to e for $p \in [1,\infty]$ with gain

$$\tilde{\gamma}(\tau) = \frac{T\exp(|A|(T-1)\tau)(\exp(|A|\tau) - 1)}{|A|(2 - \exp(|A|T\tau))}. \tag{41}$$

Proof: The proofs are deferred until Section VII-A and further technical results needed in the proof can be found in the Appendix. ■

Remark 1: The inequality (39) is the vector analogue of the dissipation-type inequality

$$\dot{V} \leq LV + \tilde{y} \quad t \in (t_i, t_{i+1})$$

used in [3]. In some respects, the function $\overline{e}$ can be viewed as a particular choice of a *vector* Lyapunov function. The additional property

$$\exists\, \rho \in [0,1) : V(e(t_i^+)) \leq \rho V(e(t_i)) \quad \forall\, i \in \mathbb{N}$$

was enough to establish the finite $\mathcal{L}_p$ gain of the e-subsystem in [3]. This strict decrease at transmission instants is not verified by general PE protocols and a different argument based on the effect of the scheduling protocol on e and, hence, $\overline{e}$ directly is used in the proof of the theorem. ◁

Remark 2: Suppose that $g(t,x,e,w) = Bx + Ce + Dw$ and let $A = [a_{ij}]$, where $a_{ij} = \max\{|c_{ij}|, |c_{ji}|\}$ and $\tilde{y}(x,w) = \overline{Bx + Dw}$. We immediately have that A and $\tilde{y}(x,w)$ satisfy condition 2 of Theorem 5.2 and $\|\tilde{y}(x,w)\|_p = \|Bx + Dw\|_p \leq \|Bx\|_p + \|Dw\|_p$. Whenever g satisfies a linear growth bound of the form $|g(t,x,e,w)| \leq L(|x| + |e| + |w|)$, it is straightforward to construct an appropriate A and $\tilde{y}$. ◁

B. $\mathcal{L}_p$ Stability Properties of NCS with PET Scheduling Protocols

The following theorem provides the closed-loop $\mathcal{L}_p$ stability result for persistently exciting scheduling protocols and is the main result of the paper. It asserts that PE protocols lead to $\mathcal{L}_p$ stability of the NCS for sufficiently small MATI. While we do not provide a closed-form expression for MATI bounds, the bounds are readily obtained in examples by numerically solving for τ^* in (42). We do, however, provide a closed-form approximation in Section VI-C for the purposes of analytic comparison and show that this lowerbound is asymptotically larger by a factor of $O(N^{1/2})$ than the MATI obtained in [3] for linear systems using the RR scheduling protocol.

Theorem 5.2: Consider NCS (18)-(22) and suppose that:

1) the hypotheses of Theorem 5.1 hold with $\tilde{y} = G(x) + H(w)$;
2) system (18) is $\mathcal{L}_p$ stable from (e, w) to $G(x)$ with gain γ for some $p \in [1, \infty]$; (19) is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from $\tilde{y}$;
3) and MATI satisfies $\tau \in (\epsilon, \tau^*)$, $\epsilon \in (0, \tau^*)$, where

$$\tau^* = \frac{\ln(z)}{|A|T}$$

and z solves

$$\gamma\kappa T z^{1+1/T} + z(|A| - \gamma\kappa T) - 2|A| = 0, \tag{42}$$

where $\kappa = \exp(-1)$ and $|A|$ comes from (39).

Then, the NCS is $\mathcal{L}_p$-stable from w to (x, e) with linear gain.

Proof: Together with the detectability assumptions, condition 1 ensures that (19) is $\mathcal{L}_p$-stable from (x, w) to e with gain $\tilde{\gamma}(\tau)$ where

$$\tilde{\gamma}(\tau) = \frac{T\exp(|A|(T-1)\tau)(\exp(|A|\tau) - 1)}{|A|(2 - \exp(|A|T\tau)).} \tag{43}$$

We note that $\tilde{\gamma}(\tau)$ is differentiable and monotonically increasing in τ for $\tau \in [0, \ln(2)/|A|T)$. By the inverse function theorem (see [14, Theorem 2-11], for instance), there exists a unique solution τ^* to $\tilde{\gamma}(\tau)\gamma = 1$. By monotonicity of $\tilde{\gamma}(\tau)$, $\tilde{\gamma}(\tau)\gamma < 1$ for any $\tau \in [0, \tau^*)$. Moreover, as $\tilde{\gamma}(0) = 0$, and by monotonicity of $\tilde{\gamma}$ and monotonicity of its (implicit) inverse, $\tau^* > 0$ for every $\gamma < \infty$. The proof is complete in light of the small-gain theorem for jump-discontinuous systems, Theorem 2.3. ■

Remark 3: Suppose that the network-free system is $\mathcal{L}_p$ stable from w to x with gain γ and the NCS satisfies the hypotheses of Theorem 5.2. Then for any $\gamma^* > \gamma$, it is possible to show that there exists a MATI τ such that the NCS is $\mathcal{L}_p$ stable from w to x with gain γ^*. This corollary of Theorem 5.2 is particularly useful in the design of optimal/robust controllers. ◁

VI. Case Studies & Analytical Bounds

For simplicity, and since $\mathcal{L}_p$ stability results are not provided in [8], we restrict the discussion to linear time-invariant system in the absence of exogenous disturbances and verify UGES, specializing $\mathcal{L}_p$ stability results via Theorem 2.5. Primarily, we will examine performance bounds of NCS employing RR scheduling as it is the only scheduling protocol that can been mutually treated by the analysis frameworks in this paper, [3] and [8].

Suppose that the simplified equations for an N-link NCS take the form

$$\dot{x} = A_{11}x + A_{12}e \quad \dot{e} = A_{21}x + A_{22}e \tag{44}$$

together with jump equations (21)-(22) and let σ_1 and σ_2 to be the largest and smallest singular value of P, respectively, that solve the Lyapunov matrix equation $A_{11}^T P + PA_{11} = -I$. We set $\tilde{y} = A_{21}x$, and let $k_h = \left|\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right|$. From the discussion in [3, Section VII], we have that the MATI bound for RR stated therein, which we will refer to as $\tau^*_{[3]}$ is given by[6]

$$\tau^*_{[3]} = \frac{1}{k_h\sqrt{N}} \ln\left(\frac{k_h + \gamma}{k_h\sqrt{(N-1)/N} + \gamma}\right), \tag{45}$$

the MATI bound, $\tau^*_{[8]}$, obtained via [8][Theorem 1] is given by

$$\tau^*_{[8]} = \min\left\{\frac{\ln(2)}{k_h T}, \frac{S}{8}, \frac{S}{16\sigma_2\sqrt{\sigma_2/\sigma_1}k_h}\right\} \tag{46}$$

where $S = [k_h\sqrt{\sigma_2/\sigma_1}\sum_{i=1}^{N}(i + T - N)]^{-1}$, and that of this paper, τ^*_{new} is given by Theorem 5.2 – readily obtained by application of Remark 2 and solving (42) numerically.

A. CH-47 Tandem-Rotor Helicopter

We consider the networked control of a CH-47 tandem-rotor helicopter in horizontal motion about a nominal airspeed of 40 knots as discussed in [15]:

$$\dot{x}_P = Ax_P + Bu; \quad y = Cx_P, \tag{47}$$

where $C_P = \left[\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{smallmatrix}\right]$,

$$A_P = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B_P = \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{bmatrix}.$$

The outputs y_1, y_2 are vertical velocity (knots/hr) and pitch altitude (radians) respectively and the inputs u_1, u_2 are collective rotor thrust and differential collective rotor thrust respectively. Let the stabilizing output feedback be $u = Ky$ where K is given by $K = \left[\begin{smallmatrix} -12.7177 & -45.0824 \\ 63.5123 & 25.9144 \end{smallmatrix}\right]$. The design of K is due to [16]. We assume that only outputs are transmitted via the network thus $e = \hat{y} - y$ with two links ($N = 2$) in the NCS and, hence, we have $A_{11} = A_P + B_P K C_P$, $A_{12} = B_P K$, $A_{21} = -C_P A_{11}$, and $A_{22} = -C_P A_{12}$, $k_h = 19482$, $\sigma_1 = 0.0009$ and $\sigma_2 = 399.22$. For the purposes of applying Remark 2, we select $A = \left[\begin{smallmatrix} 550.78 & 239.09 \\ 239.09 & 0 \end{smallmatrix}\right]$ with $|A| = 640.09$ and $A_{22} \preceq A$. We can readily compute the $\mathcal{L}_2$ gain from $\tilde{y}$ to e and have $\gamma = 769$.

From the perspective of MATI bounds, neither the results of this paper nor those in [8] distinguish between different PE_T scheduling protocols. The comparison results are summarized below:

[6] In principle, different choices of Lyapunov function and "output" $\tilde{y}$ in the framework presented in [3] may lead to improved MATI bounds. We have followed [3, Section VII] in choices of Lyapunov function and $\tilde{y}$ – these were choices that lead to the best previously obtainable MATI bounds for RR.

1) The MATI bounds are shown in Table I with the bounds in this paper larger than those obtained using the results of [8] by a factor of 10^{14} and larger than the bound obtained by the results of [3] by factor of 23. The bounds τ^*_{new} and $\tau^*_{[8]}$ apply to any PE_T protocol for the original two-link system (44). The bound $\tau^*_{[3]}$ only applies to RR ($T = N = 2$).
2) The improvements are significant and to put them into perspective, when using RR, τ^*_{new} that achieves UGES is equivalent to a network throughput of 3.7 Mbps (assuming 128 byte frames), achievable on current 802.11g wireless networks, while $\tau^*_{[3]}$ requires an effective network throughput of 84 Mbps.
3) We formally fix σ_1, σ_2, γ, k_h and $|A|$ and plot $\tau^*_{[3]}$ and $\tau^*_{[8]}$ with $T = N \in [1, 1000]$ in Figure 2 to examine the behavior of the bounds as the number of links grow. We also fix $N = 2$ and allow $T \geq 2$ to vary for $\tau^*_{[8]}$ and τ^*_{new}.[7] The differences are marked on the $\log_{10}(T) \times \log_{10}(\tau^*)$ scale used in Figure 2.

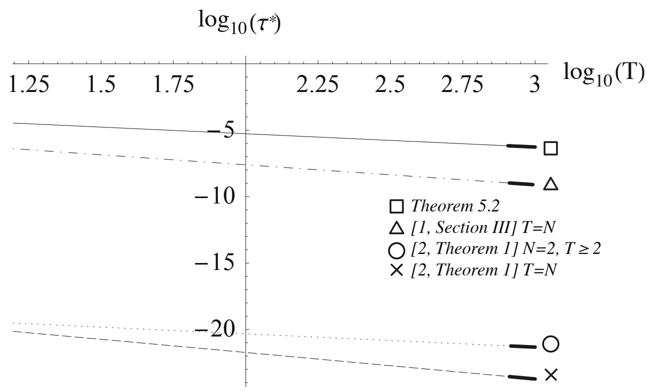

Fig. 2. CH-47 Tandem-Rotor Helicopter MATI bounds comparison for PE_T protocols, $T \in [1, 1000]$.

	$T = 2$	$T = 6$	$T = 50$
τ^*_{new}	2.81×10^{-4}	9.12×10^{-5}	1.08×10^{-5}
$\tau^*_{[3]}$	1.20×10^{-5}	N/A	N/A
$\tau^*_{[8]}$	3.13×10^{-19}	8.52×10^{-20}	9.47×10^{-21}
$\tau^*_{\text{new}}/\tau^*_{[8]}$	8.97×10^{14}	1.07×10^{15}	1.14×10^{15}
$\tau^*_{\text{new}}/\tau^*_{[3]}$	23.4	N/A	N/A

TABLE I
MATI BOUNDS ACHIEVING UGES FOR THE CH-47 TANDEM-ROTOR HELICOPTER WITH PE_T PROTOCOLS.

B. Batch Reactor

The linearized model of an unstable batch reactor is a two-input-two-output NCS that can be written as:

$$\dot{x}_P = A_P x_P + B_P u \quad y = C_P x_P$$

[7] For the purposes of applying Theorem 5.2, it is immaterial as to whether N is fixed and $T \geq N$ is varied or whether $T = N$ is varied.

where $C_P = \left[\begin{smallmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right]$

$$A_P = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix} B_P = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}.$$

The system is controlled by a PI controller with a state-space realization prescribed by

$$\dot{x}_C = A_C x_C + B_C y \quad u = C_C x_C + D_C y$$

and

$$A_C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad B_C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$-C_C = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \qquad -D_C = \begin{bmatrix} 0 & 2 \\ -5 & 0 \end{bmatrix}.$$

Assuming that only the outputs are transmitted via the network, we have a two link NCS ($N = 2, N_1 = N_2 = 1$) with error and state equations

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{48}$$

where

$$A_{11} = \begin{bmatrix} A_P + B_P D_C C_P & B_P C_C \\ B_C C_P & A_C \end{bmatrix} \quad A_{12} = \begin{bmatrix} B_P D_C \\ B_C \end{bmatrix}$$
$$A_{21} = -\begin{bmatrix} C_P & 0 \end{bmatrix} A_{11} \quad A_{22} = -\begin{bmatrix} C_P & 0 \end{bmatrix} A_{12}.$$

The error equation is given by

$$\dot{e} = A_{22} e + A_{21} x \tag{49}$$

and, in light of Remark 2, we have

$$\dot{\overline{e}} \preceq A\overline{e} + \tilde{y}, \tag{50}$$

where $\tilde{y} = \overline{A_{21}x}$ and $A = A_{22}$, as A_{22} is diagonal and has all nonnegative entries.

We compute the $\mathcal{L}_2$ gain for the x subsystem from the input e to an auxiliary output $A_{21}x$ which is $\gamma \approx 15.9222$ however we note that the "gain" from $A_{21}x$ to $\tilde{y}$ is unity (see Remark 2), hence, γ is also the gain from input e to output $\tilde{y}$. Lastly, we note that $|A| = 15.73$, $\sigma_1 = 11.09$, $\sigma_2 = 0.0245$ and $k_h = 61.46$.

As in Section VI-A, neither the results of this paper nor those in [8] distinguish between different PE_T scheduling protocols with respect to MATI bounds. The comparison results are summarized below:

1) The MATI bounds are shown in Table II with the bounds in this paper larger than those obtained using the results of [8] by a factor of 10^7 and larger than the bound obtained by the results of [3] by factor of 1.5. The bounds τ^*_{new} and $\tau^*_{[8]}$ apply to any PE_T protocol for the original two-link system (44). The bound $\tau^*_{[3]}$ only applies to RR ($T = N = 2$).
2) The improvements are not as dramatic as those realized in Section VI-A, essentially, since the system is "slower". When using RR, τ^*_{new} that achieves UGES is equivalent to a network throughput of 84 kbps (assuming 128 byte frames), achievable on current 802.11g and 802.11b wireless networks and $\tau^*_{[3]}$ requires an effective network throughput of approximately 125 kbps.

3) We formally fix σ_1, σ_2, γ, k_h and $|A|$ and plot $\tau^*_{[3]}$ and $\tau^*_{[8]}$ with $T = N \in [1, 1000]$ in Figure 2 to examine the behavior of the bounds as the number of links grow. We also fix $N = 2$ and allow $T \geq 2$ to vary for $\tau^*_{[8]}$ and τ^*_{new}. Despite the relatively modest improvements for the nominal two-link system using RR, the differences are significant on the $\log_{10}(T) \times \log_{10}(\tau^*)$ scale used in Figure 3 when we formally increase T or, equivalently, the number of links.

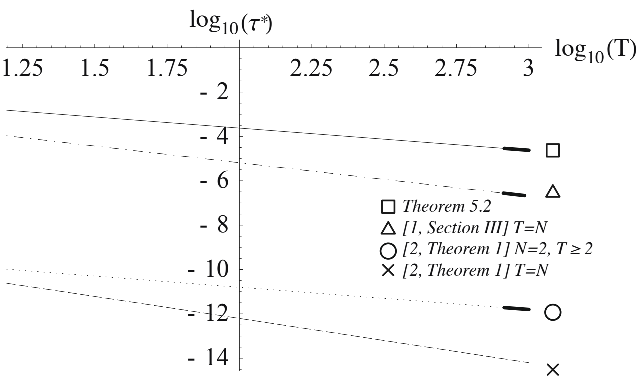

Fig. 3. Batch Reactor MATI bounds comparison for PE_T protocols, $T \in [1, 1000]$.

	$T = 2$	$T = 6$	$T = 50$
τ^*_{new}	0.0123	0.004	4.75×10^{-4}
$\tau^*_{[3]}$	0.0082	N/A	N/A
$\tau^*_{[8]}$	1.05×10^{-9}	2.86×10^{-10}	3.18×10^{-11}
$\tau^*_{\text{new}}/\tau^*_{[8]}$	1.18×10^7	1.40×10^7	1.49×10^7
$\tau^*_{\text{new}}/\tau^*_{[3]}$	1.50	N/A	N/A

TABLE II

MATI BOUNDS ACHIEVING UGES FOR THE BATCH REACTOR WITH PE_T PROTOCOLS.

C. Analytic Comparison With Existing Results

We focus on comparing performance bounds of RR scheduling as it is the only scheduling protocol that can been mutually treated by the analysis frameworks in this paper, [3] and [8].

The bounds in [3] are, in turn, analytically better than the bounds provided in [8] for every scheduling protocol considered therein.

Consider the NCS dynamics equations

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{51}$$

and suppose the RR scheduling protocol is used in an NCS consisting of N links. By Theorem 5.1, the error subsystem is $\mathcal{L}_p$ stable from $\tilde{y}$ to e if $\tau \in [0, \tau^*_{e,new})$ where

$$\tau^*_{e,new} = \frac{\ln(2)}{|A|N}.$$

We require A to be a symmetric, nonnegative matrix with nonnegative entries such that for all $e \in \mathbb{R}^{n_e}_+$, $(A - A_{22})e \in \mathbb{R}^{n_e}_+$. A procedure for constructing such an A was given in Remark 2. The analogous bound for error subsystem stability in [3] is given as $\tau^*_{e,old}$ where, following [3, Section VII],

$$\tau^*_{e,old}\frac{1}{2\sqrt{N}k_h}\ln\left(\frac{N}{N-1}\right)$$

for LTI systems employing RR scheduling, where $k_h = \left|\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{smallmatrix}\right|$. We have that,

$$\frac{\tau^*_{e,new}}{\tau^*_{e,old}} \geq \frac{2k_h\ln(2)}{|A|\sqrt{N}\ln\left(\frac{N}{N-1}\right)}. \tag{52}$$

In the limit, we have

$$\lim_{N\to\infty}\frac{\tau^*_{e,new}}{\tau^*_{e,old}} \geq \frac{2k_h\ln(2)}{|A|}\sqrt{N}.$$

That is, $\tau^*_{e,new} = O(N^{1/2}) \cdot \tau^*_{e,old}$.

The remarks toward the end of Theorem 5.2 establish that there is indeed a "best" MATI to look for, that is, there exists a unique τ^*_{new} that satisfies (42). Although (42) is transcendental in τ^*_{new}, we seek a lower bound by assuming it has the form:

$$\tau^*_{new} = \frac{\ln(z)}{|A|T}.$$

We substitute this into the small-gain condition (42) (43) yielding a new equation in z:

$$R(z) = \gamma\exp(-1)Tz^{1+1/T} + z(|A| - \gamma\exp(-1)T) - 2|A| = 0,$$

where $z \in [1, 2]$ and γ is the gain from (e, w) to x. As we are establishing a MATI bound for RR, we henceforth set $T = N$, the number of links in the NCS. We seek to approximate $R(z)$ by a degree 1 polynomial $R_*(z) = Mz + c$ for which we can write the MATI bound

$$\tau^*_{new} \geq \frac{\ln(-c/M)}{|A|T}.$$

Conservatively, we can upperbound M by

$$\begin{aligned} M &= \max_{z\in[1,2]}\frac{\partial R}{\partial z} \\ &= \max_{z\in[1,2]}\Big(\gamma\exp(-1)N(1+1/N)z^{1/N} \\ &\qquad + |A| - \gamma\exp(-1)N\Big). \end{aligned}$$

It is obvious that the maximum is attained at $z = 2$ so we have

$$M = \gamma\exp(-1)N(1 + 1/N)2^{1/N} + |A| - \gamma\exp(-1)N.$$

We also have that $R(1) = -|A|$ for all $\gamma > 0$ and $N \geq 1$ so we impose the condition $R_*(1) = M + c = -|A|$ and have that the solution to $R_*(z) = 0$ is given by $z = |A|/M + 1$. Our lowerbound for τ^*_{new} becomes $\tau^*_{new} \geq \frac{1}{|A|N}\ln(\eta)$, where

$$\eta = 1 + \frac{|A|}{|A| - N\exp(-1)\gamma + 2^{1/N}(N+1)\exp(-1)\gamma}. \tag{53}$$

The analogous result for RR scheduling in [3] has that

$$\tau^*_{old} = \frac{1}{\sqrt{N}k_h}\ln\left(\frac{k_h + \gamma}{k_h\sqrt{(N-1)/N} + \gamma}\right).$$

It is clear that $\lim_{N\to\infty}\tau^*_{old}=0$ and $\lim_{N\to\infty}\tau^*_{new}=0$ and the arguments of the logarithm in each MATI expression tend to unity. As the function $\ln(1+\zeta)$ is analytic in the region $|\zeta|<1$, we can differentiate τ^*_{old} and τ^*_{new} with respect to N to evaluate the limit $\lim_{N\to\infty}\tau^*_{new}/\tau^*_{old}$ via l'Hôpital's rule and it can be shown that

$$\lim_{N\to\infty}\frac{\tau^*_{new}}{\tau^*_{new}}=\lim_{N\to\infty}\alpha N^{1/2},$$

where

$$\alpha \;=\; \frac{4(\gamma+k_h)}{3|A|}\ln\left(\frac{|A|}{|A|+\exp(-1)\gamma(1+\ln(2))}+1\right).$$

That is, $\tau^*_{new}=O(N^{1/2})\cdot\tau^*_{old}$.

D. Comparison With Simulation-based Bounds

Simulations and alternative techniques for calculating MATI are a key test of the practicality of the MATI bounds and stability results produced in this paper and in the literature. For linear systems with equidistant transmission times employing RR scheduling, an actual analytic MATI bound can be computed as discussed in [3, Section VII-A]. For general protocols, however, simulations are the only resort and, as such, no firm conclusions can be drawn vis-a-vis the theoretical bounds for arbitrary NCS.

We revisit the example of Section VI-A and simulate the system employing the silent-time variant of TOD (CP-TOD with zero penalty) described in Example 4.6. As discussed, for an N-link NCS, the protocol is PE_T with $T=N+M-1$, where M denote the silent-time threshold. Determining the MATI via simulation involves iterating over an initial search space, $\tau\in[0,1]$ seconds, for example, and successively integrating (44) for some choice of τ and applying the protocol map. The NCS is deemed to be unstable when $|(x_i,e_i)|$ exceeds a large constant multiple, K, of the norm of the initial condition (x_0,e_0), and exponentially stable if the norm of (x,e) satisfies $|x_i,e_i|<R|(x_0,e_0)|\exp(-\lambda i\tau)$ at the ith transmission instant. In practice, various heuristics are used to "guess" appropriate value of λ, K and R and the simulation is repeated for a large number of randomly selected initial conditions. This process itself is iterated for various $\tau\in[0,1]$ using Algorithm 1, where the parameter δ determines the coarseness of the search.

Algorithm 1 Binary search for a MATI upperbound.

$\tau_u\leftarrow 1, \tau_l\leftarrow 0, \tau_m\leftarrow 0.5$
while $\tau_u-\tau_l>\delta$ **do**
  run simulation transmitting every τ_m seconds
  if system is exponentially stable **then**
    $\tau_l\leftarrow\tau_m$
  else {system is not exponentially stable}
    $\tau_u\leftarrow\tau_m$
  end if
  $\tau_m=\frac{1}{2}(\tau_u+\tau_l)$
end while

The simulation-based MATI bounds, τ_m, are compared to those obtained via Theorem 5.2, τ^*_{new}, for various values of the silent-time threshold M (through the effect on T) in Table III to one significant figure.

T	10	10^2	10^3	10^4
τ^*_{new}	5.4×10^{-5}	5.4×10^{-6}	5.4×10^{-7}	5.4×10^{-8}
τ_m	2.0×10^{-4}	1.9×10^{-5}	8.0×10^{-6}	8.7×10^{-8}
$\tau_m/\tau^*_{\text{new}}$	3.7	3.5	15	1.6

TABLE III
THEORETICAL AND SIMULATION-BASED MATI BOUNDS ACHIEVING UGES FOR THE CH-47 TANDEM-ROTOR HELICOPTER USING THE CP-TOD PROTOCOL.

Importantly, both τ^*_{new} and τ_m are $O(1/T)$ over the range of T considered and τ^*_{new} is within an order of a magnitude of τ_m. Intuitively, beyond the parameter T of the PE property, the remaining attributes of the protocol that are not captured by our analyis framework contribute no more than a multiplicative constant to the "true" MATI. In the case of the bounds obtained via Theorem 5.2, it can be seen from (53) that for $|A|\gg N$, $\tau^*_{\text{new}}\approx\ln(2)/(|A|T)$, hence the approximately constant multiples in Table III.

We emphasize that this particular set of simulations only suggest that Theorem 5.2 may not be overly conservative when used to find MATI bounds and that general properties of scheduling protocol beyond uniform persistency of excitation may be relatively fine.

VII. PROOFS OF MAIN RESULTS

We will first need the following technical lemma to conclude that the NCS error decreases over T transmission instants for small enough MATI.

Lemma 7.1: Suppose that $A\in\mathcal{A}_n^+$ and $\{Q_i\}_{i\in\mathbb{N}}\in\mathscr{S}_n(T)$, arbitrary. Then

$$(\forall n\in\mathbb{N})\qquad\left|\prod_{i=n}^{n+T-1}Q_i\exp(A\tau)\right|<1 \tag{54}$$

for all $\tau\in[0,\tau^*)$ where $\tau^*=\frac{\ln(2)}{|A|T}$.

Proof: We have that $\mathbf{0}\preceq Q_i\preceq I$. By using Properties 2 and 3 of Lemma 1.4 we can upperbound instances of Q_n with I where need be, hence,

$$\begin{aligned}
&Q_{n+1}\exp(A\tau)Q_n\exp(A\tau)=\\
&(Q_{n+1}+Q_{n+1}(\exp(A\tau)-I))(Q_n+Q_n(\exp(A\tau)-I))\\
&=Q_{n+1}Q_n+Q_{n+1}(\exp(A\tau)-I)Q_n\\
&\quad+Q_{n+1}(\exp(A\tau)-I)Q_n(\exp(A\tau)-I)\\
&\quad+Q_{n+1}Q_n(\exp(A\tau)-I)\\
&\preceq Q_{n+1}Q_n+2(\exp(A\tau)-I)+\exp(2A\tau)+I-2\exp(A\tau)\\
&=Q_{n+1}Q_n+\exp(2A\tau)-I.
\end{aligned}$$

This establishes the base case for a proof by induction. To continue, we assume that

$$\prod_{i=n}^{n+K-1}Q_i\exp(A\tau)\preceq\left(\prod_{i=n}^{n+K-1}Q_i\right)+(\exp(AK\tau)-I) \tag{55}$$

and show that

$$\prod_{i=n}^{n+K} Q_i \exp(A\tau) \preceq \left(\prod_{i=n}^{n+K} Q_i\right) + (\exp(A(K+1)\tau) - I).$$

We first note that

$$\prod_{i=n}^{n+K} Q_i \exp(A\tau) = Q_{n+K}\exp(A\tau) \prod_{i=n}^{n+K-1} Q_i \exp(A\tau) \tag{56}$$

and, by assumption, we can use (55) to bound the right-hand side of (56) to obtain:

$$\begin{aligned}
&\prod_{i=n}^{n+K} Q_i \exp(A\tau) \\
&\preceq Q_{n+K}\exp(A\tau)\left(\left(\prod_{i=n}^{n+K-1} Q_i\right) + \exp(AK\tau) - I\right) \\
&= (Q_{n+K} + Q_{n+K}(\exp(A\tau) - I))\times \\
&\qquad \left(\left(\prod_{i=n}^{n+K-1} Q_i\right) + \exp(AK\tau) - I\right) \\
&= \left(\prod_{i=n}^{n+K} Q_i\right) + Q_{n+K}(\exp(AK\tau) - I) \\
&\qquad + Q_{n+K}(\exp(A\tau) - I)\left(\prod_{i=n}^{n+K-1} Q_i\right) \\
&\qquad + Q_{n+K}(\exp(A\tau) - I)(\exp(AK\tau) - I) \\
&\preceq \left(\prod_{i=n}^{n+K} Q_i\right) + \exp(A(K+1)\tau) - \exp(A\tau) - \exp(AK\tau) \\
&\qquad + I + \exp(AK\tau) - I + \exp(A\tau) - I \\
&= \left(\prod_{i=n}^{n+K} Q_i\right) + \exp(A(K+1)\tau) - I.
\end{aligned}$$

Hence, by induction we have shown that

$$\begin{aligned}
\prod_{i=n}^{n+T-1} Q_i \exp(A\tau) &\preceq \left(\prod_{i=n}^{n+T-1} Q_i\right) + \exp(AT\tau) - I \\
&= \left(\prod_{i=n}^{n+T-1} Q_i\right) + p(AT\tau),
\end{aligned}$$

where $p(s) := \sum_{k=1}^{\infty} \frac{s^k}{k!}$. As $\{Q_i\}_{i\in\mathbb{N}} \in \mathscr{S}_n(T)$, we have

$$\begin{aligned}
\left|\left(\prod_{i=n}^{n+T-1} Q_i\right) + p(AT\tau)\right| &\leq \left|\prod_{i=n}^{n+T-1} Q_i\right| + |p(AT\tau)| \\
&\leq |p(AT\tau)|
\end{aligned}$$

and by Lemma 1.3, for all $\tau \geq 0$, we have

$$|p(AT\tau)| = |p(|A|T\tau)| = \exp(|A|T\tau) - 1. \tag{57}$$

Define

$$\tilde{\rho}(\tau) = \exp(|A|T\tau) - 1 \tag{58}$$

and it is clear that we wish to choose τ such that $\tilde{\rho}(\tau) < 1$. We let $\tilde{\rho}(\tau^*) = 1$ in (58) and solve for τ^* to immediately yield

$$\exp(|A|T\tau^*) = 2 \Rightarrow \tau^* = \frac{\ln(2)}{|A|T}. \tag{59}$$

By monotonicity of $\tilde{\rho}(\tau)$ in τ, $\tilde{\rho}(\tau) < 1$ for all $\tau \in [0, \tau^*)$. ■

A. Proof of Theorem 5.1

We write $\tilde{y}(s)$ in place of $\tilde{y}(\overline{x}(s), \overline{w}(s))$ and let $Q_i = I - \Psi(i, \hat{e}(t_i))$ for each $i \in \mathbb{N}$ and note that $\{Q_i\}_{i\in\mathbb{N}} \in \mathscr{S}_{n_e}(T)$. By hypothesis, we have

$$\overline{g}(t, x, e, w) = \overline{\dot{e}} \preceq A\overline{e} + \tilde{y}(t), \tag{60}$$

on each interval $[t_{i-1}, t_i]$ and the ith component of $\overline{\dot{e}}$ is given by:

$$\begin{aligned}
\left|\frac{d}{dt}e_i(t)\right| &= \left|\lim_{h\to 0, h<0} \frac{e_i(t+h) - e_i(t)}{h}\right| \\
&\geq \lim_{h\to 0, h<0} \frac{|e_i(t+h)| - |e_i(t)|}{h} = D\overline{e}_i(t),
\end{aligned}$$

hence,

$$D\overline{e} \preceq A\overline{e} + \tilde{y}(t) \tag{61}$$

Since the right-hand side of (61) is globally Lipschitz, uniformly in t, by applying Corollary 1.8 with the initial condition $\overline{e}(t_{i-1})$ we can establish the bound:

$$\begin{aligned}
\overline{e}(t_i^+) &\preceq Q_i \exp(A(t_i - t_{i-1}))\overline{e}(t_{i-1}^+) \\
&\qquad + Q_i \int_{t_{i-1}}^{t_i} \exp(A(t_i - s))\tilde{y}(s)ds
\end{aligned} \tag{62}$$

for all $i \in \mathbb{N}$.

Since $I \preceq \exp(At)$ for all $t \geq 0$ and with Properties 2 and 3 of Lemma 1.4, we can upperbound (62) with

$$\begin{aligned}
&\overline{e}(t_i^+) \preceq Q_i \exp(A\tau)\times \\
&\left(\overline{e}(t_{i-1}^+) + \exp(-A\tau)\int_{t_{i-1}}^{t_i} \exp(A(t_i - s))\tilde{y}(s)ds\right)
\end{aligned} \tag{63}$$

for all $i \in \mathbb{N}$.

For brevity, define $R_i = Q_i \exp(A\tau)$. With a MATI τ, we can immediately solve the linear recurrence (63) to produce the bound:

$$\begin{aligned}
&\overline{e}(t_k^+) \preceq \left(\prod_{i=0}^{k} R_i\right)\overline{e}(t_s) \\
&\quad + \exp(-A\tau)\sum_{i=0}^{k}\left(\prod_{n=i}^{k} R_n\right)\int_{t_{i-1}}^{t_i} \exp(A(t_i - s))\tilde{y}(s)ds
\end{aligned} \tag{64}$$

for all $k \in \mathbb{N}$.

Fix $\tau \in (0, \tau^*)$, where τ^* comes from (59), and let $\lambda = \tilde{\rho}(\tau)$, where $\tilde{\rho}(\cdot)$ is defined in (58). By Lemma 7.1,

$$1 > \lambda \geq \left|\prod_{i=n}^{n+T-1} Q_i \exp(A\tau)\right| = \left|\prod_{i=n}^{n+T-1} R_i\right|$$

for every $\tau \in (0, \tau^*)$. We set the disturbance term $\tilde{y} \equiv 0$ and have that

$$|\overline{e}(t_{mT-1}^+)| \leq \left|\prod_{i=0}^{mT-1} R_i\right| |\overline{e}(t_s)| \leq \lambda^m |\overline{e}(t_s)| \quad \left(\forall m \in \mathbb{N}^+\right). \tag{65}$$

With $\tilde{y} = 0$, $D\overline{e} \preceq A\overline{e}$ and for the initial condition $\overline{e}(s_0) = \overline{e}_0$, we have

$$\overline{e}(s) \preceq \exp(A(s - s_0))\overline{e}_0. \tag{66}$$

Taking the norm of the left and right hand sides of (66) and using the bound in (65) as the initial condition, we have that for all $m \in \mathbb{N}^+$, $\theta \in \left(t_{mT-1}, t_{(m+1)T-1}\right)$, the following bound on $|\overline{e}|$ holds:

$$|\overline{e}(\theta)| \leq \exp(|A|(\theta - t_{mT-1}))\lambda^m |\overline{e}(t_s)|. \tag{67}$$

Raising to the pth power and integrating over each interval $[t_{mT-1}, t_{(m+1)T-1}]$, we obtain

$$\|\overline{e}[t_{mT-1}, t_{(m+1)T-1}]\|_p^p \leq \frac{\lambda^{mp}}{p|A|} \left(\exp(|A|pT\tau) - 1\right) |\overline{e}(t_s)|^p \tag{68}$$

for all for all $m \in \mathbb{N}^+$ and all $p \in [1, \infty)$. We can also bound $|\overline{e}|$ on the interval $[t_s, t_{T-1}]$ by $|\overline{e}(\theta)| \leq \exp(|A|(\theta - t_s))|\overline{e}(t_s)|$ and obtain an $\mathcal{L}_p$ bound as above:

$$\|\overline{e}[t_s, t_T]\|_p^p \leq \frac{1}{p|A|} \left(\exp(|A|pT\tau) - 1\right) |\overline{e}(t_s)|^p \tag{69}$$

for any $p \in [1, \infty)$.

Taking the pth root in (68) and (69) and summing with $m \to \infty$, we have that, for all $t \geq t_s$,

$$\begin{aligned} \|\overline{e}[t_s, t]\|_p &\leq \sum_{i=0}^{\infty} \lambda^i \left(\frac{\exp(|A|pT\tau) - 1}{p|A|}\right)^{1/p} |\overline{e}(t_s)| \\ &= \frac{1}{1-\lambda} \left(\frac{\exp(|A|pT\tau) - 1}{p|A|}\right)^{1/p} |\overline{e}(t_s)|, p \in [1, \infty). \end{aligned} \tag{70}$$

The $\mathcal{L}_\infty$ bound is easily obtained by taking $\lim_{p\to\infty} \|\overline{e}[t_s, t]\|_p$ in (70):

$$\|\overline{e}[t_s, t]\|_\infty \leq \frac{1}{1-\lambda} \exp(|A|T\tau)|\overline{e}(t_s)|. \tag{71}$$

We now set $\overline{e}(t_s) = 0$ in (64) and estimate the contribution from the disturbance term to yield:

$$\overline{e}(t_k^+) \preceq \exp(-A\tau) \sum_{i=0}^{k} \left(\prod_{n=i}^{k} R_n\right) \int_{t_{i-1}}^{t_i} \exp(A(t_i - s))\tilde{y}(s)ds. \tag{72}$$

Applying the variations of parameters formula to (72), we have

$$\begin{aligned} \overline{e}(\theta) \preceq{}& \exp(-A\tau)\exp(A(\theta - t_k))\times \\ & \sum_{i=0}^{k} \left(\prod_{n=i}^{k} R_n\right) \int_{t_{i-1}}^{t_i} \exp(A(t_i - s))\tilde{y}(s)ds \\ & + \int_{t_k}^{\theta} \exp(A(\theta - t_k))\tilde{y}(s)ds \end{aligned} \tag{73}$$

for $\theta \in [t_k, t_{k+1}]$. Consider the term $\prod_{n=i}^{k} R_n$. By the division algorithm, we can always write $k - i + 1 = qT + r$ where $q, r \in \mathbb{Z}$ and $r \leq T - 1$. In particular, $q = \left\lfloor \frac{k+1-i}{T} \right\rfloor$. For $q > 0$, we can now rewrite the product in consideration as:

$$\prod_{n=i}^{k} R_n = \left(\prod_{n=k'+1}^{k} R_n\right) \left(\prod_{n=i}^{k'} R_n\right), \tag{74}$$

where $k - k' = r$ and $k' + 1 - i = qT$. For $q = 0$, we have $k - i + 1 = r$ and for notational convenience, we can set $k' = i - 1$ in (74). By the PET property (32),

$$\begin{aligned} \left|\prod_{n=i}^{k'} R_n\right| &\leq \lambda^q = \lambda^{\lfloor (k+1-i)/T \rfloor}, \\ \left|\prod_{n=k'+1}^{k} R_n\right| &\leq \exp(|A|r) \leq \exp(|A|(T-1)\tau). \end{aligned}$$

With this observation, together with the estimate (73), we can bound $|\overline{e}(\theta)|$ by:

$$\begin{aligned} |\overline{e}(\theta)| \leq{}& \exp(|A|(\theta - t_k))\exp(|A|(T-2)\tau)\times \\ & \sum_{i=0}^{k} \lambda^{\left\lfloor \frac{k+1-i}{T} \right\rfloor} \int_{t_{i-1}}^{t_i} \exp(|A|(t_i - s))|\tilde{y}(s)|ds \\ & + \int_{t_k}^{\theta} \exp(|A|(\theta - t_k))|\tilde{y}(s)|ds \end{aligned} \tag{75}$$

for all $\theta \in [t_k, t_{k+1}]$. Let $\varphi(s) = \exp(|A|s)$. Integrating (75) and using Young's inequality[8] yields the $\mathcal{L}_1$-norm estimate:

$$\begin{aligned} \|\overline{e}[t_k, t_{k+1}]\|_1 \leq{}& \|\varphi[0, \tau]\|_1 \exp(|A|(T-2)\tau)\times \\ & \sum_{i=0}^{k} \lambda^{\left\lfloor \frac{k+1-i}{T} \right\rfloor} \int_{t_{i-1}}^{t_i} \exp(|A|(t_i - s))|\tilde{y}(s)|ds \\ & + \|\varphi[0, \tau]\|_1 \|\tilde{y}[t_k, t_{k+1}]\|_1. \end{aligned} \tag{76}$$

Applying Hölder's inequality to the term $\int_{t_{i-1}}^{t_i} \exp(|A|(t_i - s))|\tilde{y}(s)|ds$ in (76), we have

$$\begin{aligned} \|\overline{e}[t_k, t_{k+1}]\|_1 \leq{}& \|\varphi[0, \tau]\|_1 \times \\ & \exp(|A|(T-1)\tau) \sum_{i=0}^{k} \lambda^{\left\lfloor \frac{k+1-i}{T} \right\rfloor} \|\tilde{y}[t_{i-1}, t_i]\|_1 \\ & + \|\varphi[0, \tau]\|_1 \|\tilde{y}[t_k, t_{k+1}]\|_1 \\ \leq{}& \|\varphi[0, \tau]\|_1 \exp(|A|(T-1)\tau) \sum_{i=0}^{k+1} \lambda^{\left\lfloor \frac{k+1-i}{T} \right\rfloor} \|\tilde{y}[t_{i-1}, t_i]\|_1. \end{aligned} \tag{77}$$

Analogously, taking suprema in (75) and applying Young's

[8] Letting $*$ denote convolution over the interval Ω, $f \in L_p(\Omega)$, $g \in L_q(\Omega)$, Young's Inequality is $\|f * g\|_r \leq \|f\|_p \|g\|_q$ for $r^{-1} = p^{-1} + q^{-1} - 1$, $p, q, r > 0$. See [17, Chap. 25, Convolution of Functions], for instance.

inequality, we can bound the $\mathcal{L}_\infty$ on the interval $[t_k, t_{k+1}]$ by

$$\begin{aligned}\|\bar{e}[t_k, t_{k+1}]\|_\infty &\leq \exp(|A|\tau)\times \\ &\exp(|A|(T-2)\tau)\sum_{i=0}^{k}\lambda^{\lfloor\frac{k+1-i}{T}\rfloor}\|\varphi[0,\tau]\|_1\|\tilde{y}[t_{i-1},t_i]\|_\infty \\ &\quad + \|\varphi[0,\tau]\|_1\|\tilde{y}[t_k,t_{k+1}]\|_\infty \\ &\leq \|\varphi[0,\tau]\|_1\exp(|A|(T-1)\tau)\sum_{i=0}^{k+1}\lambda^{\lfloor\frac{k+1-i}{T}\rfloor}\|\tilde{y}[t_{i-1},t_i]\|_\infty.\end{aligned} \tag{78}$$

We can regard $\bar{e}$ in the left-hand side of (73) as the image of $\tilde{y}$ under the action of a linear operator G with bounds for the norms $\|G\|_1 \leq \|G\|_1^*$ and $\|G\|_\infty \leq \|G\|_\infty^*$ given by (77) and (78). As $\|G\|_1^* = \|G\|_\infty^*$, by the Riesz-Thorin Interpolation Theorem (see Appendix 1.2), $\|G\|_p \leq \|G\|_1^* = \|G\|_\infty^*$ for all $p \in [1,\infty]$. Summing (76) (equivalently, (78)), we have

$$\begin{aligned}\|\bar{e}[t_s, t_M]\|_p &\leq \|\varphi[0,\tau]\|_1\exp(|A|(T-1)\tau)\times \\ &\sum_{k=-1}^{M-1}\sum_{i=0}^{k+1}\lambda^{\lfloor\frac{k+1-i}{T}\rfloor}\|\tilde{y}[t_{i-1},t_i]\|_p \quad p\in[1,\infty].\end{aligned} \tag{79}$$

Applying Lemma 1.1 to (79), and taking the limit as $M \to \infty$ in the summation, the $\mathcal{L}_p$ norms can be estimated by

$$\begin{aligned}\|\bar{e}[t_s, t_M]\|_p &\leq \|\varphi[0,\tau]\|_1\exp(|A|(T-1)\tau)\times \\ &\qquad \|\tilde{y}[t_s,t_M]\|_p\sum_{k=0}^{\infty}\lambda^{\lfloor\frac{k}{T}\rfloor} \\ &= \|\varphi[0,\tau]\|_1\exp(|A|(T-1)\tau)\times \\ &\qquad \|\tilde{y}[t_s,t_M]\|_p\frac{T}{1-\lambda} \quad p\in[1,\infty].\end{aligned} \tag{80}$$

Either $\|\tilde{y}[t_s, t_M]\|_p = 0$ or the ratio $\|\bar{e}[t_s, t_M]\|_p/\|\tilde{y}[t_s, t_M]\|_p$ is bounded by an expression that is independent of M, hence, (80) remains true with t in lieu of t_M for any $t \geq t_s$.

Using the definition of φ, substituting the value of λ from (58) and adding the contributions of (80) and (70) yields:

$$\begin{aligned}\|\bar{e}[t_s, t]\|_p &\leq \frac{|\bar{e}(t_s)|}{2-\exp(|A|T\tau)}\left(\frac{\exp(|A|pT\tau)-1}{p|A|}\right)^{1/p} + \\ &\frac{T\exp(|A|(T-1)\tau)(\exp(|A|\tau)-1)}{|A|(2-\exp(|A|T\tau))}\|\tilde{y}[t_s,t]\|_p.\end{aligned} \tag{81}$$

for $p \in [1,\infty)$. Making similar substitutions but this time adding the contributions of (80) and (71), the $\mathcal{L}_\infty$ bound is given by

$$\begin{aligned}\|\bar{e}[t_s, t]\|_\infty &\leq \frac{\exp(|A|T\tau)|\bar{e}(t_s)|}{2-\exp(|A|T\tau)} + \\ &\frac{T\exp(|A|(T-1)\tau)(\exp(|A|\tau)-1)}{|A|(2-\exp(|A|T\tau))}\|\tilde{y}[t_s,t]\|_\infty.\end{aligned} \tag{82}$$

VIII. Conclusion

This paper presented a general framework for the study of general nonlinear control systems with disturbances that relies upon properties of the network-free system and the uniform persistency of excitation interval of the scheduling protocol used. We provided a proof for the following qualitative statement that intuition suggests: *for high enough transmission rates, a scheduling protocol that regularly visits every NCS node within a fixed period of time ought to preserve stability properties of the network-free system.* In particular, the order and the precise times at which nodes are visited are immaterial to the analysis so long as the persistency of excitation property of the protocol is preserved uniformly. Quantitatively, this paper provides sharp bounds for the MATI of the RR and hybrid scheduling protocols, significantly improving upon bounds provided in [3] and [1]. However, the PE approach to the analysis of protocol performance neglects any improvement that might be won by using estimation schemes that do not directly influence the periodicity of the uniform PE property and this is reflected in the disparity between theoretical MATI values and those obtained by simulations for protocols that are PE over a large number of transmissions but employ estimation schemes for "inter-PE" transmission instants.

With the analysis focussed on the periodicity of the PE property, there are no theoretical arguments to support using protocols other than RR and, in special cases, TOD. Simulations and experiments suggest otherwise and the development of frameworks that are able to address the performance of scheduling protocols using properties complementary to PE would be an important step to a systematic approach to the design of NCS and scheduling protocols.

Appendix I
Linear and Nonnegative Analysis

The following is a collection of definitions and results from linear and convex analysis and the theory of differential equations in Banach spaces needed to carry out the main proofs in this paper. These are also useful results in their own right. Lemma 1.3 is a new result to the best of the authors' knowledge and is novel in combining two notions of positivity of matrices to conclude a tight bound on matrix norms of functions of matrices.

A. Linear Analysis & Spectral Theory

Lemma 1.1 (Discrete Young's Inequality): Suppose $M \in [0,\infty]$, and $u_n \geq 0$ and $h_n \geq 0$ for all $n \in \mathbb{N}$, then

$$\sum_{t=0}^{M}\left(\sum_{n=0}^{t}h_{t-n}u_n\right)^p \leq \left(\sum_{n=0}^{M}h_n\right)^p\left(\sum_{n=0}^{M}u_n^p\right). \tag{83}$$

This is based on [18, Theorem 8.14] and [19, Example 5.2], following almost immediately from a generalized Hölder's inequality.

Theorem 1.2 (Riesz-Thorin): Let $F : \mathcal{A}_n \to \mathcal{A}_n$ be a linear operator and suppose that $p_0, p_1, q_0, q_1 \in [1,\infty]$ satisfy $p_0 < p_1$ and $q_0 < q-1$. For any $t \in [0,1]$ define p_t, q_t by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$$

and

$$\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Then

$$\|F\|_{p_t \to q_t} \le \|F\|_{p_0 \to q_0}^{1-t} \|F\|_{p_1 \to q_1}^{t}.$$

In particular, if $\|F\|_{p_0 \to q_0} \le M_0$ and $\|F\|_{p_1 \to q_1} \le M_1$, then

$$\|T\|_{p_t \to q_t} \le M_0^{1-t} M_1^t.$$

For any $A \in \mathcal{A}_n$, we denote the spectrum of A by $\sigma(A)$ and the spectral radius by $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|$.

A matrix calculus for self-adjoint $A \in \mathcal{A}_n$ can be defined for (real) analytic functions $f(s) = \sum_{k=0} c_k s^n$ in the natural way:

$$f(A) = \sum_{k=0} c_k A^k. \tag{84}$$

The series converges absolutely (in the matrix norm) and satisfies

$$\sigma(f(A)) = f(\sigma(A)) \tag{85}$$

$$|f(A)| \ge r(f(A)). \tag{86}$$

This matrix calculus is a specialization of the Riesz-Dunford calculus for linear operators discussed in [20] and equation (85) is known as the spectral mapping theorem. With appropriate conditions on A and f, we may obtain equality in (86). This motivates the following lemma:

Lemma 1.3: Let $A \in \mathcal{A}_n^+$ and $f : \mathbb{R} \to \mathbb{R}$ analytic everywhere with $f \in \mathcal{K}$. Then

$$|f(A)| = f(|A|). \tag{87}$$

Proof: Write $f(A) = \sum_{k=0}^{\infty} c_k A^k$ with $c_k \in \mathbb{R}$ for all $k \in \mathbb{N}$. Let $S_M = \sum_{k=0}^{M} c_k A^k$ and $S = \lim_{M \to \infty} S_M$. We have that

$$|S_M^* - S^*| = |(S_M - S)^*| = |S_M - S|,$$

since $|B^*| = |B|$ for any $B \in \mathcal{A}_n$. We also have that

$$S_M^* = \sum_{k=0}^{M} c_k (A^k)^* = \sum_{k=0}^{M} c_k A^k,$$

and passing to the limit $M \to \infty$ shows that $f(A)^* = f(A)$. Recall that for any self-adjoint $B \in \mathcal{A}_n$, $|B| = r(B)$, hence

$$f(A)^* = \sum_{k=0}^{\infty} c_k (A^*)^k = \sum_{k=0}^{\infty} c_k A^k = f(A)$$
$$\Rightarrow |f(A)| = r(f(A)).$$

Since A is symmetric and non-negative, $\sigma(A) \subset [0, \infty)$. Since A consists of non-negative entries, by [21, Chapter 1, Theorem 3.2], $r(A)$ is a spectral point of A. Hence, by the continuity of f and the spectral mapping theorem, we have

$$\begin{aligned} |f(A)| = r(f(A)) &= \sup_{\lambda \in \sigma(f(A))} |\lambda| \\ &= \sup_{\lambda \in \sigma(A)} |f(\lambda)| = f(r(A)) = f(|A|). \end{aligned} \tag{88}$$

The last equality in (88) follows from the fact that A is self-adjoint and f, being continuous and increasing, attains its maximum at $r(A)$ when restricted to $\sigma(A)$. ∎

B. The Partial Order $\preceq$ and its Properties

Let $\mathbb{R}_+^{n\circ}$ denote the interior of the positive orthant. The partial order $\preceq$ discussed in the Preliminaries can be formalized by

$$x \preceq y \iff y - x \in \mathbb{R}_+^n \quad (x, y \in \mathbb{R}^n), \tag{89}$$

an ordering on $\mathbb{R}^n$. We write

$$x \prec y \iff y \succ x \iff y - x \in \mathbb{R}_+^{n\circ}. \tag{90}$$

Let $\pi(\mathbb{R}_+^n)$ denote the set of $n \times n$ matrices with non-negative entries. We note that $\mathcal{A}_n^+ \subset \pi(\mathbb{R}_+^n)$ and that elements of $\pi(\mathbb{R}_+^n)$ are $\mathbb{R}_+^n$-invariant and this property completely characterizes $\pi(\mathbb{R}_+^n)$. That is,

$$A \in \pi(\mathbb{R}_+^n) \iff (\forall x \in \mathbb{R}_+^n) Ax \in \mathbb{R}_+^n. \tag{91}$$

Let $A, B \in \pi(\mathbb{R}_+^n)$. We write

$$A \preceq B \iff B - A \in \pi(\mathbb{R}_+^n). \tag{92}$$

This induced partial order $\preceq$ defined on elements of $\pi(\mathbb{R}_+^n)$ satisfies two key properties:

Lemma 1.4: Let $A, B \in \pi(\mathbb{R}_+^n)$, $A \preceq B$. Then

1) $(\forall x \in \mathbb{R}_+^n) Ax \preceq Bx$;
2) $(\forall C \in \pi(\mathbb{R}_+^n)) AC \preceq BC$.
3) $(\forall C \in \pi(\mathbb{R}_+^n)) CA \preceq CB$.

Proof: Property 1 follows from the $\mathbb{R}_+^n$-invariance characterization of elements of $\pi(\mathbb{R}_+^n)$:

$$\begin{aligned} A \preceq B &\Rightarrow (B - A) \in \pi(\mathbb{R}_+^n) \\ &\Rightarrow (\forall x \in \mathbb{R}_+^n)(B - A)x \in \mathbb{R}_+^n \\ &\Rightarrow (\forall x \in \mathbb{R}_+^n) Ax \preceq Bx. \end{aligned}$$

Property 2 is similar: Fix $x \in \mathbb{R}_+^n$ and by invariance, $(\forall C \in \pi(\mathbb{R}_+^n)) Cx \in \mathbb{R}_+^n$. Let $Cx = y_C$ to emphasize that C is the free variable. We have

$$\begin{aligned} &(\forall C \in \pi(\mathbb{R}_+^n))(BC - AC)x \\ &\quad \iff (\forall C \in \pi(\mathbb{R}_+^n))((B - A)Cx \\ &\quad \iff (\forall C \in \pi(\mathbb{R}_+^n))(B - A)y_C \in \mathbb{R}_+^n, \end{aligned}$$

where the last equivalence follows from invariance of $B - A$. Since invariance is also a sufficient condition for characterizing elements of $\pi(\mathbb{R}_+^n)$, we have that:

$$\begin{aligned} &(\forall C \in \pi(\mathbb{R}_+^n))(B - A)y_C \in \mathbb{R}_+^n \\ &\quad \iff (\forall C \in \pi(\mathbb{R}_+^n))(B - A)Cx \in \mathbb{R}_+^n \\ &\quad \iff (\forall C \in \pi(\mathbb{R}_+^n))(BC - AC) \in \pi(\mathbb{R}_+^n) \\ &\quad \iff (\forall C \in \pi(\mathbb{R}_+^n)) \iff AC \preceq BC. \end{aligned}$$

Now release x and the result follows. Property 3 is proved in much the same way. ∎

C. Quasimonotonicity and results on (vector) Differential Inequalities

Let $(\mathbb{R}_+^n)^*$, the dual space of $\mathbb{R}_+^n$ be given by

$$(\mathbb{R}_+^n)^* = \{\varphi \in (\mathbb{R}^n)^* : \varphi(x) \ge 0 \quad \forall x \in \mathbb{R}_+^n\}. \tag{93}$$

Definition 1.5: A function $H : \mathbb{R}^n \to \mathbb{R}^n$ is said to be quasimonotone increasing if for all $x, y \in \mathbb{R}^n$ and all $\varphi \in (\mathbb{R}^n_+)^*$ we have

$$x \preceq y, \varphi(x) = \varphi(y) \Rightarrow \varphi(Hx) \leq \varphi(Hy). \quad (94)$$

Quasimonotonicity also admits the following characterization presented in [22]:

Lemma 1.6: Let $|\cdot|_1$ denote the 1-norm on $\mathbb{R}^n$ and $\Psi(x) = \sum_{i=1}^n x_k$. Note that whenever $x \in \mathbb{R}^n_+$ we have that $\Psi(x) = |x|_1$. Let m_+ denote the one-sided directional derivative of the norm:

$$m_+[x, y] = \lim_{h \to 0^+} \frac{|x + hy|_1 - |x_1|}{h}.$$

Equivalent are:

1) $f : \mathbb{R}^n \to \mathbb{R}^n$ is quasimonotone increasing;
2) $m_+[y - x, f(y) - f(x)] = \Psi(f(y) - f(x)) \quad (x \preceq y)$.

Remark 4: Note that a corollary of Lemma 1.6 is that A is quasimonotone for all $A \in \pi(\mathbb{R}^n_+)$. To see this, note that $x - y \in \mathbb{R}^n_+$ and $A \in \pi(\mathbb{R}^n_+)$ implies that $A\mathbb{R}^n_+ \subset \mathbb{R}^n_+$, hence

$$\Psi(Ax - Ay) = \Psi(A(x - y)) = |A(x - y)|_1.$$

We also have that

$$\begin{aligned} &m_+[y - x, f(y) - f(x)] = \\ &\quad \lim_{h \to 0^+} \frac{|x - y + hA(x - y)|_1 - |x - y|_1}{h} \\ &\quad = \lim_{h \to 0^+} \frac{h|A(x - y)|_1}{h} = |A(x - y)|_1. \end{aligned}$$

Theorem 1.7: Let $I = [t_0, t_1]$. Suppose $v, w : I \to \mathbb{R}^n$ are any continuous functions such that $v(t_0) \prec w(t_0)$, $Dv(t), Dw(t)$ exist for $t \in I$ and

$$Dv(t) - f(t, v(t)) \prec Dw(t) - f(t, w(t)) \quad (\forall t \in I), \quad (95)$$

then $v(t) \prec w(t)$ for $t \in I$ if $f(t, \cdot)$ is quasimonotone for all $t \in I$.

Proof: The proof is similar to that of [23, Theorem 1] and [24, Theorem 1]. Let $d(t) = w(t) - v(t)$. Immediately we have that $d(t_0) \in \mathbb{R}^{n\circ}_+$. For a contradiction, suppose that we had $w(t) \preceq v(t)$. Then there exists $\kappa \in (t_0, t_1]$ such that $d(t) \in \mathbb{R}^{n\circ}_+$ for $t \in [t_0, \kappa)$ and $d(\kappa) = 0$. By the Hahn-Banach Theorem, there exists $\varphi \in (\mathbb{R}^n_+)^*$ such that $\varphi(d(\kappa)) = 0$ and $\varphi(x) > 0$ for all $x \in \mathbb{R}^{n\circ}_+$. We then have

$$\begin{aligned} D\varphi(d(\kappa)) &= \lim_{h \to 0, h < 0} \frac{\varphi(d(\kappa + h)) - \varphi(d(\kappa)}{h} \\ &= \lim_{h \to 0, h < 0} \frac{\varphi(d(\kappa + h))}{h} \leq 0. \end{aligned} \quad (96)$$

Moreover, as φ is linear, we have that $D\varphi(d(\kappa)) = \varphi(Dd(\kappa))$ and, hence, $\varphi(Dd(\kappa)) \leq 0$. As $d(\kappa) = 0$ we have $w(\kappa) = v(\kappa)$ and by the quasimonotonicity of f, we must have

$$\varphi(f(\kappa, v(\kappa)) \leq \varphi(f(\kappa, w(\kappa)). \quad (97)$$

By (95) and (97) we have that

$$\begin{aligned} &\varphi(Dv(\kappa)) - \varphi(f(\kappa, v(\kappa)) < \varphi(Dw(\kappa)) - \varphi(f(\kappa, w(\kappa)) \\ &\qquad \Rightarrow \varphi(Dw(\kappa)) - \varphi(Dv(\kappa)) > 0. \end{aligned}$$

Finally, this is in contradiction with (96) and so it must be the case that $v(t) \prec w(t)$ for $t \in I$. ■

Remark 5: With a slight abuse of notation, we can rephrase this as: Suppose that f is quasimonotone and consider the system of ordinary differential equations

$$Du = f(t, u) \quad (\forall t \in I), \quad (98)$$

where we assume u and $Du(t)$ to exist for $t \in I$. Let $v(t)$ be a continuous function where $Dv(t)$ exists for all $t \in I$ and satisfies the inequality

$$Dv \prec f(t, v(t)) \quad (\forall t \in I)(v(t_0) \prec u(t_0)). \quad (99)$$

We then have $v(t) \prec u(t)$ for $t \in I$.

Remark 6: (c.f. [24][Theorem 1].) We can relax the strict inequality (95) to a non-strict inequality when f is locally Lipschitz, uniformly in t. Let $Dv(t) \preceq f(t, v(t))$, $v(t_0) \preceq u(t_0)$ and define $e_n = \frac{1}{n}(1, \ldots, 1)^T \in \mathbb{R}^n$. Let $Du_n = f_n(t, u_n)$, where $f_n(t, u_n) = f(t, u_n) + e_n$, with initial condition $u_n(t_0) = u(t_0) + e_n$. We note that if f is quasimonotone, then f_n is quasimonotone for each n since

$$\begin{aligned} &x \preceq y, \varphi(x) = \varphi(y) \\ &\qquad \Rightarrow \varphi(f(t, x)) \leq \varphi(f(t, y)) \\ &\qquad \Rightarrow \varphi(f(t, x)) + \varphi(e_n) \leq \varphi(f(t, y)) + \varphi(e_n) \\ &\qquad \Rightarrow \varphi(f_n(t, x)) \leq \varphi(f_n(t, y)). \end{aligned}$$

We then apply Theorem 1.7 with $Dv(t) \prec f_n(t, v(t))$ and $v(t_0) \prec u_n(t_0)$ to show that $v(t) \prec u_n(t)$ for each n. We have that, for all $t \in I$, $u_n(t) - v(t) \in \mathbb{R}^{n\circ}_+$ for each n and so $\lim_{n \to \infty} u_n(t) - v(t) \in \text{closure}(\mathbb{R}^{n\circ}_+) = \mathbb{R}^n_+$, hence, $v(t) \preceq \lim_{n \to \infty} u_n(t)$. It is clear that $\lim_{n \to \infty} u_n(t) = u(t)$ from, for instance, [19, Theorem 3.5].

Corollary 1.8: (c.f. [24][Example 2].) Let $v \in \mathbb{R}^n$ and consider the inequality

$$Dv \preceq Av + d(t), \quad v(t_0) = v_0, \quad (\forall t \in I), \quad (100)$$

where $A \in \pi(\mathbb{R}^n_+)$ and $d(t) : I \to \mathbb{R}^n$ is continuous. Then, for all $t \in I$, $v(t)$ is bounded by

$$v(t) \preceq \exp(A(t - t_0))v_0 + \int_{t_0}^t \exp(A(t - s))d(s)ds. \quad (101)$$

Proof: Let $g(t, v) = Av + d(t)$. Let $v_2, v_1 \in \mathbb{R}^n$, $v_1 \preceq v_2$. We have

$$\begin{aligned} &m_+[v_2 - v_1, g(t, v_2) - g(t, v_1)] = \\ &\lim_{h \to 0^+} \frac{|v_2 - v_1 + hAv_2 + hd(t) - hAv_1 - hd(t))|_1 - |v_2 - v_1|}{h} \\ &\qquad = |A(v_2 - v_1)|_1 = \Psi(g(t, v_2) - g(t, v_1)), \end{aligned}$$

and hence, by Lemma 1.6, $g(t, v)$ is quasimonotone increasing. In light of Remark 6, $v(t) \preceq u(t)$, where $u(t)$ is the solution to

$$Du(t) = g(t, u) \quad u(t_0) = v(t_0). \quad (102)$$

Applying the variation of parameters formula to (102) immediately yields

$$u(t) = \exp(A(t - t_0))u_0 + \int_{t_0}^t \exp(A(t - s))h(s)ds,$$

and the result follows. ■

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