Strong Lyapunov functions for systems satisfying the conditions of La Salle*

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Abstract. We present a construction of a (strong) Lyapunov function whose derivative is negative definite along the solutions of the system using another (weak) Lyapunov function whose derivative along the solutions of the system is negative semi-definite. The construction can be carried out if a Lie algebraic condition that involves the (weak) Lyapunov function and the system vector field is satisfied. Our main result extends to general nonlinear systems the strong Lyapunov function construction presented in [2] that was valid only for homogeneous systems.

Key words. La Salle principle; Lyapunov functions; Stability.

1 Introduction

Lyapunov functions are an indispensable tool in analysis and controller design of nonlinear systems as illustrated by books [6, 7, 13] and numerous references cited therein. In particular, positive definite Lyapunov functions whose derivative is negative definite along solutions of the system (strong Lyapunov functions) are typically more useful than the ones whose derivative is only negative semi-definite (weak Lyapunov functions). Indeed, in the latter case we can only conclude if the system is stable using the La Salle invariance principle [10] while in the former case we can also guarantee certain form of stability robustness. This is probably the main reason why strong Lyapunov functions have found a widespread use in controller design methods, such as backstepping [7], forwarding [11, 13] and universal stabilizing controllers via control Lyapunov functions [14]. While it is very desirable to obtain a strong Lyapunov function, this is highly non trivial to achieve in general. On the other hand, for large classes of systems it is much easier to obtain a weak Lyapunov function although it is not as useful as a strong Lyapunov function. For instance, the storage (energy) function of passive nonlinear systems, such as electro-mechanical systems, typically turns out to be a weak Lyapunov function for the closed loop

*This work was supported by the Australian Research Council under the Discovery Grants Scheme while the first author was visiting The University of Melbourne.

system if the system is stabilized using linear output feedback. Another example are systems that are stabilized using the Jurdjevic-Quinn Theorem [2, 5, 12].

Hence, there is a strong motivation in providing constructions of strong Lyapunov functions (that are usually more useful) from weak Lyapunov functions (that are often easier to construct). This problem has attracted attention of several researchers[1], such as [1, 2]. The construction in [1] requires a weak Lyapunov function $V_1(\cdot)$ and an auxiliary Lyapunov function $V_2(\cdot)$ that verifies appropriate detectability properties of the system with respect to an appropriately defined output. The construction is very general but its main drawback is that the construction of $V_2(\cdot)$ may be still very hard in general and there is no general procedure that achieves this. A different approach was pursued in [2] where the construction of a strong Lyapunov function is based entirely on a weak Lyapunov function $V_1(\cdot)$ and its iterated Lie derivatives along solutions of an auxiliary system with a scaled vector field. This approach appears to be more direct than the approach in [1] but it can be used under slightly different (possibly stronger) conditions. However, results in [2] are applicable only to homogeneous systems and we are not aware of any similar results for general nonlinear systems.

In this paper, we present a construction of a strong Lyapunov function for general nonlinear systems borrowing some ideas from [2]. The Lie bracket conditions under which we can carry out our construction is very similar to the conditions from [2] but our result applies to general nonlinear systems, as opposed to [2] that only applies to homogeneous systems. Furthermore, our construction is different from the one used in [2] and, hence, our Lyapunov functions will be in general different from the ones obtained in [2] when applied to the class of homogeneous systems.

The paper is organized as follows. First we present preliminaries in Section 2. The main results are stated in Section 3 and proved in Section 4. Conclusions and suggestions for further work are presented in the last Section. Several technical lemmas are presented in the Appendix.

2 Preliminaries

The set of natural numbers (not including zero) is denoted as $\mathbb{N}$. Sets of real, nonnegative real and strictly positive real numbers are denoted respectively as $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$. A function $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, zero at zero and unbounded. Given $a, b \in \mathbb{N}$, with $a \leq b$ we use the notation $[a, b]$ to denote the set $\{z \in \mathbb{N} : a \leq z \leq b\}$. We use the convention that if $a > b$ then $[a, b] = \emptyset$. Also, given an arbitrary sequence c_i, we let $\sum_{i=a}^{b} c_i = 0$ if $a, b \in \mathbb{N}$ are such that $b < a$. If a real-valued function $k : \mathbb{R} \to \mathbb{R}$ has continuous derivatives of arbitrary order we say that the function is smooth and we denote its i-th derivative by $k^{(i)}(\cdot)$. Given $V : \mathbb{R}^n \to \mathbb{R}$

[1] Results in [1, 2] are presented for slightly different situations from ours. Indeed, the results in [1] are presented for analysis of the more general property of input-to-state stability (ISS) for systems with disturbance inputs. The results in [2] are presented for controller design via the control Lyapunov functions (CLF) for systems that satisfy the Jurdjevic-Quinn conditions.

and $f : \mathbb{R}^n \to \mathbb{R}^n$, we use the following notation:

$$L_f^1 V(x) = L_f V(x) := \frac{\partial V}{\partial x} f(x); \qquad L_f^{i+1} V(x) := L_f(L_f^i V(x)), \qquad \forall i \in \mathbb{N} .$$

We say that $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. It is positive semi-definite if $V(0) = 0$ and $V(x) \geq 0$ for all $x \neq 0$. The function $V(\cdot)$ is negative definite (semi-definite) if $-V(\cdot)$ is positive definite (semi-definite). The function $V(\cdot)$ is radially unbounded if $|x| \to \infty$ implies that $V(x) \to \infty$.

In this paper we consider systems of the form:

$$\dot{x} = f(x) , \tag{1}$$

with $f(0) = 0$. We will carry out analysis of (1) via an auxiliary system of the form:

$$\dot{x} = f_\lambda(x) := \lambda(V(x)) f(x) , \tag{2}$$

where $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is a strictly positive function that will be defined later and $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is positive definite. We will always assume that all functions are differentiable sufficiently many times.

In the sequel we use the following definition:

Definition 2.1 *A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a* weak Lyapunov function *for the system (1) if it is positive definite, radially unbounded and*

$$\dot{V} = L_f V(x) \leq 0 \qquad \forall x \in \mathbb{R}^n . \tag{3}$$

A function $V(\cdot)$ is a strong Lyapunov function *for the system (1) if it is positive definite, radially unbounded and the function $-L_f V(x)$ is positive definite. If there exists a neighbourhood of the origin $E \subset \mathbb{R}^n$ such that $V(\cdot)$ is positive definite on E and the condition (3) holds for all $x \in E$, we say that $V(\cdot)$ is a weak Lyapunov function for the system (1) on the set E. We define strong Lyapunov functions on the set E in a similar manner.*

We use the standard definitions of exponential, global exponential, asymptotic and global asymptotic stability (see [6]).

3 Main result

In this section we state the main result of the paper which is proved in the next section. The result provides a construction of a strong Lyapunov function for the system (1) under the following conditions: (i) a weak Lyapunov function is available for the system (1); (ii) the weak Lyapunov function and the system (1) satisfy a Lie algebraic condition (item (ii) of Theorem 3.1).

Theorem 3.1 *Suppose that there exists $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the following conditions hold:*

(i) $V(\cdot)$ is a weak Lyapunov function for the system (1);

(ii) There exists $\ell \in \mathbb{N}$ such that for all $x \neq 0$, there exists $i \in [1, \ell]$, such that $L_f^i V(x) \neq 0$.

Then, there exists a nonincreasing strictly positive function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that the function

$$U(x) \;=\; V(x)\left[1 + V(x) - \sum_{i=1}^{\ell-1} L_{f_\lambda}^i V(x) \cdot \left(L_{f_\lambda}^{i+1} V(x)\right)^{3^i}\right] \tag{4}$$

is a strong Lyapunov function for the system (1), where $f_\lambda = f_\lambda(x) := \lambda(V(x)) f(x)$. In particular, we can take any $\lambda(\cdot)$ such that

$$|L_{f_\lambda}^j V(x)| \;\leq\; \frac{1}{3^{\ell+1}} \qquad \forall j \in [1, \ell+1] \;,\; \forall x \in \mathbb{R}^n . \tag{5}$$

A local version of Theorem 3.1 can also be stated. Due to space reasons the proof of this fact is omitted.

Corollary 3.2 *Suppose that all conditions of Theorem 3.1 hold on a given neighbourhood of the origin $E \subset \mathbb{R}^n$. Then, there exists a neighbourhood of the origin E_1 with $E_1 \subset E$ such that the Lyapunov function (4) is a strict Lyapunov function for the system (1) on the set E_1.*

In the case of globally asymptotically stable analytic systems for which the condition (i) of Theorem 3.1 holds with an analytic function $V(\cdot)$, it can be proved that the condition (ii) holds on arbitrarily large compact sets which do not contain the origin. The proof of this result is provided in the next section.

Proposition 3.3 *Consider an analytic system (1) and assume that it is globally asymptotically stable. Suppose that the condition (i) of Theorem 3.1 holds with an analytic function $V(\cdot)$. Then, given an arbitrary compact set E which does not contain the origin there exists $\ell \in \mathbb{N}$ such that for any $x \in E$ with $x \neq 0$ we have that $L_f^i V(x) \neq 0$ for some $i \in [1, \ell]$.*

We note that the conditions (i) and (ii) in Theorem 3.1 are not necessary for global asymptotic stability of the system. Indeed, there may exist a system (1) that satisfies the condition (i) but not the condition (ii) and which can be proved to be globally asymptotically stable using the La Salle invariance principle. The following example that is taken from [12] illustrates this situation.

Example 3.4 *Consider the system $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_2 B(x_2)$, where $B(s) = e^{-\frac{1}{s^2-1}}$ when $|s| \neq 1$ and $B(1) = B(-1) = 0$. Note that $B(\cdot)$ is a smooth function. Note that item (i) of Theorem 3.1 is satisfied with $V(x_1, x_2) = x_1^2 + x_2^2$ since $\dot{V} = -2x_2^2 B(x_2)$. Moreover, by applying the La Salle invariance principle, one can prove that this system is globally asymptotically stable (see [12]). However, the condition (ii) of Theorem 3.1 does not hold since for $x^* = (0\ 1)^T$ and all $i \in \mathbb{N}$ we have that $L_f^i V(x^*) = 0$.*

Remark 3.5 *We note that when the system (1) is locally exponentially stable, the derivative of (4) along the trajectories of (1) is typically not locally upper bounded by a negative definite quadratic function. Moreover, our construction usually does not produce a Lyapunov function $U(\cdot)$ using which we can verify local or global exponential stability. However, one can use $U(\cdot)$ to obtain another strong Lyapunov function $W(\cdot)$ using which exponential stability can be verified. For example, if the system satisfies conditions of Theorem 3.1 and, moreover, its linearization is exponentially stable, then one can construct a Lyapunov function $W(\cdot)$ such that $\alpha_1(|x|) \leq W(x) \leq \alpha_2(|x|)$, $\dot{W} \leq -\alpha_3(|x|)$ for all $x \in \mathbb{R}^n$ and some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and, moreover, there exists $\delta, a, b \in \mathbb{R}_{>0}$ such that $\alpha_1(s) = as^2$ and $\alpha_3(s) = bs^2$ for all $s \in [0, \delta]$ (see [4, Lemma 10.1.5]).*

Remark 3.6 *The construction in [2] applies only to homogeneous systems whereas our construction can be used for general nonlinear systems (1). We note that the Example of the TORA system in [2, Section 8] illustrates how results of [2] can be sometimes used for non-homogeneous systems. However, [2] does not provide a systematic construction of strong Lyapunov functions for general non-homogeneous system, which is what we do.*

Remark 3.7 *It is possible to use our result in a range of controller design situations, similar to the one presented in [2] for systems satisfying the Jurdjevic-Quinn conditions. Due to space reasons we have not pursued these controller design questions in the current paper.*

Remark 3.8 *An important application of strong Lyapunov functions is robustness analysis of stability of nonlinear systems (see for example [6, Chapter 5]). In order to analyze stability robustness we typically need also that the strong Lyapunov function $U(\cdot)$ satisfies the following condition $\left|\frac{\partial U}{\partial x}\right| \leq \alpha(|x|), \forall x \in \mathbb{R}^n$, for some class $\mathcal{K}_\infty$ function $\alpha(\cdot)$. We note that if the weak Lyapunov function $V(\cdot)$ is continuously differentiable sufficiently many times, then the function $U(\cdot)$ constructed using (4) will satisfy this extra condition.*

Remark 3.9 *An alternative construction of strong Lyapunov functions for systems satisfying conditions of La Salle is presented in [1], where a weak Lyapunov function is combined with another auxiliary Lyapunov function. More precisely, it is assumed that there exist two positive definite and radially unbounded functions $V_1(\cdot)$ and $V_2(\cdot)$ satisfying[2] for all $x \in \mathbb{R}^n$:*

$$\dot{V}_1 \leq -\alpha_1(|y|)\ ,\ \dot{V}_2 \leq -\alpha_2(|x|) + \gamma(|y|)\ , \tag{6}$$

where $\alpha_1, \alpha_2, \gamma$ are $\mathcal{K}_\infty$ functions and $y = h(x)$. Note that $V_1(\cdot)$ in (6) is typically a weak Lyapunov function since $|h(x)|$ is typically a positive semi-definite function. The function $V_2(\cdot)$ in (6) is an output-to-state Lyapunov function (see [8]) that characterizes a particular form of detectability of x from the

[2]Actually, results in [1] are presented for the more general case of input-to-state stability (ISS) for systems with inputs.

output y. A strong Lyapunov function in [1] takes the form $U(x) = V_1(x) + \rho(V_2(x))$, where ρ is a $\mathcal{K}_\infty$ function. The main difference between our approach and [1] is that our conditions appear to be stronger but easier to check than those in [1]. Indeed, searching for $V_2(\cdot)$ satisfying (6) is much harder than checking our Lie bracket condition (ii) in Theorem 3.1 and then using the formula (4).

4 Proofs of main results

Proof of Theorem 3.1. Note first that since $V(\cdot)$ is radially unbounded and positive definite and $L_f^i V(x)$ are continuous functions of x for all $i \in [1, \ell+1]$, all conditions of Lemma A.2 in the Appendix hold and the lemma guarantees that we can always find $\lambda(\cdot)$ such that (5) holds. Moreover, one possible construction of $\lambda(\cdot)$ is presented in Lemma A.2. Consider the auxiliary system (2). Item 2 of Lemma A.1 states that if we construct a strong Lyapunov function for the system (2), then the same function is a strong Lyapunov function for the system (1). Hence, in the rest of the proof we concentrate only on the system (2). Moreover, items 1 and 3 of Lemma A.1 imply that since conditions (i) and (ii) of Theorem 3.1 hold for the system (1), the same conditions hold for the system (2).

We now show that (4) is a strong Lyapunov function for the system (2) that satisfies the conditions (i) and (ii) of Theorem 3.1 and for which (5) holds.

To simplify the notation we introduce the functions

$$M_i(x) \;:=\; -L_{f_\lambda}^i V(x) \cdot \left(L_{f_\lambda}^{i+1} V(x)\right)^{3^i} \qquad i \in [1, \ell-1] \tag{7}$$

and

$$S(x) \;:=\; V(x) + \sum_{i=1}^{\ell-1} M_i(x) \; . \tag{8}$$

Direct calculations show that the derivative of M_i along solutions of (2) is:

$$\dot{M}_i\Big|_{(2)} \;=\; -\left(L_{f_\lambda}^{i+1} V(x)\right)^{3^i+1} - 3^i \cdot L_{f_\lambda}^i V(x) \cdot \left(L_{f_\lambda}^{i+1} V(x)\right)^{3^i-1} \cdot L_{f_\lambda}^{i+2} V(x) \; , \; i \in [1, \ell-1] \; . \tag{9}$$

Note that for any $a, b \in \mathbb{R}_{\geq 0}$ and any $i \in \mathbb{N}$ we can write (consider the cases $a \leq \frac{1}{2}b^2$ and $a > \frac{1}{2}b^2$):

$$ab^{3^i-1} \;\leq\; \frac{1}{2} b^{3^i+1} + \frac{1}{2}(2a)^{\frac{3^i+1}{2}} \tag{10}$$

and using (9) and (10) with $b = |L_{f_\lambda}^{i+1} V(x)|$ and $a = 3^i \cdot |L_{f_\lambda}^i V(x)| \cdot |L_{f_\lambda}^{i+2} V(x)|$ we can write

$$3^i \cdot L_{f_\lambda}^i V(x) \cdot \left(L_{f_\lambda}^{i+1} V(x)\right)^{3^i-1} \cdot L_{f_\lambda}^{i+2} V(x) \leq \frac{1}{2}\left(L_{f_\lambda}^{i+1} V(x)\right)^{3^i+1} + \frac{1}{2}\left(2 \cdot 3^i \cdot |L_{f_\lambda}^i V(x)| \cdot |L_{f_\lambda}^{i+2} V(x)|\right)^{\frac{3^i+1}{2}} \; . \tag{11}$$

Using (9) and (11) we can write for all $i \in [1, \ell-1]$:

$$\dot{M}_i\Big|_{(2)} \;\leq\; -\tfrac{1}{2}\left(L_{f_\lambda}^{i+1} V(x)\right)^{3^i+1} + \tfrac{1}{2}\left[2 \cdot 3^i \cdot |L_{f_\lambda}^i V(x)| \cdot |L_{f_\lambda}^{i+2} V(x)|\right]^{\frac{3^i+1}{2}} \; . \tag{12}$$

Then, using (8), (12) and (5),

$$\dot{S}\Big|_{(2)} \le L_{f_\lambda}V(x) - \tfrac{1}{2}\sum_{i=1}^{\ell-1}\left(L_{f_\lambda}^{i+1}V(x)\right)^{3^i+1} + \sum_{i=1}^{\ell-1}\frac{1}{2}\left[2\cdot 3^i\frac{1}{3^{\ell+1}}|L_{f_\lambda}^{i}V(x)|\right]^{\frac{3^i+1}{2}} . \tag{13}$$

But

$$\begin{aligned}\sum_{i=1}^{\ell-1}\frac{1}{2}\left[2\cdot 3^i\frac{1}{3^{\ell+1}}|L_{f_\lambda}^{i}V(x)|\right]^{\frac{3^i+1}{2}} &= \tfrac{1}{2}\left[2\cdot 3\tfrac{1}{3^{\ell+1}}|L_{f_\lambda}V(x)|\right]^2 + \sum_{i=2}^{\ell-1}\frac{1}{2}\left[2\cdot\frac{3^i}{3^{\ell+1}}|L_{f_\lambda}^{i}V(x)|\right]^{\frac{3^i+1}{2}}\\ &\le \tfrac{1}{2}\left(\tfrac{2}{3}\right)^2(L_{f_\lambda}V(x))^2 + \sum_{i=1}^{\ell-2}\frac{1}{2}\left[2\cdot\frac{3^i}{3^{\ell}}|L_{f_\lambda}^{i+1}V(x)|\right]^{\frac{3^{i+1}+1}{2}}\\ &\le \tfrac{1}{2}\left(\tfrac{2}{3}\right)^2(L_{f_\lambda}V(x))^2 + \sum_{i=1}^{\ell-2}\frac{1}{2}\left(\frac{2}{3}\right)^{\frac{3^{i+1}+1}{2}}|L_{f_\lambda}^{i+1}V(x)|^{\frac{3^{i+1}+1}{2}} .\end{aligned} \tag{14}$$

This implies

$$\begin{aligned}\dot{S}\Big|_{(2)} &\le L_{f_\lambda}V(x) - \tfrac{1}{2}\sum_{i=1}^{\ell-1}\left(L_{f_\lambda}^{i+1}V(x)\right)^{3^i+1} + \frac{1}{2}\left(\frac{2}{3}\right)^2(L_{f_\lambda}V(x))^2\\ &+\sum_{i=1}^{\ell-2}\frac{1}{2}\left(\frac{2}{3}\right)^{\frac{3^{i+1}+1}{2}}|L_{f_\lambda}^{i+1}V(x)|^{\frac{3^{i+1}+1}{2}} .\end{aligned} \tag{15}$$

The condition (i) of Theorem 3.1 and the item 1 of Lemma A.1 guarantee that $L_{f_\lambda}V(x)$ is negative semi-definite and since $\frac{3^{i+1}+1}{2} \ge 3^i+1$ and $|L_{f_\lambda}^{i}V(x)| \le 1$ for all $i = 1, 2, \dots, \ell+1$, we get

$$\begin{aligned}\dot{S}\Big|_{(2)} &\le \tfrac{1}{2}L_{f_\lambda}V(x) - \tfrac{1}{2}\sum_{i=1}^{\ell-1}\left(L_{f_\lambda}^{i+1}V(x)\right)^{3^i+1} + \sum_{i=1}^{\ell-2}\frac{1}{2}\left(\frac{2}{3}\right)^5\left(L_{f_\lambda}^{i+1}V(x)\right)^{3^i+1}\\ &\le \tfrac{1}{2}L_{f_\lambda}V(x) - \tfrac{1}{4}\sum_{i=1}^{\ell-1}\left(L_{f_\lambda}^{i+1}V(x)\right)^{3^i+1} .\end{aligned} \tag{16}$$

Observe that the function $U(x)$ defined in (4) can be written as:

$$U(x) = V(x)\cdot(1+S(x)) \tag{17}$$

and the inequality (16) ensures that its derivative along the trajectories of (2) satisfies

$$\dot{U}\Big|_{(2)} \le L_{f_\lambda}V(x)\cdot(1+S(x)) + V(x)\cdot\left(\frac{1}{2}L_{f_\lambda}V(x) - \frac{1}{4}\sum_{i=1}^{\ell-1}\left(L_{f_\lambda}^{i+1}V(x)\right)^{3^i+1}\right) . \tag{18}$$

¿From (5), it follows that for all $x \in \mathbb{R}^n$ we have:

$$1+S(x) \ge 1+\sum_{i=1}^{\ell-1}M_i(x) \ge 1-\sum_{i=1}^{\ell-1}\frac{1}{3^{\ell+1}}\cdot\left(\frac{1}{3^{\ell+1}}\right)^{3^i} \ge 1-\frac{\ell-1}{9^{\ell+1}} \ge \frac{1}{2} \tag{19}$$

Hence, since $L_{f_\lambda}V(x)$ is negative semi-definite and using (19) we can obtain using (18) the following:

$$\dot{U}\Big|_{(2)} \le \frac{1}{2}L_{f_\lambda}V(x) + V(x)\cdot\left(\frac{1}{2}L_{f_\lambda}V(x) - \frac{1}{4}\sum_{i=1}^{\ell-1}\left(L_{f_\lambda}^{i+1}V(x)\right)^{3^i+1}\right) \qquad \forall x \in \mathbb{R}^n . \tag{20}$$

It is easy to see that the right hand side of (20) is negative definite. Indeed, note first that both terms on the right hand side of (20) are non-positive functions. Moreover, if for some $x \neq 0$ we have $L_{f_\lambda} V(x) < 0$ then the first term is negative and hence $\dot{U}\Big|_{(2)} < 0$. On the other hand, if $L_{f_\lambda} V(x) = 0$ then from the condition (ii) of the Theorem and Lemma A.2 we have that $-\frac{1}{4} \cdot V(x) \cdot \sum_{i=1}^{\ell-1} \left(L_{f_\lambda}^{i+1} V(x) \right)^{3^i+1} < 0$ and hence $\dot{U}\Big|_{(2)} < 0$.

Finally, note that (17) and (19) imply that

$$U(x) \geq \frac{1}{2} V(x) \qquad \forall x \in \mathbb{R}^n \ , \tag{21}$$

which implies that $U(x)$ is positive definite and radially unbounded (since $V(\cdot)$ is positive definite and radially unbounded). Hence, $U(\cdot)$ is a strong Lyapunov function for the system (2). According to item 2 of Lemma A.1 in the Appendix, $U(\cdot)$ is also a strong Lyapunov function for the system (1), which completes the proof.

Proof of Proposition 3.3: Let all conditions of Proposition 3.3 be satisfied.

First, we show that for any $x \in \mathbb{R}^n$, with $x \neq 0$ there exists an integer $n = n(x) > 0$ such that $L_f^n V(x) \neq 0$. Consider an arbitrary $x \in \mathbb{R}^n$ with $x \neq 0$. Since the system is assumed to be GAS, the solution $\phi(t, x)$ is defined for all $t \in [0, \infty)$. For the purpose of showing contradiction suppose that $L_f V(\phi(t, x)) = 0$ for all $t \in [0, \infty)$. This implies that $\dot{V}(\phi(t, x)) \equiv 0$ and hence $V(\phi(t, x)) = V(x) > 0$ for all $t \in [0, \infty)$. Since $V(\cdot)$ is positive definite and radially unbounded, this contradicts the assumptions that the system is GAS. Hence, there exists $t_c \in (0, \infty)$ such that

$$L_f V(\phi(t_c, x)) \neq 0 \ . \tag{22}$$

Since $V(\cdot)$ and $f(\cdot)$ are analytic functions, we have that $L_f V(\phi(t, x))$ is an analytic function of t and we can write:

$$L_f V(\phi(t, x)) \ = \ \sum_{i=0}^{\infty} L_f^{i+1} V(x) \frac{t^j}{j!} \ , \qquad \forall t \in [0, \infty) \ . \tag{23}$$

Now if we assume that $L_f^i V(x) = 0$ for all $i \in \mathbb{N}$ we have from (23) with $t = t_c$ that $L_f V(\phi(t_c, x)) = \sum_{i=0}^{\infty} L_f^{i+1} V(x) \frac{t_c^j}{j!} = 0$, but this contradicts (22). Consequently, there must exist an integer $n = n(x)$ such that $L_f^n V(x) \neq 0$.

Now we show that for the given set E there exists $\ell \in \mathbb{N}$ such that for any $x \in E$ we have $L_f^i V(x) \neq 0$ for some $i \in [1, \ell]$. Assume for the purpose of showing contradiction that there exists a sequence $x_p \in E$ and a strictly increasing sequence of positive integers n_p such that

$$L_f^i V(x_p) \ = \ 0 \ , \ \forall i \in [1, n_p - 1]; \qquad L_f^{n_p} V(x_p) \ \neq \ 0 \ . \tag{24}$$

Since E is a compact set, there exists a subsequence of x_p, that we still denote as x_p and $x^* \in E$ such that $\lim_{p\to+\infty} x_p = x^*$. ¿From the first part of the proof we know that there exists an integer $n^* = n(x^*)$ such that $L_f^{n^*} V(x^*) \neq 0$. Since $L_f^{n^*} V(x)$ is a continuous function of x, there exists $\delta > 0$ such that

$$L_f^{n^*} V(x) \neq 0 \qquad \forall x \in \{z : |x^* - z| \leq \delta\} \ . \tag{25}$$

Since x_p converges to x^*, there exists $P_1 \in \mathbb{N}$ such that $|x_p - x^*| \leq \delta$ for all $p \geq P_1$ and hence

$$L_f^{n^*} V(x_p) \neq 0 \qquad \forall p \geq P_1 \ . \tag{26}$$

The sequence n_p is strictly increasing and such that $\lim_{p\to+\infty} n_p = +\infty$. It follows that there exists $P_2 \in \mathbb{N}$ with $P_2 \geq P_1$ such that

$$L_f^{n^*} V(x_p) = 0 \qquad \forall p \geq P_2 \ . \tag{27}$$

But (26) and (27) yield a contradiction and this completes the proof.

5 Conclusions

We have presented a partial construction of a strong Lyapunov function using a weak Lyapunov function. The construction can be carried out if a mild Lie algebraic condition that involves the weak Lyapunov function holds. Our construction would be useful for analysis of stability robustness for important classes of systems, such as passive nonlinear systems since the storage function of the passive system typically turns out to be a weak Lyapunov function. Also, using our result in design or redesign of controllers for systems such as the ones satisfying Jurdjevic-Quinn conditions is an interesting topic for further research.

Acknowledgment. The authors would like to thank J.B. Pomet for useful discussions.

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A Technical lemmas

Lemma A.1 *Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a positive definite function and $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ a strictly positive function. Then, the following holds:*

1. *$V(\cdot)$ is a weak Lyapunov function for the system (1) if and only if it is a weak Lyapunov function for the system (2).*
2. *$V(\cdot)$ is a strong Lyapunov function for the system (1) if and only if it is a strong Lyapunov function for the system (2).*
3. *The system (1) satisfies the condition (ii) of Theorem 3.1 if and only if the system (2) satisfies the condition (ii) of Theorem 3.1*

Proof of Lemma A.1: Note that

$$\dot{V}|_{(1)} = \frac{\partial V}{\partial x}(x)f(x)\ ,\ \dot{V}|_{(2)} \ = \ \lambda(V(x))\frac{\partial V}{\partial x}(x)f(x)\ . \tag{28}$$

Then the proof of the first two statements of Lemma A.1 follows from the following observation. Since $\lambda(V(x))$ is strictly positive for all $x \in \mathbb{R}^n$, we have that $\frac{\partial V}{\partial x}(x)f(x)$ is negative definite (semi-definite) if and only if $\lambda(V(x))\frac{\partial V}{\partial x}(x)f(x)$ is negative definite (semi-definite).

Now we prove the third claim in Lemma A.1. By induction, one can show that for any $i \in \mathbb{N}$ there exist polynomials $Q_i : \mathbb{R}^i \to \mathbb{R}$ and $P_i : \mathbb{R}^i \to \mathbb{R}$ with $P_i(0,0,\dots,0) = Q_i(0,0,\dots,0) = 0$ and for all $j \in [0,i]$ there exist continuous functions $\rho^i_j : \mathbb{R}^n \to \mathbb{R}$ and $\theta^i_j : \mathbb{R}^n \to \mathbb{R}$ such that

$$L^{i+1}_{f_\lambda}V(x) \ = \ \lambda(V(x))\left[L^{i+1}_f V(x) + P_i\left(\rho^i_1(x)L_f V(x),\dots,\rho^i_i(x)L^i_f V(x)\right)\right] =: \bar{P}_{i+1}(x) \tag{29}$$

$$L^{i+1}_f V(x) \ = \ \frac{1}{\lambda(V(x))}\left[L^{i+1}_{f_\lambda}V(x) + Q_i\left(\theta^i_1(x)L_{f_\lambda}V(x),\dots,\theta^i_i(x)L^i_{f_\lambda}V(x)\right)\right] =: \bar{Q}_{i+1}(x)\ . \tag{30}$$

According to (28), the property is satisfied at the step $i = 0$. If one assumes that it is satisfied at the step i, then direct calculations on the equalities (29) and (30) show that it is satisfied at the step $i+1$.

The next part of the proof consists in proceeding by contradiction. Suppose that there exists $\ell \in \mathbb{N}$ such that: (i) for any $x \in \mathbb{R}^n$, there exists an integer $n = n(x)$ with $n \in [1,\ell]$ such that $L^n_f V(x) \neq 0$; (ii) there exists $x_c \in \mathbb{R}^n$ such that, for all $m \in [1,\ell]$, $L^m_{f_\lambda}V(x_c) = 0$. According to (30), for all $i \in [0, l-1]$,

$$L^{i+1}_f V(x_c) \ = \ \frac{1}{\lambda(V(x_c))}\left[L^{i+1}_{f_\lambda}V(x_c) + Q_i\left(\theta^i_1(x_c)L_{f_\lambda}V(x_c),\dots,\theta^i_i(x_c)L^i_{f_\lambda}V(x_c)\right)\right]\ . \tag{31}$$

Since for all $m \in [1,\ell]$, $L^m_{f_\lambda}V(x_c) = 0$, it follows that for all $i \in [0, l-1]$,

$$L^{i+1}_f V(x_c) \ = \ 0\ . \tag{32}$$

This contradicts the assumptions that for any $x \in \mathbb{R}^n$, there exists an integer $n = n(x)$ with $n \in [1,\ell]$ such that $L^n_f V(x) \neq 0$. This proves that if item (ii) of Theorem 3.1 holds for the system (1), then the same condition holds for the system (2). The statement in the opposite direction is obtained in the same way by using (29).

Lemma A.2 *Consider $f : \mathbb{R}^n \to \mathbb{R}^n$ and a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Suppose that for some $N \in \mathbb{N}$ and all $i \in [1,N]$ there exist $\kappa_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that*

$$\left|L^i_f V(x)\right| \leq \kappa_i(V(x)), \qquad \forall i \in [1,N],\ \forall x \in \mathbb{R}^n\ . \tag{33}$$

Then, given any $c > 0$ there exists a strictly positive nonincreasing function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such $f_\lambda(x) := \lambda(V(x))f(x)$ satisfies

$$\left|L^i_{f_\lambda}V(x)\right| \leq c, \qquad \forall i \in [1,N], \forall x \in \mathbb{R}^n\ . \tag{34}$$

Proof of Lemma A.2: Let $N \in \mathbb{N}$ come from the lemma. To simplify notation we use $V = V(x)$ in this proof. Note first that given any function λ and for any $i \in [1, N]$ there exists a polynomial $G_i : \mathbb{R}^{2i} \to \mathbb{R}$ such that

$$L^i_{f_\lambda} V = \lambda(V) \cdot G_i(L_f V, \dots, L^i_f V, \lambda(V), \dots, \lambda^{(i-1)}(V)) \ , \tag{35}$$

where $\lambda^{(j)}(V) := \frac{d^j \lambda}{dv^j}(V)$. This implies that there exist two polynomials $\Gamma_1 : \mathbb{R}^N \to \mathbb{R}$ and $\Gamma_2 : \mathbb{R}^N \to \mathbb{R}$ with positive coefficients and $\min\{\Gamma_1(0, 0, \dots, 0), \Gamma_2(0, 0, \dots, 0)\} > 0$ such that

$$|L^i_{f_\lambda} V| \le \lambda(V) \cdot \Gamma_1(|L_f V|, \dots, |L^N_f V|) \cdot \Gamma_2(|\lambda(V)|, \dots, |\lambda^{(N-1)}(V)|) \qquad \forall i \in [1, N],\ \forall x \in \mathbb{R}^n \ . \tag{36}$$

Since (33) holds, there exists a nondecreasing function $\kappa : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ such that such that

$$|L^i_{f_\lambda} V| \le \lambda(V) \cdot \kappa(V) \cdot \Gamma_2(|\lambda(V)|, \dots, |\lambda^{(N-1)}(V)|) \qquad \forall i \in [1, N],\ \forall x \in \mathbb{R}^n \ . \tag{37}$$

We denote $\sigma := \sup_{z_1 \in [0,1], \dots, z_N \in [0,1]} \Gamma_2(z_1, \dots, z_N)$. Let $c > 0$ be given and define

$$\lambda(v) = \eta \int_v^{v+1} \left(\int_{s_1}^{s_1+1} \cdots \int_{s_N}^{s_N+1} \frac{1}{\kappa(s_{N+1})} ds_{N+1} \right) \dots ds_1 \tag{38}$$

where $\eta := \min\left\{\kappa(0), \frac{c}{1+\sigma}\right\}$. We show below that (34) holds.

Note that the function $\kappa(\cdot)$ is positive and nondecreasing. It follows that the function $\eta \int_{s_N}^{s_N+1} \frac{1}{\kappa(s_{N+1})} ds_{N+1}$ is a nonnegative nonincreasing function of s_N such that

$$\eta \int_{s_N}^{s_N+1} \frac{1}{\kappa(s_{N+1})} ds_{N+1} \le \frac{\eta}{\kappa(s_N)} \ .$$

It follows by induction that $\lambda(\cdot)$ is nonincreasing and such that

$$\lambda^{(j)}(v) \le \frac{\eta}{\kappa(v)} \qquad \forall j \in [0, N],\ \forall v \ge 0. \tag{39}$$

So in particular, since $\eta \le \kappa(0)$, then we have

$$\lambda^{(j)}(v) \ \le\ 1 \qquad \forall j \in [0, N],\ \forall v \ge 0. \tag{40}$$

It follows from (37), (39), (40) and the definition of η and σ that for all $x \in \mathbb{R}^n$ we have:

$$\begin{array}{rcl} |L^i_{f_\lambda} V(x)| & \le & \lambda(V) \cdot \kappa(V) \cdot \Gamma_2(|\lambda(V)|, \dots, |\lambda^{(N-1)}(V)|) \\ & \le & \eta \cdot \Gamma_2(|\lambda(V)|, \dots, |\lambda^{(N-1)}(V)|) \\ & \le & \frac{c}{1+\sigma} \cdot \sigma \\ & \le & c \ , \end{array} \tag{41}$$

which completes the proof.