Lyapunov functions for time varying systems satisfying generalized conditions of Matrosov theorem*

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Abstract

The classical Matrosov theorem concludes uniform asymptotic stability of time varying systems via a *weak Lyapunov function* (positive definite, decrescent, with *negative semi-definite derivative* along solutions) and another auxiliary function with derivative that is strictly non-zero where the derivative of the Lyapunov function is zero [M1]. Recently, several generalizations of the classical Matrosov theorem that use a finite number of Lyapunov-like functions have been reported in [LPPT2]. None of these results provides a construction of a *strong Lyapunov function* (positive definite, decrescent, with *negative definite derivative* along solutions) that is a very useful analysis and controller design tool for nonlinear systems. We provide a construction of a strong Lyapunov function via an appropriate weak Lyapunov function and a set of Lyapunov-like functions whose derivatives along solutions of the system satisfy inequalities that have a particular triangular structure. Our results will be very useful in a range of situations where strong Lyapunov functions are needed, such as robustness analysis and Lyapunov function based controller redesign. We illustrate our results by constructing a strong Lyapunov function for a simple Euler-Lagrange system controlled by an adaptive controller.

Key words. Lyapunov functions, Matrosov Theorem, Nonlinear, Stability, Time-Varying.

1 Introduction

Lyapunov second method is ubiquitous in stability and robustness analysis of nonlinear systems. In recent years, its different versions were used for controller design, e.g. control Lyapunov functions, nonlinear damping, backstepping, forwarding, and so on [K, SJK, ST, FP, M2]. While it is often useful to obtain a *strong Lyapunov function* (positive definite, decrescent, with *negative definite derivative* along solutions) to analyze robustness or redesign the given controller, it is often the case that only a *weak Lyapunov function* (positive definite, decrescent,

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with *negative semi-definite derivative* along solutions) can be constructed for a problem at hand [A1, A2, JQ, L1, L2, NA]. For example, controller design methods that are based on the passivity property typically require the use of the La Salle invariance principle [L1] which exploits weak Lyapunov functions to conclude asymptotic stability.

The La Salle Theorem in its original form applies only to time-invariant systems. On the other hand, the Matrosov Theorem [M1] concludes uniform asymptotic stability of time-varying systems via a *weak Lyapunov function* and another auxiliary function with derivative that is strictly non-zero where the derivative of the Lyapunov function is zero [M1]. Different generalizations of the Matrosov theorem that use an arbitrary number of auxiliary functions to conclude uniform asymptotic stability have been recently reported in [LPPT2]. Moreover, results in [LPPT2] make use of the recently proposed notion of uniform δ persistency of excitation (uδ-PE condition) [LPT2] that allows to further relax the original Matrosov conditions. The proofs presented in [LPPT2, M1] do not provide a construction of a strong Lyapunov function and they conclude uniform asymptotic stability by considering directly the behavior of the trajectories of the system.

The main purpose of this paper is to construct strong Lyapunov functions using appropriate generalized Matrosov conditions that are inspired by main results in [LPPT2]. In particular, each of our results assumes existence of an appropriate weak Lyapunov function and a set of Lyapunov-like functions, similar to [LPPT2], to provide explicit formulas for constructing a strong Lyapunov function. Moreover, our results parallel main results in [LPPT2] and we present constructions that exploit the uδ-PE condition. Constructions provided in this paper will be useful in a range of situations when the knowledge of a strong Lyapunov function is useful, such as robustness analysis and Lyapunov based controller redesign. Observe, in particular, that an ISS Lyapunov characterization was obtained in [SW] and that strong Lyapunov functions have been used to design stabilizing feedback laws that render asymptotically controllable systems ISS (as defined in [S3]) to actuator errors and small observation noise (see [S4]). Such control laws are expressed in terms of gradients of Lyapunov functions, and therefore require explicit strong Lyapunov functions to be implemented. We illustrate our main results by constructing a strong Lyapunov function for the pendulum equations controlled by an adaptive controller and, in a second step, by using this Lyapunov function to determine a feedback rendering the closed loop system globally ISS with respect to an additive disturbance in the input. We also comment on how our constructions apply to cases discussed in [LPPT2] where an appropriate weak Lyapunov function is known and certain uniform observability conditions are satisfied. Our results can also be applied to a class of nonholonomic systems studied, for instance, in [S1] and [LPPT2]. We note that our results provide an alternative construction of a strong Lyapunov function to the one presented in [MN] for time-invariant systems and a special case of our results also generalizes the constructions of strong Lyapunov functions given in [M3] and [MM].

The paper is organized as follows. In Section 2 we present mathematical preliminaries and assumptions that are needed in the sequel. Section 3 is devoted to the case where the assumptions of the classical Matrosov theorem are satisfied. Section 4 contains main results. An illustration of our main results is presented in Section 5 and the proofs of all main results are given in Section 6. Conclusions and some auxiliary results are given respectively in the last section and the appendix.

2 Preliminaries

Unless otherwise stated, we assume throughout the paper that the functions encountered are sufficiently smooth. We often omit arguments of functions to simplify notation. Throughout this paper, $|\cdot|$ stands for the Euclidean norm vectors and induced norm matrices. A continuous function $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if $k(0) = 0$ and k is increasing. It is said to belong to class $\mathcal{K}_\infty$ if it is unbounded. A function $\kappa : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r, and for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\lim\limits_{s\to+\infty} \beta(r,s) = 0$. A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is positive semi-definite if $V(0) = 0$ and $V(x) \geq 0$ for all $x \in \mathbb{R}^n$. It is positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. It is negative semi-definite (definite) if $-V$ is positive semi-definite (definite).

Consider the time varying system:

$$\dot{x} = f(t,x) \tag{1}$$

with $t \in \mathbb{R}, x \in \mathbb{R}^n$. For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, we will denote by $x(t; t_0, x_0)$, or simply by $x(t)$, the unique solution of (1) that satisfies $x(t_0; t_0, x_0) = x_0$. In order to simplify the notation, we use the following notation:

$$DV := \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \ ,$$

where $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

We need the definitions and assumptions given below. The following definition is a slightly modified version of [S2, Definition 5.14].

Definition 1 *A continuous function $\phi(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^p$ is called a function decrescent in norm if, there exists a function $\beta(\cdot)$ of class $\mathcal{K}$, such that such that for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$ the following holds*

$$|\phi(t,x)| \ \leq \ \beta(|x|) \ . \tag{2}$$

Definition 2 *The system (1) is* uniformly globally asymptotically stable *provided there exists $\beta \in \mathcal{KL}$ such that $|x(t; t_0, x_0)| \leq \beta(|x_0|, t - t_0)$ for all $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and $t \in \mathbb{R}$.*

Definition 3 *Suppose that there exist functions $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $\alpha_1, \alpha_2, \alpha_4 \in \mathcal{K}_\infty$ and $\alpha_3 : \mathbb{R}^n \to \mathbb{R}$ such that for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$, the following holds:*

$$\begin{aligned} \alpha_1(|x|) \ &\leq \ V(t,x) \ \leq \ \alpha_2(|x|) \ , && (3) \\ DV \ &\leq \ -\alpha_3(x) \ , && (4) \\ \left| \frac{\partial V}{\partial x}(t,x) \right| \ &\leq \ \alpha_4(|x|) \ . && (5) \end{aligned}$$

If the function α_3 is positive semi-definite, then we say that V is a weak Lyapunov function for the system (1). If, on the other hand, α_3 is positive definite, then V is referred to as a strong Lyapunov function for the system (1).

Assumption 4 *The function f in (1) is locally Lipschitz uniformly in t, $f(t,0) = 0$ for all $t \in \mathbb{R}$, and a weak Lyapunov function V_1 for the system (1) is known.*

Assumption 5 *The following functions are known:* $V_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i = 2, 3, \ldots, j$, *such that* V_i *and* $\frac{\partial V_i}{\partial x}(t, x)$ *are decrescent in norm; positive semi-definite functions* $N_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *for* $i = 2, \ldots, j$, *decrescent in norm; continuous functions* $\chi_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{i-2} \to \mathbb{R}$ *for* $i = 3, \ldots, j$ *such that,*

$$|\chi_i(t, x, N_2, \ldots, N_{i-1})| \leq \lambda_i(N_2, ..., N_{i-1})\rho_i(x) \tag{6}$$

where the functions $\rho_i : \mathbb{R}^n \to \mathbb{R}$ *are positive and the functions* $\lambda_i : \mathbb{R}^{i-1} \to \mathbb{R}$ *are positive semi-definite, continuous but not necessarily of class* C^1. *Moreover, for all* $t \in \mathbb{R}$ *and all* $x \in \mathbb{R}^n$, *we have:*

$$\begin{array}{rcl} DV_2 & \leq & -N_2 \ , \\ DV_3 & \leq & -N_3 + \chi_3(t, x, N_2) \ , \\ DV_4 & \leq & -N_4 + \chi_4(t, x, N_2, N_3) \ , \\ \vdots & \vdots & \vdots \\ DV_j & \leq & -N_j + \chi_j(t, x, N_2, \ldots, N_{j-1}) \ . \end{array} \tag{7}$$

Remark 6 *According to [S2, Theorem 5.16], when the vector field of the system (1) is locally Lipschitz uniformly in* t, *satisfies* $f(t, 0) = 0$ *for all* $t \in \mathbb{R}$ *and admits a strong Lyapunov function, then it admits the origin as a globally uniformly asymptotically stable equilibrium point.*

Remark 7 *All our main results will be using Assumptions 4 and 5, as well as some other conditions. We note that Assumption 4 assumes existence of a weak Lyapunov function, whereas Assumption 5 assumes existence of a set of auxiliary functions. We note that these auxiliary functions do not have to be positive definite in general. Moreover, we note that the references [LKT, LPPT2, LPPT1, LPT1, LPT2] present a range of different situations where Assumptions 4 and 5 hold. Moreover, the functions* V_i *are constructed for the cases of model reference control [LKT, LPPT2], classical Matrosov theorem [LPT1], a class of nonholonomic systems [LPPT1] and systems satisfying appropriate uniform observability conditions [LPPT2]. We will recall and revisit this last construction in Section 4, Proposition 17.*

Remark 8 *If a function* χ_i *satisfies the inequality*

$$|\chi_i(t, x, N_2, \ldots, N_{i-1})| \leq \rho_m(x)$$

where ρ_m *is a positive function and is such that, for all* t, x, $\chi(t, x, 0, \ldots, 0) = 0$, *then functions* λ_i *and* ρ_i *such that (6) is satisfied can be determined. We do not give the proof of this result, because, on the one hand, it is very technical and, on the other hand, it is easy, in pratice, to determine the required functions* λ_i *and* ρ_i.

3 Introductory result

The objective of this section, is to familiarize the reader with the technique used thoughout our work. We explicitly construct a family of strong Lyapunov functions in the simple case where the system (1) satisfies the conditions of the classical Mastrosov theorem. This construction is the first construction of a strong Lyapunov function under the conditions of the Matrosov theorem. Due to its introductory interest, we give it in this section, instead of putting it in Section 6.

Theorem 9 *Consider the system (1) and suppose that Assumptions 4 and 5 hold with $j = 3$, that $f(t, x)$ is decrescent in norm and that $V_1 = V_2$. Suppose also that:*

$$N_2(t, x) + N_3(t, x) \geq \omega(x) \tag{8}$$

where ω is a positive definite function. Then, one can determine two nonnegative functions p_1, p_3 such that the following function:

$$W(t, x) = p_1(V_1(t, x))V_1(t, x) + p_3(V_1(t, x))V_3(t, x) \tag{9}$$

is a strong Lyapunov function for system (1).

Proof. Let

$$S_a(t, x) = V_1(t, x) + V_3(t, x) \ . \tag{10}$$

From Assumption 5, we deduce that

$$DS_a = DV_1 + DV_3 \leq -N_2 - N_3 + \chi_3(t, x, N_2) \ . \tag{11}$$

Using Assumption 5, one can determine a function ϕ, of class $\mathcal{K}_\infty$ such that

$$|\chi_3(t, x, N_2)| \leq \phi(N_2)\rho_3(x) \ . \tag{12}$$

(For instance, one can choose $\phi(r) = r + \sup_{\{0 \leq l \leq r\}} \lambda_2(l)$). This inequality and (8) yield

$$DS_a \leq -\omega(x) + \phi(N_2)\rho_3(x) \ . \tag{13}$$

Let

$$S_b(t, x) = p_3(V_1(t, x))S_a(t, x) \tag{14}$$

where p_3 is a positive definite function to be specified later. A simple calculation yields

$$DS_b \leq -p_3(V_1)\omega(x) + p_3(V_1)\phi(N_2)\rho_3(x) + p_3'(V_1)S_a DV_1 \ . \tag{15}$$

Let us distinguish between two cases:
<u>*First case:*</u> $N_2 \leq p_3(V_1)$. Since ϕ is nondecreasing, the inequality

$$p_3(V_1)\phi(N_2)\rho_3(x) \leq p_3(V_1)\phi(p_3(V_1))\rho_3(x) \tag{16}$$

is satisfied.
<u>*Second case:*</u> $N_2 \geq p_3(V_1)$. Then the inequality

$$p_3(V_1)\phi(N_2)\rho_3(x) \leq N_2\phi(N_2)\rho_3(x) \tag{17}$$

is satisfied. It follows that, for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$,

$$p_3(V_1)\phi(N_2)\rho_3(x) \leq N_2\phi(N_2)\rho_3(x) + p_3(V_1)\phi(p_3(V_1))\rho_3(x) \ . \tag{18}$$

From Lemma 22, we deduce that one can construct a positive definite function p_3 such that

$$p_3(V_1) \leq \phi^{-1}\left(\frac{\omega(x)}{2\rho_3(x)}\right) \ . \tag{19}$$

For such a choice, the inequality

$$p_3(V_1)\phi(N_2)\rho_3(x) \leq N_2\phi(N_2)\rho_3(x) + \frac{1}{2}p_3(V_1)\omega(x) \tag{20}$$

is satisfied. Combining (15) and (20), we obtain

$$DS_b \leq -\frac{1}{2}p_3(V_1)\omega(x) + N_2\phi(N_2)\rho_3(x) + p_3'(V_1)S_a DV_1 \ . \tag{21}$$

Since V_1 is a weak Lyapunov function, there is a function α_1 of class $\mathcal{K}_\infty$ such that the inequality

$$\alpha_1(|x|) \quad \leq \quad V(t,x) \tag{22}$$

is satisfied (see Definition 3). On the other hand, since $\phi(N_2)\rho_3(V_1)$ and $p_3'(V_1)S_a$ are decrescent in norm, one can determine a function Γ, positive and nondecreasing, such that

$$2\phi(N_2)\rho_3(x) \leq \Gamma(\alpha_1^{-1}(|x|)) \ , \ 2|p_3'(V_1)S_a| \leq \Gamma(\alpha_1^{-1}(|x|)) \ . \tag{23}$$

It follows that

$$2\phi(N_2)\rho_3(x) \leq \Gamma(V_1) \ , \ 2|p_3'(V_1)S_a| \leq \Gamma(V_1) \ . \tag{24}$$

Using these inequalities and $DV_1 = DV_2 \leq -N_2$, we obtain

$$DS_b \leq -\frac{1}{2}p_3(V_1)\omega(x) + \frac{1}{2}(N_2 - DV_1)\Gamma(V_1) \leq -\frac{1}{2}p_3(V_1)\omega(x) - DV_1\Gamma(V_1) \ . \tag{25}$$

We deduce that the function W given in (9) with $p_1(r) = \frac{1}{r}\displaystyle\int_0^r \Gamma(l)dl + p_3(r)$ and p_3 satisfying (19) satisfies

$$DW \leq -\frac{1}{2}p_3(V_1)\omega(x) \ . \tag{26}$$

Since V_1 is a weak Lyapunov function, we deduce from (3) that there exists a positive definite function α_3 such that $\frac{1}{2}p_3(V_1)\omega(x) \geq \alpha_3(x)$. Therefore the requirement (4) is satisfied. Besides, W is decrescent in norm and

$$\Gamma(0)\alpha_1(|x|) \leq \Gamma(0)V_1(t,x) \leq W(t,x) \tag{27}$$

and $\frac{\partial V_1}{\partial x}(t,x)$, $\frac{\partial V_3}{\partial x}(t,x)$ are decrescent in norm. Therefore W also satisfies the requirement (3) and (5). It follows that W is a strong Lyapunov function for system (1).

4 Main results

In this section, we establish main results of this paper that are summarized in Theorem 10, Theorem 12, Corollary 14, Theorem 15 and Proposition 17. Each of these results provides a construction of a strong Lyapunov function using an existing weak Lyapunov function from Assumption 4, a set of Lyapunov-like functions from Assumption 5 and other appropriate conditions.

The first result of this section is an extention of Theorem 9 to the case where, instead of only one auxiliary function, several auxiliary functions are available.

Theorem 10 *Consider the system (1) and suppose that Assumptions 4 and 5 hold and that $f(t,x)$ is decrescent in norm. Suppose also that:*

$$\sum_{i=2}^{j} N_i(t,x) \geq \omega(x) \tag{28}$$

where $\omega(x)$ is a positive definite function. Then, one can determine nonnegative functions p_i such that the following function:

$$W(t,x) = \sum_{i=1}^{j} p_i(V_1(t,x))V_i(t,x) \tag{29}$$

is a strong Lyapunov function for system (1).

Remark 11 *We note that a construction of the functions p_i in (29) is provided in the proof of Theorem 10. Moreover, we emphasize that there is some flexibility in terms of choosing functions p_i in (29). This flexibility can be seen from the proof of Theorem 10. The same comment applies to all the results of this section.*

To state the second main result, we will suppose that the system (1) admits the decomposition:

$$\dot{x}_1 = f_1(t,x)\ ,\ \dot{x}_2 = f_2(t,x) \tag{30}$$

with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$. Note that we allow for the cases when either $n_1 = n$ or $n_2 = n$ that correspond to $x_1 = x$ and $x_2 = x$, respectively.

Theorem 12 *Consider the system (30) and suppose that Assumptions 4 and 5 hold. Suppose also that the following conditions hold:*

C1. *There exist a positive definite real-valued function ω, and a positive semi-definite continuously differentiable function $M : \mathbb{R} \times \mathbb{R}^{n_2} \to \mathbb{R}$ such that $M(t,x_2)$ and $\frac{\partial M}{\partial x_2}(t,x_2)$ are decrescent in norm and the following holds for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,*

$$\sum_{i=2}^{j} N_i(t,x) \geq \omega(|x_1|) + M(t,x_2) \tag{31}$$

and

$$|f_2(t,x)| \leq \chi_f(t,x,N_2,N_3,\ldots,N_{j-1})\ , \tag{32}$$

where χ_f is so that

$$0 \leq \chi_f(t,x,N_2,\ldots,N_{j-1}) \leq \lambda_f(N_2,...,N_{j-1})\rho_f(x) \tag{33}$$

where the function ρ_f is positive and the function λ_f is positive semi-definite.

C2. *There exist a differentiable function $\theta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a positive definite function $\gamma : \mathbb{R} \to \mathbb{R}$ such that for all $(t,x_2) \neq (t,0)$, we have:*

$$\int_{t-\theta(|x_2|^2)}^{t} M(s,x_2)ds \geq \gamma(|x_2|)\ . \tag{34}$$

Then, one can determine nonnegative functions p_i and a positive definite function δ such that the following function:

$$W(t,x) = \sum_{i=1}^{j} p_i(V_1(t,x))V_i(t,x) + p_{j+1}(V_1(t,x))\delta(|x_2|^2)A(t,x_2) \tag{35}$$

with

$$A(t,x_2) = \int_{t-\theta(|x_2|^2)}^{t} \left(\int_s^t M(l,x_2)dl \right) ds \tag{36}$$

when $x_2 \neq 0$ and

$$A(t,0) = 0 \ , \ \forall t \tag{37}$$

is a strong Lyapunov function for system (30).

Remark 13 *Conditions of Theorem 12 can be regarded as generalized Matrosov theorem conditions and they are directly related to conditions used in [LPPT2, Theorem 1]. Indeed, our Assumption 4 corresponds to [LPPT2, Assumption 1]. Our Assumption 5 corresponds to [LPPT2, Assumptions 2 and 3], and so on. In particular, our condition* **C2** *corresponds to the so called $u\delta$-PE condition introduced in [LPT2]. Note, however, that our conditions are stronger in that we assume that we know all the bounding functions since they are required in the construction of the strong Lyapunov function W. For instance, we assume that we know the functions θ and γ in the condition* **C2** *of Theorem 12 whereas this is not needed in main results of [LPPT2]. This is the main difference between our conditions and those given in [LPPT2]. A consequence of our stronger assumptions is that we construct a strong Lyapunov function W, which was not done in [LPPT2].*

The following corollary is devoted to the case where $x_2 = x$. In particular, we can state:

Corollary 14 *Consider the system (30) and suppose that Assumptions 4 and 5 hold. Suppose also there exists a positive semi-definite continuously differentiable function $M : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $M(t,x)$ and $\frac{\partial M}{\partial x}(t,x)$ are decrescent in norm and the following holds for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$*

$$\sum_{i=2}^{j} N_i(t,x) \quad \geq \quad M(t,x) \tag{38}$$

and

$$|f(t,x)| \leq \chi_f(t,x,N_2,N_3,\ldots,N_{j-1}) \ , \tag{39}$$

where χ_f is so that

$$0 \leq \chi_f(t,x,N_2,\ldots,N_{j-1}) \leq \lambda_f(N_2,...,N_{j-1})\rho_f(x) \tag{40}$$

where the function ρ_f is positive and the function λ_f is positive semi-definite. Moreover, there exist a differentiable function $\theta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a positive definite function $\gamma : \mathbb{R} \to \mathbb{R}$ such that for all $(t,x) \neq (t,0)$, we have:

$$\int_{t-\theta(|x|^2)}^{t} M(s,x)ds \geq \gamma(|x|) \ . \tag{41}$$

Then, one can determine nonnegative functions p_i, $i = 1$ to $j+1$ and a positive definite function δ such that the following function defind by

$$W(t,x) = \sum_{i=1}^{j} p_i(V_1(t,x))V_i(t,x) + p_{j+1}(V_1(t,x))\delta(|x|^2)\int_{t-\theta(|x|^2)}^{t}\left(\int_s^t M(l,x)dl\right)ds \qquad (42)$$

when $x \neq 0$ and

$$W(t,0) = 0\ ,\ \forall t \in \mathbb{R} \qquad (43)$$

is a strong Lyapunov function for system (30).

It is possible to strengthen the persistency condition (41) and at the same time relax the condition (39) to provide a similar Lyapunov function construction that is presented in the next corollary. Observe that the strong Lyapunov functions we obtain are given by expressions slightly simpler than (35).

Theorem 15 *Consider the system (1) and suppose that Assumptions 4 and 5 hold and that $f(t,x)$ is decrescent in norm. Suppose also that the following holds for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$*

$$\sum_{i=2}^{j} N_i(t,x) \geq M(t,x) = \overline{p}(t)\mu(x) \qquad (44)$$

where μ is a positive definite function and $\overline{p}(t)$ is a nonnegative function such that, for all $t \in \mathbb{R}$,

$$\int_{t-\tau}^{t} \overline{p}(l)dl \geq p_m \quad , \quad \overline{p}(t) \leq p_M \qquad (45)$$

where $\tau > 0, p_m > 0, p_M > 0$.

Then, one can determine nonnegative functions p_i such that the following function:

$$W(t,x) = \sum_{i=1}^{j} p_i(V_1(t,x))V_i(t,x) + p_{j+1}(V_1(t,x))\left(\int_{t-\tau}^{t}\left(\int_s^t \overline{p}(l)dl\right)ds\right) \qquad (46)$$

is a strong Lyapunov function for system (1).

Remark 16 *We note that it is not clear how to construct a locally or globally quadratic Lyapunov function under appropriate conditions that would guarantee uniform local or global exponential stability. More generally, it is not clear how to construct homogeneous Lyapunov functions under appropriate conditions for homogeneous time varying systems. Constructing Lyapunov functions with special properties under different stronger conditions is outside the scope of this paper but they are very important and are left for further research.*

The result below is an extension of the main result of [MN] to time-varying systems. Similar observability conditions were used in [LPPT2, Section 3.3] to conclude uniform asymptotic stability of time-varying systems.

Proposition 17 *Consider the system (1) for which Assumption 4 holds, that is we have:*

$$DV_1(t,x) =: b_1(t,x) \leq 0\ . \qquad (47)$$

Define the following functions $b_{i+1}(t,x) := Db_i(t,x), i \geq 1$. Assume that all, for $i \geq 1$, the functions b_i and $\frac{\partial b_i}{\partial x}(t,x)$ are decrescent in norm. Then, given any integer $N \geq 3$, the following functions satisfy Assumption 5:

$$\begin{array}{ll} V_2(t,x) := V_1(t,x) \ , & \\ V_i(t,x) := -b_{i-2}(t,x)b_{i-1}(t,x) & i \in \{3,\ldots,N\} \ , \end{array} \tag{48}$$

with $N_2 = -b_1$, $\chi_3(t,x,N_2) = |b_1||b_3|$, and, for $i \geq 3$, $N_i = b_{i-1}^2$, $\chi_i(t,x,N_2,...,N_{i-1}) = \sqrt{N_{i-1}}|b_i| = |b_{i-2}||b_i|$.

Remark 18 *It is obvious that if the functions V_i constructed in Proposition 17 further satisfy conditions of one of the theorems 10, 12, 15, or Corollary 14, we can use V_i to construct a strict Lyapunov function as outlined in the previous section. We do not state all of these corollaries for space reasons.*

5 Illustration

In this section, we illustrate our main results by means of a system resulting from an adaptive tracking control problem for the well-known pendulum equations with an unknown friction coefficient (see [K, Section 1.1.1]). First, we recall how an adaptive control law can be constructed, using a classical approach, which relies on the construction of a weak Lyapunov function. In a second step, we use Theorem 15 to determine a strong Lyapunov function. At last, we exploit this strong Lyapunov function to obtain a control law which renders the system ISS with respect to additive disturbance in the input.

The system we consider is given by the equations

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2 \ , \\ \dot{x}_2 & = & -\frac{g}{l}\sin(x_1) - \theta x_2 - \frac{1}{ml^2}(T+d) \ , \end{array} \right. \tag{49}$$

with θ unknown but constant, with g, m, l known, where T is the input and where d is a disturbance. The controller will be designed to track the trajectory

$$x_1^*(t) = \frac{1}{2}\sin(t) \ , \ x_2^*(t) = \frac{1}{2}\cos(t) \ . \tag{50}$$

This very simple adaptive control problem can been solved by classical Lyapunov-based design techniques, presented for instance in [KKK]. The proof relies on a dynamic extention and the construction of a weak Lyapunov function, which ensure convergence of the state variables to the reference trajectory (50), when $d \equiv 0$. But the construction of a strong Lyapunov function is still an open problem and therefore so is the problem of constructing a control law so that the corresponding closed-loop system is globally ISS with respect to the additive disturbance d. This absence of strong Lyapunov function in the broad literature devoted to mechanical systems and adaptive control and the advantages inherent to the knowledge of strong Lyapunov functions, such as the possibility of constructing a robust control law, are motivations for our choice of illustrating example.

Step 1. Solution of the adaptive problem when $d = 0$.

First, we briefly recall a solution to the adaptive control problem in the absence of disturbance d, based on the Lyapunov technique of [KKK, Section 4.3].

Lemma 19 *Consider the system (49) with, for all $t \in \mathbb{R}$, $d(t) = 0$ and the adaptive controller*

$$\begin{array}{rcl} T(t, x_1, x_2, \hat{\theta}) & = & -mlg\sin(x_1) + ml^2\left[e_1 + e_2 + \frac{1}{2}\sin(t) - \hat{\theta}x_2\right] \\ \dot{\hat{\theta}} & = & -x_2[2e_2 + e_1] \end{array} \tag{51}$$

with

$$e_1 = x_1 - x_1^*(t) \ , \ e_2 = x_2 - x_2^*(t) \ . \tag{52}$$

Then this adaptive controller guarantees that global asymptotic tracking is achieved:

$$\lim_{t\to+\infty}[x_1(t) - x_1^*(t)] = 0 \ , \ \lim_{t\to+\infty}[x_2(t) - x_2^*(t)] = 0 \ . \tag{53}$$

Besides,

$$\lim_{t\to+\infty}[\theta - \hat{\theta}(t)] = 0 \ . \tag{54}$$

Proof. Using the error variables e_1, e_2 and the expression of the control law in (51), we obtain

$$\dot{e}_1 = e_2 \tag{55}$$

and

$$\begin{array}{rcl} \dot{e}_2 & = & -\frac{g}{l}\sin(x_1) - \theta x_2 - \dot{x}_2^*(t) \\ & & -\frac{1}{ml^2}[-mlg\sin(x_1) + ml^2[e_1 + e_2 + \frac{1}{2}\sin(t) - \hat{\theta}x_2] \\ & = & -e_1 - e_2 - \theta x_2 + \hat{\theta}(e_2 + x_2^*(t)) \\ & = & -e_1 - e_2 - \theta(e_2 + x_2^*(t)) + \hat{\theta}(e_2 + x_2^*(t)) \ . \end{array} \tag{56}$$

Hence, using the notation $\tilde{\theta} = \hat{\theta} - \theta$, we obtain the system

$$\left\{ \begin{array}{rcl} \dot{e}_1 & = & e_2 \ , \\ \dot{e}_2 & = & -e_1 - e_2 + \tilde{\theta}(e_2 + x_2^*(t)) \ , \\ \dot{\tilde{\theta}} & = & -(e_2 + x_2^*(t))[2e_2 + e_1] \ . \end{array} \right. \tag{57}$$

To simplify the notations, let $Z = (e_1, e_2, \tilde{\theta})$. The derivative of the positive definite and radially unbounded function

$$V_1(Z) = e_1^2 + e_2^2 + e_1e_2 + \frac{1}{2}\tilde{\theta}^2 \tag{58}$$

along the trajectories of (57) satisfies

$$\begin{array}{rcl} DV_1 & = & 2e_1e_2 + (2e_2 + e_1)[-e_1 - e_2 + \tilde{\theta}(e_2 + x_2^*(t))] + e_2^2 - \tilde{\theta}(e_2 + x_2^*(t))[2e_2 + e_1] \\ & = & -e_1^2 - e_1e_2 - e_2^2 \end{array} \tag{59}$$

and therefore $DV_1 < 0$ when $(e_1, e_2) \neq (0, 0)$. Since the system (57) is periodic in time, the LaSalle Invariance Principle applies and ensures that (53), (54) are satisfied. Observe that V_1 is a weak Lyapunov function for the system (57).

Step 2. Construction of a strong Lyapunov function.

By using Theorem 15, we construct now a strong Lyapunov function for the system (57). Since V_1 is a weak Lyapunov function a natural choice for V_2 and N_2 is $V_2 = V_1$ and

$$N_2(Z) = e_1^2 + e_1e_2 + e_2^2 \ . \tag{60}$$

We select as auxiliary function V_3 the function

$$V_3(t,Z) = -\frac{1}{2}\tilde{\theta}\cos(t)e_2 \tag{61}$$

because its derivative along the trajectories of (57) satisfies

$$\begin{aligned} DV_3 &= -\tfrac{1}{2}\tilde{\theta}\cos(t)\left[-e_1 - e_2 + \tilde{\theta}\left(e_2 + \tfrac{1}{2}\cos(t)\right)\right] \\ &\quad +\tfrac{1}{2}\left[\left(e_2 + \tfrac{1}{2}\cos(t)\right)(2e_2 + e_1)\cos(t) + \tilde{\theta}\sin(t)\right]e_2 \\ &= -N_3(t,Z) + R_3(t,Z) \end{aligned} \tag{62}$$

with

$$N_3(t,Z) = \frac{1}{4}\cos^2(t)\tilde{\theta}^2 \geq 0 \tag{63}$$

and

$$\begin{aligned} R_3(t,Z) &= \tfrac{1}{2}\tilde{\theta}\cos(t)e_1 + \tfrac{1}{2}\tilde{\theta}\cos(t)e_2 - \tfrac{1}{2}\cos(t)\tilde{\theta}^2 e_2 \\ &\quad +\tfrac{1}{2}\left[\left(e_2 + \tfrac{1}{2}\cos(t)\right)(2e_2 + e_1)\cos(t) + \tilde{\theta}\sin(t)\right]e_2 \ . \end{aligned} \tag{64}$$

Observe that $DV_3 < 0$ when $N_2(Z) = 0$ and $\tilde{\theta} \neq 0$, $\cos(t) \neq 0$. More precisely, one can check that, with our choice of functions V_1, V_2, V_3, Theorem 15 applies:
i) Assumption 4 is satisfied because V_1 defined in (58) is a weak Lyapunov function for the system (57).
ii) One can easily prove that

$$|R_3(t,Z)| \leq \tfrac{1}{2}|\tilde{\theta}e_1| + |\tilde{\theta}e_2| + \tfrac{1}{2}\tilde{\theta}^2|e_2| + |e_2|^3 + \tfrac{1}{2}|e_1 e_2^2| + \tfrac{1}{2}e_2^2 + \tfrac{1}{4}|e_1 e_2| \ . \tag{65}$$

Using successively the inequality $N_2(Z) \geq \frac{1}{2}[e_1^2 + e_2^2]$ and the inequality $V_1(Z) \geq \frac{1}{2}\left[e_1^2 + e_2^2 + \tilde{\theta}^2\right]$, we deduce that

$$\begin{aligned} |R_3(t,Z)| &\leq \tfrac{3}{2}|\tilde{\theta}|\sqrt{2N_2} + \tfrac{1}{2}\tilde{\theta}^2\sqrt{2N_2} + \left[\tfrac{5}{2}|e_2| + \tfrac{5}{4}\right]N_2 \\ &\leq \chi_3(t,Z,N_2) \end{aligned} \tag{66}$$

with

$$\begin{aligned} \chi_3(t,Z,N_2) &= \left[\tfrac{3}{\sqrt{2}} + \sqrt{V_1}\right]|\tilde{\theta}|\sqrt{N_2} + \tfrac{5}{2}\left[1 + V_1\right]N_2 \\ &\leq \left[\left(\tfrac{3}{\sqrt{2}} + \sqrt{V_1}\right)|\tilde{\theta}| + \tfrac{5}{2}\left(1 + V_1\right)\right]\left[\sqrt{N_2} + N_2\right] \ . \end{aligned} \tag{67}$$

It follows that Assumption 5 is satisfied.
iii) The inequality

$$N_2(Z) + N_3(t,Z) \leq \overline{p}(t)\mu(Z) \tag{68}$$

is satisfied with

$$\overline{p}(t) = \frac{1}{4}\cos^2(t) \ , \ \mu(Z) = e_1^2 + e_2^2 + \tilde{\theta}^2 \tag{69}$$

and $\displaystyle\int_{t-\pi}^{t} \overline{p}(l)dl = \frac{1}{8}\pi$, $0 \leq \overline{p}(l) \leq \frac{1}{4}$.

Hence, all the conditions of Theorem 15 are satisfied and therefore one can construct a strong Lyapunov function for the system (57). By performing explicitly this construction, we obtain the following result:

Lemma 20 *The function*

$$W(t,Z) \quad = \quad \tfrac{\pi}{2}\left[\tfrac{\sin(2t)}{4}+\tfrac{\pi}{4}+79\right]V_1(Z)+\tfrac{21\pi}{4}V_1(Z)^2+\pi V_3(t,Z) \tag{70}$$

where V_1 is the function defined in (58) and V_3 is the function defined in (61), is a strong Lyapunov function of the system (57). Its derivative along the trajectories of this system satisfies

$$DW \leq -\frac{\pi}{8}V_1(Z)\ . \tag{71}$$

Proof. Observe that the function μ defined in (69) satisfies

$$\mu(Z) \geq \frac{1}{2}V_1(Z)\ . \tag{72}$$

This inequality and the proof of Theorem 15 lead us to consider the function

$$\begin{aligned} C(t,Z) &= \tfrac{1}{2}\left(\int_{t-\pi}^{t}\left(\int_{s}^{t}\cos^2(l)dl\right)ds\right)V_1(Z) \\ &= \tfrac{\pi}{8}\left[\sin(2t)+\pi\right]V_1(Z) \end{aligned} \tag{73}$$

whose derivative along the trajectories of (57) is

$$DC = \frac{\pi}{4}\cos(2t)V_1(Z)+\frac{\pi}{8}\left[\sin(2t)+\pi\right]DV_1\ . \tag{74}$$

Since $DV_1 \leq 0$, $[\sin(2t)+\pi] \geq 0$ and $\cos(2t) = 2\cos^2(t)-1$, it follows that

$$DC \quad \leq \quad -\tfrac{\pi}{4}V_1(Z)+\tfrac{\pi}{2}\cos^2(t)V_1(Z)\ . \tag{75}$$

Moreover, one can prove easily that derivative along the trajectories of (57) of V_1+V_3 satisfies the inequality

$$DV_1+DV_3 \quad \leq \quad -\tfrac{1}{2}\cos^2(t)V_1(Z)+R_3(t,Z)\ , \tag{76}$$

where R_3 is the function defined in (64). It follows that the derivative of

$$V_4(t,Z) = C(t,Z)+\pi V_1(Z)+\pi V_3(t,Z) \tag{77}$$

along the trajectories of (57) satisfies

$$DV_4 \leq -\frac{\pi}{4}V_1(Z)+\pi R_3(t,Z)\ . \tag{78}$$

By using (66), the expression of χ_3 in (67) and the triangular inequality, we deduce that

$$\begin{aligned} \chi_3(t,Z,N_2) &\leq \tfrac{1}{16}\tilde{\theta}^2+4\left(\tfrac{3}{\sqrt{2}}+\sqrt{V_1}\right)^2+\tfrac{5}{1}(1+V_1)N_2 \\ &\leq \tfrac{1}{16}\tilde{\theta}^2+\tfrac{77}{2}N_2+\tfrac{21}{2}V_1N_2\ . \end{aligned} \tag{79}$$

Combining (78) and (79), we obtain

$$DV_4 \quad \leq \quad -\tfrac{\pi}{8}V_1+\tfrac{77\pi}{2}N_2+\tfrac{21\pi}{2}V_1N_2\ . \tag{80}$$

Since $N_2 = -DV_1$, it follows that the derivative of the function (70) along the trajectories of (57) satisfies (71). By using the fact that V_1 is a weak Lyapunov function and that the functions V_3 and $\frac{\partial V_3}{\partial x}$ are decrescent in norm, and that $W \geq V_1$, one can check easily that W

is a strong Lyapunov function.

Step 3. Control law yielding the ISS property.

In this part we use our Lyapunov construction to robustify our controller and obtain the desirable ISS property. Observe that the ISS property, introduced by E. Sontag in [S3] plays a central role in modern non-linear control analysis, controller design and robustness analysis.

Theorem 21 *Consider the system (49) with the adaptive controller*

$$\begin{aligned} T(t,x_1,x_2,\hat{\theta}) &= -mlg\sin(x_1) + ml^2\left[e_1 + e_2 + \tfrac{1}{2}\sin(t) - \hat{\theta}x_2\right] \\ &\quad + \left[\tfrac{\pi\sin(2t)+\pi^2}{16} + \tfrac{79\pi}{4} + \tfrac{21\pi}{4}V_1(e_1,e_2,\tilde{\theta})\right]\left[e_2 + \tfrac{e_1}{2}\right] - \tfrac{\pi}{8}\tilde{\theta}\cos(t) \\ \dot{\hat{\theta}} &= -x_2[2e_2 + e_1] \end{aligned} \tag{81}$$

where V_1 is the function defined in (58), e_1, e_2 defined in (52), $\tilde{\theta} = \hat{\theta} - \theta$. Then this adaptive controller guarantees that there are a $\mathcal{KL}$ functions β and a function γ of class $\mathcal{K}$ (see the preliminaries for the definitions of functions of class $\mathcal{K}$ and class $\mathcal{KL}$) such that, for all $t_0 \in \mathbb{R}$, $Z_0 \in \mathbb{R}^n$, $t \geq t_0$,

$$|Z(t;t_0,Z_0)| = \beta(|Z_0|, t-t_0) + \gamma\left(\sup_{\{s\in[t_0,t]\}} d(s)\right) . \tag{82}$$

Proof. We deduce directly from Lemma 20 and its proof that, when the adaptive controller is

$$T(t,x_1,x_2,\hat{\theta}) = -mlg\sin(x_1) + ml^2[e_1 + e_2 - \dot{x}_2^*(t) - \hat{\theta}(e_2 + x_2^*(t))] + v \tag{83}$$

where v is an input to be specified later and when there is a disturbance d, then

$$DW \leq -\frac{\pi}{8}V_1(Z) - \frac{1}{ml^2}\frac{\partial W}{\partial e_2}(t,Z)(v+d) . \tag{84}$$

The choice

$$v = \frac{1}{4}\frac{\partial W}{\partial e_2}(t,Z) \tag{85}$$

gives

$$DW \leq -\frac{\pi}{8}V_1(Z) - \frac{1}{4ml^2}\left(\frac{\partial W}{\partial e_2}(t,Z)\right)^2 - \frac{1}{ml^2}\frac{\partial W}{\partial e_2}(t,Z)d . \tag{86}$$

Thanks to the triangular inequality, we deduce that

$$DW \leq -\frac{\pi}{8}V_1(Z) + \frac{1}{ml^2}d^2 . \tag{87}$$

Since V_1 is a weak Lyapunov function, it follows that W is a ISS Lyapunov function (see [ASW], [SW] for the definition of ISS Lyapunov function). From the results of [S4] or [SW] (see also [K, Theorem 5.2]), one can deduce that the closed-loop system is ISS.

To conclude, one can prove after lengthy but simple calculations that the function T given in (83) with v defined in (85) admits the expression (81).

6 Proofs of Main Results

The proof of main results is carried out by first proving Theorem 10. Then, the proof of Theorem 12 is carried out by showing that under the given conditions it is possible to construct a function V_{j+1} such that the functions $V_2, \ldots, V_{j+1}$ satisfy all conditions of Theorem 10. Theorem 15 is proved in a similar way.

Proof of Theorem 10: We prove this result by induction on the number of the auxiliary functions. The result of Theorem 10 holds in the case where its assumptions are satisfied with only one auxiliary function, i.e. when $j = 2$ because in that case $N_2 = \omega(x)$ and one can construct a strong Lyapunov function by following the proof of Theorem 9. Assume that the result of Theorem 10 holds when its assumptions are satisfied with $j-1$ auxiliary functions with $j \geq 2$. Let us prove that it is true as well when the assumptions are satisfied with j auxiliary functions. To prove this, let us consider a system (1) satisfying the assumptions of Theorem 10 with j auxiliary functions, with $j \geq 2$, and let us construct a new set of $j-1$ auxiliary functions for which the assumptions of Theorem 10 are satisfied.

Let us define

$$S_a(t,x) := \sum_{i=2}^{j+1} V_i(t,x) \ . \tag{88}$$

Then, according to Assumption 5 and (28),

$$DS_a \leq -\sum_{i=2}^{j+1} N_i + \sum_{i=3}^{j+1} \chi_i(t,x,N_2,\ldots,N_{i-1}) \leq -\omega(x) + \sum_{i=3}^{j+1} \chi_i(t,x,N_2,\ldots,N_{i-1}) \ . \tag{89}$$

Using Assumption 5, one can determine a function ϕ, of class $\mathcal{K}_\infty$ and a positive function ρ such that

$$\left| \sum_{i=3}^{j+1} \chi_i(t,x,N_2,\ldots,N_{i-1}) \right| \leq \phi\left(\sum_{i=2}^{j} N_i \right) \rho(x) \ . \tag{90}$$

It follows that

$$DS_a \leq -\omega(x) + \phi\left(\sum_{i=2}^{j} N_i \right) \rho(x) \ . \tag{91}$$

By following verbatim the proof of Theorem 9 from (15) to (25), one can determine a positive definite function p_j and a function Γ_a, positive and nondecreasing, such that the derivative of the function

$$S_b(t,x) = p_j(V_1(t,x)) S_a(t,x) \tag{92}$$

along the trajectories of (1) satisfies

$$DS_b \leq -\frac{1}{2} p_j(V_1)\omega(x) + \frac{1}{2}\left(\sum_{i=2}^{j} N_i \right) \Gamma_a(V_1) - \frac{1}{2}\Gamma_a(V_1) DV_1 \ . \tag{93}$$

Let

$$\nu_a(t,x) = S_b(t,x) + \frac{1}{2}\Gamma_a(V_1(t,x)) V_j(t,x) \ . \tag{94}$$

Simple calculations yield

$$\begin{aligned} D\nu_a \ \leq \ & -\tfrac{1}{2} p_j(V_1)\omega(x) + \tfrac{1}{2}\left(\sum_{i=2}^{j} N_i \right) \Gamma_a(V_1) - \tfrac{1}{2}\Gamma_a(V_1) DV_1 + \tfrac{1}{2}\Gamma'_a(V_1) V_j DV_1 \\ & + \tfrac{1}{2}\Gamma_a(V_1) DV_j \ . \end{aligned} \tag{95}$$

Since the function $-\frac{1}{2}\Gamma_a(V_1) + \frac{1}{2}\Gamma'_a(V_1)V_j$ is decrescent in norm, one can determine a function Γ_b, positive and nondecreasing, such that

$$\left| -\frac{1}{2}\Gamma_a(V_1) + \frac{1}{2}\Gamma'_a(V_1)V_j \right| \leq \Gamma_b(V_1) \ . \tag{96}$$

Let

$$\nu_b(t,x) = \nu_a(t,x) + \int_0^{V_1(t,x)} \Gamma_b(l)dl \ . \tag{97}$$

Then, using (96), we straightforwardly obtain

$$D\nu_b \ \leq \ -\tfrac{1}{2}p_j(V_1)\omega(x) + \tfrac{1}{2}\left(\sum_{i=2}^{j} N_i\right)\Gamma_a(V_1) + \tfrac{1}{2}\Gamma_a(V_1)DV_j \ . \tag{98}$$

Using Assumption 5, we deduce that

$$D\nu_b \ \leq \ -\tfrac{1}{2}p_j(V_1)\omega(x) + \tfrac{1}{2}\left(\sum_{i=2}^{j-1} N_i\right)\Gamma_a(V_1) + \tfrac{1}{2}\Gamma(V_1)\chi_j(t,x,N_2,\ldots,N_{j-1}) \ . \tag{99}$$

One can easily prove that ν_b is decrescent in norm. It follows that the system (1) satisfies the assumptions of Theorem 10 with $j-1$ auxiliary functions, V_2,...,V_{j-1}, ν_b. According to our induction assumption, it follows that one can construct explicitly a strong Lyapunov function. Consequently, our induction assumption is satisfied at the step j.

Proof of Theorem 12: The proof of this result consits in constructing a function V_{j+1} such that the condition (28) of Theorem 10 is satisfied.

The function $A(t,x_2)$, defined in (36), is continuously differentiable, except at $x_2 = 0$. This function is not necessarily bounded but the condition C1 ensures that both $M(t,x_2)$ and $\frac{\partial M}{\partial x_2}(t,x_2)$ are decrescent in norm which guarantees the existence of a function σ_2 of class $\mathcal{K}_\infty$ such that, for all $t \in \mathbb{R}, x_2 \in \mathbb{R}^{n_2}$,

$$|M(t,x_2)| \leq \sigma_2(|x_2|) \ , \ \left|\frac{\partial M}{\partial x_2}(t,x_2)\right| \leq \sigma_2(|x_2|) \tag{100}$$

and therefore, for all $(t,x_2) \neq (t,0)$,

$$0 \leq A(t,x_2) \leq \theta(|x_2|^2)^2\sigma_2(|x_2|) \ . \tag{101}$$

On the other hand, the derivative of $A(t,x_2)$ along the trajectories of (30) satisfies, when $x_2 \neq 0$,

$$\begin{aligned} DA \ &= \ \theta(|x_2|^2)M(t,x_2) - \left[\int_{t-\theta(|x_2|^2)}^{t} M(l,x)dl\right]\left[1 - 2\theta'(|x_2|^2)x_2^\top f_2(t,x)\right] \\ &\quad + \int_{t-\theta(|x_2|^2)}^{t}\left(\int_s^t \frac{\partial M}{\partial x_2}(l,x_2)f_2(t,x)dl\right)ds \ . \end{aligned} \tag{102}$$

Using (100) and (32) in the condition C1, we deduce that, when $x_2 \neq 0$,

$$\begin{aligned} DA \ &\leq \ \theta(|x_2|^2)M(t,x_2) - \left[\int_{t-\theta(|x_2|^2)}^{t} M(l,x_2)dl\right] \\ &\quad +2\theta(|x_2|^2)\sigma_2(|x_2|)|\theta'(|x_2|^2)||x_2|\chi_f(t,x,N_2,N_3,\ldots,N_{j-1}) \\ &\quad +\theta(|x_2|^2)^2\sigma_2(|x_2|)\chi_f(t,x,N_2,N_3,\ldots,N_{j-1}) \ . \end{aligned} \tag{103}$$

Grouping the terms and using the inequality (34) in the condition C2, we obtain, when $x_2 \neq 0$,

$$\begin{array}{rcl} DA & \leq & \theta(|x_2|^2)M(t,x_2) - \gamma(|x_2|) \\ & & +\theta(|x_2|^2)\sigma_2(|x_2|)\left[2|\theta'(|x_2|^2)||x_2| + \theta(|x_2|^2)\right]\chi_f(t,x,N_2,N_3,\dots,N_{j-1}) \ . \end{array} \tag{104}$$

We define now a function B as follows

$$B(t,0) = 0 \ , \ \forall t \tag{105}$$

$$B(t,x_2) = \delta(|x_2|^2)A(t,x_2) \ , \ \forall (t,x_2) \neq (t,0) \tag{106}$$

where δ is a positive definite function such that, for all $s \geq 0$,

$$\begin{array}{rcl} 0 \leq \delta(s) & \leq & \frac{s}{\sqrt{1+s^2}} \min\left\{ \frac{1}{\theta(s)[\sigma_2(\sqrt{s})+1]}, \frac{1}{2\sigma_2(\sqrt{s})\left[2\theta(s)|\theta'(s)|\sqrt{s}+\theta(s)^2\right]} \right\} \ , \\ |\delta'(s)| & \leq & \frac{s}{\sqrt{1+s^2}} \frac{1}{4\sqrt{s}\theta(s)^2\sigma_2(\sqrt{s})} \ . \end{array} \tag{107}$$

The inequalities (101) and (107) imply that, for all (t,x_2),

$$0 \leq B(t,x_2) \leq \delta(|x_2|^2)\theta(|x_2|^2)^2\sigma_2(|x_2|) \leq \frac{|x_2|^2}{\sqrt{1+|x_2|^4}} \ . \tag{108}$$

This inequality and (37) imply that B is continuous. Moreover, by taking advantage of (107) and (108), one can show that $B(t,x_2)$ is continuously differentiable on $\mathbb{R} \times \mathbb{R}^{n_2}$ by showing that, for any (t,x_2),

$$\lim_{l \to 0} \frac{B(t,lx_2) - B(t,0)}{l} = 0$$

and, for all $(t,x_2) \neq (t,0)$,

$$\left|\frac{\partial B}{\partial t}(t,x_2)\right|^2 + \left|\frac{\partial B}{\partial x_2}(t,x_2)\right|^2 \leq (4+n_2)\frac{|x_2|^4}{1+|x_2|^4} \ . \tag{109}$$

The derivative of $B(t,x_2)$ along the along the trajectories of (30) is given by

$$DB = \delta(|x_2|^2)DA + \delta'(|x_2|^2)2x_2^\top f_2(t,x)A(t,x_2) \tag{110}$$

when $x_2 \neq 0$. From (104), (101) and (32) in the condition C1 we deduce that

$$\begin{array}{rcl} DB & \leq & \delta(|x_2|^2)\theta(|x_2|^2)M(t,x_2) - \delta(|x_2|^2)\gamma(|x_2|) \\ & & +\delta(|x_2|^2)\theta(|x_2|^2)\sigma_2(|x_2|)\left[2|\theta'(|x_2|^2)||x_2| + \theta(|x_2|^2)\right]\chi_f(t,x,N_2,N_3,\dots,N_{j-1}) \\ & & +\delta'(|x_2|^2)2|x_2|\chi_f(t,x,N_2,N_3,\dots,N_{j-1})\theta(|x_2|^2)^2\sigma_2(|x_2|) \end{array} \tag{111}$$

when $x_2 \neq 0$. We deduce from the inequalities (107) that, for all (t,x_2),

$$DB \ \leq \ M(t,x_2) - \delta(|x_2|^2)\gamma(|x_2|) + \chi_f(t,x,N_2,N_3,\dots,N_{j-1}) \ . \tag{112}$$

We define now the following function

$$V_{j+1}(t,x) = \sum_{i=2}^{j} V_i(t,x) + B(t,x_2) \ . \tag{113}$$

Using Assumption 5 and (109), one can conclude that $\frac{\partial V_{j+1}}{\partial x}$ is decrescent in norm Then, from Assumption 5 and (112), it follows that

$$\begin{array}{rcl} DV_{j+1} & \leq & -\sum_{i=2}^{j} N_i(t,x) + \sum_{i=2}^{j} \chi_i(t,x,N_2,N_3,\ldots,N_{i-1}) + M(t,x_2) \\ & & -\delta(|x_2|^2)\gamma(|x_2|) + \chi_f(t,x,N_2,N_3,\ldots,N_{j-1}) \ . \end{array} \tag{114}$$

Thanks to (31) in the condition C1, we deduce that

$$\dot{V}_{j+1} \ \leq \ -N_{j+1}(x) + \chi_{j+1}(t,x,N_2,N_3,\ldots,N_{j-1},N_j) \tag{115}$$

with

$$N_{j+1}(x) = \omega(|x_1|) + \delta(|x_2|^2)\gamma(|x_2|) \tag{116}$$

and

$$\chi_{j+1}(t,x,N_2,N_3,\ldots,N_{j-1},N_j) = \sum_{i=2}^{j} \chi_i(t,x,N_2,N_3,\ldots,N_{i-1}) + \chi_f(t,x,N_2,N_3,\ldots,N_{j-1}) \ . \tag{117}$$

One can check readily that Theorem 10 applies. This theorem provides a strong Lyapunov function for the system (1) with the features of (35).

Proof of Theorem 15: The function μ is positive definite. Therefore, from Lemma 22, one can deduce that on can determine a positive definite real-valued function γ of class C^1 such that

$$\mu(x) \geq \gamma(V_1(t,x)) \ , \ |\gamma'(V_1(t,x))| \leq 1 \ , \ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n \ . \tag{118}$$

Next, let us consider the function

$$C(t,x) = \left(\int_{t-\tau}^{t} \left(\int_{s}^{t} \overline{p}(l) dl \right) ds \right) \gamma(V_1(t,x)) \ . \tag{119}$$

This function and $\frac{\partial C}{\partial x}$ are decrescent in norm and the derivative of C along (1) satisfies

$$\begin{array}{rcl} DC & = & \tau\overline{p}(t)\gamma(V_1(t,x)) - \left(\int_{t-\tau}^{t} \overline{p}(l) dl \right) \gamma(V_1(t,x)) \\ & & + \left(\int_{t-\tau}^{t} \left(\int_{s}^{t} \overline{p}(l) dl \right) ds \right) \gamma'(V_1(t,x)) DV_1 \ . \end{array} \tag{120}$$

Thanks to (45) and (118), we deduce that

$$DC \leq \tau\overline{p}(t)\mu(x) - p_m\gamma(V_1(t,x)) + \tau^2 p_M |DV_1| \ . \tag{121}$$

Consider now the function

$$V_{j+1}(t,x) := C(t,x) + \tau^2 p_M V_1(t,x) + \tau \sum_{i=2}^{j} V_i(t,x) \tag{122}$$

which is decrescent in norm as long as $\frac{\partial V_{j+1}}{\partial x}$. From (121), we deduce that its derivative along (1) satisfies

$$DV_{j+1} \ \leq \ \tau\overline{p}(t)\mu(x) - p_m\gamma(V_1) + \tau^2 p_M |DV_1| + \tau^2 p_M DV_1 + \tau \sum_{i=2}^{j} DV_i \ . \tag{123}$$

Using the fact that DV_1 is nonpositive and Assumption 5, we deduce that

$$DV_{j+1} \quad \leq \quad \tau\overline{p}(t)\mu(x) - p_m\gamma(V_1) - \tau\sum_{i=2}^{j} N_j + \tau\sum_{i=3}^{j}\chi_i(t,x,N_2,\ldots,N_{i-1}) \ . \tag{124}$$

Using (44) we obtain

$$DV_{j+1} \quad \leq \quad -p_m\gamma(V_1) + \tau\sum_{i=3}^{j}\chi_i(t,x,N_2,\ldots,N_{i-1}) \ . \tag{125}$$

One can check readily that Theorem 10 applies. This theorem provides with a strong Lyapunov function for the system (1) with the features of (46).
Proof of Proposition 17: Simple calculations yield

$$DV_2 = DV_1 = b_1 = -N_2 \leq 0 \tag{126}$$

and, for $i \in \{3,\ldots,N\}$,

$$\begin{array}{rcl} DV_i & = & -Db_{i-2}b_{i-1} - b_{i-2}Db_{i-1} \\ & \leq & -b_{i-1}^2 + |b_{i-2}||b_i| \\ & = & -N_i + \chi_i(t,x,N_{i-1}) \ . \end{array} \tag{127}$$

In addition, the functions V_1 and b_i for $i \geq 1$ are decrescent in norm as long as $\frac{\partial V_1}{\partial x}$ and $\frac{\partial b_i}{\partial x}$ for $i \geq 1$. It follows that all the functions V_i, $\frac{\partial V_i}{\partial x}$ and the functions N_i, χ_i are decrescent in norm. Therefore Assumption 5 is satisfied.

7 Conclusion

We provided several constructions of strong Lyapunov functions for time-varying systems that satisfy generalized conditions of the Matrosov theorem. We expect that our results will have significant implications in several areas of nonlinear control, especially in the areas of tracking and adaptive control. We will address these issues in our future work.

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A Technical lemma

Lemma 22 *Let $w_i : \mathbb{R}^n \to \mathbb{R}$ $i = 1, 2$ be two positive definite functions; $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and γ_1, γ_2 of class $\mathcal{K}_\infty$ such that for all (t, x) we have:*

$$\gamma_1(|x|) \leq V(t, x) \leq \gamma_2(|x|) \ . \tag{128}$$

Then, one can construct a real-valued function L of class C^N, where $N \geq 1$ is an integer, such that $L(0) = 0$, $L(s) > 0$ for all $s > 0$ and for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, we have:

$$L(V(t, x)) \leq w_1(x) \ , \tag{129}$$

$$|L'(V(t, x))| \leq w_2(x) \ . \tag{130}$$

Proof. We will prove at the end of this proof that one can construct a function ρ, positive, increasing and of class C^N, and a function α of class $\mathcal{K}_\infty$ and of class C^N such that

$$\alpha(V(t, x)) \leq w_1(x)\rho(V(t, x)) \ , \tag{131}$$

$$\alpha(V(t, x)) \leq w_2(x)\rho(V(t, x)) \ . \tag{132}$$

We introduce now the following function

$$L(s) := \int_{\frac{s}{2}}^{s} \frac{\alpha(l)}{2(1 + l^2)(1 + \rho(2l)^2)} dl \ . \tag{133}$$

Then $L(0) = 0$, $L(s) > 0$ for all $s > 0$, L is of class C^N and, since both α and ρ are increasing, for all $s \geq 0$,

$$L(s) \leq \int_{\frac{s}{2}}^{s} \frac{\alpha(s)}{2\left(1 + \left(\frac{s}{2}\right)^2\right)(1 + \rho(s)^2)} dl \leq \frac{\alpha(s)}{4(1 + \rho(s)^2)} \leq \frac{\alpha(s)}{\rho(s)} \ . \tag{134}$$

It follows that

$$L(V(t, x)) \leq \frac{\alpha(V(t, x))}{\rho(V(t, x))} \leq w_1(x) \ . \tag{135}$$

Therefore (129) is satisfied. On the other hand, the first derivative of L is

$$L'(s) = \frac{\alpha(s)}{2(1+s^2)(1+\rho(2s)^2)} - \frac{1}{2}\frac{\alpha\left(\frac{s}{2}\right)}{2\left(1+\left(\frac{s}{2}\right)^2\right)(1+\rho(s)^2)} \ . \tag{136}$$

Since both α and ρ are increasing, it follows that

$$\begin{aligned} |L'(s)| &\leq \tfrac{\alpha(s)}{2(1+s^2)(1+\rho(2s)^2)} + \tfrac{1}{2}\tfrac{\alpha\left(\frac{s}{2}\right)}{2\left(1+\left(\frac{s}{2}\right)^2\right)(1+\rho(s)^2)} \\ &\leq \tfrac{\alpha(s)}{2(1+s^2)(1+\rho(s)^2)} + \tfrac{1}{2}\tfrac{\alpha(s)}{2\left(1+\left(\frac{s}{2}\right)^2\right)(1+\rho(s)^2)} \\ &\leq \tfrac{\alpha(s)}{2(1+\rho(s)^2)} + \tfrac{\alpha(s)}{4(1+\rho(s)^2)} \\ &\leq \tfrac{\alpha(s)}{\rho(s)} \ . \end{aligned} \tag{137}$$

Consequently, the inequality

$$|L'(V(t,x))| \quad \leq \quad \tfrac{\alpha(V(t,x))}{\rho(V(t,x))} \leq w_2(x) \tag{138}$$

is satisfied and therefore (130) is satisfied.

We end this proof by constructing a function ρ, positive, increasing and of class C^N, and a function α of class $\mathcal{K}_\infty$ and of class C^N such that (131) and (132) are satisfied.

We introduce the constant

$$W_f = \inf_{\{z:|z|=1\}} w(z) \tag{139}$$

and define four functions:

$$w(x) = \inf\{w_1(x), w_2(x)\} \ , \tag{140}$$

$$\delta_l(r) = \begin{cases} \inf\limits_{\{z:|z|\in[1,r]\}} w(z) & \text{if} \quad r \geq 1 \ , \\ W_f \quad \text{if} \quad r \in [0,1] \ , \end{cases} \tag{141}$$

$$\delta_s(r) = \begin{cases} \inf\limits_{\{z:|z|\in[r,1]\}} w(z) & \text{if} \quad r \in [0,1] \ , \\ W_f \quad \text{if} \quad r \geq 1 \ , \end{cases} \tag{142}$$

$$\delta(r) = \frac{1}{W_f}\delta_s(r)\delta_l(r) \ . \tag{143}$$

Observe that

i) If $|x| \leq 1$, then $\delta(|x|) = \frac{1}{W_f}\delta_s(|x|)\delta_l(|x|) = \delta_s(|x|) = \inf\limits_{\{z:|z|\in[|x|,1]\}} w(z) \leq w(x)$.

ii) If $|x| \geq 1$, then $\delta(|x|) = \frac{1}{W_f}\delta_s(|x|)\delta_l(|x|) = \delta_l(|x|) = \inf\limits_{\{z:|z|\in[1,|x|]\}} w(z) \leq w(x)$.

It follows that, for all $x \in \mathbb{R}^n$,

$$w(x) \geq \delta(|x|) = \frac{1}{W_f}\delta_s(|x|)\delta_l(|x|) \ . \tag{144}$$

Since w is a positive definite function, δ_l is a positive function on $[0, +\infty)$. Therefore, from (144), it follows that, for all $x \in \mathbb{R}^n$,

$$\delta_s(|x|) \leq w(x)\frac{W_f}{\delta_l(|x|)} \ . \tag{145}$$

We introduce two functions

$$\alpha_a(r) = r\delta_s(r) \ , \ \rho_a(r) = \frac{W_f(1+r)}{\delta_l(r)} \quad , \quad \forall r \geq 0 \ . \tag{146}$$

Then, from (145), we deduce that, for all $x \in \mathbb{R}^n$,

$$\alpha_a(|x|) \leq w(x)\rho_a(|x|) \ . \tag{147}$$

Since w is positive definite and at least continuous, one can prove easily that $\delta_s(0) = 0$, $\delta_s(r) > 0$ if $r > 0$ and δ_s is nondecreasing and continuous. It follows that α_a is of class $\mathcal{K}_\infty$. For similar reasons, δ_l is continuous, positive and nonincreasing. It follows that ρ_a is well-defined, positive and increasing. Using these properties of α_a and ρ_a and (128), we deduce that, for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\alpha_a(\gamma_2^{-1}(V(t,x)) \leq w(x)\rho_a(\gamma_1^{-1}(V(t,x))) \ . \tag{148}$$

As an immediate consequence, we have

$$V(t,x)^N\alpha_b(V(t,x)) \leq w(x)[V(t,x)+1]^N\rho_b(V(t,x)) \ . \tag{149}$$

with

$$\alpha_b(r) = \alpha_a(\gamma_2^{-1}(r)) \ , \ \rho_b(r) = \rho_a(\gamma_1^{-1}(r)) \quad , \quad \forall r \geq 0 \ . \tag{150}$$

We define now, for all $r \geq 0$, two functions

$$\alpha(r) = \int_0^r \left(\int_0^{s_1} \dots \int_0^{s_{N-1}} \alpha_b(s_N) ds_N \right) \dots ds_1 \ , \tag{151}$$

$$\rho(r) = \int_0^{r+1} \left(\int_0^{s_1+1} \dots \int_0^{s_{N-1}+1} (s_N+1)^N \rho_b(s_N) ds_N \right) \dots ds_1 \ . \tag{152}$$

Observe that

$$\alpha(r) \leq r^N\alpha_b(r) \ , \ \rho(r) \geq (r+1)^N\rho_b(r) \ . \tag{153}$$

These inequalities and (149) yield

$$\alpha(V(t,x)) \leq w(x)\rho(V(t,x)) \ . \tag{154}$$

Since $0 \leq w_1(x) \leq w(x), 0 \leq w_2(x) \leq w(x)$, we deduce that (131) and (132) are satisfied. One can check readily that ρ is positive, increasing and of class C^N, and α is of class $\mathcal{K}_\infty$ and of class C^N.