Analysis of Input to State Stability for Discrete Time Nonlinear Systems via Dynamic Programming*

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April 8, 2004

Abstract

The Input-to-state stability (ISS) property for systems with disturbances has received considerable attention in the last ten years, with many applications and characterisations reported in the literature. The main purpose of this paper is to present novel analysis results for ISS that utilise dynamic programming techniques to characterise minimal ISS gains and transient bounds. These characterisations naturally lead to computable necessary and sufficient conditions for ISS. Our results make a connection between ISS and optimisation problems in nonlinear dissipative systems theory (including L_2-gain analysis and nonlinear H_∞ theory). As such, the results presented address an obvious gap in the literature.
Keywords: Analysis; Disturbances; Dynamic Programming; Input-to-state stability.

1 Introduction

Among the many stability properties for systems with disturbances that have been proposed in the literature, the input-to-state stability (ISS) property proposed by Sontag in 1989 [18] deserves special attention. Indeed, ISS is fully compatible with Lyapunov stability theory [20] while its other equivalent characterizations relate it to robust stability, dissipativity and input-output stability theory [19, 21, 24]. The ISS property has found its main application in the ISS small gain theorem that was first proved by Jiang,

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Teel and Praly in [11]. Several different versions of the ISS small gain theorem that use different (equivalent) characterizations of the ISS property and their various applications to nonlinear controller design can be found in [12, 13, 25] and references defined therein.

The ISS property and the ISS small gain theorems naturally lead to the concept of nonlinear disturbance gain functions or simply "nonlinear gains". In this context, obtaining sharp estimates for the nonlinear gains is an important issue. Indeed, the better the nonlinear gain estimate that we can obtain, the larger the class of systems to which the ISS small gain results can be applied. Currently, the main tool for estimating the nonlinear gains are the so called ISS Lyapunov functions that typically produce rather conservative estimates (over bounds) for the ISS nonlinear gains.

It is the main purpose of this paper to present several results that provide a computational framework based on dynamic programming for obtaining *minimum ISS nonlinear gains*. These results are related to optimization based methods in nonlinear dissipative systems theory, such as L_2-gain analysis and nonlinear H_∞ theory (see [5] and references defined therein), as well as recently developed optimization based L_∞ methods (see [3, 6, 8, 14]). Needless to say, the optimisation approach that we take in this paper can inflict a heavy (and sometimes infeasible) computational burden on the user. However, this is not due to the approach we take but rather to the intrinsic complexity of the problem that we are trying to solve. For technical reasons we present results only for discrete-time nonlinear systems since many calculations are in this way simplified.

The paper is organised as follows. In Section 2 we present several equivalent definitions of the ISS property and state a result from the literature that motivates our definitions and results. A fundamental dynamic programming equation that we need to state our main results is given in Section 3. Sections 4, 5 and 6 contain results on minimum nonlinear gains for different equivalent definitions of the ISS property. Two related ISS properties are analysed in Section 7 using the techniques of Sections 5 and 6. Several illustrative examples are presented in Section 8 and the paper is closed with conclusions in Section 9.

2 Preliminaries

Sets of real numbers, integers and nonnegative integers are denoted respectively as $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{Z}_+$. A function $\gamma : [0,\infty) \to [0,\infty)$ is of class $\bar{\mathcal{K}}$ if it is nondecreasing, satisfies $\gamma(0) = 0$ and is right continuous at 0. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\bar{\mathcal{K}}\bar{\mathcal{L}}$ if for each fixed $t \geq 0$, $\beta(\cdot,t)$ is of class $\bar{\mathcal{K}}$ and for each fixed $s \geq 0$, $\lim_{t\to+\infty} \beta(s,t) = 0$. Denote $l_\infty = \{u : \mathbf{Z}_+ \to \mathbf{R}^m : \|u\|_\infty = \sup\limits_{k\in\mathbf{Z}_+} |u_k| < \infty\}$ where $|\cdot|$ is the Euclidean norm.

Consider the following dynamical system

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

where $x_k \in \mathbf{R}^n, u_k \in \mathbf{R}^m$, and $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is continuous and satisfies $f(0,0) = 0$. For any $x_0 \in \mathbf{R}^n$ and any input $u : Z_+ \to \mathbf{R}^m$, we denote by $x(\cdot, x_0, u)$ the solution of (1) with initial state x_0 and input u.

The following definitions are taken from ISS related literature. It was shown in [9] that these definitions of ISS are qualitatively equivalent. However, the gains in different definitions are not the same and since we are interested in minimum disturbance gains for different characterizations, we find it useful to introduce different notation for each of the different characterizations. In all the definitions below we assume that $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.

Definition 2.1 *(Input-to-state stability with + formulation)*
System (1) is ISS_+ (with (β, γ)) if

$$|x(k, x_0, u)| \leq \beta(|x_0|, k) + \gamma(\|u\|_\infty) \tag{2}$$

for all $x_0 \in \mathbf{R}^n$, all $u \in l_\infty$ and all $k \in \mathbf{Z}_+$.

Definition 2.2 *(Asymptotic gain property)*
System (1) is AG (with gain γ) if for all $x_0 \in \mathbf{R}^n$ and all $u \in l_\infty$,

$$\limsup_{k \to +\infty} |x(k, x_0, u)| \leq \gamma(\|u\|_\infty). \tag{3}$$

Remark 2.3 Using arguments as in Lemma II.1 of [21], we can show that the above definition is equivalent to the following: for all $x_0 \in \mathbf{R}^n$ and all $u \in l_\infty$,

$$\limsup_{k \to +\infty} |x(k, x_0, u)| \leq \gamma(\limsup_{k \to +\infty} |u_k|), \tag{4}$$

which is the definition of asymptotic gain property in [9].

Definition 2.4 *(Zero global asymptotic stability property)*
System (1) is 0-GAS (with β) if the state trajectories with $u \equiv 0$ satisfy

$$|x(k, x_0, 0)| \leq \beta(|x_0|, k). \tag{5}$$

for all $x_0 \in \mathbf{R}^n$ and all $k \in \mathbf{Z}_+$.

Definition 2.5 *(Input-to-state stability with asymptotic gain formulation)*
The system (1) is ISS_{AG} (with (β, γ)) if it is AG (with gain γ) and 0-GAS (with β).

Remark 2.6 The above definition is motivated by the result proved in [21] which shows for continuous-time systems that $ISS_+ \Leftrightarrow AG + 0\text{-}GAS$. A similar result for discrete-time systems was proved in [4, 9]. This result is restated below in Theorem 2.9 for convenience.

Definition 2.7 *(Input-to-state stability with* max *formulation)*
System (1) is $ISS_{\max}$ (with (β, γ)) if

$$|x(k, x_0, u)| \leq \max\{\beta(|x_0|, k), \gamma(\|u\|_\infty)\} \tag{6}$$

for all $x_0 \in \mathbf{R}^n$, all $u \in l_\infty$ and all $k \in \mathbf{Z}_+$.

Remark 2.8 It is more common in the literature to use the classes of functions $\mathcal{K}$ and $\mathcal{KL}$ when defining ISS and related properties. A function $\gamma : [0,\infty) \to [0,\infty)$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. A continuous function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$ $\beta(s, \cdot)$ decreases to zero.

It is not hard to see that the stability definitions that we use are qualitatively equivalent to the stability definitions when the classes of functions $\bar{\mathcal{K}}$ and $\bar{\mathcal{K}}\bar{\mathcal{L}}$ are replaced respectively by $\mathcal{K}$ and $\mathcal{KL}$. This follows from the following three facts: (i) $\mathcal{K} \subset \bar{\mathcal{K}}$ and $\mathcal{KL} \subset \bar{\mathcal{K}}\bar{\mathcal{L}}$; (ii) given any $\gamma \in \bar{\mathcal{K}}$, there exists $\gamma_1 \in \mathcal{K}$ such that $\gamma(s) \leq \gamma_1(s), \forall s \geq 0$; (iii) given any $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, there exists $\beta_1 \in \mathcal{KL}$ such that $\beta(s,k) \leq \beta_1(s,k), \forall s \geq 0, \forall k \in \mathbf{Z}_+$. Consequently, most results that were proved in the literature for classes of functions $\mathcal{K}$ and $\mathcal{KL}$ are still true when stated with function classes $\bar{\mathcal{K}}$ and $\bar{\mathcal{K}}\bar{\mathcal{L}}$.

Finally, we note that our relaxed function class definitions are necessitated by the fact that the minimal ISS gain for some systems can be of class $\bar{\mathcal{K}} \setminus \mathcal{K}$, as is demonstrated in Section 8.1, Example 1.

The following theorem has been proved in the context of function classes $\mathcal{K}$ and $\mathcal{KL}$ for continuous-time systems in [21] and for discrete-time systems in [4, 9]. However, this result remains valid for function classes $\bar{\mathcal{K}}$ and $\bar{\mathcal{K}}\bar{\mathcal{L}}$.

Theorem 2.9 *The following statements are equivalent:*

1. *There exist $\beta_{AG} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_{AG} \in \bar{\mathcal{K}}$ such that the system (1) is ISS$_{AG}$ with $(\beta_{AG}, \gamma_{AG})$;*
2. *There exist $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_+ \in \bar{\mathcal{K}}$ such that the system (1) is ISS$_+$ with (β_+, γ_+);*
3. *There exist $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_{\max} \in \bar{\mathcal{K}}$ such that the system (1) is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$.*

In the sequel we use the non-standard notation from Theorem 2.9 since it is important to distinguish between different characterizations and the related functions. Indeed, the functions $\beta_{AG}, \beta_+, \beta_{\max}$ (respectively functions $\gamma_{AG}, \gamma_+, \gamma_{\max}$) in the above theorem are all different in general. Note that although notation β_{AG} characterizing 0-GAS seems counterintuitive, it is consistent with the definition of ISS$_{AG}$ in Definition 2.5.

Remark 2.10 We note that each of the properties ISS$_{AG}$, ISS$_+$ and ISS$_{\max}$ has been used in the literature. In particular, there exist small gain theorems that use each of these different characterizations (see, for instance, [10, 11, 12, 13, 25]). Computing the smallest possible functions β, γ (or their estimates) in each of these properties is an important problem for the following reasons: (i) the smaller the estimates of gains functions, the larger the class of systems to which the small gain theorem can be applied; (ii) better estimates of the functions β, γ for subsystems produce (via the small gain theorems) sharper bounds on solutions of the composite system; (iii) the smallest functions will be different in general for each of the properties ISS$_{AG}$, ISS$_+$ and ISS$_{\max}$ (this further motivates our notation). In the sequel, we provide a framework for the computation of minimum functions $\beta_{AG}, \beta_+, \beta_{\max}$ and $\gamma_{AG}, \gamma_+, \gamma_{\max}$ via dynamic programming.

3 Dynamic Programming

In this section we define a value function that is used in the derivation of our subsequent results, and present a dynamic programming equation to compute it. The dynamic programming equation can be used in developing numerical algorithms for testing each of the characterizations of the ISS property that were defined in the previous section. In particular, we can obtain minimum disturbance gains and/or the minimal bounds on the transients by using this technique.

For $x \in \mathbf{R}^n$, $\delta \geq 0$, integer $k \in \mathbf{Z}_+$, denote

$$V^\delta(x,k) := \sup_{\|u\|_\infty \leq \delta} \{|x(k,x_0,u)| : x_0 = x\} \ . \tag{7}$$

The Dynamic Programming Equation (DPE) for $V^\delta(x,k)$ is

$$V^\delta(x,k) = \sup_{|u|\leq\delta} V^\delta(f(x,u),k-1) \tag{8}$$

with the initial condition

$$V^\delta(x,0) = |x| \, . \tag{9}$$

In subsequent sections, we show how $V^\delta(x,k)$ can be used to compute the functions β,γ needed in different characterizations of ISS.

4 Necessary and sufficient conditions for ISS$_{AG}$

The main results of this section are necessary and sufficient conditions for ISS$_{AG}$. The results do not require a Lyapunov function but rather use the value function $V^\delta(x,k)$ to generate γ_{AG} and β_{AG} directly. More importantly, we show that the computed functions are minimal. This type of result is not possible to obtain via Lyapunov techniques since they involve a certain conservatism in estimating γ_{AG} and β_{AG}.

Using $V^\delta(x,k)$ we introduce

$$V_a^\delta(x) := \limsup_{k\to+\infty} V^\delta(x,k) \tag{10}$$

and

$$\gamma_\infty(\delta) := \sup_{x\in\mathbf{R}^n} V_a^\delta(x) \tag{11}$$

Denote

$$\beta_a(s,k) := \sup_{|x|\leq s} V^0(x,k) \ . \tag{12}$$

Using the above definitions, we can state the main result of this section:

Theorem 4.1 *If the system (1) is ISS$_{AG}$ with (β_{AG},γ_{AG}) then $\gamma_\infty \in \bar{\mathcal{K}}$, $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and*

$$\begin{aligned} \gamma_\infty(s) &\leq \gamma_{AG}(s), & \forall s \geq 0 \\ \beta_a(s,k) &\leq \beta_{AG}(s,k), & \forall s \geq 0, \forall k \in \mathbf{Z}_+ \ . \end{aligned}$$

If, on the other hand, $\gamma_\infty \in \bar{\mathcal{K}}$ and $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, then the system (1) is ISS$_{AG}$ with (β_a,γ_∞).

Proof. Suppose the system (1) is ISS_{AG} with $(\beta_{AG}, \gamma_{AG})$. Then, the system is AG with $\gamma_{AG} \in \bar{\mathcal{K}}$.

Since f is continous, by Lemma 10 in [4], $\forall \delta \geq 0, \forall x_0 \in \mathbf{R}^n$, we can prove the following property: $\forall \varepsilon > 0, \exists K$ (depend only on x_0 and ε), such that

$$|x(k, x_0, u)| \leq \gamma_{AG}(\delta) + \varepsilon, \quad \forall\, k \geq K, \forall\, \|u\|_\infty \leq \delta. \tag{13}$$

which implies

$$V^\delta(x_0, k) \leq \gamma_{AG}(\delta) + \varepsilon, \quad \forall k \geq K. \tag{14}$$

Since ε is arbitrary, we have

$$V_a^\delta(x_0) \leq \gamma_{AG}(\delta). \tag{15}$$

Hence

$$0 \leq \gamma_\infty(\delta) \leq \gamma_{AG}(\delta) < +\infty, \ \forall \delta \geq 0.$$

Since $\gamma_{AG}(0) = 0$ and γ_{AG} is right continuous at 0, we have $\gamma_\infty(0) = 0$ and γ_∞ is right continuous at 0. Hence $\gamma_\infty \in \bar{\mathcal{K}}$.

Since the system is ISS_{AG} with $(\beta_{AG}, \gamma_{AG})$, it is 0-GAS with $\beta_{AG} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$. Hence, when $u \equiv 0$, the trajectories satisfy

$$|x(k, x_0, 0)| \leq \beta_{AG}(|x_0|, k), \ \forall x_0 \in \mathbf{R}^n, \forall k \in \mathbf{Z}_+.$$

Consequently, $\forall s \geq 0, \forall k \in \mathbf{Z}_+$, for any initial state x_0 such that $|x_0| \leq s$, we have

$$|x(k, x_0, 0)| \leq \beta_{AG}(|x_0|, k) \leq \beta_{AG}(s, k).$$

By (7) and (12),

$$\beta_a(s, k) \leq \beta_{AG}(s, k) < +\infty, \quad \forall s \geq 0, \forall k \in \mathbf{Z}_+.$$

For fixed $k \in \mathbf{Z}_+$, since $0 \leq \beta_a(s,k) \leq \beta_{AG}(s,k)$ and $\beta_{AG}(s,k)$ is right continuous at 0 with $\beta_{AG}(0,k) = 0$, $\beta_a(s,k)$ must be right continuous at 0 with $\beta_a(0,k) = 0$. So $\beta_a(\cdot, k) \in \bar{\mathcal{K}}$. Moreover, for fixed $s \geq 0$, since $0 \leq \beta_a(s,k) \leq \beta_{AG}(s,k)$ and $\beta_{AG}(s,k)$ tends to zero as $k \to \infty$, $\beta_a(s,k)$ also tends to zero as $k \to \infty$. Thus, we have proved that $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.

The sufficiency part of the proof follows directly from the definitions of ISS_{AG}, AG, 0-GAS, the gain γ_∞ and the function β_a. □

Remark 4.2 It is clear from the above proof that system (1) is AG if and only if $\gamma_\infty \in \bar{\mathcal{K}}$. Moreover, system (1) is 0-GAS if and only if $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.

5 Necessary and sufficient conditions for ISS_+

In this section we show how the value function $V^\delta(x,k)$ can be used in analysing the ISS_+ property. Results of this section are slightly weaker than the results of the previous section since they do not produce minimal β_+ and γ_+ simultaneously. Instead, we show that given a fixed γ_+ it is possible to compute a minimal β_+ corresponding to the given

γ_+ and vice versa. Consequently, results of this section are divided into two subsections addressing respectively the case when β_+ is fixed and the case when γ_+ is fixed.

We note that the gain γ_∞ which was used in characterizing the ISS$_{AG}$ property is not appropriate for results in this section. For this reason, we introduce a new function γ_a. Define

$$\gamma_a(\delta) := \max\{\gamma_\infty(\delta), \sup_{k\geq 0} V^\delta(0,k)\}. \tag{16}$$

We first show that β_a and γ_a are respectively lower bounds for β_+ and γ_+.

Lemma 5.1 *If the system (1) is ISS$_+$ with (β_+, γ_+), then $\gamma_a \in \bar{\mathcal{K}}, \beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and*

$$\begin{aligned} \gamma_a(\delta) &\leq \gamma_+(\delta), \qquad \forall \delta \geq 0 \\ \beta_a(s,k) &\leq \beta_+(s,k), \qquad \forall s \geq 0, \forall k \in \mathbf{Z}_+. \end{aligned}$$

Proof. Since system (1) is ISS$_+$ with (β_+, γ_+), it is AG with γ_+ and 0-GAS with β_+. From Theorem 4.1, we only need to prove that $\sup_{k\geq 0} V^\delta(0,k) \leq \gamma_+(\delta), \forall \delta \geq 0$. Choosing $x_0 = 0$, the ISS$_+$ property implies that $\forall k \in \mathbf{Z}_+$,

$$\sup_{\|u\|_\infty \leq \delta} |x(k,0,u)| \leq \gamma_+(\delta).$$

Hence $V^\delta(0,k) \leq \gamma_+(\delta), \forall k \in \mathbf{Z}_+$ and hence $\sup_{k\geq 0} V^\delta(0,k) \leq \gamma_+(\delta)$. $\square$

5.1 Minimal β_+ for fixed γ_+

In this subsection we address the question of constructing the minimal transient bound β_+ for a fixed gain γ_+ such that the system is ISS$_+$ with (β_+, γ_+).

For $\gamma_+ \in \bar{\mathcal{K}}$, we define

$$\beta^{\gamma_+}(\delta, s, k) := \max\left\{\sup_{|x|\leq s} V^\delta(x,k) - \gamma_+(\delta), 0\right\} \tag{17}$$

and

$$\beta_a^{\gamma_+}(s,k) := \sup_{\delta\geq 0} \beta^{\gamma_+}(\delta, s, k). \tag{18}$$

The main result of the subsection is presented below.

Theorem 5.2 *For fixed $\gamma_+ \in \bar{\mathcal{K}}$, if there exists $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ such that system (1) is ISS$_+$ with (β_+, γ_+), then $\beta_a^{\gamma_+} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and*

$$\beta_a^{\gamma_+}(s,k) \leq \beta_+(s,k), \qquad \forall s \geq 0, k \in \mathbf{Z}_+ \ . \tag{19}$$

Conversely, if $\beta_a^{\gamma_+} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, then the system (1) is ISS$_+$ with $(\beta_a^{\gamma_+}, \gamma_+)$.

PROOF. Let $\gamma_+ \in \bar{\mathcal{K}}$ be fixed, if there exists $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ such that system (1) is ISS$_+$ with (β_+, γ_+), then $\forall \delta \geq 0$,

$$|x(k, x_0, u)| \leq \beta_+(|x_0|, k) + \gamma_+(\delta), \ \forall x_0 \in \mathbf{R}^n, \forall \|u\|_\infty \leq \delta, \forall k \in \mathbf{Z}_+.$$

Hence

$$V^\delta(x, k) - \gamma_+(\delta) \leq \beta_+(|x|, k), \ \forall x \in \mathbf{R}^n, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0.$$

Since $\beta_+(s, k)$ is nondecreasing in s (for fixed k), we have

$$\sup_{|x| \leq s} V^\delta(x, k) - \gamma_+(\delta) \leq \beta_+(s, k), \ \forall s \geq 0, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0.$$

Noting that $\beta_+(s, k)$ is nonnegative, by (17) we have

$$0 \leq \beta^{\gamma_+}(\delta, s, k) \leq \beta_+(s, k), \ \forall s \geq 0, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0.$$

Since δ is arbitrary, we have

$$0 \leq \beta_a^{\gamma_+}(s, k) \leq \beta_+(s, k), \ \forall s \geq 0, \forall k \in \mathbf{Z}_+.$$

It is easy to see that $\beta_a^{\gamma_+} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, as $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.

The sufficiency part of the proof follows from the definitions of $\beta_a^{\gamma_+}$ and ISS$_+$. $\square$

5.2 Minimal γ_+ for fixed β_+

The purpose of this section is to find the minimum gain γ_+ when the transient bound β_+ is fixed so that the system is ISS$_+$ with (β_+, γ_+). To this end we define

$$\gamma_a^{\beta_+}(\delta) := \sup_{x \in \mathbf{R}^n} \sup_{k \in \mathbf{Z}_+} \max \left\{ V^\delta(x, k) - \beta_+(|x|, k), 0 \right\}. \tag{20}$$

The main result of the subsection is presented below.

Theorem 5.3 *For fixed $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, if there exists $\gamma_+ \in \bar{\mathcal{K}}$ such that system (1) is ISS$_+$ with (β_+, γ_+), then $\gamma_a^{\beta_+} \in \bar{\mathcal{K}}$ and*

$$\gamma_a^{\beta_+}(\delta) \leq \gamma_+(\delta), \qquad \forall \delta \geq 0 \ . \tag{21}$$

Conversely, if $\gamma_a^{\beta_+} \in \bar{\mathcal{K}}$, then system (1) is ISS$_+$ with $(\beta_+, \gamma_a^{\beta_+})$.

PROOF. Let $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ be fixed, if there exists $\gamma_+ \in \bar{\mathcal{K}}$ such that system (1) is ISS$_+$ with (β_+, γ_+), then $\forall \delta \geq 0$,

$$|x(k, x_0, u)| \leq \beta_+(|x_0|, k) + \gamma_+(\delta), \qquad \forall x_0 \in \mathbf{R}^n, \forall \|u\|_\infty \leq \delta, \forall k \in \mathbf{Z}_+.$$

Hence

$$V^\delta(x, k) - \beta_+(|x|, k) \leq \gamma_+(\delta), \qquad \forall x \in \mathbf{R}^n, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0.$$

Noting that $\gamma_+(\delta)$ is nonnegative, by (20) we have

$$0 \leq \gamma_a^{\beta_+}(\delta) \leq \gamma_+(\delta), \qquad \forall \delta \geq 0.$$

Since $\gamma_+ \in \bar{\mathcal{K}}$, we have $\gamma_a^{\beta_+} \in \bar{\mathcal{K}}$.

The sufficiency part of the proof follows from the definitions of $\gamma_a^{\beta_+}$ and ISS$_+$. $\square$

6 Necessary and sufficient conditions for ISS$_{\max}$

In this section we present necessary and sufficient conditions for ISS$_{\max}$ and moreover, we obtain in a similar manner as in the previous section, a minimum gain $\gamma_{\max}$ for a fixed transient bound $\beta_{\max}$ and vice versa. The constructions of the minimal functions are different from the constructions in the previous section although the ideas are the same.

Lemma 6.1 *If the system (1) is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$, then $\gamma_a \in \bar{\mathcal{K}}, \beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and*

$$\begin{aligned} \gamma_a(\delta) &\leq \gamma_{\max}(\delta), \qquad \forall \delta \geq 0 \\ \beta_a(s,k) &\leq \beta_{\max}(s,k), \qquad \forall s \geq 0, \forall k \in \mathbf{Z}_+ \ . \end{aligned}$$

PROOF. Follows directly from the definitions of β_a, γ_a and the property ISS$_{\max}$.

6.1 Minimal $\beta_{\max}$ for fixed $\gamma_{\max}$

In this subsection we present results for a fixed gain $\gamma_{\max} \in \bar{\mathcal{K}}$. In particular we find the minimal transient bound $\beta_{\max}$ such that the system is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$.

To this end, for $\gamma_{\max} \in \bar{\mathcal{K}}$, we define

$$\tilde{\beta}^{\gamma_{\max}}(\delta, s, k) := \begin{cases} \sup\limits_{|x|\leq s} V^\delta(x,k) & \text{if } \sup\limits_{|x|\leq s} V^\delta(x,k) > \gamma_{\max}(\delta), \\ 0 & \text{if } \sup\limits_{|x|\leq s} V^\delta(x,k) \leq \gamma_{\max}(\delta). \end{cases} \tag{22}$$

and

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k) := \sup_{\delta \geq 0} \tilde{\beta}^{\gamma_{\max}}(\delta, s, k) \tag{23}$$

The main result of this subsection is presented next.

Theorem 6.2 *For a fixed $\gamma_{\max} \in \bar{\mathcal{K}}$, if there exists $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ such that the system (1) is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$, then $\tilde{\beta}_a^{\gamma_{\max}} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and*

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k) \leq \beta_{\max}(s,k), \qquad \forall s \geq 0, k \in \mathbf{Z}_+ \ . \tag{24}$$

Conversely, if $\tilde{\beta}_a^{\gamma_{\max}} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, then the system is ISS$_{\max}$ with $(\tilde{\beta}_a^{\gamma_{\max}}, \gamma_{\max})$.

PROOF. Let $\gamma_{\max} \in \bar{\mathcal{K}}$ be fixed. If there exists $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ such that system (1) is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$, then $\forall \delta \geq 0$,

$$|x(k, x_0, u)| \leq \max\{\beta_{\max}(|x_0|, k), \gamma_{\max}(\delta)\}, \qquad \forall x_0 \in \mathbf{R}^n, \forall \|u\|_\infty \leq \delta, \forall k \in \mathbf{Z}_+.$$

Hence

$$V^\delta(x,k) \leq \max\{\beta_{\max}(|x|, k), \gamma_{\max}(\delta)\}, \qquad \forall x \in \mathbf{R}^n, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0.$$

Since $\beta_{\max}(s,k)$ is nondecreasing in s (for fixed k), we have

$$\sup_{|x|\leq s} V^{\delta}(x,k) \leq \max\{\beta_{\max}(s,k), \gamma_{\max}(\delta)\}, \qquad \forall s \geq 0, \forall k \in \mathbf{Z}_+, \forall \delta \geq 0. \tag{25}$$

By (22), if $\sup_{|x|\leq s} V^{\delta}(x,k) > \gamma_{\max}(\delta)$, then

$$\tilde{\beta}^{\gamma_{\max}}(\delta,s,k) = \sup_{|x|\leq s} V^{\delta}(x,k) \leq \max\{\beta_{\max}(s,k), \gamma_{\max}(\delta)\} = \beta_{\max}(s,k).$$

If $\sup_{|x|\leq s} V^{\delta}(x,k) \leq \gamma_{\max}(\delta)$, then

$$\tilde{\beta}^{\gamma_{\max}}(\delta,s,k) = 0 \leq \beta_{\max}(s,k).$$

So, in either case we have

$$0 \leq \tilde{\beta}^{\gamma_{\max}}(\delta,s,k) \leq \beta_{\max}(s,k).$$

Since δ is arbitrary,

$$0 \leq \tilde{\beta}_a^{\gamma_{\max}}(s,k) \leq \beta_{\max}(s,k), \qquad \forall s \geq 0, \forall k \in \mathbf{Z}_+.$$

It is easy to see that $\tilde{\beta}_a^{\gamma_{\max}} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, since $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$.

The sufficiency part of the proof follows from the definitions of $\tilde{\beta}_a^{\gamma_{\max}}$ and ISS$_{\max}$. $\square$

6.2 Minimal $\gamma_{\max}$ for fixed $\beta_{\max}$

In this section we present results for a fixed transient bound $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$. In particular, we find the minimal gain $\gamma_{\max}$ such that the system is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$. We define

$$\tilde{\gamma}^{\beta_{\max}}(\delta,s,k) := \begin{cases} \sup_{|x|\leq s} V^{\delta}(x,k) & \text{if } \sup_{|x|\leq s} V^{\delta}(x,k) > \beta_{\max}(s,k), \\ 0 & \text{if } \sup_{|x|\leq s} V^{\delta}(x,k) \leq \beta_{\max}(s,k). \end{cases} \tag{26}$$

and

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) := \sup_{s\geq 0} \sup_{k\in\mathbf{Z}_+} \tilde{\gamma}^{\beta_{\max}}(\delta,s,k). \tag{27}$$

The main result of this subsection is presented next.

Theorem 6.3 *For a fixed $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, if there exists $\gamma_{\max} \in \bar{\mathcal{K}}$ such that the system (1) is ISS$_{\max}$ with $(\beta_{\max}, \gamma_{\max})$ for some $\gamma_{\max} \in \bar{\mathcal{K}}$, then $\tilde{\gamma}_a^{\beta_{\max}} \in \bar{\mathcal{K}}$ and*

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) \leq \gamma_{\max}(\delta), \qquad \forall \delta \geq 0. \tag{28}$$

Conversely, if $\tilde{\gamma}_a^{\beta_{\max}} \in \bar{\mathcal{K}}$, then the system is ISS$_{\max}$ with $(\beta_{\max}, \tilde{\gamma}_a^{\beta_{\max}})$.

PROOF. Let $\beta_{\max} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ be fixed, if there exists $\gamma_{\max} \in \bar{\mathcal{K}}$ such that system (1) is $\mathrm{ISS}_{\max}$ with $(\beta_{\max}, \gamma_{\max})$, then from (25) in the proof of Theorem 6.2,

$$\sup_{|x|\le s} V^{\delta}(x,k) \le \max\{\beta_{\max}(s,k), \gamma_{\max}(\delta)\}, \qquad \forall s \ge 0, \forall k \in \mathbf{Z}_+, \forall \delta \ge 0.$$

By (26), if $\sup_{|x|\le s} V^{\delta}(x,k) > \beta_{\max}(s,k)$, then

$$\tilde{\gamma}^{\beta_{\max}}(\delta,s,k) = \sup_{|x|\le s} V^{\delta}(x,k) \le \max\{\beta_{\max}(s,k), \gamma_{\max}(\delta)\} = \gamma_{\max}(\delta).$$

If $\sup_{|x|\le s} V^{\delta}(x,k) \le \beta_{\max}(s,k)$, then

$$\tilde{\gamma}^{\beta_{\max}}(\delta,s,k) = 0 \le \gamma_{\max}(\delta).$$

So, in either case we have

$$0 \le \tilde{\gamma}^{\beta_{\max}}(\delta,s,k) \le \gamma_{\max}(\delta), \qquad \forall s \ge 0, k \in \mathbf{Z}_+, \delta \ge 0.$$

Since s, k are arbitrary,

$$0 \le \tilde{\gamma}_a^{\beta_{\max}}(\delta) \le \gamma_{\max}(\delta), \qquad \forall \delta \ge 0.$$

It is easy to see that $\tilde{\gamma}_a^{\beta_{\max}} \in \bar{\mathcal{K}}$, since $\gamma_{\max} \in \bar{\mathcal{K}}$.

The sufficiency part of the proof follows from the definitions of $\tilde{\gamma}_a^{\beta_{\max}}$ and $\mathrm{ISS}_{\max}$. □

Remark 6.4 It can be seen from Theorem 4.1, Lemmas 5.1 and 6.1 (see also equation (16)) that the minimal ISS_{AG} gain γ_∞ defined by (11) is a lower bound of both the minimal ISS_+ gain and the minimal $\mathrm{ISS}_{\max}$ gain (this is also clear from the different ISS definitions). However, we do not have clear formulas for the the minimal ISS_+ gain and the minimal $\mathrm{ISS}_{\max}$ gain. In fact, there is a tradeoff between the minimal ISS gain and the minimal transient bound for the ISS_+ and $\mathrm{ISS}_{\max}$ cases. Moreover, our examples (see Examples 2 and 3 in Section 8) shows that the limit of some good $\mathrm{ISS}_{\max}$ gains may not be a good $\mathrm{ISS}_{\max}$ gain itself. Our results (see (20) and (27)) also show that for a fixed transient bound $\beta_+ = \beta_{\max}$, the minimal ISS_+ gain $\gamma_a^{\beta_+}$ is not greater than the minimal $\mathrm{ISS}_{\max}$ gain $\tilde{\gamma}_a^{\beta_{\max}}$ if they both exist. The minimal transient bounds of different ISS definitions enjoy a similar property.

7 Analysis of related ISS like properties

It is possible to analyse several other ISS like properties using techniques of Sections 5 and 6. In particular, we sketch below how one can analyse input-to-output stability (IOS) and incremental input-to-state stability (Δ-ISS) that were respectively considered in [22, 23] and [1]. Other ISS like properties can be analysed using similar techniques, but we have omitted those results for space reasons.

Consider the system (1) with the output

$$y_k = h(x_k). \tag{29}$$

We introduce the following two IOS properties:

Definition 7.1 *The system (1) with the output (29) is IOS$_+$ (with (β, γ)) if there exists $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, such that*

$$|h(x(k, x_0, u))| \leq \beta(|x_0|, k) + \gamma(\|u\|_\infty), \qquad \forall x_0 \in \mathbf{R}^n, u \in l_\infty, k \in \mathbf{Z}_+. \tag{30}$$

Definition 7.2 *The system (1) with the output (29) is IOS$_{\max}$ (with (β, γ)) if there exists $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, such that*

$$|h(x(k, x_0, u))| \leq \max\{\beta(|x_0|, k), \gamma(\|u\|_\infty)\}, \qquad \forall x_0 \in \mathbf{R}^n, u \in l_\infty, k \in \mathbf{Z}_+. \tag{31}$$

For $x \in \mathbf{R}^n$, $\delta \geq 0$, integer $k \in \mathbf{Z}_+$, denote

$$U^\delta(x, k) := \sup_{\|u\|_\infty \leq \delta} \{|h(x(k, x_0, u))| : x_0 = x\} . \tag{32}$$

The Dynamic Programming Equation (DPE) for $U^\delta(x, k)$ is

$$U^\delta(x, k) = \sup_{|u| \leq \delta} U^\delta(f(x, u), k - 1) \tag{33}$$

with the initial condition

$$U^\delta(x, 0) = |h(x)| . \tag{34}$$

Results similar to those in Sections 5 and 6 still hold for IOS properties defined above. It should be noted that the results in Section 4 do not hold since we do not have an appropriate asymptotic gain characterization of IOS.

Another property that can be treated in a similar way is incremental ISS (Δ-ISS) considered in [1]. In particular, we can define the following two characterizations of Δ-ISS:

Definition 7.3 *The system (1) is Δ-ISS$_+$ (with (β, γ)) if there exists $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, such that any two solutions $x(k, x_0, u)$ and $x(k, z_0, v)$ satisfy:*

$$|x(k, x_0, u) - x(k, z_0, v)| \leq \beta(|x_0 - z_0|, k) + \gamma(\|u - v\|_\infty) , \tag{35}$$

for all $x_0, z_0 \in \mathbf{R}^n$, all $u, v \in l_\infty$ and all $k \in \mathbf{Z}_+$.

Definition 7.4 *The system (1) is Δ-ISS$_{\max}$ (with (β, γ)) if there exists $\gamma \in \bar{\mathcal{K}}$ and $\beta \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, such that any two solutions $x(k, x_0, u)$ and $x(k, z_0, v)$ satisfy:*

$$|x(k, x_0, u) - x(k, z_0, v)| \leq \max\{\beta(|x_0 - z_0|, k), \gamma(\|u - v\|_\infty)\} \tag{36}$$

for all $x_0, z_0 \in \mathbf{R}^n$, all $u, v \in l_\infty$ and all $k \in \mathbf{Z}_+$.

In order to state the appropriate dynamic programming equation for Δ-ISS, we introduce the following $2n$ dimensional auxiliary system containing system (1) and an augmented exact copy:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k), \\ z_{k+1} &= f(z_k, v_k). \end{aligned}$$

Here $x_k, z_k \in \mathbf{R}^n$ and $u_k, v_k \in \mathbf{R}^m$. Then, we introduce for $x, z \in \mathbf{R}^n$, $\delta \geq 0$, integer $k \in \mathbf{Z}_+$

$$W^\delta(x,z,k) := \sup_{\|u-v\|_\infty \leq \delta} \{|x(k,x_0,u) - x(k,z_0,v)| : x_0 = x, z_0 = z\} \ . \tag{37}$$

The Dynamic Programming Equation (DPE) for $W^\delta(x,z,k)$ is

$$W^\delta(x,z,k) = \sup_{|u-v|\leq\delta} W^\delta(f(x,u), f(z,v), k-1) \tag{38}$$

with the initial condition

$$W^\delta(x,z,0) = |x-z| \, . \tag{39}$$

Results similar to those in Sections 5 and 6 still hold for Δ-ISS while the results in Section 4 do not hold since we do not have an appropriate asymptotic gain characterization of Δ-ISS.

8 Examples

In this section, we present four examples to which the results of Sections 3, 4, 5 and 6 are applied. The first example shows that the minimal asymptotic gain for an ISS system may be discontinuous. The second and third examples consider respectively scalar linear and nonlinear systems, whilst the fourth example considers a second order nonlinear system.

Where necessary in analysing these examples, a numerical scheme is applied to solve DPE (8) approximately. This scheme utilizes a bounded discretized input bound space Δ, state space X and input space U. In terms of notation, these spaces are denoted respectively by

$$\begin{aligned} \Delta &= \{\delta \in \mathbf{R} : \delta_{\min} \leq \delta \leq \delta_{\max}\}_{N_\Delta} \, , \\ X &= \{x \in \mathbf{R} : |x| \leq x_{\max}\}_{N_X} \, , \\ U^\delta &= \{u \in \mathbf{R} : |u| \leq \delta\}_{N_U} \, , \ \ \delta \in \Delta. \end{aligned} \tag{40}$$

Here, N_Δ, N_X and N_U respectively refer to the number of points in each of the discretized spaces Δ, X and U^δ. The result of applying DPE (8) over these discretized spaces is an approximation for V^δ. With $V^\delta(x,k)$ computed for all $\delta \in \Delta$, computation of approximations for the remaining quantities is then possible.

We acknowledge that, while straightforward in principle, these approximations can be computationally expensive to obtain. Aside from this observation, we stress that while the details of the attendant numerical scheme are important, the scheme itself is not fundamental to understanding the concepts presented in this paper. Consequently, a detailed discussion of possible numerical schemes is postponed for inclusion in a later paper.

8.1 Example 1: A system with discontinuous minimal asymptotic gain

Consider the one dimensional system

$$x_{k+1} \ = \ \frac{1}{2} x_k \left(1 + \phi(|x_k|)\, a(|u_k|)\right) \tag{41}$$

where

$$\phi(s) \;=\; \begin{cases} 1, & s \in [0,20), \\ 21-s, & s \in [20,21), \\ 0, & s \in [21,\infty), \end{cases} \qquad \text{and} \qquad a(s) \;=\; \begin{cases} 0, & s \in [0,9), \\ s-9, & s \in [9,10), \\ 1, & s \in [10,\infty). \end{cases} \tag{42}$$

ISS$_{AG}$ property: To demonstrate that system (41) satisfies the ISS$_{AG}$ property, we show that the AG and 0-GAS properties hold.

(i) AG property: First, we can show that the dynamics of (41) are attracted to an input invariant set as $k \to \infty$. Consider two cases for $|x_k|$.

(a) $|x_k| \geq 21$: For any u_k, $|x_k| \geq 21$ implies that $|x_{k+1}| \leq \frac{1}{2}|x_k|$.

(b) $|x_k| < 21$: For any u_k, $|x_k| < 21$ implies that $|x_{k+1}| \leq \frac{1}{2}|x_k| + \frac{1}{2}a(|u_k|)\,|x_k|\,\phi(|x_k|) \leq \frac{1}{2}|x_k| + \frac{1}{2}|x_k| = |x_k|$.

Hence, combining these two cases, for any input u,

$$\sup_{x_\circ \in \mathbf{R}} \limsup_{k\to\infty} |x(k,x_0,u)| \;\leq\; 21. \tag{43}$$

Next, consider four cases defined by different bounds on $\|u\|_\infty$ to bound the candidate asymptotic gain.

(a) $\|u\|_\infty < 9$: By definition, $a(|u_k|) = 0$. So, for any x_k, $|x_{k+1}| \leq \frac{1}{2}|x_k|$. That is,

$$\|u\|_\infty < 9 \;\Rightarrow\; \sup_{x_\circ \in \mathbf{R}} \limsup_{k\to\infty} |x(k,x_0,u)| = 0.$$

(b) $\|u\|_\infty \in [9,10)$: There exists $\delta \in (0,1)$ such that $\|u\|_\infty \leq 10-\delta$, which implies that $a(|u_k|) \leq 1-\delta$. So, for any x_k,

$$|x_{k+1}| \;\leq\; \tfrac{1}{2}|x_k| + \tfrac{1}{2}(1-\delta)\,|x_k|\,\phi(|x_k|) \;\leq\; \tfrac{1}{2}|x_k| + \tfrac{1}{2}(1-\delta)\,|x_k| \;=\; (1-\tfrac{1}{2}\delta)\,|x_k|\,. \tag{44}$$

That is,

$$\|u\|_\infty \in [9,10) \;\Rightarrow\; \sup_{x_\circ \in \mathbf{R}} \limsup_{k\to\infty} |x(k,x_0,u)| = 0.$$

(c) $\|u\|_\infty = 10$: First, consider $x_0 = 20$. By inspection, $x_{k+1} = \frac{1}{2}x_k + \frac{1}{2}x_k = x_k = 20$. Hence, for $x_0 = 20$, $\limsup_{k\to\infty}|x(k,x_0,u)| = 20$, and so $\sup_{x_\circ\in\mathbf{R}}\limsup_{k\to\infty}|x(k,x_0,u)| \geq 20$. Combining this limit with (43), implies that

$$\|u\|_\infty = 10 \;\Rightarrow\; \sup_{x_\circ \in \mathbf{R}} \limsup_{k\to\infty} |x(k,x_0,u)| \in [20,21].$$

(d) $\|u\|_\infty \in (10,\infty)$: In this case, we defer to the bound (43). That is,

$$\|u\|_\infty \in (10,\infty) \;\Rightarrow\; \sup_{x_\circ \in \mathbf{R}} \limsup_{k\to\infty} |x(k,x_0,u)| \leq 21.$$

Combining these four cases along with (3) implies that system (41) satisfies the AG property with gain γ_{AG}, where γ_{AG} is any $\bar{\mathcal{K}}$ function such that $\gamma_{AG}(s) \geq \gamma_1(s)$, and

$$\gamma_1(s) \;=\; \begin{cases} 0, & s \in [0,9), \\ 21, & s \in [9,\infty). \end{cases}$$

(ii) 0-GAS property: As $a(0) = 0$, $u_k = 0$ implies that $x_{k+1} = \frac{1}{2}x_k$ for all $k \geq 0$. Hence, applying inequality (5), the 0-GAS property holds with transient bound $\beta_{AG}(s, k) = s\left(2^{-k}\right)$.

This demonstrates that system (41) satisfies the ISS_{AG} property.

(iii) Minimal asymptotic gain: Referring to AG property above, it also follows that any candidate asymptotic gain $\gamma_{AG} \in \mathcal{K}$ for system (41) must satisfy the inequality $\gamma_{AG}(s) \geq \gamma_0(s)$ for all $s \geq 0$, where

$$\gamma_0(s) = \begin{cases} 0, & s \in [0, 10), \\ 20, & s \in [10, \infty). \end{cases}$$

Hence, the minimal asymptotic gain γ_∞ defined by (11) must satisfy $\gamma_0(s) \leq \gamma_\infty(s) \leq \gamma_1(s)$ for all $s \geq 0$, which implies a jump discontinuity in γ_∞ at $s = 10$.

(iv) Minimal asymptotic gain via dynamic programming: In computing an approximation of γ_∞ via the dynamic programming approach, we selected the following numerical parameters as per (40):

$$\begin{array}{llll} N_\Delta = 41, & \delta_{\min} = 9, & \delta_{\max} = 11, \\ N_X = 201, & x_{\max} = 20, & \\ N_U = 21. & & \end{array}$$

The obtained approximation of γ_∞ is shown in Figure 1. Although computed on a finite grid, this approximation clearly demonstrates the jump discontinuity (at $s = 10$) in γ_∞. An example of a numerical approximation of V^δ is shown in Figure 2.

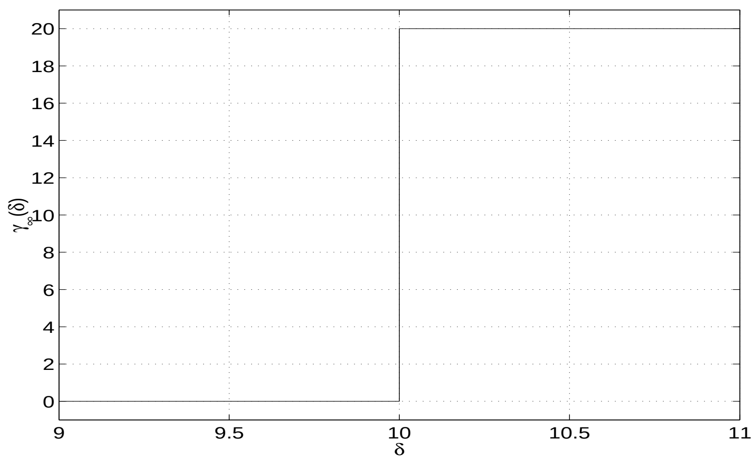

Figure 1: Approximation of γ_∞ obtained by dynamic programming (Example 1).

<u>**ISS_+ Property:**</u> System (41) satisfies the ISS_+ property for $\beta_+(s,k) = s\left(2^{-k}\right)$ and any $\mathcal{K}$ function γ_+ with $\gamma_+(9) \geq 21$. However, system (41) can not be ISS_+ for γ_∞ and any $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, as it is impossible to find a $\beta_+ \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ such that $\beta_+(s,k) \geq (1 - \frac{1}{2}\delta)^k s$ for all $\delta \in (0, 1)$ (see inequality (44)).

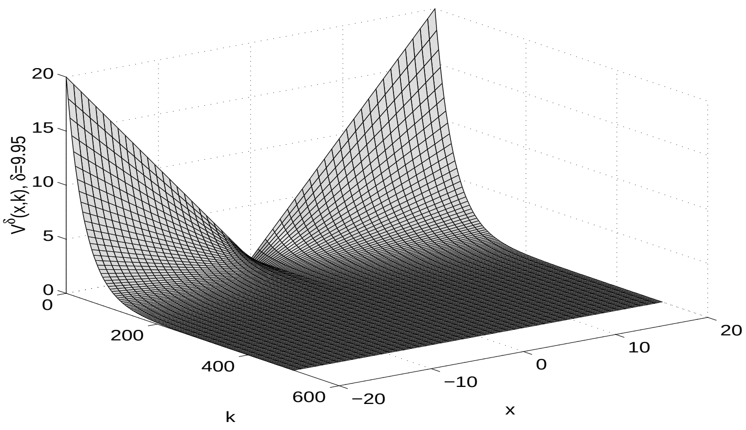

Figure 2: Approximation of V^δ for $\delta = 9.95$ (Example 1).

Remark 8.1 This example demonstrates that for some systems, the minimal asymptotic gain γ_∞ can be of class $\bar{\mathcal{K}} \setminus \mathcal{K}$. Furthermore, γ_∞ may not be a candidate gain for the ISS$_+$ property, even though it is the minimal asymptotic gain. Consequently, we cannot use γ_∞ to determine a minimal candidate transient bound for the ISS$_+$ property in this example.

8.2 Example 2: A class of scalar linear systems

Consider the class of scalar linear systems given by

$$x_{k+1} \;=\; a x_k + b u_k, \tag{45}$$

where $0 < a < 1$ and $b \geq 0$. By direct calculation, DPE (8) and initialisation (9) for V^δ imply that for any $k \geq 0$,

$$V^\delta(x,k) \;=\; a^k |x| + \left(\frac{1-a^k}{1-a}\right) b\delta. \tag{46}$$

<u>**ISS$_{AG}$ property:**</u> Applying definitions (10), (11), and (12) of respectively $V_a^\delta(x)$, $\gamma_\infty(\delta)$ and $\beta_a(s,k)$,

$$\begin{aligned} V_a^\delta(x) \;&=\; \limsup_{k\to\infty} V^\delta(x,k) \;=\; \left(\frac{b}{1-a}\right)\delta, \\ \gamma_\infty(\delta) \;&=\; \sup_{x\in\mathbf{R}^n} V_a^\delta(x) \;=\; \left(\frac{b}{1-a}\right)\delta, \\ \beta_a(s,k) \;&=\; \sup_{|x|\leq s} V^0(x,k) \;=\; s a^k. \end{aligned}$$

Since $\gamma_\infty \in \bar{\mathcal{K}}$ and $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, Theorem 4.1 implies that system (45) is ISS$_{AG}$ with (β_a, γ_∞).

<u>**ISS_+ property:**</u> Applying definition (16) of $\gamma_a(\delta)$,

$$\gamma_a(\delta) \;=\; \max\left\{\gamma_\infty(\delta), \sup_{k\geq 0} V^\delta(0,k)\right\} \;=\; \max\left\{\gamma_\infty(\delta), \left(\frac{b}{1-a}\right)\delta\right\} \;=\; \gamma_\infty(\delta).$$

(i) Minimal β_+ for fixed γ_+: Using γ_a as a candidate (fixed) gain in testing ISS_+ (i.e. $\gamma_+ = \gamma_a$), the definition (17) of the minimal corresponding transient bound β_+ for $\|u\|_\infty \leq \delta$ yields

$$\beta^{\gamma_+}(\delta,s,k) \;=\; \max\left\{\left(s - \frac{b\delta}{1-a}\right)a^k, 0\right\}.$$

Taking the supremum over all input bounds δ yields the minimal ISS_+ transient bound for gain γ_+,

$$\beta_a^{\gamma_+}(s,k) \;=\; \sup_{\delta\geq 0}\beta^{\gamma_+}(\delta,s,k) \;=\; sa^k,$$

which in this case is the same as $\beta_a(s,k)$. By inspection, $\gamma_+ = \gamma_a \in \bar{\mathcal{K}}$ and $\beta_a^{\gamma_+} \in \bar{\mathcal{K}}\bar{\mathcal{L}}$. Theorem 5.2 then implies that system (45) is ISS_+ with $(\beta_a^{\gamma_+}, \gamma_+)$, where $\gamma_+ = \gamma_a$ and $\beta_a^{\gamma_+}$ is the minimal corresponding transient bound.

(ii) Minimal γ_+ for fixed β_+: Using β_a as a candidate (fixed) transient bound in testing ISS_+ (i.e. $\beta_+ = \beta_a$), the definition (20) of the minimal corresponding gain yields

$$\gamma_a^{\beta_+}(\delta) \;=\; \left(\frac{b}{1-a}\right)\delta \;=\; \gamma_a(\delta)$$

in this case. Since $\gamma_a \in \bar{\mathcal{K}}$ and $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$, Theorem 5.3 implies that system (45) is ISS_+ with $(\beta_+, \gamma_a^{\beta_+})$, where $\beta_+ = \beta_a$ and $\gamma_a^{\beta_+}$ is the minimal corresponding gain.

Remark 8.2 Calculations (i) and (ii) above highlight an important property of scalar linear systems. In particular, (i) shows that the minimal ISS_+ transient bound $\beta_a^{\gamma_+}$ determined using the minimal candidate ISS_+ gain $\gamma_+ = \gamma_a$ is exactly the minimal candidate ISS_+ transient bound β_a. Similarly, (ii) shows that the minimal candidate ISS_+ gain bound γ_a is recovered as the minimal ISS_+ gain. That is, both approaches yield that the ISS_+ property holds with the transient bound / gain *pair* defined by the minimal candidate transient bound β_a and the minimal candidate gain γ_a. We note that this is not in general the case, either for other classes of systems or other equivalent ISS properties. This is illustrated below in the $\mathrm{ISS}_{\max}$ case.

<u>**$\mathrm{ISS}_{\max}$ property:**</u> Based on the candidates β_a and γ_a for $\beta_{\max}$ and $\gamma_{\max}$, it is also possible (in this case) to explicitly test the $\mathrm{ISS}_{\max}$ property. Unlike the ISS_+ property however, we find that (for this example) the $\mathrm{ISS}_{\max}$ property does not hold for the *pair* defined by the minimal candidate transient bound β_a and the minimal candidate gain γ_a.

(i) Minimal $\beta_{\max}$ for fixed $\gamma_{\max}$: Using γ_a as a candidate (fixed) gain in testing $\mathrm{ISS}_{\max}$ (i.e. $\gamma_{\max} = \gamma_a$), the definition (22) of the minimal corresponding transient bound $\beta_{\max}$ for $\|u\|_\infty \leq \delta$ yields

$$\tilde{\beta}^{\gamma_{\max}}(\delta,s,k) \;=\; \begin{cases} \left(s - \frac{b\delta}{1-a}\right)a^k + \left(\frac{b\delta}{1-a}\right) & s > \frac{b\delta}{1-a}, \\ 0 & s \leq \frac{b\delta}{1-a}. \end{cases}$$

Taking the supremum over all input bounds δ yields the minimal ISS$_{\max}$ transient bound (23), given by

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k) \;=\; s,$$

which is not of class $\bar{\mathcal{K}}\bar{\mathcal{L}}$. Hence, the gain γ_a is too small to be a gain candidate for computing the minimal transient bound. To illustrate this point further, suppose a slightly larger candidate gain is chosen, namely

$$\gamma_{\max}(\delta) \;=\; (1+\varepsilon)\gamma_a(\delta)$$

where $\varepsilon > 0$ is fixed and small. Repeating the above computation yields that

$$\tilde{\beta}_a^{\gamma_{\max}}(s,k) \;=\; \left(\frac{1+\varepsilon}{a^k+\varepsilon}\right) s a^k \;=\; \left(\frac{1+\varepsilon}{a^k+\varepsilon}\right) \beta_a(s,k),$$

which is of class $\bar{\mathcal{K}}\bar{\mathcal{L}}$ for any $\varepsilon > 0$. Hence, by Theorem 6.2, system (45) is ISS$_{\max}$ with $(\tilde{\beta}_a^{\gamma_{\max}}, \gamma_{\max})$, $\gamma_{\max} = (1+\varepsilon)\gamma_a$.

(ii) Minimal $\gamma_{\max}$ for fixed $\beta_{\max}$: Using β_a as a candidate (fixed) transient bound in testing ISS$_{\max}$ (i.e. $\beta_{\max} = \beta_a$), the minimal corresponding gain (26) for $\|u\|_\infty \le \delta$ is

$$\tilde{\gamma}^{\beta_{\max}}(\delta,s,k) \;=\; \begin{cases} s a^k + \left(\frac{1-a^k}{1-a}\right) b\delta & \delta > 0, \\ 0 & \delta = 0. \end{cases}$$

Taking the supremum over all time $k \in \mathbf{Z}_+$ and all $s \ge 0$ yields the corresponding minimal ISS$_{\max}$ gain (27),

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) \;\ge\; \sup_{s\ge 0} s \;=\; \infty, \tag{47}$$

for all $\delta > 0$, which is clearly not of class $\bar{\mathcal{K}}$. As in the minimal transient bound case, this implies that the transient bound β_a is too small to be a candidate transient bound for ISS$_{\max}$. To illustrate that this system is ISS$_{\max}$, choose the slightly larger transient bound

$$\beta_{\max}(s,k) \;=\; (1+\varepsilon)\beta_a(s,k)$$

where $\varepsilon > 0$. Repeating the above computation yields that

$$\tilde{\gamma}_a^{\beta_{\max}}(\delta) \;=\; \left(1+\frac{1}{\varepsilon}\right)\frac{b\delta}{1-a} \;=\; \left(1+\frac{1}{\varepsilon}\right)\gamma_a(\delta),$$

which is of class $\bar{\mathcal{K}}$. Theorem 6.3 then implies that system (45) is ISS$_{\max}$ with $(\beta_{\max}, \tilde{\gamma}_a^{\beta_{\max}})$, $\beta_{\max} = (1+\varepsilon)\beta_a$.

8.3 Example 3: A scalar nonlinear system

Consider the scalar nonlinear system

$$x_{k+1} \;=\; \frac{x_k^3}{2(1+x_k^2)} + u_k^3 \tag{48}$$

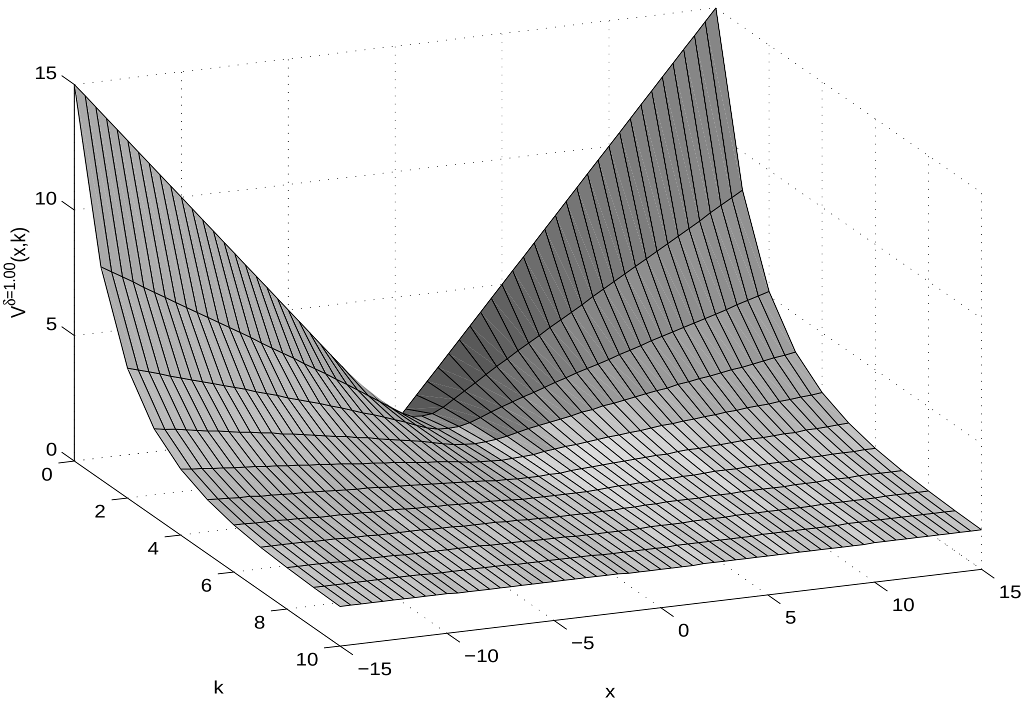

Figure 3: Approximation of $V^{\delta}(x,k)$ with $\delta = 1.00$ (Example 3).

In computing an approximate solution to the DPE (8), we selected the following numerical parameters as per (40):

$$
\begin{aligned}
N_\Delta &= 301, \quad \delta_{\min} = 0, \quad \delta_{\max} = 3,\\
N_X &= 1501, \quad x_{\max} = 15,\\
N_U &= 201.
\end{aligned}
$$

The result of the applying DPE (8) over these discretized spaces is an approximation for V^δ. For example, $V^\delta(x,k)$ for $\delta = 1.00$ obtained in this way is illustrated in Figure 3.

ISS$_{AG}$ property: Computing $V_a^\delta(x)$ from definition (10) yields (in this case) functions that are independent of x (see for example $k = 10$ in Figure 3). γ_∞ then follows from definition (11). Note that since $\left|\frac{x_k^3}{2(1+x_k^2)}\right| \le \left|\frac{x_k}{2}\right|$, it is easy to prove that $\gamma_\infty(\delta) \le 2\delta^3$, thereby providing a useful upper bound for this gain. β_a likewise follows from definition (12). The resulting approximations are illustrated in Figures 4 and 5 respectively. As $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$ and $\gamma_\infty \in \bar{\mathcal{K}}$ (at least over the discretized spaces used in the computation), Theorem 4.1 implies that system (48) is ISS$_{AG}$ with (β_a, γ_∞).

ISS$_+$ property: Computing γ_a based on definition (16) yields an identical gain to γ_∞, as shown in Figure 4.

(i) Minimal β_+ for fixed γ_+: Utilising $\gamma_+ = \gamma_a$ allows the computation of the minimal transient bound $\beta_a^{\gamma_+}$ from definition (18). This function is illustrated in Figure 6, and is of class $\bar{\mathcal{K}}\bar{\mathcal{L}}$ on the domain of computation. We also verified that $\beta_a^{\gamma_+} \ge \beta_a$. Theorem 5.2 then implies that system (48) is ISS$_+$ with $(\beta_a^{\gamma_+}, \gamma_+)$, $\gamma_+ = \gamma_a$.

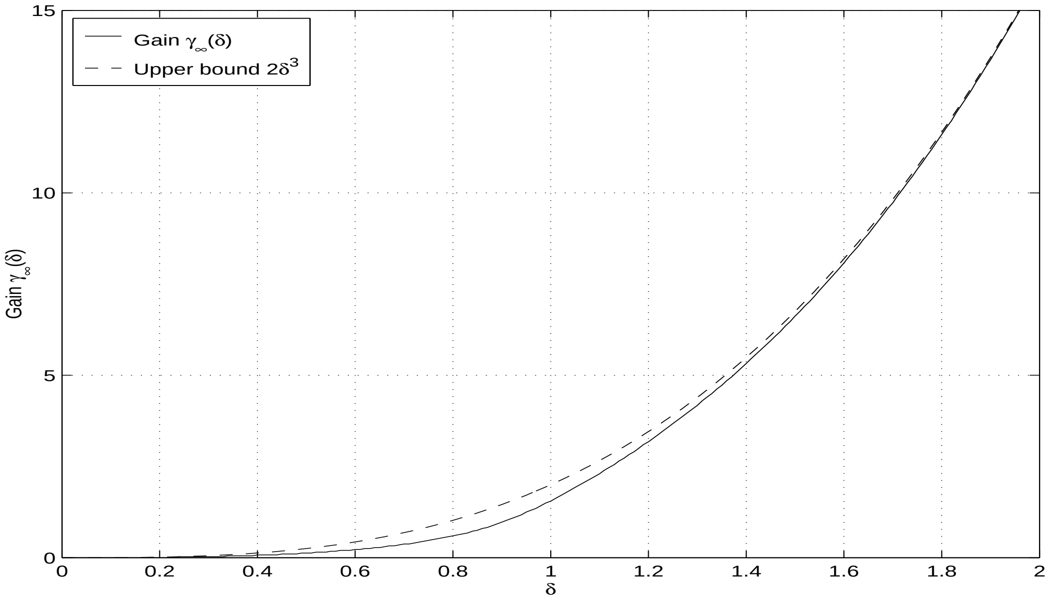

Figure 4: Approximation of $\gamma_\infty(\delta)$ (Example 3).

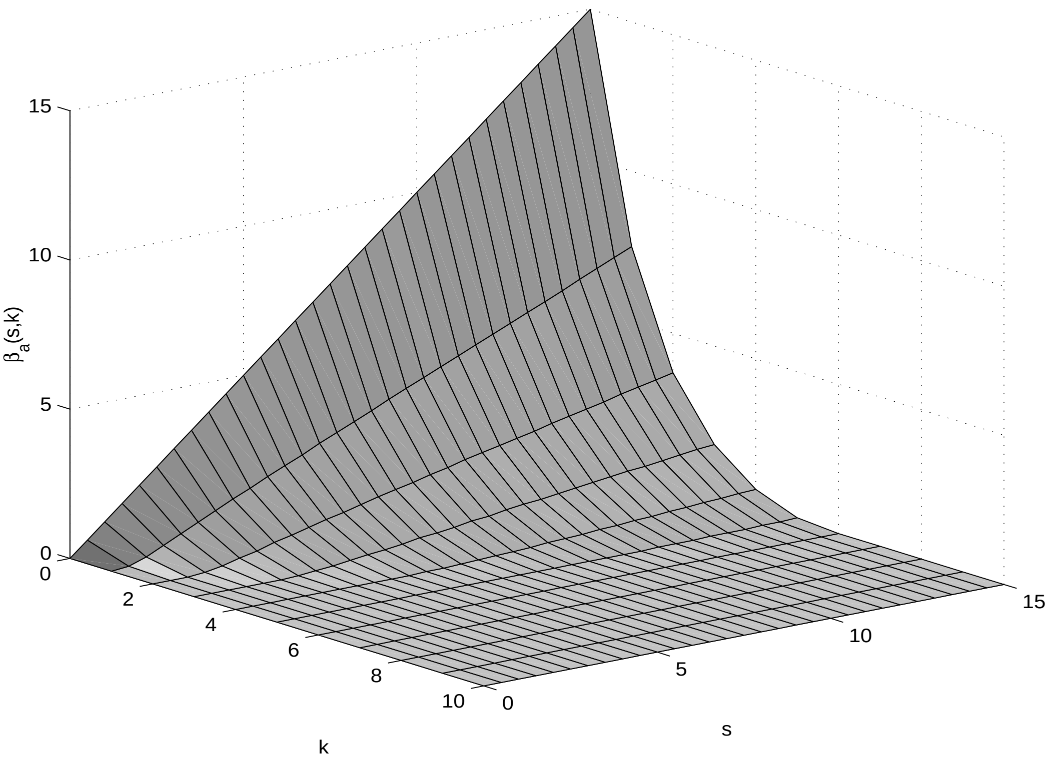

Figure 5: Approximation of $\beta_a(s, k)$ (Example 3).

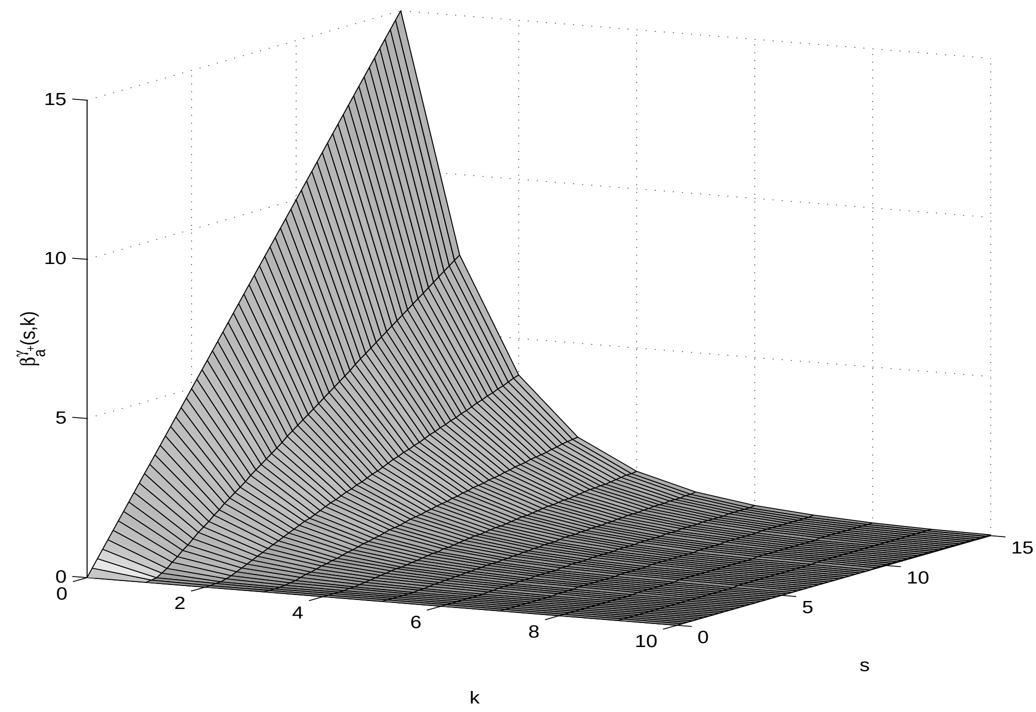

Figure 6: Approximation of $\beta_a^{\gamma_+}(s,k)$ with $\gamma_+ = \gamma_a$ (Example 3).

(ii) Minimal γ_+ for fixed β_+: Utilising $\beta_+ = \beta_a$ allows the computation of the minimal gain $\gamma_a^{\beta_+}$ from definition (20). This function is of class $\bar{\mathcal{K}}$ as illustrated in Figure 7. Hence, Theorem 5.3 implies that system (48) is ISS$_+$ with $(\beta_+, \gamma_a^{\beta_+})$, $\beta_+ = \beta_a$.

<u>**ISS$_{\max}$ property:**</u> As in the ISS$_+$ case, it is possible to compute an approximation to the minimal transient bound given a fixed gain, or vice versa. However, unlike the ISS$_+$ case, it is not possible to compute these bounds where the fixed function is minimal (for example, $\tilde{\gamma}_a^{\beta_{\max}}$ with $\beta_{\max} = \beta_a$), as the numerical approximation becomes very sensitive to the parameters (40). This suggests that either the ISS$_{\max}$ property does not hold for system (48) where the fixed function in minimal, or the numerical approximation is too coarse. Given that it was the ISS$_{\max}$ property that did not hold (as was shown explicitly) for the scalar linear system (45), it is plausible that this also the case for system (48).

Regardless of this, it is still possible to demonstrate that the ISS$_{\max}$ property holds, but a larger gain or transient bound must be used. This is demonstrated below.

(i) Minimal $\beta_{\max}$ for fixed $\gamma_{\max}$: Recall that both $\gamma_\infty(\delta)$ and $\gamma_a(\delta)$ are bounded above by $2\delta^3$. Consequently, a suitable choice of a candidate gain is $\gamma_{\max} = (2+\varepsilon)\delta^3$, $\varepsilon > 0$. ***Case (i)(a):*** With $\varepsilon = 0.1$, computation of the minimal transient bound $\tilde{\beta}_a^{\gamma_{\max}}$ from definition (23) is then possible. This function is illustrated in Figure 8, and is of class $\bar{\mathcal{K}}\bar{\mathcal{L}}$ on the domain of computation. We also verified that $\tilde{\beta}_a^{\gamma_{\max}} \geq \beta_a$. Theorem 6.2 then implies that system (48) is ISS$_{\max}$ with $(\tilde{\beta}_a^{\gamma_{\max}}, \gamma_{\max})$, $\gamma_{\max} = (2+\varepsilon)\delta^3$, $\varepsilon = 0.1$.

Case (i)(b): The tightness of the aforementioned gain bound $2\delta^3$ may be tested by repeating Case (i)(a) with $\varepsilon = 0$. In this case, illustrated in Figure 9, it is evident that

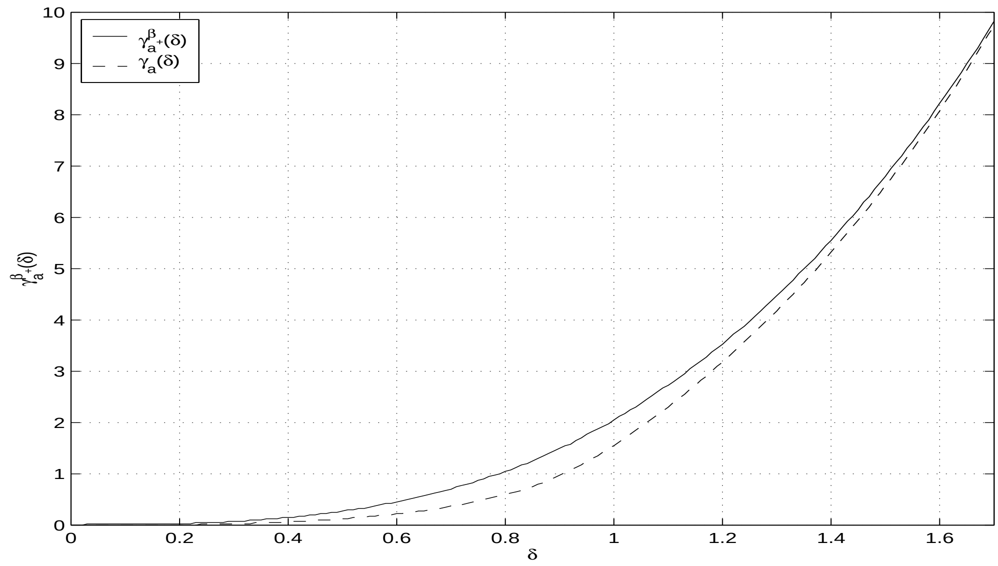

Figure 7: Approximation of $\gamma_a^{\beta_+}(\delta)$ with $\beta_+ = \beta_a$ (Example 3).

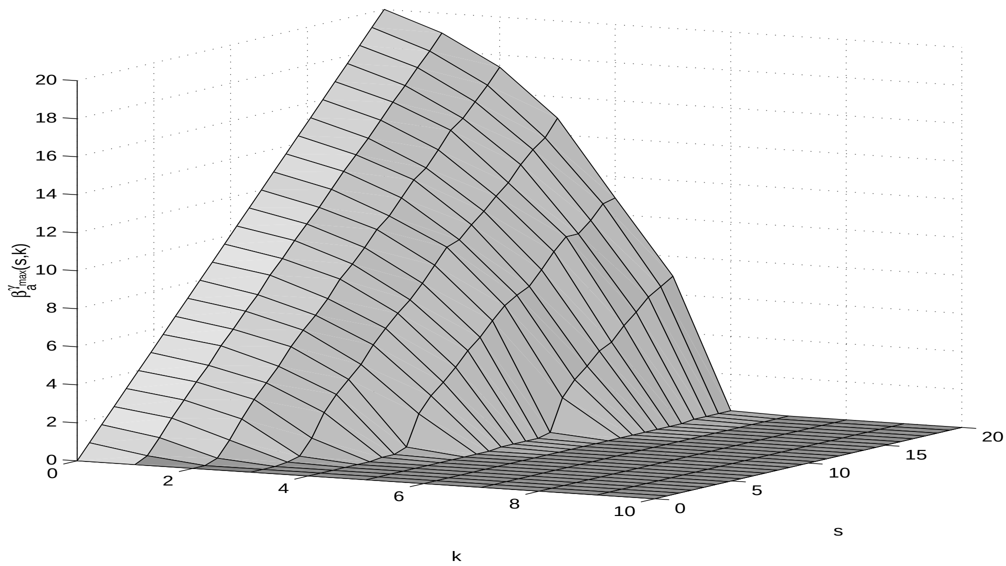

Figure 8: Approximation of $\tilde{\beta}_a^{\gamma_{\max}}(s,k)$ with $\gamma_{\max} = (2+\varepsilon)\delta^3$, $\varepsilon = 0.1$ (Example 3).

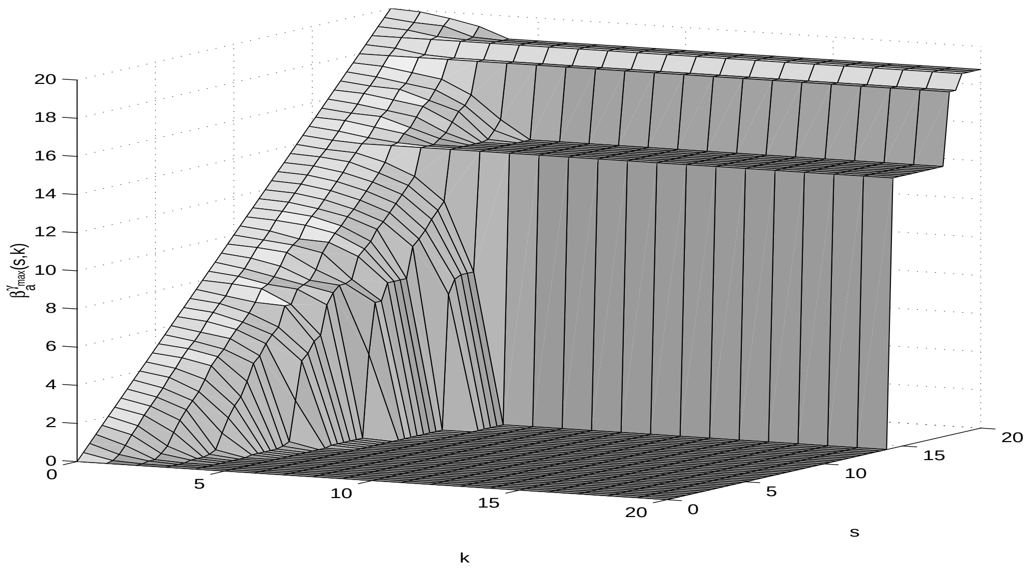

Figure 9: Approximation of $\tilde{\beta}_a^{\gamma_{\max}}(s,k)$ with $\gamma_{\max} = (2+\varepsilon)\delta^3$, $\varepsilon = 0.0$ (Example 3).

the resulting function $\tilde{\beta}_a^{\gamma_{\max}}$ is not of class $\bar{\mathcal{K}}\bar{\mathcal{L}}$. Hence, we cannot conclude that the ISS$_{\max}$ property holds for the pair $(\tilde{\beta}_a^{\gamma_{\max}}, \gamma_{\max})$, where $\gamma_{\max} = (2+\varepsilon)\delta^3$, $\varepsilon = 0$.

(ii) Minimal $\gamma_{\max}$ for fixed $\beta_{\max}$: In choosing a candidate function $\beta_{\max}$, it is reasonable to choose a $\bar{\mathcal{K}}\bar{\mathcal{L}}$ function that is the result of Case(i)(a) above. Alternatively, we may choose an arbitrary class $\bar{\mathcal{K}}\bar{\mathcal{L}}$ function. These two cases are considered in Case (ii)(a) and Case (ii)(b) below.

Case (ii)(a): Choosing $\beta_{\max} = \tilde{\beta}_a^{\gamma_{\max}}$ (the result of Case (i)(a)), $\gamma_{\max} = (2+\varepsilon)\delta^3$, $\varepsilon = 0.1$, yields a minimal gain bound $\tilde{\gamma}_a^{\beta_{\max}}$ equal to the original function $\gamma_{\max}(\delta) = (2+\varepsilon)\delta^3$. As such, the corresponding figure is omitted.

Case (ii)(b): Lemma 6.1 indicates that any candidate $\bar{\mathcal{K}}\bar{\mathcal{L}}$ transient bound $\beta_{\max}$ must be bounded below by β_a. One such candidate function is $\beta_{\max}(s,k) = se^{-k/3}$. Utilising this choice of transient bound allows the computation of the minimal corresponding gain $\tilde{\gamma}_a^{\beta_{\max}}$ from definition (27). This function is of class $\bar{\mathcal{K}}$ as illustrated in Figure 10. Hence, Theorem 6.3 implies that system (48) is ISS$_{\max}$ with $(\beta_{\max}, \tilde{\gamma}_a^{\beta_{\max}})$, $\beta_{\max} = se^{-k/3}$.

Iterated computation of minimal gains and transient bounds: Given that it is now possible to compute the minimal gain given a candidate transient bound, and the minimal transient bound given a candidate gain, it is intuitively clear that iterated computation of minimal gains and minimal transient bounds is possible. The following case illustrates this idea for the ISS$_{\max}$ property, and follows on from Case (ii)(b) above.

The minimal gain $\tilde{\gamma}_a^{\beta_{\max}}$ obtained in Case (ii)(b) can be used as the candidate gain $\gamma_{\max}$ to repeat Case (i)(a), thereby yielding a necessarily smaller transient bound $\tilde{\beta}_a^{\gamma_{\max}}$ (by Lemma 6.1). That is, we expect $\tilde{\beta}_a^{\gamma_{\max}}(s,k) \leq se^{-k/3}$. This is indeed the case, as illustrated in Figures 11 and 12. Further iterations yield the same minimal gain and minimal transient bound, and as such, are omitted.

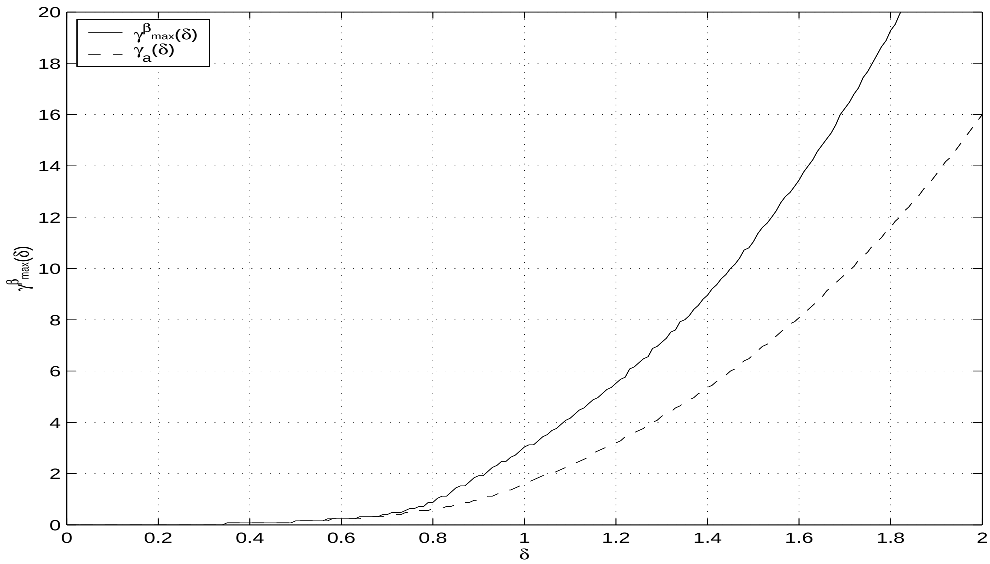

Figure 10: Approximation of $\tilde{\gamma}_a^{\beta_{\max}}(\delta)$ with $\beta_{\max} = se^{-k/3}$ (Example 3).

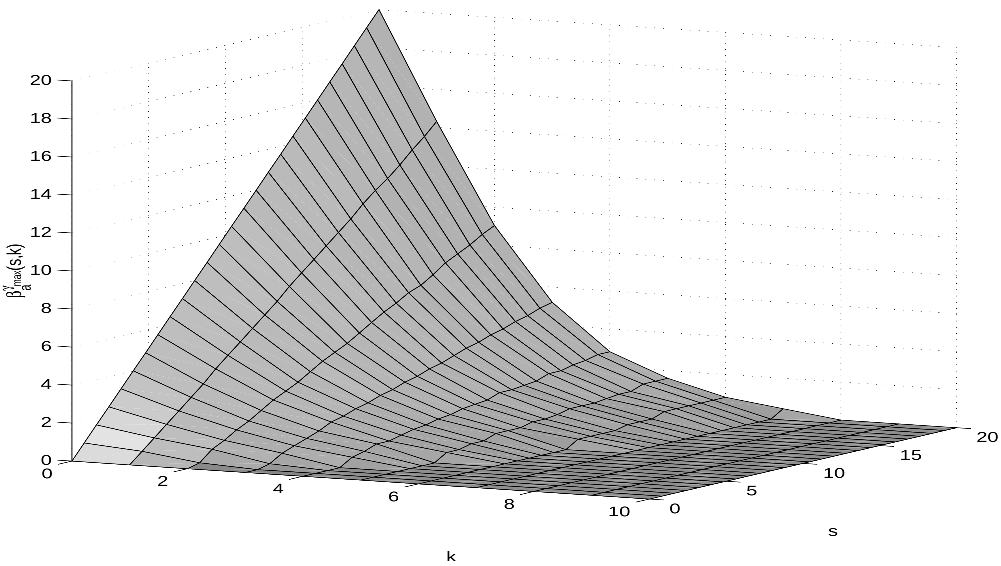

Figure 11: $\tilde{\beta}_a^{\gamma_{\max}}(s,k)$ with $\gamma_{\max} = \tilde{\gamma}_a^{\beta_{\max}}$ as obtained from Case (ii)(b) (Example 3).

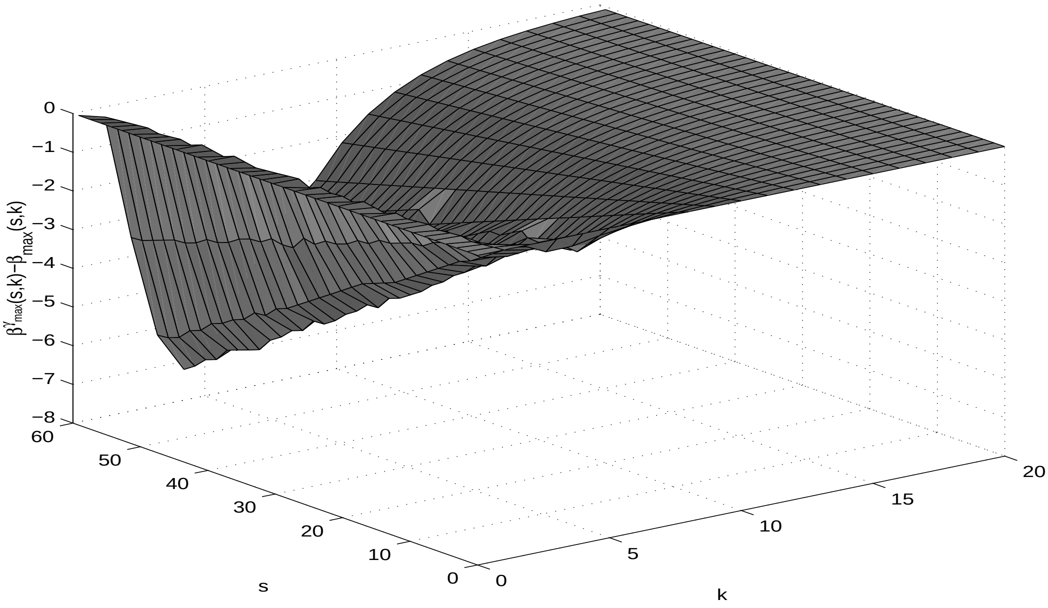

Figure 12: Approximation of $\tilde{\beta}_a^{\gamma_{\max}}(s,k) - se^{-k/3}$ with $\gamma_{\max} = \tilde{\gamma}_a^{\beta_{\max}}$ obtained from Case (ii)(b) (Example 3).

8.4 Example 4: A two dimensional nonlinear system

In this example, the ISS$_{AG}$ property is examined for a two dimensional nonlinear system. In particular, consider the (asymptotically stable) closed loop system

$$\begin{aligned} x_{1,k+1} &= x_{2,k} + \sqrt[3]{x_{1,k}} \\ x_{2,k+1} &= x_{2,k} - \tfrac{1}{4}x_{1,k} + \sqrt[3]{x_{1,k}} - \sqrt[3]{x_{2,k} + \sqrt[3]{x_{1,k}}} + w_k, \end{aligned} \tag{49}$$

obtained via backstepping from the corresponding open loop system with $x_{2,k+1} = u_k + w_k$. Here, $u_k \in \mathbf{R}$ and $w_k \in \mathbf{R}$ represent respectively control and disturbance inputs at time $k \in \mathbf{Z}_{\geq 0}$. The Lyapunov function utilized in the backstepping procedure was

$$V(x_1, x_2) = \frac{1}{2}|x_1| + \frac{3}{2}\left|x_2 - \frac{x_1}{2} + \sqrt[3]{x_1}\right|. \tag{50}$$

The aim is to determine the minimal asymptotic gain and transient bound for which the ISS$_{AG}$ property holds (from disturbance to state) for this closed loop system.

ISS Lyapunov characterizations: We present Lyapunov characterizations of ISS for two state space representations of the closed loop system (49). The state vector used in each case is respectively $x := [x_1\ x_2]^T$ and $\xi := [x_1\ y_2]^T$, where $y_2 := x_2 - \frac{x_1}{2} + \sqrt[3]{x_1}$. Using the Lyapunov function V given by (50), we find (in both characterizations) the functions α_1, α_2, α_3, $\sigma \in \mathcal{K}_\infty$ such that for all $x \in \mathbf{R}^2$ (equivalently $\xi \in \mathbf{R}^2$),

$$\begin{aligned} \alpha_1(|x|) &\leq V(x) \leq \alpha_2(|x|) \\ V(f(x,w)) - V(x) &\leq -\alpha_3(|x|) + \sigma(|w|). \end{aligned} \tag{51}$$

These characterizations yield two conservative estimates for the asymptotic gain. A further less conservative estimate is obtained from the second Lyapunov characterization, based on a bound for the asymptotic gain of another state space representation of the same system.

(i) Characterization 1: (State vector $x = [x_1\ x_2]^T$.) System (49) satisfies the Lyapunov characterization (51) of ISS, with Lyapunov function (50). In particular, (51) holds with

$$\begin{aligned} \alpha_1(s) &= \min\left(\tfrac{1}{10}s,\ \tfrac{1}{16}s^3\right), \\ \alpha_2(s) &= \tfrac{5\sqrt{2}}{4}s + \tfrac{3\sqrt[6]{2}}{2}\sqrt[3]{s}, \\ \alpha_3(s) &= \min\left(\tfrac{1}{20}s,\ \tfrac{1}{32}s^3\right), \\ \sigma(s) &= \tfrac{3}{2}s. \end{aligned} \tag{52}$$

(ii) Characterization 2: (State vector $\xi = [x_1\ y_2]^T$.) System (49) can be expressed in the coordinates $\xi := [x_1\ y_2]^T$, where $y_2 := x_2 - \frac{x_1}{2} + \sqrt[3]{x_1}$. Then, the closed loop dynamics are linear, with

$$\xi_{k+1} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \xi_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k. \tag{53}$$

Consequently, system (53) satisfies the Lyapunov characterization (51) of ISS (in the ξ coordinates) with Lyapunov function (50), $V(x_1, y_2) = \frac{1}{2}|x_1| + \frac{3}{2}|y_2|$, and

$$\begin{aligned} \alpha_1^{lin}(s) &= \tfrac{1}{4}s, \\ \alpha_2^{lin}(s) &= \tfrac{3}{\sqrt{2}}s, \\ \alpha_3^{lin}(s) &= \tfrac{1}{4}s, \\ \sigma^{lin}(s) &= \tfrac{3}{2}s. \end{aligned} \tag{54}$$

Applying results from [16], there exists κ_1,$\kappa_2 \in \mathcal{K}_\infty$ such that

$$\kappa_1(|x|) \ \le\ |\xi| \ \le\ \kappa_2(|x|). \tag{55}$$

In this case,

$$\begin{aligned} \kappa_1(s) &= \sqrt{\tilde{\kappa}_1\left(\tfrac{1}{\sqrt{2}}s\right)}, \\ \kappa_2(s) &= \sqrt{3}s + 2\sqrt[3]{s}, \end{aligned} \tag{56}$$

where

$$\tilde{\kappa}_1(s) = \min\left(\frac{1}{4}s^2,\ \left(\tilde{\varphi}_1^{-1}\left(\frac{1}{2}s\right)\right)^2\right)$$

and $\tilde{\varphi}_1(s) := s + \sqrt[3]{s}$. Note that

$$\tilde{\varphi}_1^{-1}(s) = s + \frac{2^{\frac{1}{3}}}{\left(27\,s+\sqrt{108+729\,s^2}\right)^{\frac{1}{3}}} - \frac{\left(27\,s+\sqrt{108+729\,s^2}\right)^{\frac{1}{3}}}{3\,(2)^{\frac{1}{3}}}.$$

Here, it can be shown that $\tilde{\varphi}_1^{-1} \in \mathcal{K}_\infty$. Combining (51), (54), (55) and (56) implies that the closed loop system (49) satisfies the Lyapunov characterization of ISS in the original x coordinates with bounds

$$\begin{aligned} \alpha_1(s) &= \alpha_1^{lin} \circ \kappa_1(s) &&= \tfrac{1}{4}\kappa_1(s), \\ \alpha_2(s) &= \alpha_2^{lin} \circ \kappa_2(s) &&= \tfrac{3}{\sqrt{2}}\kappa_2(s), \\ \alpha_3(s) &= \alpha_3^{lin} \circ \kappa_1(s) &&= \tfrac{1}{4}\kappa_1(s), \\ \sigma(s) &= \sigma^{lin}(s) &&= \tfrac{3}{2}s. \end{aligned} \tag{57}$$

__**ISS$_{AG}$ gains:**__ Applying results from [9], the Lyapunov characterization (51) of ISS implies that the ISS$_{\max}$ property holds with gain

$$\gamma_{\max}(s) \;=\; \alpha_1^{-1} \circ \hat{\gamma}(s), \tag{58}$$

where

$$\begin{array}{rcll} \hat{\gamma}(s) & = & \hat{\alpha}_4^{-1}(k\sigma(s)), & k > 1, \\ \hat{\alpha}_4(s) & \le & \alpha_4(s), & \mathrm{Id} - \hat{\alpha}_4 \in \mathcal{K}, \\ \alpha_4(s) & = & \alpha_3 \circ \alpha_2^{-1}(s). & \end{array} \tag{59}$$

Here Id refers to the identity map, $\mathrm{Id}(s) = s$. Where $\hat{\alpha}_4$ can be selected as α_4, (58) simplifies to

$$\gamma_{\max}(s) \;=\; \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(k\sigma(s)), \qquad k > 1, \tag{60}$$

as $\hat{\alpha}_4^{-1}(s) = \alpha_2 \circ \alpha_3^{-1}(s)$. As the ISS$_{\max}$ property implies the ISS$_{AG}$ property (with identical gains), (60) is an upper bound for the minimal ISS$_{AG}$ gain γ_∞. That is,

$$\gamma_\infty(s) \;\le\; \gamma_{AG}(s) \;:=\; \gamma_{\max}(s). \tag{61}$$

(i) Upper bound 1: By direct calculation using Characterization 1,

$$\alpha_2^{-1}(s) \;=\; \left(\frac{2\sqrt{2}}{5}\right) s - \frac{108\,(2)^{\frac{5}{6}}}{\left(-72900000\,s + \sqrt{2125764000000000 + 5314410000000000\,s^2}\right)^{\frac{1}{3}}} + \frac{\left(\;-72900000\,s + \sqrt{2125764000000000 + 5314410000000000\,s^2}\right)^{\frac{1}{3}}}{375\,(2)^{\frac{5}{6}}} \tag{62}$$

Applying (59) yields α_4, which can be shown to be a suitable candidate for $\hat{\alpha}_4$. Hence, (60) can be applied using (52), where

$$\begin{array}{rcl} \alpha_1^{-1}(s) & = & \max\left(10s,\; \sqrt[3]{16s}\right), \\ \alpha_3^{-1}(s) & = & \max\left(20s,\; \sqrt[3]{32s}\right). \end{array} \tag{63}$$

This yields a conservative ISS$_{\max}$ gain $\gamma_{\max,1}$ which provides an upper bound (61) for the minimal asymptotic gain γ_∞:

$$\gamma_\infty(s) \;\le\; \gamma_{\max,1}(s) \;=\; \max\left(2^{\frac{4}{3}}\left(\frac{3\max(30\,k\,s, 2\,(6k\,s)^{\;\frac{1}{3}})^{\frac{1}{3}}}{2^{\frac{5}{6}}} + \frac{5\;\max(30\,k\,s, 2\,(6k\,s)^{\frac{1}{3}})}{2\sqrt{2}}\right)^{\frac{1}{3}},\right.$$
$$\left. 10\left(\frac{3\max(30\,k\,s, 2\,(6k\,s)^{\;\frac{1}{3}})^{\frac{1}{3}}}{2^{\frac{5}{6}}} + \frac{5\;\max(30\,k\,s, 2\,(6k\,s)^{\frac{1}{3}})}{2\sqrt{2}}\right)\right), \tag{64}$$

where $k > 1$.

(ii) Upper bound 2: We repeat the calculation of (58) using the Characterization 2. First, note that by direct calculation,

$$\kappa_1^{-1}(s) \;=\; \sqrt{2}\tilde{\kappa}_1^{-1}(s^2) \;=\; 2\sqrt{2}\max\left(s,\; \varphi_1(s)\right) \;=\; 2\sqrt{2}\left(s + \sqrt[3]{s}\right),$$
$$\kappa_2^{-1}(s) \;=\; \frac{s}{\sqrt{3}} + \frac{8\,(2)^{\frac{1}{3}}}{\left(1944\sqrt{3}\,s + \sqrt{4478976\sqrt{3} + 11337408\,s^2}\right)^{\frac{1}{3}}} - \frac{\left(1944\sqrt{3}\,s + \sqrt{4478976\sqrt{3} + 11337408\,s^2}\right)^{\frac{1}{3}}}{9\,(2)^{\frac{1}{3}}\sqrt{3}} \tag{65}$$

Hence, using (57) and (65),

$$\begin{array}{rcl}\alpha_1^{-1}(s) = \alpha_3^{-1}(s) & = & \kappa_1^{-1}(4s) \;=\; 2\sqrt{2}\left(4s + \sqrt[3]{4s}\right), \\ \alpha_2^{-1}(s) = \kappa_2^{-1}\left(\frac{s\sqrt{2}}{3}\right). & & \end{array} \tag{66}$$

Applying (59) reveals that α_4 is a suitable candidate for $\hat{\alpha}_4$, so that (60) can be applied. So, (60) and (66) yield a conservative $\mathrm{ISS}_{\max}$ gain $\gamma_{\max,2}$ which provides an upper bound (61) for the minimal asymptotic gain γ_∞:

$$\begin{aligned}\gamma_\infty(s) \;\le\; \gamma_{\max,2}(s) \;=\; 4\Bigg(6\sqrt{2}\left(6\,k\,s + (6k\,s)^{\frac{1}{3}}\right)^{\frac{1}{3}} + 6\sqrt{6}\left(6\,k\,s + (6k\,s)^{\frac{1}{3}}\right)\Bigg)^{\frac{1}{3}} + \\ 24\left(2\sqrt{2}\left(6\,k\,s + (6k\,s)^{\frac{1}{3}}\right)^{\frac{1}{3}} + 2\sqrt{6}\left(6\,k\,s + (6k\,s)^{\frac{1}{3}}\right)\right)\end{aligned} \tag{67}$$

where $k > 1$.

(iii) Upper bound 3: Suppose that $V^\delta(\xi, k)$ is defined for system (53). Then, from (7) and (55),

$$V^\delta(x,k) \;\le\; \sup_{\|u\|_\infty \le \delta} \left\{\kappa_1^{-1}\left(|\xi(k,\xi_0,u)|\right) : \xi_0 = \xi\right\} \;=\; \kappa_1^{-1}\left(V^\delta(\xi,k)\right). \tag{68}$$

The minimal asymptotic gain γ_∞ for system (49) is then

$$\gamma_\infty(\delta) \;=\; \sup_{x\in\mathbf{R}^2} \limsup_{k\to\infty} V^\delta(x,k) \;\le\; \kappa_1^{-1}\left(\sup_{\xi\in\mathbf{R}^2} \limsup_{k\to\infty} V^\delta(\xi,k)\right) \;=\; \kappa_1^{-1}\circ\gamma_\infty^{lin}(\delta), \tag{69}$$

where γ_∞^{lin} is the minimal asymptotic gain for system (53). It can be shown that a candidate asymptotic gain γ_{AG}^{lin} for system (53) is $\gamma_{AG}^{lin}(s) = 5s$. Hence, an upper bound (69) for minimal asymptotic gain γ_∞ is

$$\gamma_\infty(s) \;\le\; \kappa_1^{-1}(5s). \tag{70}$$

(iv) Lower bound 1: A lower bound γ_- for the minimal ISS_{AG} gain γ_∞ follows from (11). In particular, for any $\tilde{u}$ satisfying $\|\tilde{u}\|_\infty \le s$, we can define

$$\gamma_-(s) := \sup_{x\in\mathbf{R}^n} \limsup_{k\to\infty} \left\{|x(k,x_0,\tilde{u})| : x_0 = x\right\} \;\le\; \gamma_\infty(s). \tag{71}$$

For this example, $\tilde{u}$ was chosen (arbitrarily) to be a square wave of amplitude s and period 10 samples.

(v) Minimal asymptotic gain via dynamic programming: An approximation to the minimal asymptotic gain $\gamma_\infty(\delta)$ for the nonlinear system (49) was computed over three overlapping intervals and combined. These intervals and the corresponding parameters as per (40) are as follows:

$$\delta \in [0.00, 0.05] \;:\; \begin{array}{rclrclrcl} N_\Delta & = & 11, & \delta_{\min} & = & 0.00, & \delta_{\max} & = & 0.05, \\ N_{X_1} & = & 201, & x_{1\max} & = & 2.00, & & & \\ N_{X_2} & = & 161, & x_{2\max} & = & 0.80, & & & \\ N_U & = & 21, & & & & & & \end{array}$$

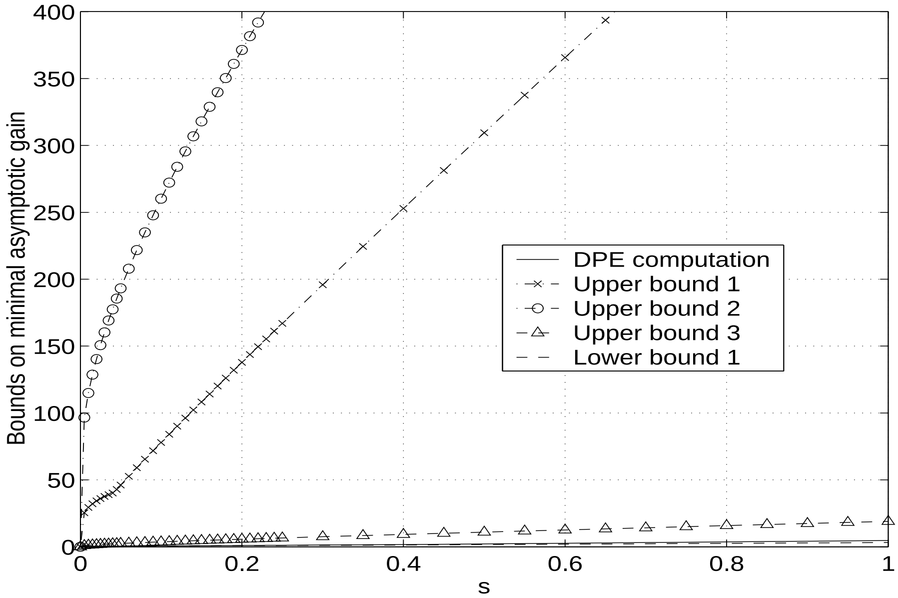

Figure 13: Comparison of γ_∞ obtained by dynamic programming, upper bounds 1-3, and lower bound 1 (Example 4).

$$\delta \in [0.05, 0.25] : \quad \begin{aligned} N_\Delta &= 21, & \delta_{\min} &= 0.05, & \delta_{\max} &= 0.25, \\ N_{X_1} &= 251, & x_{1\max} &= 2.50, \\ N_{X_2} &= 201, & x_{2\max} &= 1.00, \\ N_U &= 21, \end{aligned}$$

$$\delta \in [0.25, 1.00] : \quad \begin{aligned} N_\Delta &= 16, & \delta_{\min} &= 0.25, & \delta_{\max} &= 1.00, \\ N_{X_1} &= 33, & x_{1\max} &= 8.00, \\ N_{X_2} &= 21, & x_{2\max} &= 5.00, \\ N_U &= 21. \end{aligned}$$

Figures 13 and 14 clearly demonstrate that asymptotic gains obtained from the Lyapunov characterization of ISS can be very conservative when transformed to other characterizations. This highlights a distinct advantage of the dynamic programming approach presented, particularly in (for example) small gain applications.

(vi) Minimal transient bound for ISS$_{AG}$: The computation outlined in (v) above also enables approximation of the minimal transient bound β_a for which the ISS$_{AG}$ property holds. This approximation is illustrated in Figure 15, which show qualitatively that $\beta_a \in \bar{\mathcal{K}}\bar{\mathcal{L}}$. Theorem 4.1 then implies that system (49) satisfies the ISS$_{AG}$ property with (β_a, γ_∞).

9 Conclusions

We have presented results for verifying different characterizations of ISS via dynamic programming. Formulas for minimum nonlinear gains and bounds on transients for different

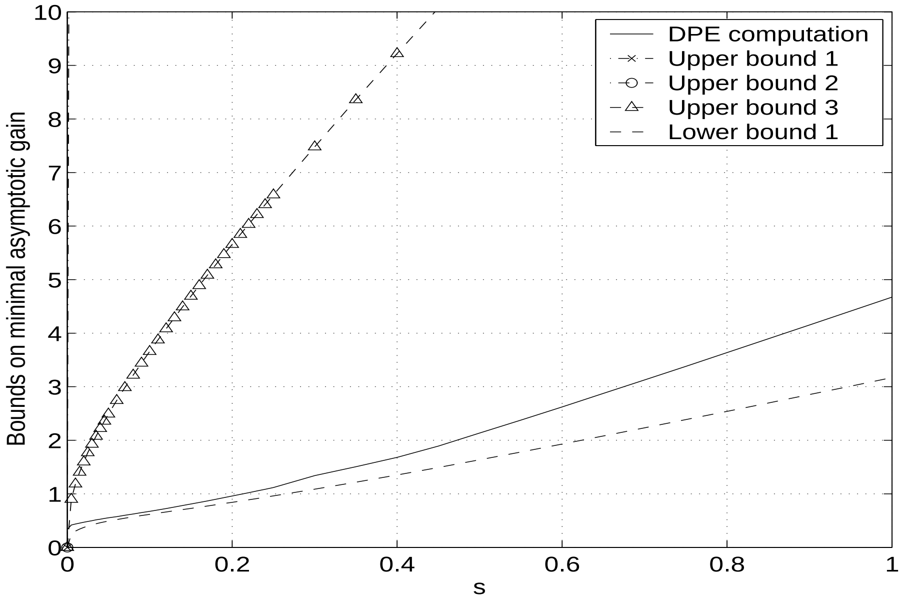

Figure 14: Enlargement of Figure 13 illustrating the DPE computation and lower bound (Example 4).

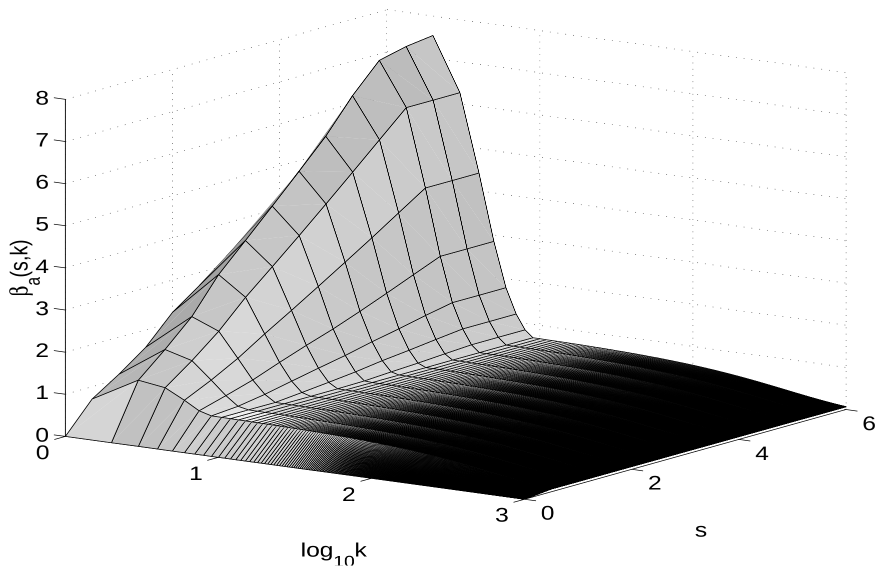

Figure 15: Approximation of β_a (Example 4).

characterizations are presented. A discussion on how these results can be used to analyse input-to-output stability and incremental input-to-state stablity is also given. We illustrated our approach by four examples.

The aim of this paper is to present a constructive formulation for finding minimal ISS gains and transient bounds. The results of this paper provide a framework for generating numerical algorithms for calculation calculating ISS gains and transient bounds. These numerical algorithms will require extra analysis since they are inherently local in nature (in the states, the inputs, and the time) as opposed to the global results that we presented. This is an important direction for our future research and we believe it is outside the scope of this paper. Our example indicate, however, potential benefits of this numerical approach and the motivate careful investigation of numerical issues.

Acknowledgements. We wish to thank Yuan Wang and Lars Grüne for indicating to us several useful references. Especially, Yuan Wang provided us the continuous-time version of Example 1 in this paper.

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