On Global Extremum Seeking In The Presence Of Local Extrema*

Y. Tan, D. Nešić, I.M.Y.Mareels†, and Alessandro Astolfi‡

Abstract

We analyze global extremum seeking in the presence of local extrema for a simple scalar extremum seeking feedback scheme. Sufficient conditions are given under which it is possible to tune the controller parameters to achieve convergence to an arbitrarily small neighborhood of the global extremum from an arbitrarily large set of initial conditions. These sufficient conditions are shown to hold always when the output equilibrium map is a 4^{th} order polynomial. However, when such a map becomes a 6^{th} order polynomial, we present an example that invalidates these conditions. On the other hand, extensive computations show that most 6^{th} order polynomials and many other functions satisfy all our conditions. Several examples provide insights and highlight the potential difficulties that one would face when trying to generalize our results.

1 Introduction

The main goal in extremum seeking (ES) control is to find an extremum value of an unknown nonlinear mapping. This is an old method but the first rigorous local stability analysis for a class of ES schemes was provided recently in (Ariyur and Krstić, 2003) and later extended to semi-global stability analysis in (Tan *et al.*, 2005). Stability of a different class of ES controllers was recently presented in (Popović *et al.*, 2003). There has been a renewed interest in this research area (Teel and Popović, 2001; Guay and Zhang, 2003; Popović *et al.*, 2003; Guay *et al.*, 2004; Peterson and Stefanopoulou, 2004) that lead to numerous practical implementations of the scheme, as well as its better theoretical understanding.

While it has been often demonstrated that ES controllers work well in simulations, experiment or real applications, a full understanding of their convergence properties and robustness is still lacking. Global extremum seeking in absence of local extrema for a class of extremum schemes was rigorously analyzed in (Tan *et al.*, 2005; Tan *et al.*, 2006*a*). On the other hand, it

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†Y. Tan, D. Nešić and I.M.Y. Mareels are with The Department of Electrical and Electronics, The University of Melbourne, Parkville, VIC 3010, Y. Tan's e-mail is `y.tan@ee.unimelb.edu.au`, D. Nesic's e-mail is `d.nesic@ee.unimelb.edu.au` and I.Mareels's e-mail is `i.mareels@ee.unimelb.edu.au`

‡A. Astolfi is with Department of Electrical and Electronic Engineering, Imperial College London United Kingdom, Phone: `+44 20 7594 6289`, Fax: `+44 20 7594 6282`, E-mail: `a.astolfi@ic.ac.uk` and Dipartimento di Informatica, Sistemi e Produzione, Universita' di Roma Tor Vergata, Via del Politecnico 1, 00133 Roma Italy

was often observed by the users of extremum seeking controllers that by tuning the amplitude of the excitation (dither) signal properly, it is possible to pass through a local extremum and converge to the global one. In other words, global extremum seeking is possible in the presence of local extrema in certain situations. However, rigorous analysis of this problem appears to be lacking in the literature.

It is the purpose of this paper to present the first analysis of global extremum seeking in the presence of local extrema for the scalar ES feedback scheme proposed in (Tan *et al.*, 2006*a*). We show that the extremum to which the ES mechanism converges depends on the averaged system of the "reduced system". This averaged system is closely related to the output equilibrium map. When there are extrema in such a map, the averaged system exhibits the bifurcation phenomenon as the amplitude of the excitation varies. On the basis of such a bifurcation diagram, we present a set of sufficient conditions under which the proposed ES scheme yields global extremum seeking despite local extrema. Nevertheless, in general, the problem is quite hard and we illustrate this by several examples. Our conditions are shown to hold always when the output equilibrium map is a 4^{th} order polynomial whereas we present example of a 6^{th} order polynomial that does not satisfy these conditions. On the other hand, extensive computations show that most 6^{th} order polynomials, many higher order polynomials and more general functions satisfy all our conditions. Furthermore, when our conditions hold, our main result outlines a tuning strategy for the ES controller that yields convergence to an arbitrarily small neighborhood of the global extremum from an arbitrarily large set of initial conditions. Hence, we believe that our results will be useful to the users of extremum seeking control and, moreover, they may motivate further research into this challenging area.

The paper is organized as follows. In Section 2 we present preliminaries and problem formulation. Main result is stated in Section 3. Discussions and examples are provided in Section 4. Summary is given in Section 5. Proofs are presented in the Appendix.

2 Preliminaries and Problem Formulation

The set of real numbers is denoted as R, the set of complex number is denoted as C and the set of integers is denoted as N. The continuous function $\beta : R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is of class $\mathcal{KL}$ if it is nondecreasing in its first argument and strictly decreasing to zero in its second argument. For a nonlinear smooth function $h : R^n \to R$ we denote $D_i^j h := \frac{\partial^j h}{\partial x_i^j}$ where $i \in \{1, 2, \ldots, n\}$ and $j \in N$. When $j = 1$ or $i = 1$, we often omit this argument, e.g. we write $D_i h := D_i^1 h$.

Consider the following single-input-single-output (SISO) nonlinear dynamic system

$$\dot{\mathbf{x}} = f(\mathbf{x}, u), \qquad y = h(\mathbf{x}), \tag{1}$$

where $\mathbf{f} : R^n \times R \to R^n$ and $h : R^n \to R$ are continuously differentiable[1]. $\mathbf{x}$ is the measurable state, u is the input and y is the output. Consider a family of control laws of the following form:

$$u = \alpha(\mathbf{x}, \theta), \tag{2}$$

[1] In the sequel, all functions are assumed to be sufficiently smooth (all derivatives that we need are continuous).

where $\theta \in R$ is a scalar parameter. The closed-loop system (1) with (2) is then

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \alpha(\mathbf{x}, \theta)). \tag{3}$$

In this paper, we consider the scalar extremum seeking scheme shown in Figure 1 that was first introduced in (Tan *et al.*, 2006*a*). The excitation signal $a\sin(t)$ is added to the dynamic system to get probing while the multiplication (modulation) of output and the excitation signal ($sin(t)$) extracts the gradient of the unknown mapping $h(\cdot)$. The dynamics of the ES

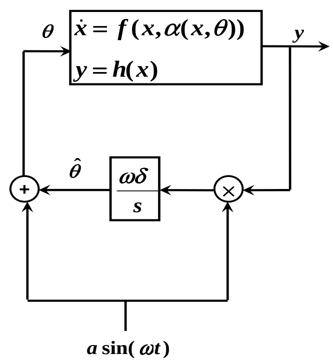

Figure 1: The scalar extremum seeking feedback scheme

system in Figure 1 can be written as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \alpha(\mathbf{x}, \hat{\theta} + a \cdot \sin(\omega \cdot t))) \\ \dot{\hat{\theta}} &= \omega \cdot \delta \cdot h(\mathbf{x}) \cdot \sin(\omega \cdot t) \end{aligned} \tag{4}$$

where a, δ and ω are positive design parameters.

The following assumptions are made for nonlinear dynamic system (3).

Assumption 1 *There exists a smooth function* $\mathbf{l} : R \to R^n$ *such that*

$$\mathbf{f}(\mathbf{x}, \alpha(\mathbf{x}, \theta)) = 0, \quad if \ \ and \ \ only \ \ if \ \ \mathbf{x} = \mathbf{l}(\theta). \tag{5}$$

Assumption 2 *For each* $\theta \in R$, *the equilibrium* $\mathbf{x} = \mathbf{l}(\theta)$ *of the system (3) is globally asymptotically stable, uniformly in* θ.

Remark 1 *Assumption 1 and Assumption 2 are the same as (Tan* et al.*, 2006*a*, Assumption 1 and 2). Assumption 2 is a stronger version of (Krstić and Wang, 2000, Assumption 2.2) in order to prove a non-local stability result.*

Assumption 3 *Denoting $Q(\cdot) = h \circ \mathbf{l}(\cdot)$, there exists a unique global maximum $\zeta^* \in R$ of $Q(\cdot)$ such that*[2]

$$
\begin{aligned}
DQ(\zeta^*) &= 0; \qquad\qquad D^2Q(\zeta^*) < 0\ , && (6)\\
Q(\zeta^*) &> Q(\zeta) \qquad\quad \forall \zeta \in R,\ \zeta \neq \zeta^* && (7)
\end{aligned}
$$

Note that (6) and (7) in Assumption 3 indicate that x^* is the unique global maximum of the nonlinear mapping $h(\cdot)$. Other than this global maximum, there may also exist local maxima that satisfy (6). This assumption is weaker than that in (Tan *et al.*, 2006*a*, Assumption 3), in which $x^* \in R$ is the unique maximum which satisfies (6).

In this paper, we discuss sufficient conditions to ensure that the global maximum would be found if the amplitude of the excitation signal is tuned adaptively. To this end, we introduce the following law for adaptation of the amplitude of the excitation signal:

$$
\dot{a} = -\delta \cdot \epsilon \cdot g(a), \quad a(0) = a_0 > 0, \tag{8}
$$

where $g(\cdot)$ is a locally Lipschitz function that is zero at zero and positive otherwise and the strictly positive parameters ϵ, δ and a_0 are to be chosen by the designer. The simplest choice is $g(a) = a$.

3 Main results

The proposed extremum seeking feedback scheme is shown in Figure 2, whose closed-loop can be written as

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \alpha(\mathbf{x}, \hat{\theta} + a \cdot \sin(\omega \cdot t)))\\
\dot{\hat{\theta}} &= \omega \cdot \delta \cdot h(\mathbf{x}) \cdot \sin(\omega \cdot t)\\
\dot{a} &= -\omega \cdot \epsilon \cdot \delta \cdot g(a), \qquad\qquad a(0) = a_0.
\end{aligned} \tag{9}
$$

Denoting $\sigma = \omega \cdot t$, the system equations (9) expanded in time σ are:

$$
\begin{aligned}
\omega \cdot \frac{d\mathbf{x}}{d\sigma} &= \mathbf{f}(\mathbf{x}, \alpha(\mathbf{x}, \hat{\theta} + a \sin(\sigma)))\\
\frac{d\hat{\theta}}{d\sigma} &= \delta \cdot h(\mathbf{x}) \cdot \sin(\sigma)\\
\frac{da}{d\sigma} &= -\epsilon \cdot \delta \cdot g(a), \qquad a(0) = a_0.
\end{aligned} \tag{10}
$$

The system (10) is in the standard singular perturbation form, where the singular perturbation parameter is ω. The stability properties of (10) therefore depend on the stability properties of its "boundary layer" system and "reduced system".

In this section, we first discuss the stability properties of the "reduced system" by showing the semi-global practical asymptotical convergence to the global extremum if the averaged

[2] In this paper we assume that $Q(\cdot)$ possesses a global maximum but we can deal with functions $Q_1(\cdot)$ with a global minimum by defining $Q(\cdot) := -Q_1(\cdot)$ and applying our results unchanged.

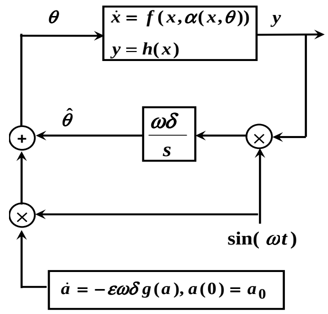

Figure 2: The proposed extremum seeking feedback scheme

system of the "reduced system" satisfies some sufficient condition (Theorem 1). Next, by showing the appropriate stability properties of the "boundary layer" system, we can show that output of the overall system (9) semi-globally practically asymptotically converges to the global extremum (Theorem 2) despite the presence of local extrema (cf. Assumption 3).

3.1 The stability of the "reduced system"

To obtain the fast and slow systems, we set $\omega = 0$ and "freeze" $\mathbf{x}$ at its "equilibrium", $\tilde{\mathbf{x}} = \mathbf{l}(\hat{\theta} + a \cdot \sin(\sigma))$ to obtain the reduced system in variables (θ_r, a_r) in the time scale $\sigma = \omega t$:

$$\frac{d\theta_r}{d\sigma} = \delta \cdot Q(\theta_r + a \cdot \sin(\sigma)) \cdot \sin(\sigma) = \delta \cdot \mu(\sigma, \theta_r, a_r) \tag{11}$$

$$\frac{da_r}{d\sigma} = -\epsilon \cdot \delta \cdot g(a_r), \qquad a_r(0) = a_0. \tag{12}$$

where $Q(\cdot) = h \circ \mathbf{l}(\cdot)$ is the output equilibrium map. To simplify the notation, we use the following second order system to represent the "reduced system" (11) and (12):

$$\dot{\zeta} = \delta \cdot Q(\zeta + \rho \cdot \sin t) \cdot \sin t = \delta \cdot \mu(t, \zeta, \rho) \tag{13}$$

$$\dot{\rho} = \epsilon \cdot \delta \cdot g(\rho), \qquad \rho(0) = \rho_0 \tag{14}$$

For the system (13), we can write its averaged system by using:

$$\mu_{av}(\zeta, \rho) := \frac{1}{2\pi} \int_0^{2\pi} \mu(t, \zeta, \rho) dt \tag{15}$$

where $\mu(\cdot, \cdot, \cdot)$ comes from (13). Indeed, using the above definition, we can analyze the closed loop system (13), (14) via the following auxiliary averaged system:

$$\dot{\zeta} = \delta \cdot \mu_{av}(\zeta, \rho) \tag{16}$$

$$\dot{\rho} = -\delta \cdot \epsilon \cdot g(\rho), \qquad \rho(0) = \rho_0 > 0 \; . \tag{17}$$

Remark 2 *By introducing the new time $\tau := \epsilon\delta t$, we can rewrite the above equations as follows:*

$$\begin{aligned} \epsilon \cdot \frac{d\zeta}{d\tau} &= \mu_{av}(\zeta, \rho) \\ \frac{d\rho}{d\tau} &= -g(\rho), \qquad \rho(0) = \rho_0 > 0 \ , \end{aligned} \tag{18}$$

which exhibits time scale separation and appears to be in standard singular perturbation form (KHalil, 2002, Section 9.1). However, there are three reasons why we do not use the standard singular perturbation techniques here. First, in our case the equation:

$$0 = \mu_{av}(\zeta, \rho) \tag{19}$$

may not have k isolated *real roots $\zeta = \ell_i(\rho)$, which is required of the standard form. Indeed, some of the real roots may only be defined for $\rho \in [0, \bar{\rho}]$ and such that for some i and j we have $\ell_i(\rho) \neq \ell_j(\rho), \rho \in [0, \bar{\rho})$ and $\ell_i(\bar{\rho}) = \ell_j(\bar{\rho})$ (see Example 4). Moreover, it will be shown in the Appendix that there exists a continuous function $p(\zeta, \rho)$ such that we can write:*

$$\mu_{av}(\zeta, \rho) = \rho \cdot p(\zeta, \rho) \tag{20}$$

and this means that we will be unable to prove stability of the boundary layer system uniform in *ρ that is a standard assumption in the singular perturbation literature. Finally, we are interested in convergence properties of this system that is initialized from a set of initial conditions satisfying $\rho(0) = \rho_0$ which is a weaker property from the standard stability properties considered in the singular perturbation literature. Hence, we will state and prove results directly without appealing to the rich literature on singular perturbations.*

Before we state our main results, we state another assumption that characterizes solutions of the equation (19).

Assumption 4 *There exists an isolated real root $\zeta = \ell(\rho) : R_{\geq 0} \to R$ of the equation (19) with the following properties:*

1. *ℓ is continuous and $D_1 p(\ell(\rho), \rho) < 0$, $\forall \rho \geq 0$, where $p(\zeta, \rho)$ is defined in (20).*
2. *There exists $\rho^* > 0$, such that for all $\rho \geq \rho^*$, $\zeta = \ell(\rho)$ is the unique real root of (19).*
3. *$\ell(0) = \zeta^*$, where ζ^* is the global extremum, which comes from Assumption 3.*

Remark 3 *Note that if the output equilibrium map $Q(\cdot)$ was a known mapping, it would be very easy to check conditions of Assumption 4. Indeed, one needs to plot the bifurcation diagram using (19) that shows how equilibria of the ζ subsystem of the average system (16) change as we vary the amplitude of the excitation signal ρ and verify the conditions by inspecting plot. However, since the standing assumption in extremum seeking control is that $Q(\cdot)$ is* unknown, *then having results that guarantee conditions of Assumption 4 for* classes of functions *$Q(\cdot)$ is more useful to the users of extremum seeking control (as in this case we do not need to plot the bifurcation diagram). Indeed, it may be known that $Q(\cdot)$ belongs to a certain class of functions but its exact description may be unknown. Conditions in Assumption 4 are hard to check in general for large classes of functions. Form extensive simulations, we found that many general functions satisfy all conditions of Assumption 4. Note that our results will apply to any system for which Assumptions 3 and 4 hold and, hence, they are general.*

Our first main result states the stability properties of closed loop system (13), (14).

Theorem 1 *Suppose that Assumptions 3 and 4 hold. Then, for any strictly positive* (Δ, ν) *and* $\rho_0 > \rho^*$ *there exist* $\beta = \beta_{\rho_0,\Delta,\nu} \in \mathcal{KL}$ *and* $\epsilon^* = \epsilon^*(\rho_0, \Delta, \nu) > 0$ *and for any* $\epsilon \in (0, \epsilon^*)$ *there exists* $\delta^* = \delta^*(\epsilon) > 0$ *such that for any such* ρ_0, ϵ *and* $\delta \in (0, \delta^*)$ *we have that for all* $(\zeta(t_0), \rho(t_0))$ *satisfying* $\rho(t_0) = \rho_0$ *and* $|\zeta(t_0) - \ell(\rho_0)| \leq \Delta$ *and all* $t \geq t_0 \geq 0$ *the solutions of the system (13), (14) satisfy:*

$$|\zeta(t) - \ell(\rho(t))| \leq \beta(|\zeta(t_0) - \ell(\rho(t_0))|, \delta \cdot (t - t_0)) + \nu \ . \tag{21}$$

Remark 4 *A consequence of Theorem 1 is that we can tune the extremum seeking controller to achieve:*

$$\limsup_{t\to\infty} |\zeta(t) - \ell(\rho(t))| \leq \nu$$

from an arbitrarily large set of initial conditions and for arbitrarily small $\nu > 0$*. Moreover, from (14) it is obvious that there exists* $\beta_\rho \in \mathcal{KL}$ *with* $\beta_\rho(s, 0) = s$*, such that for all* $\rho(t_0) = \rho_0 \in R$ *we have:*

$$|\rho(t)| \leq \beta_\rho(\rho(t_0), \epsilon \cdot \delta \cdot (t - t_0)), \ \forall t \geq t_0 \geq 0 \ , \tag{22}$$

and since $\ell(\cdot)$ *is continuous and* $\ell(0) = \zeta^*$*, this implies*

$$\lim_{t\to\infty} \ell(\rho(t)) = \zeta^* \ .$$

Hence, we can conclude that

$$\limsup_{t\to\infty} |\zeta(t) - \zeta^*| \leq \nu \ ,$$

which implies semi-global practical extremum seeking since ζ^* *is the global extremum of* $Q(\cdot)$*. We believe that this is the first rigorous result of this kind in the literature.*

Remark 5 *The averaged model (18) suggest a two-time-scale dynamics when* ϵ *is very small. This is indeed the case, as can be seen from (21) and (22). Indeed, the solutions first converge with the rate proportional to* δ *to a small neighborhood of the set* $\mathcal{L} := \{(\zeta, \rho) : \zeta - \ell(\rho) = 0\}$ *(fast transient given by (21)) and then with the speed proportional to* $\epsilon\delta$ *to a neighborhood of the point* $(\zeta, \rho) = (\zeta^*, 0)$ *(slow transient given by (22)). Moreover, during the slow transient, the solutions stay in the* ν*-neighborhood of the set* $\mathcal{L}$*.*

Remark 6 *The ES mechanism in Theorem 1 is more general than the ES mechanism used in the main results of (Tan* et al.*, 2006*a*) where local extrema were not allowed in the analysis (compare Figure 1 and 2). The main difference between the two mechanisms is that the amplitude of the excitation signal in Theorem 1 is time varying, whereas in main results of (Tan* et al.*, 2006*a*) the amplitude is* fixed.

Remark 7 *The stability results of Theorem 1 is different from the stability result in (Tan* et al.*, 2006*a*, Theorem 1), where the "reduced system" is proved to be SPA stable, uniformly in tuning parameter: the amplitude of the sine wave dither and* δ*. First of all, the initial value of the amplitude of the sine wave dither plays an important role in the proposed ES feedback* *scheme to guarantee that the output of the "reduced system" converges to the global extremum semi-globally practically asymptotically. Such an initial value ρ_0 is also a tuning parameter in the proposed ES feedback scheme. Hence, the system (13) and (14) cannot be written in the form of a parameterized system as follows*

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(t, \mathbf{x}, \boldsymbol{\epsilon}),$$

which is independent of the initial condition. Therefore, we cannot get the similar uniform SPA stability properties as in (Tan et al.*, 2006*a*). Secondly, ρ_0 does affect the convergence speed as β is dependent on the choice of ρ_0, though analyzing how ρ_0 affects the convergence speed is much more difficult than other parameters. Example 1 shows how the choice of the ρ_0 affect the convergence speed of the system. Thirdly, Theorem 1 clearly indicates that, the choice of ϵ^* depends on the choice of ρ_0 and the choice of δ^* depends on the choice of ϵ. This implies that the SPA stability properties are not uniform for tuning parameters $(\rho_0, \epsilon, \delta)$.*

Remark 8 *Theorem 1 presents a tuning mechanism for the controller parameters (choice of ϵ, δ) and its initialization (choice of ρ_0) that guarantees semi-global practical convergence to the global extremum despite the presence of local extrema. Simulations in our examples illustrate that such convergence is indeed achieved.*

We note that since the static mapping $Q(\cdot)$ is not known, it is in general not possible to check a priori *whether Assumption 4 holds, let alone analytically compute the values of ρ^*, ϵ^* and δ^*. However, our result suggests that if there is some evidence that Assumption 3 and 4 may hold, then increasing sufficiently ρ_0 and reducing sufficiently ϵ and δ will indeed result in global convergence. In practice, determining how large ρ_0, ϵ, δ should be, may have to be determined through experimenting.*

Remark 9 *Note that from the robustness point of view, it is not desirable to reduce the amplitude of the excitation signal to zero since small perturbations may force the solutions to diverge far from the global extremum. Our results can be restated for the case when we have that $\lim_{t\to\infty} \rho(t) = \underline{\rho} > 0$ but they are omitted for reasons of brevity.*

Remark 10 *We note that Assumption 4 imposes conditions on the bifurcation diagram that is defined by the equation (19). The bifurcation diagram is a real algebraic variety in the case when μ_{av} is a polynomial or a more general set for general functions μ_{av}. The idea behind this condition is to ensure that for large amplitudes ρ the static map (19) has a unique equilibrium globally stable, given by $\zeta = \ell(\rho)$. Moreover, in order for our proposed control strategy to be successful we require this branch to be continuous and connected to the global maximum, i.e. $\ell(0) = \zeta^*$. We will discuss about the Assumption 4 in details in Section 4.*

Remark 11 *Suppose that there is no plant dynamics and the reference to output map is static $y = h(x)$, as in (Meerkov, 1967*a*; Meerkov, 1967*b*; Meerkov, 1967*c*; Morosanov, 1957; Ostrovskii, 1957). In this case, the extremum seeking scheme, as shown in Figure 3, becomes (13) and (14). Note that in this case we do not need the parameter ω and $Q = h$. Theorem 1 implies that the output of the static ES system semi-globally practically asymptotically converges to the global extremum ζ^* if the parameters ρ_0, ϵ, δ are tuned properly.*

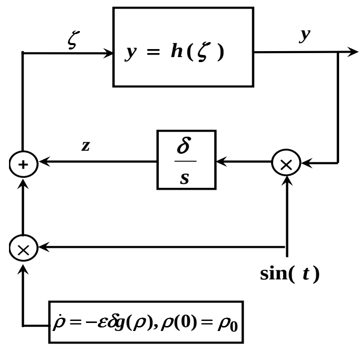

Figure 3: The extremum seeking controller

3.2 The stability properties of the overall system

With the SPA stability properties of the "reduced system" (11),(12), the stability of the overall system (4) is stated in the following theorem.

Theorem 2 *Suppose that Assumptions 1, 2 3 and 4 hold. Then, for any strictly positive* (Δ, ν) *and* $a_0 > a^*$ *there exist Class-*$\mathcal{KL}$ *functions* $\beta_{\mathbf{x}} = \beta_{\Delta,\nu}$, $\beta_\theta = \beta_{a_0,\Delta,\nu}$, β_a *and* $\epsilon^* = \epsilon^*(a_0, \Delta, \nu) > 0$ *and for any* $\epsilon \in (0, \epsilon^*)$ *there exists* $\delta^* = \delta^*(\epsilon) > 0$ *such that for any* $\delta \in (0, \delta^*(\epsilon)$ *there exists* $\omega^* = \omega^*(\delta) > 0$ *such that for any such* a_0, ϵ, $\delta \in (0, \delta^*)$ *and* $\omega \in (0, \omega^*)$, *we have that for all* $(x(t_0), \hat{\theta}(t_0), a(t_0))$ *satisfying* $a(t_0) = a_0$, $|\hat{\theta}(t_0) - \ell(a_0)| \leq \Delta$, $|\mathbf{x}(t_0) - \mathbf{l}(\hat{\theta}(t_0))| \leq \Delta$ *and all* $t \geq t_0 \geq 0$ *the solutions of the system (9) satisfy:*

$$|\mathbf{x}(t) - \mathbf{l}(\hat{\theta}(t))| \leq \beta_{\mathbf{x}}\left(|\mathbf{x}(t_0) - \mathbf{l}(\hat{\theta}(t_0))|, (t - t_0)\right) + \nu, \tag{23}$$

$$|\hat{\theta}(t) - \ell(a(t))| \leq \beta_\theta\left(|\hat{\theta}(t_0) - \ell(a(t_0))|, \omega \cdot \delta \cdot (t - t_0)\right) + \nu, \tag{24}$$

$$|a(t)| \leq \beta_a\left(|a(t_0)|, \omega \cdot \epsilon \cdot \delta \cdot (t - t_0)\right). \tag{25}$$

Remark 12 *From Remark 4, (24) indicates that, given any positive triple* (a_0, Δ, ν), *where* $a_0 > a^*$, *there exists appropriate parameters* ϵ, δ, ω *and* $\nu_1 > 0$ *such that*

$$\limsup_{t\to\infty} |\hat{\theta}(t) - \theta^*| \leq \nu_1 \tag{26}$$

$$|\hat{\theta}(t) - \theta^*| \leq \nu_1 \Longrightarrow |Q(\hat{\theta}(t)) - Q(\theta^*)| = |y(t) - y^*| \leq \nu\ . \tag{27}$$

since $Q = h \circ \mathbf{l}$ *is a sufficiently smooth function. Theorem 2 can be interpreted as follows. For any* (a_0, Δ, ν), *where* $a_0 > a^*$, *we can adjust* ϵ, δ *and* ω *appropriately so that for all* $|\mathbf{z}(t_0)| \leq \Delta$, *where* $\mathbf{z} \triangleq \begin{bmatrix} \mathbf{x} - \mathbf{l}(\hat{\theta}) \\ \hat{\theta} - \ell(a) \end{bmatrix}$, *we have that* $\limsup_{t\to\infty} |y(t) - y^*| \leq \nu$. *In other words, the output of the system can be regulated arbitrarily close to the global extremum value* y^* *from an arbitrarily large set of initial conditions by adjusting the parameters* $(\omega, \delta, \epsilon, a_0)$ *in the controller.*

4 Discussions and Examples

In this part, discussions and examples are provided. It is worthwhile to note that Assumption 4 is crucial to prove the global convergence of the proposed ES feedback scheme. From the result of Theorem 2, $\hat{\theta}$ converges to $\ell(0)$. If $\ell(0)$ is not the global extremum, the output of the overall system (4) would converge to a local maximum. Therefore, we explore the situations when Assumption 4 holds by providing discussions and examples. These examples illustrate and highlight the important issues and provide intuition that the users of the extremum seeking control will find useful.

We will show that Assumption 4 always holds under the Assumption (3) when the output equilibrium map Q is a 2^{nd} or a 4^{th} order polynomial. (see Proposition 1). We also present an example of a 6^{th} order polynomial where this does not hold (Example 4). Moreover, after extensive plotting of bifurcation diagrams we observed that for 6^{th} order polynomials all conditions in Assumption 4 hold most of the time (i.e. counterexamples are hard to construct). Similarly, these conditions also hold for many higher order polynomials or general functions $h(\cdot)$, which indicates that our results are quite general.

First, we present an example when the output equilibrium map $Q(\cdot)$ in (13) is a 2^{nd} polynomial that satisfies all conditions of Assumptions 3 and 4 and to which our main result in Theorem 1 applies. We also show in this example that the initial condition a_0 does affect the convergence speed of the system (4).

Example 1 *Consider the following dynamic system:*

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= x_2 + u \\ y &= h(\mathbf{x}) \end{aligned} \tag{28}$$

where $h(\mathbf{x}) = -(x_1 + 3x_2)^2 + 2(x_1 + 3x_2) + 1$*. The control input is chosen as*

$$u = -x_1 - 4x_2 + \theta, \tag{29}$$

which ensures that Assumption 1 and 2 are satisfied. By choosing $g(a) = a$*, the closed loop of the extremum seeking feedback scheme is thus*

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\hat{\theta} + a\sin(\omega \cdot t)\right) \\ \dot{\hat{\theta}} &= \omega \cdot \delta \cdot h(\mathbf{x}) \cdot \sin(\omega \cdot t) \\ \dot{a} &= -\omega \cdot \delta \cdot \epsilon \cdot a \quad\quad a(0) = a_0 \end{aligned} \tag{30}$$

Obviously, $Q(\theta) = -\theta^2 + 2\theta + 1$*, which has a unique maximum at* $\theta^* = 1$*.* $\ell(a) = 1$ *is a constant and* $a^* = 0$*. The simulation result is shown in Figure 4, when* $a_0 = 5$*,* $\omega = 0.1$*,* $\delta = 0.1$*,* $\epsilon = 1$ *and* $(x_1(0), x_2(0), \hat{\theta}(0)) = (0, 0, -5)$*. In Figure 4,* $\theta_h = \hat{\theta}$*.*

Let $a_0 = 2$*, while keeping other parameters* $(\omega, \delta, \epsilon, x_1(0), x_2(0), \hat{\theta}(0))$ *the same as above, the simulation result is shown in Figure 5. The different choice of the initial condition* a_0 *does affect the convergence speed of* $\hat{\theta}$ *(different* β *function).*

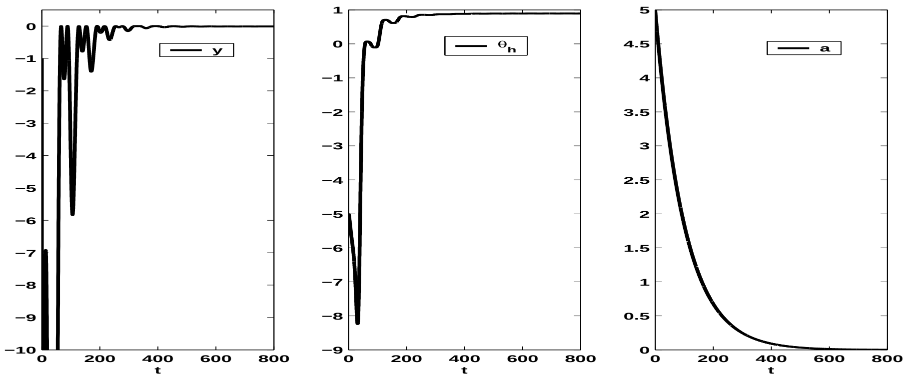

Figure 4: The output of the proposed extremum seeking feedback scheme, initial condition b

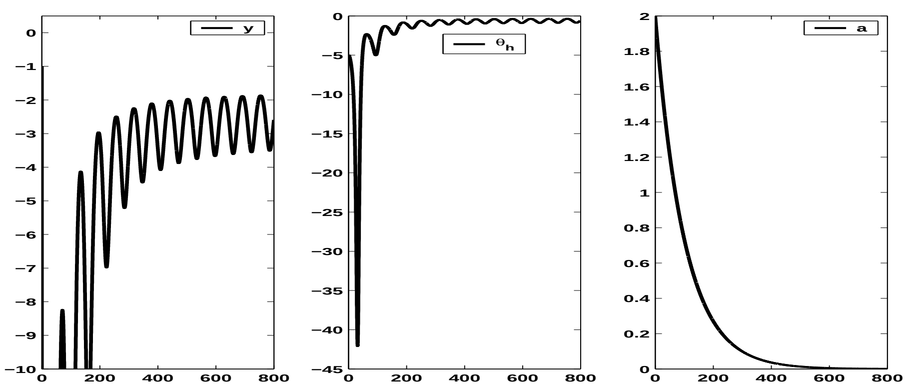

Figure 5: The output of the proposed extremum seeking feedback scheme, initial condition b

It is apparent that when there exists a unique maximum, as in (Tan *et al.*, 2006*a*, Assumption 3), the result of Theorem 2 always holds with an additional freedom in the choice of the amplitude of the dither signal by tuning a_0 and ϵ.

Secondly, we show in next example that the Assumption (4) holds for a 4^{th}-order polynomial. Moreover, this example also indicates that the choice of ϵ^* does depend on the choice of a_0.

Example 2 *Consider the dynamic system (28), where* $h(\mathbf{x}) = -(x_1+3x_2)^4 + \frac{8}{15}(x_1+3x_2)^3 + \frac{5}{6}(x_1+3x_2)^2 + 10$. *When the control input takes the form as in (29), Assumption 1 and 2 hold. Moreover, we have* $Q(\theta) = -\theta^4 + \frac{8}{15}\theta^3 + \frac{5}{6}\theta^2 + 10$ *that has a global maximum at* $\theta = 1$ *and a local maximum at* $\theta = -0.6$ *as seen in Figure 6. Hence, all conditions in Assumption 3 hold. Similarly to (20), the averaged system of the "reduced system" can be written as :*

$$\mu_{av}(\zeta,\rho) = \frac{\rho}{2}\left[-4\zeta^3 + \frac{24}{15}\zeta^2 + \frac{5}{3}\zeta + \rho^2\left(-24\zeta + \frac{48}{15}\right)\right] .$$

The bifurcation diagram is shown in Figure 6 with $\rho^* = 0.877$. *Note that there exists a continuous root* $\ell(a)$ *of (19) approaching 1 when* $\rho \to 0$. *Moreover, by inspecting the plot in Figure 6, it is not hard to check that this root satisfies all conditions of Assumption 4*

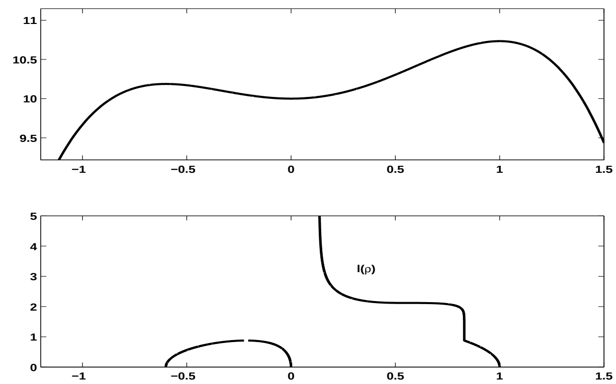

Figure 6: A 4^{th}-order polynomial and the corresponding zero plot of $\mu_{av}(\cdot,\cdot)=0$

and, hence, Theorem 2 applies. Suppose $g(a)=a$*, consider the initial condition* $\hat{\theta}_0=-1$ *which is such that the local maximum* $\hat{\theta}=-0.6$ *lays between the initial condition and the global maximum. Nevertheless, by choosing* $a_0=3>0.877$*,* $\epsilon=1$*,* $\delta=0.005$*,* $\omega=0.1$*,* $x_1(0)=x_2(0)=0$*, the output of the system converges to a small neighborhood of the global maximum* $y^*=10.734$ *as seen in Figure 7.*

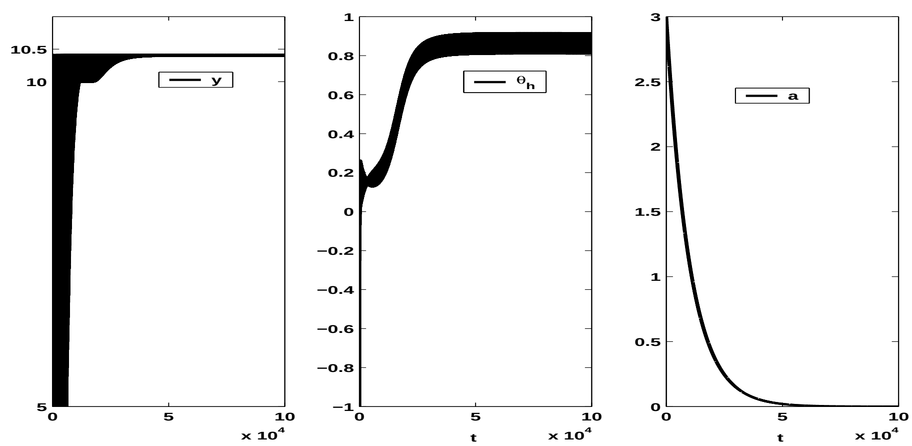

Figure 7: The performance of the proposed ES feedback scheme

However, by choosing $a_0=1.0>0.877$*, while keeping other parameters are same as above, the simulation result is shown in Figure 8. The output of the system converges to a neighborhood of the local maximum* 10.187*.*

Intuitively, the smaller ρ_0 is, where $\rho_0>\rho^*$, the smaller ϵ would be chosen in order to ensure that Fact 2 holds as indicated in the proof of Theorem 1 in Appendix.

Thirdly, we show that when Q is an arbitrary 4^{th}-order polynomial,

$$Q(x)=\alpha_0x^4+\alpha_1x^3+\alpha_2x^2+\alpha_3x+\alpha_4,\quad \alpha_0=-1 \tag{31}$$

satisfying Assumption 3 also satisfy conditions in Assumption 4:

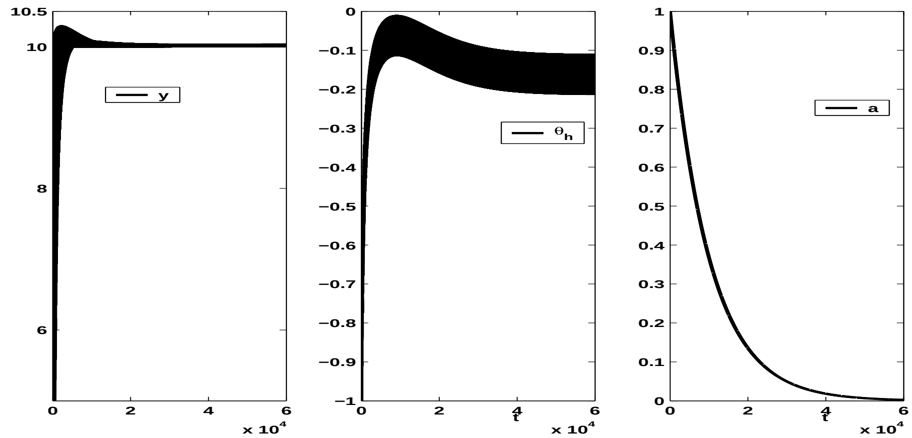

Figure 8: The performance of the proposed ES feedback scheme

Proposition 1 *Consider $Q(\cdot)$ in (31). Suppose that Assumption 3 holds. Then, all conditions in Assumption 4 hold.*

Note that since $\alpha_0 = -1$ in (31), we have that (6) in Assumption 3 always holds. However, this does not guarantee that (7) holds and we still need to assume this. Actually, if (7) in Assumption 3 does not hold while (6) holds, then we can not prove in general that Assumption 4 holds, as the following example illustrates.

Example 3 *Consider the polynomial $Q(\theta) = -\theta^4 + 2\theta^2 + 10$ that is such that there exist two points $\theta_1^* = \theta_2^*$ such that $Q(\theta_1^*) = Q(\theta_2^*)$ and $Q(\theta) < Q(\theta_1^*)$ for all $\theta \neq \theta_1^*$ and $\theta \neq \theta_2^*$. The plot of this function and its (pitchfork) bifurcation diagram are given in Figure 9. It is easy to see that there does not exist an* isolated *root of (19) that satisfies Assumption 4. Under such a situation, the results in Theorem 2 do not hold. Simulation of the dynamic system (28) when*

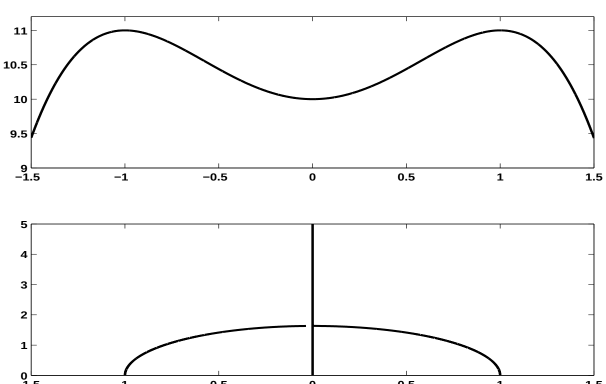

Figure 9: Bifurcation diagram of a function that does not satisfy condition (7).

$h(\mathbf{x}) = -(x_1 + 3x_2)^4 + 2(x_1 + 3x_2)^2 + 10$, *$u$ is from (29), and* $a_0 = 3$, $\epsilon = .2$, $\delta = 0.005$,

$\omega = 0.1$, $\begin{bmatrix} x_1(0) \\ x_2(0) \\ \hat{\theta}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}$ *is shown in Figure 10. The output of the system converges to the neighborhood of the* local minimum.

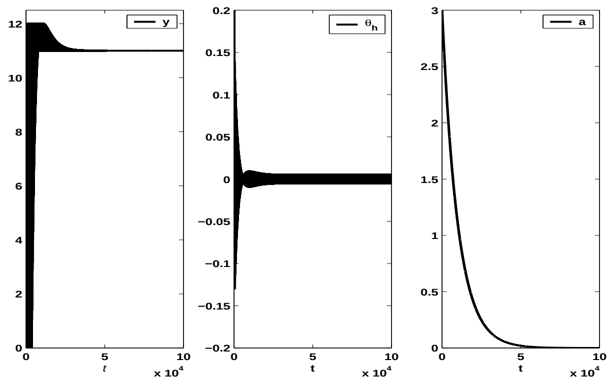

Figure 10: The output of the proposed ES feedback scheme

Fourthly, we consider 6^{th}-order polynomials. Obviously, the bifurcation diagram when $Q(\cdot)$ is a 6^{th}-order polynomial is more complicated than that of the 4^{th}-order polynomial. We have observed in simulations that for most of the cases (more than 80%), Assumption 4 holds true when Assumption 3 holds. However, this is not true in general as the following example shows.

Example 4 *For the following 6^{th}-order polynomial (see Figure 11):*

$$Q(\theta) = -\theta^6 + \frac{1}{10}\theta^5 + \frac{623}{400}\theta^4 - \frac{659}{4000}\theta^3 - \frac{11287}{20000}\theta^2 + \frac{259}{4000}\theta + \frac{637}{20000}. \tag{32}$$

which has maxima at -0.8985, $.5$, 0.8951. The global maximum occurs at $\theta^* = -0.8985$. Assumption 3 holds. The function $\mu_{av}(\zeta, \rho)$ is:

$$\rho \left[\frac{1}{2} DQ(\zeta) + \frac{\rho^2}{16} D^3 Q(\zeta) + \frac{\rho^4}{384} D^5 Q(\zeta) \right] \tag{33}$$

whose bifurcation diagram is shown in Figure 11. It is clear that item 3 of Assumption 4 does not hold since $\ell(0) = 0.8951$ which is not a global maximum. This implies that our extremum seeking controller if tuned like in Theorem 2 would yield semi-global practical convergence to the local maximum.

Remark 13 *It is not crucial that a sinusoidal signal is used as excitation (dither) in the extremum seeking controller. Indeed, one may use different dither signals, such as a square-wave or a sawtooth signals. Interestingly, changing the dither leads to a different average system μ_{av} and the bifurcation diagram defined by (19) changes. An interesting consequence of this fact is that Assumption 4 may not be satisfied for one dither signal whereas it may be*

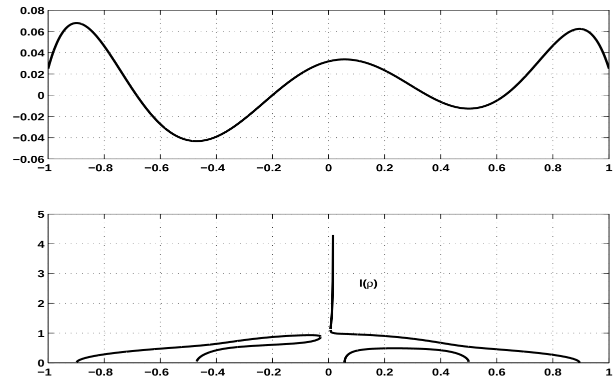

Figure 11: The 6^{th} order polynomial and its bifurcation diagram for which Assumption 4 does not hold (sine wave dither).

satisfied for a different dither. In our next example we revisit the system in Example 4 that did not satisfy Assumption 4 with a sinusoidal dither and show that the same system with a square wave dither satisfies Assumption 4. More analysis of how the choice of dither affects the convergence properties of extremum seeking controllers is given in (Tan et al.*, 2006*b*).*

Example 5 *Consider again the function $Q(\cdot)$ in Example 4 (see the top plot in Figure 11). Suppose that instead of the sine wave, we use a square wave dither in our controller, which is defined as follows*

$$sq(t) := \begin{cases} -1 & t \in [kT, kT + T/2) \\ 1 & t \in [kT + T/2, kT + T) \end{cases},$$

where $k = 0, 1, \dots$ and $T > 0$. Direct calculations yield the following $\mu_{av}(\zeta, \rho)$:

$$\rho \left[DQ(\zeta) + \frac{\rho^2}{6} D^3 Q(\zeta) + \frac{\rho^4}{120} D^5 Q(\zeta) \right], \tag{34}$$

which is different from (33). The bifurcation diagram is shown in Figure 12. Assumption 4 holds since $\ell(\rho)$ satisfies $\ell(0) = x^* = -0.8985$. Hence, if we use our controller and the tuning strategy from Theorem 2 we will obtain semi-global practical convergence to the global maximum*.*

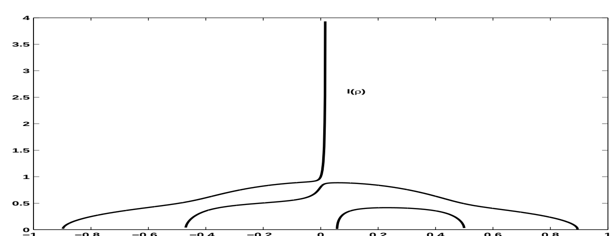

Figure 12: The bifurcation diagram for $Q(\cdot)$ from Example 4 for which Assumption 4 holds (square wave dither).

When $Q(\cdot)$ becomes a 8^{th}-order polynomial, more counterexamples appear, though it is still true that most of the cases (60%), Assumption 4 holds when Assumption 3 holds.

Finally, we come to a general nonlinear mapping Q. It is intuitively clear that the bifurcation diagram of a general nonlinear mapping may be far more complicated than of the polynomials. For example, $\ell(\rho)$ may be discontinuous, i.e, there may be a "jump" in $\ell(\rho)$ at some point ρ. Under such a situation, it is very difficult to design a suitable ES mechanism to ensure the global convergence. However, there are still many nonlinear output equilibrium maps, which ensure that Assumption 3 and 4 hold as shown in the following example.

Example 6 *Consider the function $Q(x) = e^{-(\frac{x+2}{10})^2} + 0.15e^{-(\frac{x-5}{0.5})^2}$. This function has a global maximum at $x = -2$ and a local maximum at $x = 5$(see Figure 13). Assumption 3 holds. The average system in this case is computed numerically and the bifurcation diagram plotted*

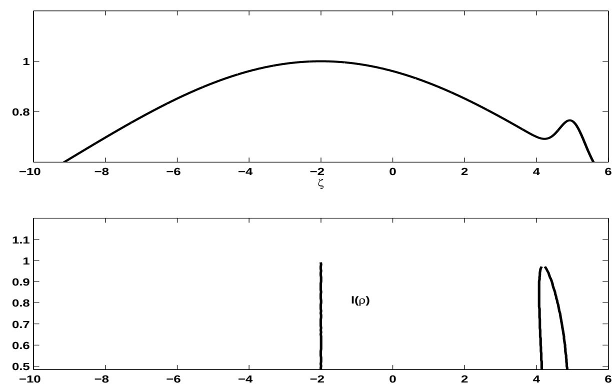

Figure 13: Nonlinear function $Q(x)$ and bifurcation diagram

directly. There exists a continuous root $\ell(\rho)$ that is unique for large ρ whereas it approaches the global maximum -2, when $\rho \to 0$. Hence, Assumption 4 holds.

5 Summary

We have presented a scalar extremum seeking feedback controller that achieves semi-global practical extremum seeking in presence of local extrema. Several examples were presented illustrating and highlighting various issues. We believe that these results will be of use to the control engineers that are using the extremum seeking control and may motivate further research into this challenging area.

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6 Appendix

6.1 Proof of Theorem 1

First, we show that (20) holds. Using the Taylor Series Expansion, noting $\int_0^{2\pi} sin^{2i-1}(t)dt = 0$, $\forall i = 1, \ldots$, we obtain that $\mu_{av}(\zeta, \rho)$ is equal to:

$$\begin{aligned}\rho \cdot \left(\sum_{i=1}^{r} \rho^{2(i-1)} \cdot c_i \cdot D^{2i-1}Q(\zeta) + \rho^{2r} \cdot c_{r+1} \cdot R \right) \\ =: \rho \cdot p(\zeta, \rho),\end{aligned} \tag{35}$$

where

$$c_i \triangleq \begin{cases} \frac{1}{(2i-1)!} \cdot \frac{1}{2\pi} \cdot \int_0^{2\pi} \sin^{2i}(t)dt & i = 1, \ldots, r \\ \frac{1}{(2r-1)!} & i = r+1 \end{cases}$$

$$R := \tfrac{1}{2\pi} \int_0^{2\pi} \int_0^1 (\sin^{2r+1} t) \cdot (1-s)^{(2r-1)} \cdot D^{2r}Q(\zeta + s \cdot \rho \cdot \sin t) ds dt$$

Before we prove the main result, we state several facts that follow directly from Assumption 4.

Proposition 2 *Suppose that Assumption 4 holds. Then, for any* $\rho_1 > 0$ *there exists* $\eta > 0$ *and* $\alpha_1 \in \mathcal{K}$ *such that*

$$\xi \cdot p(\xi + \ell(\rho), \rho) \leq -\alpha_1(|\xi|), \tag{36}$$

for all $|\xi| \leq \eta$ *and* $\rho \in [0, \rho_1]$.

Proof of Proposition 2: Introduce $\xi := \zeta - \ell(\rho)$ in (19) and note that the root $\zeta = \ell(\rho)$ of (19) corresponds to $\xi = 0$. Since $\zeta = \ell(\rho)$ is an isolated real root of (19), we have that for any $\rho_1 > 0$ there exists $\eta > 0$ such that for all $|\xi| \leq \eta$ and $\rho \in [0, \rho_1]$, $\xi = 0$ is the only root of

$$p(\xi + \ell(\rho), \rho) = 0$$

in this set. Moreover, since $p(\ell(\rho), \rho) = 0$, p is smooth and from the item 1 of the assumption we have $D_1 p(\ell(\rho), \rho) < 0$ for all $a \geq 0$, we conclude that

$$\xi \cdot p(\xi + \ell(\rho), \rho) < 0 \tag{37}$$

for all $|\xi| \leq \eta, \xi \neq 0$ and $\rho \in [0, \rho_1]$. Since the set $|\xi| \leq \eta$ $\rho \in [0, \rho_1]$ is compact, we can define:

$$\kappa(s) := \min_{|\xi| \in [s, \eta]} \min_{\rho \in [0, \rho_1]} \left[-\xi \cdot p(\xi + \ell(\rho), \rho) \right],$$

which is nondecreasing, continuous and $\kappa(0) = 0$ and, hence, we can find $\alpha_1 \in \mathcal{K}$ such that $\kappa(s) \geq \alpha_1(s)$ for all $s \in [0, \eta]$ that satisfies (36). $\circ$

Proposition 3 *Suppose that Assumption 4 holds. Then, for any $\rho_2 > \rho^*$ and any $\Delta > 0$ there exists $\alpha_2 \in \mathcal{K}$ such that*

$$\xi \cdot p(\xi + \ell(\rho), \rho) \leq -\alpha_2(|\xi|), \tag{38}$$

for all $|\xi| \leq \Delta$ and $\rho \in [\rho^*, \rho_2]$.

Proof of Proposition 3: The proof of this proposition follows in a similar manner as the proof of Proposition 2, by noting that we also have that for arbitrary $\rho_2 > \rho^*$ and arbitrary Δ we have from the item 2 of the assumption that $\xi = 0$ is a unique root of (37) on the set $\rho \in [\rho^*, \rho_2]$ and $|\xi| \leq \Delta$. ◦

Proof of Theorem 1: First, we show that an appropriate bound can be obtained for the averaged system (16) if the (ρ_0, ϵ) are appropriately adjusted. Then, the conclusion of the theorem follows from recent results on averaging (Teel and Nešić, 2000; Teel *et al.*, 2003; Teel *et al.*, 1999) and this part is omitted.

Consider the average system (16). We show for this system that for any strictly positive (D, d) and $\rho_0 > \rho^*$ there exist $\tilde{\beta} = \tilde{\beta}_{\rho_0,D,d} \in \mathcal{KL}$ and $\epsilon^* = \epsilon^*(\rho_0, D, d) > 0$ such that for any $\epsilon \in (0, \epsilon^*)$, all $(\zeta(0), \rho(0))$ satisfying $\rho(0) = \rho_0$ and $|\zeta(0) - \ell(\rho(0))| \leq D$ and all $t \geq 0$ the solutions of the system (16) satisfy:

$$|\zeta(t) - \ell(\rho(t))| \leq \tilde{\beta}(|\zeta(0) - \ell(\rho(0))|, \delta \cdot t) + d\ . \tag{39}$$

Let (D, d) and $\rho_0 > \rho^*$ be given and without loss of generality assume that $D \geq d$. Let $\rho_1 := \rho_0$ generate η and $\alpha_1 \in \mathcal{K}$ via Proposition 2. Let $\Delta := D$ and $\rho_2 := \rho_0$ generate $\alpha_2 \in \mathcal{K}$ via Proposition 3. Let $\alpha := \min\{\alpha_1, \alpha_2\}$. Let $c := \min\{\eta, \frac{1}{2}d\}$. To complete the proof, we introduce the following sets:

$$\mathcal{S}_1 := \{(\xi, \rho) : |\xi| \leq D + d/2, \rho \in [\rho^*, \rho_0]\}$$

$$\mathcal{S}_2 := \{(\xi, \rho) : |\xi| \leq c, \rho \in [0, \rho_0]\}$$

and we let $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$. These sets are shown in Figure 14.

Let $\tilde{\beta}(s, t) \in \mathcal{KL}$ be the solution of the following differential equation:

$$\dot{\beta} = -\frac{\rho^*}{2}\alpha\left(\sqrt{2\beta}\right),\ \beta(0) = s\ ,$$

$\tilde{\beta}(s, t) := \sqrt{2\beta\left(\frac{s^2}{2}, t\right)}$ and let $T > 0$ be such that

$$\tilde{\beta}\left(D + \frac{d}{2}, T\right) = c\ . \tag{40}$$

We introduce a differentiable function $\hat{\ell}(\rho)$ such that $\forall (\xi, \rho) \in \mathcal{S}$ and $t \in [0, T]$ we have:

$$\left|\xi \cdot [p(\xi + \ell(\rho), \rho) - p(\xi + \hat{\ell}(\rho), \rho)]\right| \leq \frac{\alpha(c)}{4}, \tag{41}$$

$$|\hat{\ell}(\rho) - \ell(\rho)| \leq \frac{d}{2}, \tag{42}$$

$$\left|\tilde{\beta}(|\xi - \hat{\ell}(\rho)|, t) - \tilde{\beta}(|\xi - \ell(\rho)|, t)\right| \leq \frac{d}{2} \tag{43}$$

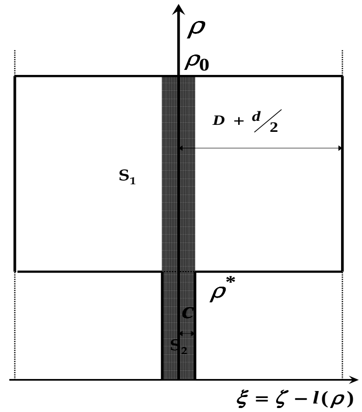

Figure 14: Sets $\mathcal{S}_1$ and $\mathcal{S}_2$.

and note that such a function always exist as we can approximate the continuous function $\ell(\cdot)$ with a differentiable function $\hat{\ell}(\cdot)$ to arbitrary accuracy on compact sets. Let $\epsilon_1^* > 0$ be such that

$$\epsilon_1^* \cdot |\xi \cdot D\hat{\ell}(\rho)| \leq \frac{1}{4}\alpha(c) \ . \tag{44}$$

Moreover, let $\epsilon_2^* > 0$ be such that:

$$\beta_\rho(\rho_0, \epsilon_2^* \cdot T) > \rho^* \ , \tag{45}$$

where β_ρ comes from (22) and note that such a number always exists because $\beta_a \in \mathcal{KL}$. Finally, we let $\epsilon^* := \min\{\epsilon_1^*, \epsilon_2^*\}$.

Next, we show that the above constructed ϵ^* and $\tilde{\beta}$ satisfy conditions in our claim. In the rest of the proof we let $\epsilon \in (0, \epsilon^*)$ be arbitrary. Moreover, we introduce the change of time $\tau := \delta \cdot t$ and a transformation of coordinates $\xi := \zeta - \hat{\ell}(\rho)$ and rewrite (16) as follows:

$$\begin{aligned} \frac{d\xi}{d\tau} &= \rho \cdot p(\xi + \hat{\ell}(\rho), \rho) - \epsilon \cdot \rho \cdot D\hat{\ell}(\rho) && (46) \\ \frac{d\rho}{d\tau} &= -\epsilon \cdot g(\rho), \quad \rho(0) = \rho_0 \ , && (47) \end{aligned}$$

where we have also used (20). We introduce the following Lyapunov function $V(\xi) := \frac{1}{2}\xi^2$ and taking its derivative along (46) we have from Propositions 2 and 3 and inequalities (41), (44) :

$$\begin{aligned} \frac{dV}{d\tau} &= \xi \cdot \left[\rho \cdot p(\xi + \hat{\ell}(\rho), \rho) - \epsilon \cdot \rho \cdot D\hat{\ell}(\rho)\right] \\ &= \rho \cdot \left[\xi \cdot p(\xi + \ell(\rho), \rho) - \epsilon \cdot \xi \cdot D\hat{\ell}(\rho)\right] \\ &\quad + \rho \cdot \left[\xi \cdot (p(\xi + \hat{\ell}(\rho), \rho) - \xi \cdot p(\xi + \ell(\rho), \rho))\right] \\ &\leq \rho \cdot \left[-\alpha(|\xi|) + \frac{1}{2}\alpha(c)\right], \ \forall (\xi, \rho) \in \mathcal{S} \ . \end{aligned} \tag{48}$$

The proof is completed by stating and proving several facts:

Fact 1: The set $\mathcal{S}_2$ is forward invariant.

Proof of Fact 1: This is straightforward from (48) and the fact that $\rho(\cdot)$ is monotonically decreasing. $\circ$

Fact 2: For the number T defined by (40) we have that for any solution initialized at $|\xi(0)| \le D + d/2$ and $\rho(0) = \rho_0$ we have $(\xi(T), \rho(T)) \in \mathcal{S}_2$.

Proof of Fact 2: If the initial state is in $\mathcal{S}_1$ we have nothing to prove since Fact 1 holds. Note that (48) guarantees that as long as we are in the set $\mathcal{S} - \mathcal{S}_1$, $\xi(\cdot)$ is monotonically decreasing and we have:

$$|\xi(\tau)| \le \tilde{\beta}(|\xi(0)|, \tau), \ \forall \tau \ge 0 \ .$$

Note that $\rho(\cdot)$ is also monotonically decreasing. Moreover, we can use the contradiction and our choice of ϵ_2^* in (45) to show that all solutions initialized at $|\xi(0)| \le D + d/2$ and $\rho(0) = \rho_0$ stay in the set $\mathcal{S}_2$ for all $\tau \in [0, T]$. Finally, if we assume that for some $\xi(0) \in \mathcal{S} - \mathcal{S}_1$ we have $|\xi(\tau)| > c$ for all $\tau \in [0, T]$, this implies:

$$\tilde{\beta}(D + d/2, \tau) \ge |\xi(\tau)| > c, \ \forall \tau \in [0, T] \ ,$$

which contradicts the choice of T in (40) and, hence, we have $(\xi(T), \rho(T)) \in \mathcal{S}_2$. $\circ$

Fact 3: If $|\xi(0)| \le D + d/2$ and $\rho(0) = \rho_0$, then solutions of the system (46), (47) satisfy

$$(\xi(\tau), \rho(\tau)) \in \mathcal{S}, \ \forall \tau \ge 0 \ .$$

Proof of Fact 3: It follows trivially from Facts 1 and 2. $\circ$

We now complete the proof of the theorem. From Fact 3, it follows that for all solutions initialized at $|\xi(0)| \le D + d/2$ and $\rho(0) = \rho_0$, the inequality (48) holds for all $\tau \ge 0$ along solutions. Moreover, Fact 2 says that for any such solution there exists $\tau_1 \in [0, T]$ such that $|\xi(\tau)| > c$ for all $\tau \in [0, \tau_1)$ and $|\xi(\tau_1)| \le c$. Since $\rho(\tau) > \rho^*$ for all $\tau \in [0, T]$, from (48) we have that for all $\tau \in [0, \tau_1)$ the solutions of the system satisfy:

$$\frac{dV}{d\tau} \le -\frac{\rho^*}{2} \alpha\left(\sqrt{2V}\right) \ ,$$

which implies:

$$|\xi(\tau)| \le \tilde{\beta}(|\xi(0)|, \tau), \ \tau \in [0, \tau_1) \ . \tag{49}$$

On the other hand, for all $\tau \ge \tau_1$ we have from Fact 1 that $(\xi(\tau), \rho(\tau)) \in \mathcal{S}_1$, which implies

$$|\xi(\tau)| \le c \le d/2, \ \forall \tau \ge \tau_1 \ . \tag{50}$$

Note now that from our choice of $\hat{\ell}$, we have that $|\zeta(0) - \ell(\rho(0))| \le D$ implies $|\xi(0)| = |\zeta(0) - \hat{\ell}(0)| \le |\zeta(0) - \ell(\rho(0))| + |\hat{\ell}(\rho(0)) - \ell(\rho(0))| \le D + \frac{d}{2}$ and this in turn implies (49) and (50). By adding and subtracting some terms to (49) and (50), we conclude that for all $\tau \in [0, \tau_1] \subseteq [0, T]$:

$$\begin{aligned} |\zeta(\tau) - \ell(\rho(\tau))| &\le \tilde{\beta}(|\zeta(0) - \ell(\rho(0))|, \tau) + |\tilde{\beta}(|\zeta(0) - \hat{\ell}(\rho(0))|, \tau) - \tilde{\beta}(|\zeta(0) - \hat{\ell}(\rho(0))|, \tau)| \\ &\quad + |\hat{\ell}(\rho(\tau)) - \ell(\rho(\tau))| \\ &\le \tilde{\beta}(|\zeta(0) - \ell(\rho(0))|, \tau) + d \end{aligned}$$

and for $\tau \geq \tau_1$ we have:

$$|\zeta(\tau) - \ell(\rho(\tau))| \leq d/2 + |\hat{\ell}(\rho(\tau)) - \ell(\rho(\tau))| \leq d \tag{51}$$

Combining these last two bounds completes the proof by noting that $\tau = \delta t$. ■

6.2 Proof of Proposition 1

From (13), where Q is given in (31), we have the following averaged system

$$\dot{\zeta} = \frac{\delta \cdot \rho}{2} \cdot \left[DQ(\zeta) + \frac{\rho^2}{8} \cdot D^3 Q(\zeta) \right] = \frac{\delta \cdot \rho}{2} \cdot p(\zeta, \rho), \tag{52}$$

which is a 3^{rd}-order polynomial. Let the three zeros of $DQ(\zeta) = 0$ be $r_1, r_2, r_3 \in C$. Without losing generality, assume that $r_3 \in R$ satisfies Assumption 3.

We rewrite $p(\zeta, \rho)$ in (52) as

$$p(\zeta, \rho) = -4(\zeta - r_1) \cdot (\zeta - r_2) \cdot (\zeta - r_3) - 3\rho^2 \cdot (\zeta - \frac{r_1 + r_2 + r_3}{3}) \tag{53}$$

By a linear transformation $w = \zeta + \frac{r_1+r_2+r_3}{3}$, we have

$$p(w, \rho) = -4 \left(w^3 + \lambda(\rho) w + \lambda_1 \right) \tag{54}$$

where $\lambda(\rho) \triangleq -\frac{3}{16}\alpha_1^2 - \frac{1}{2}\alpha_2 + \frac{3}{4}\rho^2$ is a continuous function with respect to ρ and $\lambda_1 \triangleq -\frac{1}{32}\alpha_1^3 - \frac{1}{4}\alpha_3 - \frac{1}{8}\alpha_2 \cdot \alpha_1$ is a constant. For all $i = 1, 2, 3$, the zeros of $p(w, \rho)$ can be represented as

$$w_i(a) \triangleq \sqrt[3]{-\frac{\lambda_1}{2} + \sqrt{\Lambda(\rho)}} \cdot p_i + \sqrt[3]{-\frac{\lambda_1}{2} - \sqrt{\Lambda(\rho)}} \cdot q_i \tag{55}$$

where $\Lambda(\rho) \triangleq \left(\frac{\lambda_1}{2}\right)^2 + \left(\frac{\lambda(\rho)}{3}\right)^3$, $p_1 = q_1 = 0$, $p_2 = q_3 = \omega$, $p_3 = q_2 = \omega^2$, and $\omega \triangleq \frac{-1+\sqrt{3}i}{2}$.

The value of zero w_i is dependent on the value of $\Lambda(\omega)$, i.e.,

1. $\Lambda(\rho) < 0$, three distinct zeros are real.
2. $\Lambda(\rho) = 0$, three zeros are real, two of them are equal.
3. $\Lambda(\rho) > 0$, there exists only one real zero.

The proof of Proposition 1 is complete by stating and proving several facts.

Fact 1: $\Lambda(\rho)$ increases monotonically with respect to ρ.

Proof of Fact 1: This is straightforward from the definition of $\Lambda(\rho)$ with $D\Lambda(\rho) = \frac{\lambda(\rho)^2}{3} \cdot \frac{3}{2}\rho \geq 0, \forall \rho \geq 0$. ○

Fact 2: There exists $\rho^* > 0$ such that there exists unique real zero of $p(w, \rho)$ in (54).

Proof of Fact 2: This is straightforward from the monotonicity of $\Lambda(\rho)$. ○

Fact 3: There exists a continuous function $\ell(\rho) : R_+ \to R$ such that $p(\ell(\rho), \rho) = 0$.

Proof of Fact 3: Note that when ρ is the coefficient of $p(w,\rho)$, the zero of the polynomial $p(w,\rho)$ are continuous function of the continuous function of ρ (Marden, 1966, Theorem 1,4), thus w_i is a continuous function in C.

When $\Lambda(0) > 0$, $\ell(\rho)$ is a continuous real function and is the unique real zero of the $p(w,\rho)$ for any ρ. When $\Lambda(0) \le 0$, three zeros of $p(w,\rho)$ are real. There exists a real continuous function $\ell(\rho)$ (may not unique), which is the zero of $p(w,\rho)$. ∘

Fact 4: Assume that $r_1 \le r_2 < r_3$ are the real zeros of $p(\zeta,\rho)$ in (53) and $Q(r_1) < Q(r_3)$, where $Q(\cdot)$ is defined in (31). Then $r_2 < \frac{r_1+r_2+r_3}{3}$.

Fact 4: We denote a continuous function $g(z) \triangleq \int_{r_1}^{r_3} -(\zeta - r_1)(\zeta - z)(\zeta - r_3)d\zeta$, which has the property $Dg(z) < 0$, $\forall z \in (r_1, r_3)$. Therefore $g(z)$ decreases monotonically for any $z \in (r_1,r_3)$. Note that $g(\frac{r_1+r_3}{2}) = 0$, $Q(r_1) < Q(r_3) \Rightarrow g(r_2) < 0$, the monotonicity of $g(z)$ implies that $r_2 < \frac{r_1+r_3}{2}$, i.e., $\frac{r_1+r_2+r_3}{3} - r_2 > 0$, which completes the proof. ∘

Fact 5: There exists at least one real zero of $p(\zeta,\rho)$ in (53) in the interval $(\frac{r_1+r_2+r_3}{3}, r_3]$ for any $\rho \ge 0$.

Proof of Fact 5: We consider two different cases:
(a):$\Lambda(0) > 0$. We have either $r_1 = r_2 = r_3 \in R$ or $r_1 = \overline{r_2} \in C$, $r_3 \in R$. When $r_1 = r_2 = r_3$, $\ell(\rho) \equiv r_3$, for any $\rho \ge 0$. When $r_1 = \overline{r_2} \in C$, $r_3 \in R$, for any $\rho \ge 0$, we have

$$
\begin{aligned}
& p(r_3,\rho)\cdot p(\frac{r_1+r_2+r_3}{3},\rho) \\
= \ & -3\rho^2(r_3 - \frac{r_1+r_2+r_3}{3})\cdot\left[-4(\frac{r_1+r_2+r_3}{3} - r_1)\cdot(\frac{r_1+r_2+r_3}{3} - r_2)\cdot(\frac{r_1+r_2+r_3}{3} - r_3)\right] \\
= \ & -3\rho^2(r_3 - \frac{r_1+r_2+r_3}{3})^2\cdot\left[4(\frac{r_1+r_2+r_3}{3} - \overline{r_2})\cdot(\frac{r_1+r_2+r_3}{3} - r_2)\right] < 0
\end{aligned}
$$

from the fact that $\frac{r_1+r_2+r_3}{3} \in R$.
(b): $\Lambda(0) \le 0$. We have $r_1 \le r_2 < r_3$ (as r_3 is the unique maximum). According to Fact 4, we have $DQ(\frac{r_1+r_2+r_3}{3}) < 0$, leading to the following fact

$$p(\frac{r_1+r_2+r_3}{3},\rho)\cdot p(r_3,\rho) < 0.$$

since $D^3Q(r_3) > 0$. The Implicit Function Theorem(N.Jacobson, 1974) indicates that Fact 5 holds ∘

Fact 6: $\ell(\rho) \in (\frac{r_1+r_2+r_3}{3}, r_3]$

Proof of Fact 6: We also consider two cases as in the proof of Fact 5.
(a):$\Lambda(0) > 0$. $\ell(\rho)$ is the unique real root, therefore, $\ell(\rho) \in (\frac{r_1+r_2+r_3}{3}, r_3]$.
(b): $\Lambda(0) \le 0$. We have $r_1 \le r_2 < r_3$. Denote $A = (r_2, \frac{r_1+r_2+r_3}{3}]$, $B = (r_3, \infty)$, we have

$$p(\zeta,\rho) > 0 \ \ \forall\zeta \in A, \quad p(\zeta,\rho) < 0, \ \ \forall\zeta \in B$$

for any $\rho \ge 0$. Therefore, for any positive ρ, there is no real roots of $p(\zeta,\rho)$ in any interval A or B. Combining Fact 3 and Fact 5 completes the proof. ∘

Fact 7: $D_1 p(\ell(\rho),\rho) < 0$

Proof of Fact 7: From Fact 6, $\ell(\rho) \in (\frac{r_1+r_2+r_3}{3}, r_3]$. When $\rho = 0$, r_3 is the unique root in the interval $(\frac{r_1+r_2+r_3}{3}, r_3]$ and $D_1 p(\ell(0), 0) < 0$ ((6) in Assumption 3). Assume that $\exists\zeta_\rho \in$

$(\frac{r_1+r_2+r_3}{3}, r_3]$ such that $D_1 p(\zeta_\rho, \rho) = 0$ for any $\rho > 0$. Note $D_1^2 p(\zeta_\rho, \rho) = \frac{\rho^2}{8} D^3 Q(\zeta_\rho) < 0, \forall \rho > 0$, ζ_ρ is a local maximum of $p(\zeta_\rho, \rho)$, i.e. $p(\zeta_\rho + \lambda, \rho) < p(\zeta_\rho, \rho)$ and $p(\zeta_\rho - \lambda, \rho) < p(\zeta_\rho, \rho)$, for a sufficiently small $\lambda > 0$. However, it contradicts the fact that ζ_ρ is the real root of $p(\zeta_\rho, \rho)$ satisfying the following inequality:

$$[p(\zeta_\rho + \lambda, \rho) - p(\zeta_\rho, \rho)] \cdot [p(\zeta_\rho - \lambda, \rho) - p(\zeta_\rho, \rho)] < 0,$$

We conclude that $D_1 p(\ell(\rho), \rho) \neq 0$ for all $\rho \geq 0$. Fact 7 holds from Fact 3 and the continuity of $D_1 p$. $\circ$

Proof Proposition 1) Combining Fact 1-7 completes the proof. $\blacksquare$

6.3 Proof of Theorem 2

Proof*: Applying Theorem 1 to (11) and (12), the following results are obtained in the time scale σ by tuning the parameter (a_0, ϵ, δ) appropriately:*

$$\begin{aligned} |\theta_r(\sigma) - \ell(a(\sigma))| &\leq \beta_\theta(|\theta_r(\sigma_0) - \ell(a(\sigma_0))|, \delta \cdot (\sigma - \sigma_0)) \\ |a_r(\sigma)| &\leq \beta_a(|a_0|, \epsilon \cdot \delta \cdot (\sigma - \sigma_0)) \end{aligned} \tag{56}$$

Next, We show the appropriate stability of the boundary layer system.

Introducing $\bar{\mathbf{x}} \triangleq \mathbf{x} - \mathbf{l}(\hat{\theta}(\sigma) + a \cdot \sin(\sigma))$, $t' = \frac{\sigma - \sigma_0}{\omega}$, *where* σ_0 *is a fixed time instant in time scale* σ, *setting* $\omega = 0$ *and denoting* $\theta_1 \triangleq \hat{\theta}(\sigma_0) + a \cdot \sin(\sigma_0)$, *the boundary layer corresponding to the overall system (10) satisfies,*

$$\frac{d\bar{\mathbf{x}}}{dt'} = \mathbf{f}(\bar{\mathbf{x}} + \mathbf{l}(\theta_1), \alpha(\bar{\mathbf{x}} + \mathbf{l}(\theta_1), \theta_1)). \tag{57}$$

Assumption 2 guarantees that the above system is globally asymptotically stable, uniformly in θ_1. *By applying (Tan* et al.*, 2005, Lemma 2), (10) is SPA stable, uniformly in* ω *(with the time scale* σ*). Noting that* $\sigma = \omega t$, *this implies directly that (23), (24) and (25) hold.* $\blacksquare$