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AVERAGING FOR A CLASS OF HYBRID SYSTEMS

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Abstract. Averaging theory for ordinary differential equations is extended to a class of hybrid systems. Rapid time variations in the flow map of a hybrid system generate solutions that are also solutions of a slightly perturbed time-invariant average hybrid system. Results relating solutions of the time-varying system to solutions of the average system ensue. In the absence of finite escape times for the average system, on compact time domains each solution of the time-varying system is close to a solution of the average system. If the average system is asymptotically stable, the time-varying system exhibits semi-global, practical asymptotic stability. These results rely on mild regularity properties for the average system. In particular, the average system is not required to exhibit unique solutions. Both periodic and non-periodic flow maps are considered. The results are partially motivated by the desire to justify a pulse-width modulated implementation of hybrid feedback control for nonlinear systems.

Keywords. Averaging, nonlinear systems, stability, pulse width modulated control.

AMS (MOS) subject classification: 93Cxx, 93Dxx

1 Introduction

Averaging theory exploits a time-scale separation between the time variations of the state of a dynamical system and the time variations of the derivative of that state. The theory justifies the use of a simplified - in particular, averaged - system to approximate the original system. A concise overview of averaging theory is provided by Hassan Khalil in the Control Handbook [13]. This article, which extends averaging theory to hybrid systems, is dedicated to Professor Khalil, who has exploited multiple time-scale phenomena cleverly throughout his illustrious research career.

Averaging theory for ordinary differential equations has a rich history, dating to back to the work of Krylov and Bogoliubov [15], and has been used extensively in engineering applications, including adaptive control [24], vibrational control [19], and to justify the implementation of feedback through pulse-width modulation [16], [28]. Books that cover averaging theory for continuous-time systems include [11], [14] and [23].

The objective of this paper is to extend averaging theory to a class of hybrid dynamical systems. In contrast to a differential equation or difference equation, the state of a hybrid system changes continuously in some part of the state space and changes instantaneously in other parts of the state space. Hybrid systems thus combine continuous-time and discrete-time systems, providing a rich modeling framework and exhibiting fascinating dynamical behavior [17], [27]. An extensive summary of recent results for hybrid systems can be found in [6].

A few papers in the literature have related averaging and hybrid systems. The work of which we are aware focuses on a subclass of hybrid systems (usually switching systems) and aims to approximate a rapidly time-varying hybrid system by a non-hybrid system [4], [12] and [22]. In the current work, even the averaged system is a hybrid system. Motivation for such results include implementing hybrid feedback control using pulse-width modulation. In this case, it is desirable to prove that the pulse-width modulated implementation produces closed-loop behavior that is similar to the behavior that would be generated by implementing the hybrid feedback directly.

Averaging theory is based on two observations: 1) through an appropriate coordinate transformation, a rapidly time-varying system can be viewed as a small perturbation of a simplified, time-invariant, average system, and 2) the qualitative behavior of the solutions to classical dynamical systems is robust - in a mathematically precise sense not specified here - to small perturbations under appropriate regularity assumptions. When it comes to extending averaging theory to hybrid systems, we profit from the fact that the second observation above has already been established for a general class of hybrid systems (see [6] and [7]). Thus, the main task in extending averaging theory to hybrid systems is to establish the first observation above. It turns out that this observation can be established for a class of hybrid systems in a manner that is analogous to that for ordinary differential equations. It is noteworthy that averaging results are established without any extra requirement on how often and where jumps occur in the average hybrid system, beyond those dictated by the jump set of the averaged hybrid system.

The paper is organized as follows. In Section 2 we review some basic results about the behavior of hybrid systems and perturbations to hybrid systems. Section 3 introduces the class of time-varying hybrid systems for which we generate averaging results. In Section 4 we present averaging results for periodic hybrid systems. These results include both statements about "closeness" of solutions on compact time domains and also statements about semi-global, practical asymptotic stability. In this section, we use pulse-width modulated hybrid feedback to demonstrate the significance of the results. In Section 5 we extend the results of Section 4 to time-varying but not necessarily periodic hybrid systems. In Section 6 we mention some simple extensions that we have not pursued here due to space constraints.

Conclusions are provided in Section 7.

2 Preliminaries

The set of real numbers is denoted as $\mathbb{R}$. $\mathbb{B}$ denotes a closed unit ball in $\mathbb{R}^n$. $|x|$ denotes the Euclidean norm of a vector, that is $|x| = \sqrt{\sum_{i=1}^{n} x_1^2}$. Given a set $\mathcal{A}$, we denote the distance of point x to the set as $|x|_{\mathcal{A}} := \inf_{z\in\mathcal{A}} |x-z|$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if: (i) it is continuous; (ii) for each fixed $t \in \mathbb{R}_{\geq 0}$ the function $\beta(\cdot, t)$ is zero at zero, continuous and strictly increasing; (iii) for each fixed $s \in \mathbb{R}_{\geq 0}$ the function $\beta(s, \cdot)$ is decreasing to zero as it argument increases.

The definitions and results in this section pertain to the hybrid system

$$\begin{array}{rcll} \dot{\xi} & \in & F(\xi) & \xi \in C \\ \xi^+ & \in & G(\xi) & \xi \in D \ , \end{array} \tag{1}$$

where ξ^+ is a shorthand notation for the value of ξ right after a jump. See [6] or [7] for examples of hybrid systems of the form (1) and for the definition of a solution to a hybrid system. The data of (1) satisfies the following assumption (see [6] and [7]):

Assumption 1 (Basic conditions) *The sets $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ are closed; the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally bounded, outer semi-continuous*[1] *and, for each $\xi \in C$, $F(\xi)$ is nonempty and convex; and the set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally bounded, outer semi-continuous and, for each $\xi \in D$, $G(\xi)$ is nonempty.*

We also consider an inflation of (1) given by

$$\begin{array}{rcll} \dot{y} & \in & F_\delta(y) & \xi \in C_\delta \\ y^+ & \in & G_\delta(y) & \xi \in D_\delta \end{array} \tag{2}$$

where

$$\begin{array}{rcl} C_\delta & := & \{y : (y + \delta\mathbb{B}) \cap C \neq \emptyset\} \\ F_\delta(y) & := & \overline{\mathrm{co}}F((y + \delta\mathbb{B}) \cap C) + \delta\mathbb{B} \\ D_\delta & := & \{y : (y + \delta\mathbb{B}) \cap D \neq \emptyset\} \\ G_\delta(y) & := & G((y + \delta\mathbb{B}) \cap D) + \delta\mathbb{B} \end{array} \tag{3}$$

and $\overline{\mathrm{co}}S$ denotes the closed convex hull of a set S. Under Assumption 1, the inflated data $(C_\delta, F_\delta, D_\delta, G_\delta)$ also satisfies Assumption 1. Given a set $K \subset \mathbb{R}^n$, we use $\mathcal{S}(K)$, respectively $\mathcal{S}_\delta(K)$, to denote the set of maximal solutions to (1), respectively (2), starting in the set K.

[1]A set-valued mapping is outer semi-continuous if its graph $\{(x, y) : x \in \mathbb{R}^n, y \in F(x)\}$ is closed.

2.1 Compact time domains

The following definition and results will be exploited when discussing closeness of the solutions of a time-varying system to the solutions of the average system.

Definition 1 (Forward completeness) *A maximal solution is said to be forward complete if its domain is unbounded. A maximal solution is said to be forward pre-complete if its domain is compact or unbounded. A hybrid system is said to be forward pre-complete from a set if each solution starting in that set is forward pre-complete.*

The importance of considering forward pre-completeness rather than just completeness has been documented in [1] and [6]. The subsequent propositions are based on results in [7] (see also [6]), which assume forward completeness in place of forward pre-completeness. However, it is trivial to extend the proofs from the forward complete case to the forward pre-complete case.

Proposition 1 ([7], Corollary 4.7) *Let $K \subset \mathbb{R}^n$ be compact. Under Assumption 1, if the system (1) is forward pre-complete from K then, for each $T \in \mathbb{R}_{\geq 0}$, the set*

$$R_T(K) := \{z = \xi(t,j) : \xi \in \mathcal{S}(K)\ ,\ t + j \leq T\}$$

is compact.

Proposition 2 ([7], Corollary 5.2) *Let Assumption 1 hold and let $K \subset \mathbb{R}^n$ be compact. If the system (1) is forward pre-complete from K then, for each $T \in \mathbb{R}_{\geq 0}$, there exists $\delta > 0$ such that the reachable set*

$$R_T(K) := \{z = y(t,j) : y \in \mathcal{S}_\delta(K + \delta\mathbb{B})\ ,\ t + j \leq T\}$$

is compact.

Definition 2 (Closeness of solutions) *Two hybrid arcs $x_1 : \mathrm{dom}\ x_1 \to \mathbb{R}^n$ and $x_2 : \mathrm{dom}\ x_2 \to \mathbb{R}^n$ are said to be (T,ρ)-close if*

(a) for each $(t,j) \in \mathrm{dom}\ x_1$ with $t + j \leq T$ there exists s such that $(s,j) \in \mathrm{dom}\ x_2$, $|t - s| \leq \rho$ and $|x_1(t,j) - x_2(s,j)| \leq \rho$;

(b) for each $(t,j) \in \mathrm{dom}\ x_2$ with $t + j \leq T$ there exists s such that $(s,j) \in \mathrm{dom}\ x_1$, $|t - s| \leq \rho$ and $|x_2(t,j) - x_1(s,j)| \leq \rho$.

Proposition 3 ([7], Corollary 5.5) *Let Assumption 1 hold and let $K \subset \mathbb{R}^n$ be compact. If the system (1) is forward pre-complete from K then for each $\rho > 0$ and $T \geq 0$ there exists $\delta > 0$ such that for any solution y to (2) starting in $K + \delta\mathbb{B}$ there exists a solution ξ to (1) starting in K such that y and ξ are (T,ρ)-close.*

2.2 Asymptotic stability

The following definitions and results will be exploited when discussing semi-global, practical asymptotic stability of a time-varying system based on asymptotic stability for the average system.

Definition 3 (Asymptotic stability) *For the system (1), the compact set $\mathcal{A}$ is said to be asymptotically stable if*

- **(stability)** *for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each solution ξ of (1), $|\xi(0,0)|_{\mathcal{A}} \leq \delta$ implies $|\xi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} \xi$;*
- **(attractivity)** *there exists $c > 0$ such that, each maximal solution of (1) satisfying $|\xi(0,0)|_{\mathcal{A}} \leq c$ is forward pre-complete and, if complete, satisfies $\lim_{t+j\to\infty} |\xi(t,j)|_{\mathcal{A}} = 0$.*

The notion above is called "pre-asymptotic stability" in [1] and [6]. We drop the prefix "pre" since the distinction is not significant in the current context.

Definition 4 (Basin of attraction) *For the system (1) with an asymptotically stable compact set $\mathcal{A}$, the basin of attraction for $\mathcal{A}$, denoted $\mathcal{B}_{\mathcal{A}}$, is the set of initial conditions having the property that each maximal solution of (1) is forward pre-complete and, if complete, satisfies $\lim_{t+j\to\infty} |\xi(t,j)|_{\mathcal{A}} = 0$.*

Notice that any point not in $C \cup D$ belongs to $\mathcal{B}_{\mathcal{A}}$. The definition of basin of attraction used here is slightly different than the definition used in [7] but agrees with the definition used in [1] and [6] (where the term "basin of pre-attraction" is used). The results quoted below from [7], which pertain to basins of attraction, apply as well with the definition used here.

Proposition 4 ([7], Proposition 6.4) *Under Assumption 1 for the system (1), if the set $\mathcal{A}$ is asymptotically stable then its basin of attraction is an open set.*

Definition 5 (Proper indicator) *Let $\mathcal{B}_{\mathcal{A}}$ be an open set containing the compact set $\mathcal{A}$. A function $\omega : \mathcal{B}_{\mathcal{A}} \to \mathbb{R}_{\geq 0}$ is said to be a proper indicator function for $\mathcal{A}$ on $\mathcal{B}_{\mathcal{A}}$ if ω is continuous, $\omega(\xi) = 0$ if and only if $\xi \in \mathcal{A}$, and if the sequence $\{\xi_i\}_{i=1}^{\infty}$, $\xi_i \in \mathcal{B}_{\mathcal{A}}$, approaches the boundary of $\mathcal{B}_{\mathcal{A}}$ or is unbounded then $\omega(\xi_i)$ is unbounded.*

Proposition 5 ([7], Theorem 6.5) *Under Assumption 1 if, for the system (1), the compact set $\mathcal{A}$ is asymptotically stable with basin of attraction $\mathcal{B}_{\mathcal{A}}$ then for each proper indicator ω for $\mathcal{A}$ on $\mathcal{B}_{\mathcal{A}}$ there exists $\beta \in \mathcal{KL}$ such that, for each solution ξ of (1) starting in $\mathcal{B}_{\mathcal{A}}$,*

$$\omega(\xi(t,j)) \leq \beta(\omega(\xi(0,0)), t+j) \qquad \forall (t,j) \in \operatorname{dom} \xi \ .$$

Proposition 6 ([7], Theorem 6.6) *Under Assumption 1 if, for the system (1), the compact set $\mathcal{A}$ is asymptotically stable with basin of attraction $\mathcal{B}_{\mathcal{A}}$ then for each proper indicator ω for $\mathcal{A}$ on $\mathcal{B}_{\mathcal{A}}$ there exists $\beta \in \mathcal{KL}$ and for any such fixed β, each compact set $K \subset \mathcal{B}_{\mathcal{A}}$ and $\nu > 0$ there exists $\delta > 0$ such that each solution y of (2) starting in K satisfies, for all $(t,j) \in \operatorname{dom} y$,*

$$\omega(y(t,j)) \leq \beta(\omega(y(0,0)), t+j) + \nu \qquad \forall (t,j) \in \operatorname{dom} y \ .$$

3 Rapidly time-varying hybrid systems

In this paper, we consider time-varying hybrid systems where the state variable x may contain logic variables, counters, timers, and so on. In particular, we consider a time-varying hybrid system of the form

$$\begin{array}{lll} \left.\begin{array}{rcl} \dot{x} & = & f_\varepsilon(x,\tau) \\ \dot{\tau} & = & 1/\varepsilon \end{array}\right\} & & (x,\tau) \in C \times \mathbb{R}_{\geq 0} \\[2ex] \left.\begin{array}{rcl} x^+ & \in & G(x) \\ \tau^+ & \in & H(x,\tau) \end{array}\right\} & & (x,\tau) \in D \times \mathbb{R}_{\geq 0} \end{array} \tag{4}$$

where ε is a small, positive parameter, $C \subset \mathbb{R}^n$, $D \subset \mathbb{R}^n$, $f_\varepsilon : C \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $G : D \rightrightarrows \mathbb{R}^n$ and $H : D \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}_{\geq 0}$. We impose the following regularity assumptions on the function f_ε; the first condition imposes boundedness of f_0 on compact sets whereas the second condition provides continuity of f_ε at $\varepsilon = 0$ uniformly on compact sets:

Assumption 2 *(C, D, G) satisfy Assumption 1; for each compact set $K \subset \mathbb{R}^n$ there exists $M(K) > 0$ and for each $\delta > 0$ there exists $\varepsilon^*(K,\delta) > 0$ such that*

$$\begin{array}{rcll} |f_0(x,\tau)| & \leq & M & (x,\tau) \in (K \cap C) \times \mathbb{R}_{\geq 0} \\ |f_\varepsilon(x,\tau) - f_0(x,\tau)| & \leq & \delta & (x,\tau,\varepsilon) \in (K \cap C) \times \mathbb{R}_{\geq 0} \times (0, \varepsilon^*] \ . \end{array}$$

For each $(x,\tau) \in D \times \mathbb{R}_{\geq 0}$, $H(x,\tau)$ is a nonempty set.

Note that our results will hold under very weak assumptions on $H(\cdot,\cdot)$ as outlined in Assumption 2. We are interested in the following stability property for the system (4).

Definition 6 (Semi-global, practical asymptotic stability) *For the system (4), the compact set $\mathcal{A}$ is said to be semi-globally (with respect to $\mathcal{B}_{\mathcal{A}}$) practically asymptotically stable as $\varepsilon \to 0^+$ if, for each proper indicator ω for $\mathcal{A}$ on $\mathcal{B}_{\mathcal{A}}$ there exists $\beta \in \mathcal{KL}$ and for any such fixed β, each compact set $K \subset \mathcal{B}_{\mathcal{A}}$ and each $\nu > 0$ there exists $\varepsilon^* > 0$ such that $\varepsilon \in (0, \varepsilon^*]$ and $x(0,0) \in K$ imply*

$$\omega(x(t,j)) \leq \beta(\omega(x(0,0)), t+j) + \nu \qquad \forall (t,j) \in \operatorname{dom} x \ .$$

Semi-global practical stability was shown to arise naturally in various singular perturbation and averaging problems for continuous-time systems, see [30], [20]. Supposing that $\mathcal{B}_{\mathcal{A}}$ is the whole state space, the definition states that we can achieve an arbitrarily large domain of attraction and an arbitrarily small ultimate bound on all trajectories if we sufficiently reduce the parameter ε.

4 Periodic systems

In this section, we present two main averaging results for periodic hybrid systems (4). First we introduce the notion of average for hybrid systems of the form (4). Our first main result (Theorem 1) establishes closeness of solutions between the actual hybrid system (4) and its average on arbitrarily long compact time intervals. We note that this result does not require the average or actual system to be stable. Our second main result (Theorem 2) demonstrates that if the average system is globally asymptotically stable in an appropriate sense, then this implies semi-global practical asymptotic stability of the actual system (4). Our results are derived for stability with respect to arbitrary compact sets $\mathcal{A}$.

4.1 Assumptions

We now state assumptions that are needed in the sequel and also introduce the notion of average for the actual system (4).

Assumption 3 (Periodicity) *For each $x \in C$, the function $f_0(x, \cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is periodic.*

According to Assumption 3, there exists a real number $T > 0$ such that

$$f_0(x, \tau + T) = f_0(x, \tau) \qquad \forall (x, \tau) \in C \times \mathbb{R}_{\geq 0} \ .$$

For each $(x, \tau) \in C \times \mathbb{R}_{\geq 0}$, we define

$$\begin{aligned} F(x) &:= \frac{1}{T} \int_0^T f_0(x, s) ds \\ \sigma(x, \tau) &:= \int_0^\tau \left[f_0(x, s) - F(x) \right] ds \ . \end{aligned} \tag{5}$$

Note that $\sigma(x, \cdot)$ is periodic with period T and $\sigma(x, kT) = 0$ for each non-negative integer k. Using $F(\cdot)$ defined in (5) we now introduce the average system for the time-varying system (4):

$$\begin{aligned} \dot{x} &= F(x) \qquad x \in C \\ x^+ &\in G(x) \qquad x \in D \ . \end{aligned} \tag{6}$$

For simplicity, we consider the case where the jump dynamics for the actual system (4) and the average system (6) are identical. On the other hand, it is not difficult to allow the jump map G and jump set D to depend on the small parameter ε with an appropriate convergence property (see Section 5 of [7]) as ε approaches zero.

Assumption 4 (Regularity; periodic case) *The functions $F : C \to \mathbb{R}^n$ and $\sigma : C \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ defined in (5) are continuous and, for each compact set $K \subset \mathbb{R}^n$, there exists $L(K) > 0$ such that, for all $(x,t),(w,s) \in (K \cap C) \times \mathbb{R}_{\geq 0}$,*

$$\begin{array}{rcl} |\sigma(x,t)| & \leq & L \\ |\sigma(x,t) - \sigma(w,s)| & \leq & L\left(|x-w| + |t-s|\right) \ . \end{array} \tag{7}$$

The first condition in (7) is guaranteed by the bound on $f_0(x,\tau)$ in Assumption 2, the continuity of F, and the periodicity of $\sigma(x,\cdot)$. The second condition in (7) implies, using results in [2], that the generalized Jacobian of σ satisfies $\left|\frac{\partial \sigma}{\partial x}(x,\tau)\right| \leq L, \forall (x,\tau) \in (K \cap C) \times \mathbb{R}_{\geq 0}$. The second condition in (7) is satisfied when, in addition, f has a continuous derivative that is locally bounded uniformly in τ. The following example illustrates that f can be discontinuous and still Assumption 4 holds.

Example 1 (Hybrid PWM control) Consider a continuous-time plant of the form:

$$\begin{array}{rcl} \dot{\xi} & = & O(\xi) + P(\xi)U \\ y & = & Q(\xi) \end{array}$$

for which we designed a dynamic hybrid controller of the form:

$$\begin{array}{rcll} \dot{\eta} & = & R(\eta, y) & (\eta, y) \in C \\ \eta^+ & = & S(\eta) & (\eta, y) \in D \\ U & = & U(\eta, y) \ , & \end{array}$$

where we assume that $O(\cdot)$ and $R(\cdot,\cdot)$ are continuous while $Q(\cdot)$, $P(\cdot)$ and $U(\cdot,\cdot)$ are locally Lipschitz. Denote $x := (\xi^T \ \eta^T)^T$ and

$$\widetilde{f}_0(x) := \begin{pmatrix} O(\xi) \\ R(\eta, Q(\xi)) \end{pmatrix}; \ g_0(x) := \begin{pmatrix} P(\xi) \\ 0 \end{pmatrix}; \ G(x) := \begin{pmatrix} \xi \\ S(\eta) \end{pmatrix};$$

and $h_0(x) := U(\eta, Q(\xi))$. Note that the state x may include physical variables together with logic variables that are used to describe a stabilizing hybrid feedback control law $h_0(x)$. Several examples of hybrid feedback for nonlinear control systems appear in [6].

To implement the above controller via PWM control, we introduce

$$f(x,\tau) = \widetilde{f}_0(x) + g_0(x)u(h_0(x) - p(\tau))$$

where by our assumptions $g_0 : C \to \mathbb{R}^n$ and $h_0 : C \to [0,1]$ are locally Lipschitz, $\widetilde{f}_0$ is continuous, $u(s) = 1$ for $s \geq 0$ and $u(s) = 0$ for $s < 0$, and p is periodic with period one and $p(t) = t$ for $t \in [0,1)$. The closed loop system with the PWM implementation of the above controller takes form (4) where $f_\varepsilon(\cdot,\cdot) := f(\cdot,\cdot)$, $G(\cdot)$ is defined above and $H(\cdot,\cdot)$ is arbitrary.

It can be verified that $F(x) = \widetilde{f}_0(x) + g_0(x)h_0(x)$. The function F is continuous and

$$\begin{aligned}\sigma(x,\tau) &:= \int_0^\tau [f(x,s) - F(x)]\, ds \\ &= \int_0^\tau g_0(x)\,[u(h_0(x) - p(s)) - h_0(x)]\, ds \\ &= g_0(x) \int_0^\tau [u(h_0(x) - p(s)) - h_0(x)]\, ds \ .\end{aligned}$$

Then a straightforward calculation gives that, for each $\tau \in [0,1)$,

$$\sigma(x,\tau) = g_0(x)\,(\min\{\tau, h_0(x)\} - h_0(x)\tau) \ .$$

The function σ is locally Lipschitz since g_0 and h_0 are locally Lipschitz. Then, since $\sigma(x,\cdot)$ is periodic, Assumption 4 holds.

The calculations of this example together with the results in the next sections justify implementing the hybrid feedback using pulse-width modulation. $\triangle$

4.2 Results for compact time domains

Solutions of the system (4) are compared to the solutions of (6) where F is defined in (5). The main result of this section is stated as follows.

Theorem 1 *Suppose the system (4) satisfies Assumptions 2-4, and the compact set $K_0 \subset \mathbb{R}^n$ is such that the average system (6) is forward pre-complete from K_0. Under these conditions, for each $\rho > 0$ and each $T \geq 0$ there exists $\varepsilon^* > 0$ such that each solution of (4) starting in $K_0 + \rho\mathbb{B}$ is (T,ρ)-close to some solution of (6) starting in K_0.*

Proof. Given $T \geq 0$ and $\rho > 0$, let Proposition 3 generate $\delta > 0$ so that for any solution y to (2) starting in $K_0 + \delta\mathbb{B}$ there exists a solution ξ to (6) starting in K_0 such that y and ξ are $(T,\rho/2)$-close. Without loss of generality, we assume $\rho < 1$ and $\delta < 1$.

Let $\mathcal{S}(K_0)$ denote the set of solutions to the average system (6) starting in K_0 and define

$$\begin{aligned} R_T(K_0) &:= \{z = x(t,j) : x \in \mathcal{S}(K_0)\ ,\ t + j \leq T\} \\ K_1 &:= R_T(K_0) + \mathbb{B} \\ K &:= K_1 \cup G(K_1 \cap D)\ . \end{aligned} \qquad (8)$$

According to Proposition 1 and Assumption 2, the set K is compact.

Let Assumption 2, the set K, and δ generate $M(K) \geq 1$ and $\varepsilon_1^* > 0$ such that

$$\begin{array}{rcll} |f_0(x,\tau)| & \leq & M & \forall (x,\tau) \in (K \cap C) \times \mathbb{R}_{\geq 0} \\ |f_\varepsilon(x,\tau) - f_0(x,\tau)| & \leq & \delta/2 & \forall (x,\tau,\varepsilon) \in (K \cap C) \times \mathbb{R}_{\geq 0} \times (0,\varepsilon_1^*] \ . \end{array} \tag{9}$$

Let Assumption 4 and the set K generate $L(K) \geq 1$ so that the bounds (7) hold with $L = L(K)$ for all $(x,t),(w,s) \in (K \cap C) \times \mathbb{R}_{\geq 0}$.

Let $\varepsilon_2^* = \frac{\delta}{2\sqrt{n}L(K)(M(K)+1)}$, $\varepsilon_3^* = \rho/(2L(K))$, and $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*\}$. Consider $\varepsilon \in (0,\varepsilon^*]$.

Let (x,τ) be a solution to the system

$$\begin{array}{ll} \left.\begin{array}{rcl} \dot{x} & = & f_\varepsilon(x,\tau) \\ \dot{\tau} & = & 1/\varepsilon \end{array}\right\} & (x,\tau) \in (C \cap K) \times \mathbb{R}_{\geq 0} \\ \\ \left.\begin{array}{rcl} x^+ & \in & G(x) \cap K \\ \tau^+ & \in & H(x,\tau) \end{array}\right\} & (x,\tau) \in (D \cap K) \times \mathbb{R}_{\geq 0} \ . \end{array} \tag{10}$$

This system agrees with (4) but with C, D, and G intersected with K. Note that $\sigma(x,\tau)$ is defined on the set $C \times \mathbb{R}_{\geq 0}$.

Using Lemma 2 we let $\widetilde{\sigma}(x,\tau)$ be a function defined on $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ that satisfies the following:

1. $\widetilde{\sigma}(x,\tau) = \sigma(x,\tau)$ for all $(x,\tau) \in (C \cap K) \times \mathbb{R}_{\geq 0}$
2. For all $(x,\tau) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$
$$|\widetilde{\sigma}(x,\tau)| \leq L \ .$$
3. For all $x,y \in \mathbb{R}^n$ and $\tau, s \geq 0$ we have
$$|\widetilde{\sigma}(x,\tau) - \widetilde{\sigma}(y,s)| \leq \sqrt{n}L\left[|x-y| + |\tau - s|\right] \ .$$

By construction, for each $(t,j) \in \operatorname{dom}(x,\tau)$, $x(t,j) \in K$. Therefore, using Lemma 2 in the Appendix, equation (7) and the definition of ε^*, for all $\varepsilon \in (0,\varepsilon^*]$ and all $(t,j) \in \operatorname{dom}(x,\tau)$,

$$|\varepsilon\widetilde{\sigma}(x(t,j),\tau(t,j))| \leq \varepsilon L(K) \leq \varepsilon L(K)M(K) \leq \delta \ . \tag{11}$$

For each $(t,j) \in \operatorname{dom}(x,\tau)$, define

$$y(t,j) = x(t,j) - \varepsilon\widetilde{\sigma}(x(t,j),\tau(t,j)) \ . \tag{12}$$

It is immediate that y is a hybrid arc. For each $(t,j) \in \operatorname{dom} y$ such that $(t,j+1) \in \operatorname{dom} y$,

$$x(t,j) = y(t,j) + \varepsilon\widetilde{\sigma}(x(t,j),\tau(t,j)) \in D \cap K$$

which, according to (11), implies $y(t,j) \in D_\delta$, and

$$\begin{aligned} y(t,j+1) &= x(t,j+1) - \varepsilon\widetilde{\sigma}(x(t,j+1),\tau(t,j+1)) \\ &\in G(x(t,j)\cap D)\cap K + \delta\mathbb{B} \\ &\subset G(x(t,j)\cap D) + \delta\mathbb{B} \\ &= G\left((y(t,j)+\varepsilon\widetilde{\sigma}(x(t,j),\tau(t,j)))\cap D\right) + \delta\mathbb{B} \\ &\subset G_\delta(y(t,j)) \ . \end{aligned}$$

Moreover, for each j such that the set $I_j := \{t : (t,j) \in \text{dom } y\}$ has nonempty interior and for all $t \in I_j$,

$$y(t,j) + \varepsilon\widetilde{\sigma}(x(t,j),\tau(t,j)) \in C \cap K \ ,$$

which implies $y(t,j) \in C_\delta$, and, since σ is Lipschitz continuous, $y(\cdot,j)$ is locally absolutely continuous and for almost all $t \in I_j$ satisfies

$$\begin{aligned} \dot{y}(t,j) &\in \dot{x}(t,j) - \varepsilon\frac{\partial\widetilde{\sigma}(x(t,j),\tau(t,j))}{\partial x}\dot{x}(t,j) - \frac{\partial\widetilde{\sigma}(x(t,j),\tau(t,j))}{\partial\tau} \\ &= f_\varepsilon(x(t,j),\tau(t,j)) - \varepsilon\frac{\partial\widetilde{\sigma}(x(t,j),\tau(t,j))}{\partial x}\dot{x}(t,j) \\ &\qquad\qquad - f_0(x(t,j),\tau(t,j)) + F(x(t,j)) \\ &\in F(y(t,j)+\varepsilon\widetilde{\sigma}(x(t,j),\tau(t,j))) + \varepsilon\sqrt{n}L(K)(M(K)+1)\mathbb{B} + \frac{1}{2}\delta\mathbb{B} \\ &\in F(y(t,j)+\delta\mathbb{B}) + \delta\mathbb{B} \\ &\subset F_\delta(y(t,j)) \ . \end{aligned}$$

In the first equation above, the term

$$\begin{bmatrix} \dfrac{\partial\widetilde{\sigma}(x,\tau)}{\partial x} & \dfrac{\partial\widetilde{\sigma}(x,\tau)}{\partial\tau} \end{bmatrix}$$

should be understood to be the generalized Jacobian of $\widetilde{\sigma}$. The sequence of equalities and inclusions is then justified by the results in [2, Section 2.6], the definition of $\widetilde{\sigma}$, $\delta < 1$, Assumption 4 and (9).

We conclude that y is $(T,\rho/2)$-close to some solution ξ of the average system. Then, by the definition of y and ε^*, we conclude that x is (T,ρ)-close to ξ.

We now use the properties of the solutions of (10) to derive conclusions about the solutions of (4) that start in K_0. Let $(\widetilde{x},\widetilde{\tau})$ be a solution of (4) starting in K_0. If $\widetilde{x}(t,j) \in K$ for all $(t,j) \in \text{dom } \widetilde{x}$ such that $t+j \leq T$ then $\widetilde{x}(t,j)$ is also (T,ρ)-close to ξ. Now suppose there exists $(t,j) \in \text{dom } \widetilde{x}$ such that $\widetilde{x}(s,i) \in K$ for all $(s,i) \in \text{dom } \widetilde{x}$ satisfying $s+i \leq t+j$ and either

1. $(t, j+1) \in \mathrm{dom}\, \widetilde{x}$ and $\widetilde{x}(t, j+1) \notin K$ or else
2. there exists a montonically decreasing sequence r_i with $\lim_{i\to\infty} r_i = t$ such that $(r_i, j) \in \mathrm{dom}\, \widetilde{x}$ and $\widetilde{x}(r_i, j) \notin K$ for each i.

The solution $\widetilde{x}$ must agree with a solution x of (10) up to time (t, j), and thus must satisfy $\widetilde{x}(t, j) \in R_T(K) + \rho\mathbb{B}$. It then follows, by the definition of K in (8) and $\rho < 1$, which implies that $R_T(K) + \rho\mathbb{B}$ is contained in the interior of K, that neither of these two cases can occur. This establishes the result. ■

4.3 Results based on asymptotic stability

The main result of this section is stated as follows.

Theorem 2 *Suppose the system (4) satisfies Assumptions 2-4, and the compact set $\mathcal{A}$ is asymptotically stable with basin of attraction $\mathcal{B}_\mathcal{A}$ for the system (6). Under these conditions, for the time-varying hybrid system (4), the set $\mathcal{A}$ is semi-globally (with respect to $\mathcal{B}_\mathcal{A}$), practically asymptotically stable as $\varepsilon \to 0^+$.*

Proof. Let $\omega : \mathcal{B}_\mathcal{A} \to \mathbb{R}_{\geq 0}$ be a proper indicator function for $\mathcal{A}$ with respect to $\mathcal{B}_\mathcal{A}$. Let $\beta \in \mathcal{KL}$ be such that each solution ξ of the system (6) starting in $\mathcal{B}_\mathcal{A}$ satisfies, for all $(t, j) \in \mathrm{dom}\, \xi$,

$$\omega(\xi(t, j)) \leq \beta(\omega(\xi(0, 0)), t + j) \ .$$

Let $K_0 \subset \mathcal{B}_\mathcal{A}$ be compact. Define

$$\begin{aligned} K_1 &:= \left\{ x \in \mathcal{B}_\mathcal{A} : \omega(x) \leq \beta\left(\max_{y\in K_0} \omega(y), 0\right) + 1 \right\} \\ K &:= K_1 \cup G(K_1 \cap D) \ . \end{aligned} \tag{13}$$

The set K is a compact subset of $\mathcal{B}_\mathcal{A}$ since ω is a proper indicator and G is an outer semi-continuous mapping that maps $\mathcal{B}_\mathcal{A} \cap D$ to $\mathcal{B}_\mathcal{A}$.

Let $\nu \in (0, 1)$. Using Proposition 6, there exists $\delta > 0$ such that each solution y of (2) starting in $K + \delta\mathbb{B}$ satisfies, for all $(t, j) \in \mathrm{dom}\, y$,

$$\omega(y(t, j)) \leq \beta(\omega(y(0, 0)), t + j) + \nu/3 \ . \tag{14}$$

Without loss of generality we assume that $\delta < 1$. Let Assumption 2, the set K, and δ generate an $M(K) \geq 1$ and $\varepsilon_1^* > 0$ such that

$$\begin{aligned} |f_0(x, \tau)| &\leq M && \forall (x, \tau) \in (K \cap C) \times \mathbb{R}_{\geq 0} \\ |f_\varepsilon(x, \tau) - f_0(x, \tau)| &\leq \delta/2 && \forall (x, \tau, \varepsilon) \in (K \cap C) \times \mathbb{R}_{\geq 0} \times (0, \varepsilon_1^*] \ . \end{aligned} \tag{15}$$

Let Assumption 4 and the set K generate an $L(K) \geq 1$ so that the bounds (7) hold with $L = L(K)$ for all $(x, t), (w, s) \in (K \cap C) \times \mathbb{R}_{\geq 0}$.

Let $\varepsilon_2^* = \frac{\delta}{2L(K)(M(K)+1)}$.

Using the continuity of ω and β and the fact that, for each $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \rho)$ converges to zero as ρ tends to infinity, let $\varepsilon_3^* > 0$ be such that for all $x \in K$ and $y \in K + \varepsilon_3^* L(K)\mathbb{B}$ satisfying $|x - y| \leq \varepsilon_3^* L(K)$ and all $\rho \in \mathbb{R}_{\geq 0}$, we have

$$\begin{array}{rcl} \omega(x) & \leq & \omega(y) + \nu/3 \\ \beta(\omega(y), \rho) & \leq & \beta(\omega(x), \rho) + \nu/3 \ . \end{array} \tag{16}$$

Define $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*\}$ and consider $\varepsilon \in (0, \varepsilon^*]$. Next we introduce the system (10) and define y via (12). Using steps identical to the steps in the proof of Theorem 1 we can show that $y(t,j)$ is a solution of the inflated system (2). By construction we have that (14) holds. In turn, using (16), for all $(t,j) \in \operatorname{dom} x$,

$$\begin{array}{rcl} \omega(x(t,j)) & \leq & \omega(y(t,j)) + \nu/3 \\ & \leq & \beta(\omega(y(0,0)), t+j) + 2\nu/3 \\ & \leq & \beta(\omega(x(0,0)), t+j) + \nu \ . \end{array} \tag{17}$$

In particular, since $\nu < 1$, each solution of (10) starting in K_0 remains in the compact set

$$K_\nu := \left\{ x \in \mathcal{B}_{\mathcal{A}} : \omega(x) \leq \beta\left(\max_{y \in K_0} \omega(y), 0 \right) + \nu \right\}$$

which, due to the continuity of ω and $\nu < 1$, is contained in the interior of the compact set K_1 defined in (13). Finally, using steps identical to the steps in the proof of Theorem 1, we use the bound (17) on the solutions of (10) to derive conclusions about the solutions of (4) that start in K_0. This establishes the result. ■

5 Not necessarily periodic systems

This section demonstrates that we can also deal in a similar manner with not necessarily periodic systems. We present two main averaging results for a larger class of not necessarily periodic hybrid systems (4). First, we introduce the notion of generalized average for hybrid systems in this case. Our first main result (Theorem 3) establishes closeness of solutions between the actual hybrid system (4) and its generalized average on arbitrarily long compact time intervals. Our second main result (Theorem 4) demonstrates that if the generalized average system is globally asymptotically stable in an appropriate sense, then this implies semi-global practical asymptotic stability of the actual system (4). Our results are derived for stability with respect to arbitrary compact sets $\mathcal{A}$.

5.1 Assumptions

In this section, the periodicity assumption on $f_0(x, \cdot)$, Assumption 3, is relaxed to the following condition:

Assumption 5 (Well-defined "generalized" average) *For each* $(x, \tau) \in C \times \mathbb{R}_{\geq 0}$*, the limit*

$$\lim_{T\to\infty} \frac{1}{T} \int_{\tau}^{\tau+T} f_0(x,s)ds =: F(x) \tag{18}$$

exists, is independent of τ*, and the convergence is uniform in* τ*. In particular, for each compact set* $K \subset \mathbb{R}^n$ *there exists a continuous, nonincreasing function* $\Theta_K : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *satisfying* $\lim_{T\to\infty} \Theta_K(T) = 0$ *such that, for all* $(x, \tau, T) \in (K \cap C) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$*,*

$$\left| \int_{\tau}^{\tau+T} \left(f_0(x,s) - F(x)\right) ds \right| \leq T\Theta_K(T) \ .$$

With this assumption in place we define, for each $(x, \tau, \mu) \in C \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$,

$$\sigma(x, \tau, \mu) := \int_0^{\tau} e^{\mu(s-\tau)} \left[f_0(x,s) - F(x)\right] ds \ . \tag{19}$$

A key property of σ is established in the following lemma that is proved in the Appendix.

Lemma 1 *If Assumption 5 holds then, for each compact set* $K \subset \mathbb{R}^n$*, there exists a continuous, nondecreasing function* $\alpha_K : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that* $\alpha_K(0) = 0$ *and, for all* $(x, \tau, \mu) \in (K \cap C) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$*,*

$$\mu|\sigma(x, \tau, \mu)| \leq \alpha_K(\mu) \ .$$

Assumption 6 (Regularity; general case) *The function* $F : C \to \mathbb{R}^n$ *defined in Assumption 5 is continuous and for each compact set* $K \subset \mathbb{R}^n$ *there exists* $L(K) > 0$ *such that, for all* $(x,t), (w,s) \in (K \cap C) \times \mathbb{R}_{\geq 0}$

$$|\sigma(x,t,0) - \sigma(w,s,0)| \quad \leq \quad L\left(|x-w| + |t-s|\right) \ . \tag{20}$$

The Lipschitz property in Assumption 6 implies a similar Lipschitz condition for $\sigma(x, \tau, \mu)$ for each $\mu > 0$. Indeed, integrating by parts in the definition of σ, we have

$$\sigma(x,\tau,\mu) = \sigma(x,\tau,0) - \mu \int_0^{\tau} e^{\mu(r-\tau)} \sigma(x,r,0) dr$$

so that (without loss of generality we assume $s \geq t$)

$$\begin{aligned}
|\sigma(x,t,\mu) - \sigma(w,s,\mu)| \quad &\leq \quad |\sigma(x,t,0) - \sigma(w,s,0)| + \\
&\qquad \mu \int_0^t e^{\mu(r-t)} |\sigma(x,r,0) - \sigma(w, r+s-t, 0)| dr \\
&\leq \quad L\left(|x-w| + |t-s|\right) \left(1 + \mu \int_0^t e^{\mu(r-t)} dr\right) \\
&\leq \quad 2L\left(|x-w| + |t-s|\right) \ .
\end{aligned} \tag{21}$$

Like in the periodic case, this Lipschitz condition is used to bound the elements of the generalized Jacobian of σ. The following example illustrates a not necessarily periodic, discontinuous function $f(x,\tau)$ for which Assumptions 5 and 6 are satisfied.

Example 2 (Multi-rate hybrid PWM control) As an extension of Example 1, we consider a multi-rate PWM implementation of hybrid feedback control for multi-input nonlinear systems. In particular, let

$$f(x,\tau) = \widetilde{f}_0(x) + \sum_{i=1}^{n} g_i(x) u(h_i(x) - p_i(\tau))$$

where $g_i : C \to \mathbb{R}^n$ and $h_i : C \to [0,1]$ are locally Lipschitz, $\widetilde{f}_0$ is continuous, $u(s) = 1$ for $s \geq 0$ and $u(s) = 0$ for $s < 0$, and p_i is periodic with period T_i and $p(t) = t/T_i$ for $t \in [0, T_i)$. This function can be used to model a multi-rate implementation of a PWM hybrid controller. Consider $T > \max\{T_1, \ldots, T_n\}$. For each $i \in \{1, \ldots, n\}$, let $k_i = k_i(T)$ be a positive integer and $\widetilde{T}_i \in [0, T_i)$ such that $T = k_i T_i + \widetilde{T}_i$. Note that $k_i(T) \to \infty$ as $T \to \infty$. The set valued function $G(\cdot)$ in (4) is given and satisfies all stated assumptions.

We have

$$\begin{aligned}
\frac{1}{T}\int_{\tau}^{\tau+T} f(x,s)ds &= \widetilde{f}_0(x) + \sum_{i=1}^{n} g_i(x) \frac{1}{T}\int_{\tau}^{\tau+T} u(h_i(x) - p_i(s))ds \\
&= \widetilde{f}_0(x) + \sum_{i=1}^{n} g_i(x) \left(\frac{1}{T}\int_{\tau}^{\tau+k_iT_i} u(h_i(x) - p_i(s))ds \right. \\
&\qquad \left. + \int_{\tau+k_iT_i}^{\tau+k_iT_i+\widetilde{T}_i} u(h_i(x) - p_i(s))ds \right) \\
&= \widetilde{f}_0(x) + \sum_{i=1}^{n} g_i(x) \left(\frac{k_iT_i}{k_iT_i + \widetilde{T}_i} h_i(x) + \frac{v_i(x,\tau)}{k_iT_i + \widetilde{T}_i} \right)
\end{aligned}$$

where $|v_i(x,\tau)| \leq \widetilde{T}_i$. It follows that Assumption 5 is satisfied with

$$F(x) = \widetilde{f}_0(x) + \sum_{i=1}^{n} g_i(x) h_i(x) \ , \quad \Theta_K(T) = \frac{\bar{\Theta}_K}{T+1}$$

for some $\bar{\Theta}_K > 0$.

Now we verify Assumption 6. Note that $\sigma(x,\tau,0)$ is the sum of n periodic terms where each term has the form of the σ that appears in Example 1. Since, according to Example 1, each such term has the appropriate Lipschitz continuity property, it follows that $\sigma(x,\tau,0)$ satisfies Assumption 6. ■

5.2 Results for compact time domains

The main result of this section is stated as follows.

Theorem 3 *Suppose the system (4) satisfies Assumptions 2, 5, and 6, and the compact set $K_0 \subset \mathbb{R}^n$ is such that the average system (6) with (18) is forward pre-complete from K_0. Under these conditions, for each $\rho > 0$ and each $T \geq 0$ there exists $\varepsilon^* > 0$ such that each solution of (4) starting in $K_0 + \rho\mathbb{B}$ is (T, ρ)-close to some solution of (6), (18) starting in K_0.*

Proof. Given $T \geq 0$ and $\rho > 0$, let Proposition 3 generate $\delta > 0$ so that for any solution y to (2) starting in $K_0 + \delta\mathbb{B}$ there exists a solution ξ to (6) starting in K_0 such that y and ξ are $(T, \rho/2)$-close. Without loss of generality, we assume $\rho < 1$ and $\delta < 1$.

Let $\mathcal{S}(K_0)$ denote the set of solutions to the average system (6) starting in K_0 and define

$$\begin{array}{rcl} R_T(K_0) & := & \{z = x(t,j) : x \in \mathcal{S}(K_0)\ ,\ t + j \leq T\} \\ K_1 & := & R_T(K_0) + \mathbb{B} \\ K & := & K_1 \cup G(K_1 \cap D)\ . \end{array} \tag{22}$$

According to Proposition 1 and Assumption 2, the set K is compact.

Let Assumption 2, the set K, and δ generate $M(K) \geq 1$ and $\varepsilon_1^* > 0$ such that

$$\begin{array}{rcll} |f_0(x,\tau)| & \leq & M & \forall (x,\tau) \in (K \cap C) \times \mathbb{R}_{\geq 0} \\ |f_\varepsilon(x,\tau) - f_0(x,\tau)| & \leq & \delta/3 & \forall (x,\tau,\varepsilon) \in (K \cap C) \times \mathbb{R}_{\geq 0} \times (0, \varepsilon_1^*]\ . \end{array} \tag{23}$$

Let Assumption 5, the set K, and Lemma 1 generate α_K and pick $\mu > 0$ so that $\alpha_K(\mu) \leq \delta/3$.

Let Assumption 4 and the set K generate $L(K) \geq 1$ so that the bound (20) holds with $L = L(K)$ and for all $(x,t), (w,s) \in (K \cap C) \times \mathbb{R}_{\geq 0}$. Note that (21) also holds for this L, the selected μ, and for all $(x,t), (w,s) \in (K \cap C) \times \mathbb{R}_{\geq 0}$.

Let $\varepsilon_2^* = \delta/(6L(K)\sqrt{n}(M(K)+1))$, $\varepsilon_3^* = (3\rho\mu)/(2\delta))$, $\varepsilon_4^* = 3\mu$ and $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*\}$. Consider $\varepsilon \in (0, \varepsilon^*]$.

Let (x,τ) be a solution to the system

$$\begin{array}{ll} \left.\begin{array}{rcl} \dot{x} & = & f_\varepsilon(x,\tau) \\ \dot{\tau} & = & 1/\varepsilon \end{array}\right\} & (x,\tau) \in (C \cap K) \times \mathbb{R}_{\geq 0} \\ \\ \left.\begin{array}{rcl} x^+ & \in & G(x) \cap K \\ \tau^+ & \in & H(x,\tau) \end{array}\right\} & (x,\tau) \in (D \cap K) \times \mathbb{R}_{\geq 0}\ . \end{array} \tag{24}$$

This system agrees with (4) but with C, D, and G intersected with K. Using Lemma 2 and the discussion below Assumption 6, we let $\widetilde{\sigma}(x,\tau,\mu)$ be a function defined on $\mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ that satisfies the following:

1. $\widetilde{\sigma}(x,\tau,\mu) = \sigma(x,\tau,\mu)$ for all $(x,\tau) \in (C \cap K) \times \mathbb{R}_{\geq 0}$

2. For all $(x, \tau) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$

$$|\widetilde{\sigma}(x, \tau, \mu)| \leq \frac{\delta}{3\mu} \ .$$

3. For all $x, y \in \mathbb{R}^n$ and $\tau, s \geq 0$ we have

$$|\widetilde{\sigma}(x, \tau, \mu) - \widetilde{\sigma}(y, s, \mu)| \leq \sqrt{n} 2L \left[|x - y| + |\tau - s|\right] \ .$$

By construction, for each $(t, j) \in \mathrm{dom}(x, \tau)$, $x(t, j) \in K$. Therefore, using Lemma 2, (7) and the definition of ε^*, for all $\varepsilon \in (0, \varepsilon^*]$ and all $(t, j) \in \mathrm{dom}(x, \tau)$,

$$|\varepsilon\widetilde{\sigma}(x(t, j), \tau(t, j), \mu)| \leq \varepsilon\delta/(3\mu) \leq \delta \ . \tag{25}$$

For each $(t, j) \in \mathrm{dom}(x, \tau)$, define

$$y(t, j) := x(t, j) - \varepsilon\widetilde{\sigma}(x(t, j), \tau(t, j), \mu) \ . \tag{26}$$

It is immediate that y is a hybrid arc. For each $(t, j) \in \mathrm{dom}\ y$ such that $(t, j + 1) \in \mathrm{dom}\ y$,

$$x(t, j) = y(t, j) + \varepsilon\widetilde{\sigma}(x(t, j), \tau(t, j), \mu) \in D \cap K$$

which, according to (25), implies $y(t, j) \in D_\delta$, and

$$\begin{array}{rcl} y(t, j + 1) & = & x(t, j + 1) - \varepsilon\widetilde{\sigma}(x(t, j + 1), \tau(t, j + 1), \mu) \\ & \in & G(x(t, j) \cap D) \cap K + \delta\mathbb{B} \\ & \subset & G(x(t, j) \cap D) + \delta\mathbb{B} \\ & = & G\left((y(t, j) + \varepsilon\widetilde{\sigma}(x(t, j), \tau(t, j), \mu)) \cap D\right) + \delta\mathbb{B} \\ & \subset & G_\delta(y(t, j)) \ . \end{array}$$

Moreover, for each j such that the set $I_j := \{t : (t, j) \in \mathrm{dom}\ y\}$ has nonempty interior and for all $t \in I_j$,

$$y(t, j) + \varepsilon\widetilde{\sigma}(x(t, j), \tau(t, j), \mu) \in C \cap K \ ,$$

which implies $y(t,j) \in C_\delta$, and, since $\widetilde{\sigma}$ is Lipschitz continuous, $y(\cdot,j)$ is locally absolutely continuous and for almost all $t \in I_j$ satisfies

$$\begin{aligned}
\dot{y}(t,j) &\in \dot{x}(t,j) - \varepsilon\frac{\partial\widetilde{\sigma}(x(t,j),\tau(t,j),\mu)}{\partial x}\dot{x}(t,j) - \frac{\partial\widetilde{\sigma}(x(t,j),\tau(t,j),\mu)}{\partial\tau} \\
&= f_\varepsilon(x(t,j),\tau(t,j)) - \varepsilon\frac{\partial\widetilde{\sigma}(x(t,j),\tau(t,j),\mu)}{\partial x}\dot{x}(t,j) \\
&\qquad -f_0(x(t,j),\tau(t,j)) + F(x(t,j)) - \mu\widetilde{\sigma}(x(t,j),\tau(t,j),\mu) \\
&\in F(y(t,j) + \varepsilon\widetilde{\sigma}(x(t,j),\tau(t,j),\mu)) + \varepsilon(2L(K)\sqrt{n}(M(K)+1))\mathbb{B} \\
&\quad +\tfrac{1}{3}\delta\mathbb{B} + \tfrac{1}{3}\delta\mathbb{B} \\
&\in F(y(t,j) + \delta\mathbb{B}) + \delta\mathbb{B} \\
&\subset F_\delta(y(t,j))\ .
\end{aligned}$$

In the first equation above, the term

$$\left[\begin{array}{cc} \dfrac{\partial\widetilde{\sigma}(x,\tau,\mu)}{\partial x} & \dfrac{\partial\widetilde{\sigma}(x,\tau,\mu)}{\partial\tau} \end{array}\right]$$

should be understood to be the generalized Jacobian of σ. The sequence of equalities and inclusions is then justified by the results in [2, Section 2.6], the definition of σ, $\delta < 1$, Assumption 6 and (23).

We conclude that y is $(T,\rho/2)$-close to some solution ξ of the average system. Then, by the definition of y and ε^*, we conclude that x is (T,ρ)-close to ξ.

We now use the properties of the solutions of (10) to derive conclusions about the solutions of (4) that start in K_0. Let $(\widetilde{x},\widetilde{\tau})$ be a solution of (4) starting in K_0. If $\widetilde{x}(t,j) \in K$ for all $(t,j) \in \mathrm{dom}\ \widetilde{x}$ such that $t+j \leq T$ then $\widetilde{x}(t,j)$ is also (T,ρ)-close to ξ. Now suppose there exists $(t,j) \in \mathrm{dom}\ \widetilde{x}$ such that $\widetilde{x}(s,i) \in K$ for all $(s,i) \in \mathrm{dom}\ \widetilde{x}$ satisfying $s+i \leq t+j$ and either

1. $(t,j+1) \in \mathrm{dom}\ \widetilde{x}$ and $\widetilde{x}(t,j+1) \notin K$ or else

2. there exists a montonically decreasing sequence r_i with $\lim_{i\to\infty} r_i = t$ such that $(r_i,j) \in \mathrm{dom}\ \widetilde{x}$ and $\widetilde{x}(r_i,j) \notin K$ for each i.

The solution $\widetilde{x}$ must agree with a solution x of (10) up to time (t,j), and thus must satisfy $\widetilde{x}(t,j) \in R_T(K) + \rho\mathbb{B}$. It then follows, by the definition of K in (22) and $\rho < 1$, which implies that $R_T(K) + \rho\mathbb{B}$ is contained in the interior of K, that neither of these two cases can occur. This establishes the result. ■

5.3 Results based on asymptotic stability

The main result of this section is stated as follows.

Theorem 4 *Suppose the system (4) satisfies Assumptions 2, 5, and 6, and the compact set $\mathcal{A}$ is asymptotically stable with basin of attraction $\mathcal{B}_{\mathcal{A}}$ for the system (6) with (18). Under these conditions, for the time-varying hybrid system (4), the set $\mathcal{A}$ is semi-globally (with respect to $\mathcal{B}_{\mathcal{A}}$), practically asymptotically stable as $\varepsilon \to 0^+$.*

Sketch of proof: First, in a manner similar to the proof of Theorem 2 we choose a sufficiently small ε^* that guarantees that solutions of the inflated system (2) satisfy an appropriate stability bound. Then, using steps like in Theorem 3 we consider $y(\cdot,\cdot)$ defined by (26) and show that they are solutions of (2). The proof is then completed using the steps identical to the proof of Theorem 2.

6 Additional research directions

There are several useful extensions that have not been pursued here. First note that it is straightforward, equipped with robustness results for hybrid systems, to allow the sets C and D and the jump map G to depend on ε and to define the average system in terms of these objects when $\varepsilon = 0$.

An interesting research topic involves providing conditions that guarantee global asymptotic stability, rather than just semi-global practical asymptotic stability. The key idea for such results is that the asymptotic stability in the average system should be robust (in the sense of [1, Theorem 7.9] or [6, Theorem 15]) to the perturbations induced by the averaging coordinate transformation. In this direction, the concepts of homogeneity [8] and conic linearizations [9] for hybrid systems should be relevant.

Like in the case of ordinary differential equations [29], [20], it is interesting to consider the problem of averaging in the presence of exogenous disturbances and the input-to-state stability notion introduced in [26]. A related topic is averaging where the flow map is a set-valued mapping. This topic has been pursued for differential inclusions (see, for instance, [3], [21]) and impulsive differential inclusions (see [25] and references cited therein).

Finally, a challenging problem is averaging theory for hybrid systems when the rapid time variations are due to some state components changing much more rapidly than other state components. This situation is closely related to singular perturbation theory but where the fast dynamics do not necessarily settle down to a neighborhood of an equilibrium manifold. See [30], [5], [10] and the references therein.

7 Conclusions

Averaging theory has been developed for a class of hybrid dynamical systems. When the flow map admits an average and the time variations of the flow map are sufficiently fast, the solutions of the average system are a good approximation of the solutions to the original, time-varying system. In particular, if the average system does not exhibit finite escape times from a given compact set then, on compact time domains, each solution of the time-varying system is close to some solution of the average system. If the average system exhibits an asymptotically stable compact set then, for the time-varying system, that set is semi-globally, practically asymptotically stable in the rate of the time variations.

The techniques used to establish these results are very similar to classical techniques used for ordinary differential equations, once it is recognized that the behavior of hybrid systems is robust in much the same way that the behavior of ordinary differential equations is robust. This robustness for hybrid systems, established in [7], is the key to the results provided herein.

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A Proof of Lemma 1

The proof follows the calculations in [14, p. 415]. First observe that, according to the definition of σ and Assumption 5, for all $(x, \tau, T) \in (K \cap C) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$,

$$|\sigma(x, \tau + T, 0) - \sigma(x, \tau, 0)| \leq T\Theta_K(T) \ . \tag{27}$$

Also, integrating by parts in the definition of σ and adding and subtracting $\mu\sigma(x, \tau, 0) \int_0^\tau e^{\mu(s-\tau)} ds$, we have

$$\begin{aligned} \sigma(x, \tau, \mu) &= \sigma(x, \tau, 0) - \mu \int_0^\tau e^{\mu(s-\tau)} \sigma(x, s, 0) ds \\ &= e^{-\mu\tau}\sigma(x, \tau, 0) + \mu \int_0^\tau e^{\mu(s-\tau)} \left[\sigma(x, \tau, 0) - \sigma(x, s, 0)\right] ds \ . \end{aligned} \tag{28}$$

It follows from (27) and the fact that $\sigma(x, 0, 0) = 0$ that

$$\begin{aligned} \mu|\sigma(x, \tau, \mu)| &\leq e^{-\mu\tau}\mu\tau\Theta_K(\tau) + \mu^2 \int_0^\tau e^{\mu(s-\tau)}(\tau - s)\Theta_K(\tau - s) ds \\ &= e^{-\mu\tau}\mu\tau\Theta_K(\tau) + \mu^2 \int_0^\tau e^{-\mu r} r\Theta_K(r) dr \\ &= e^{-\mu\tau}\mu\tau\Theta_K(\tau) + \int_0^{\mu\tau} e^{-z} z\Theta_K(z/\mu) dz \ . \end{aligned}$$

Now consider the two cases: $\mu\tau \leq \sqrt{\mu}$ and $\mu\tau \geq \sqrt{\mu}$. In the first case, we have

$$e^{-\mu\tau}\mu\tau\Theta_K(\tau) + \int_0^{\mu\tau} e^{-z} z\Theta_K(z/\mu) dz \leq \Theta_K(0) \left(\sqrt{\mu} + \mu/2\right) \ . \tag{29}$$

In the second case, using $e^{-\eta}\eta \leq e^{-1}$ for all $\eta \geq 0$ and $\int_0^\infty e^{-z} z dz = 1$, we have

$$\begin{aligned} &e^{-\mu\tau}\mu\tau\Theta_K(\tau) + \int_0^{\mu\tau} e^{-z} z\Theta_K(z/\mu) dz \\ &\quad\leq e^{-1}\Theta_K(1/\sqrt{\mu}) + \Theta_K(0) \int_0^{\sqrt{\mu}} z dz + \Theta_K(1/\sqrt{\mu}) \int_{\sqrt{\mu}}^\infty e^{-z} z dz \\ &\quad\leq (e^{-1} + 1)\Theta_K(1/\sqrt{\mu}) + \Theta_K(0)\mu/2 \ . \end{aligned}$$

The result follows with $\alpha_K(0) = 0$ and, for $\mu > 0$,

$$\alpha_K(\mu) := \frac{\Theta_K(0)\mu}{2} + \max\left\{\Theta_K(0)\sqrt{\mu}, \Theta_K(1/\sqrt{\mu})\left(e^{-1} + 1\right)\right\} \ .$$

Since Θ_K is continuous, nonincreasing, and $\lim_{T\to\infty} \Theta_K(T) = 0$, it follows that α_K is continuous and nondecreasing. ■

The following claim, based on [18, Theorem 1], is a Lipschitz extension theorem that is useful in proving our main results.

Lemma 2 *Let $J \subset \mathbb{R}^n$ be closed, $L > 0$, and $M > 0$. For a vector-valued function $f := (f_1, \cdots, f_n)$ where $f_i : J \to \mathbb{R}$ are real-valued functions, define*

$$\tilde{g}_i(x) := \sup_{z \in J}\{f_i(z) - L|x - z|\}. \tag{30}$$

Let

$$\mathrm{sat}(s) := \frac{Ms}{\max\{M, |s|\}}, \tag{31}$$

and $g(x) := \mathrm{sat}(\tilde{g}(x))$ with $\tilde{g} := (\tilde{g}_1, \cdots, \tilde{g}_n)$. Then, the function g satisfies the following properties:

1. *$|g(x)| \le M$ for all $x \in \mathbb{R}^n$,*
2. *$|g(x) - g(y)| \le \sqrt{n}L|x - y|$ for all $x, y \in \mathbb{R}^n$ and*
3. *if, for all $i \in \{1, \cdots, n\}$, $x, y \in J$, $|f(x)| \le M$ and $|f_i(x) - f_i(y)| \le L|x - y|$, then $g(x) = f(x)$ for all $x \in J$.*

$\square$

Proof of Lemma 2: Noting $g(x) = \mathrm{sat}(\tilde{g}(x))$ and (31), it is straightforward that the first property is satisfied.

Let $\bar{N} = \{1, \cdots, n\}$. Let $k \in \bar{N}$ satisfy $|\tilde{g}_k(x) - \tilde{g}_k(y)| = \max_{i \in \bar{N}} |\tilde{g}_i(x) - \tilde{g}_i(y)|$. Without loss of generality, assume $\tilde{g}_k(x) \ge \tilde{g}_k(y)$. Using the fact $|\mathrm{sat}(\xi) - \mathrm{sat}(\psi)| \le |\xi - \psi|$ for all $\xi, \psi \in \mathbb{R}^n$, the extended function g satisfies

$$\begin{aligned}
&|g(x) - g(y)| \\
&\quad = |\mathrm{sat}(\tilde{g}(x)) - \mathrm{sat}(\tilde{g}(y))|, \\
&\quad \le |\tilde{g}(x) - \tilde{g}(y)|, \\
&\quad = \left(\sum_{i=1}^{n} |\tilde{g}_i(x) - \tilde{g}_i(y)|^2\right)^{\frac{1}{2}}, \\
&\quad \le \left(n \cdot |\tilde{g}_k(x) - \tilde{g}_k(y)|^2\right)^{\frac{1}{2}}, \\
&\quad \le \left(n \cdot \left|\sup_{a \in J}[(f_k(a) - L|a - x|) - (f_k(a) - L|a - y|)]\right|^2\right)^{\frac{1}{2}}, \\
&\quad \le \left(n \cdot \sup_{a \in J} |L|a - x| - L|a - y||^2\right)^{\frac{1}{2}}, \\
&\quad \le \left(nL^2 \cdot \sup_{a \in J} |a - x - a + y|^2\right)^{\frac{1}{2}} = \sqrt{n}L|x - y|,
\end{aligned}$$

for all $x, y \in \mathbb{R}^n$. Now, consider the third property. Let $M > 0$ be such that

$$|f(x)| \le M \quad \forall\, x \in J. \tag{32}$$

Let $L > 0$ be such that

$$|f_i(x) - f_i(y)| \leq L|x - y| \quad \forall\, x, y \in J, i \in \bar{N}. \tag{33}$$

Using (33), for $x \in J$, we have

$$\begin{aligned} f_i(x) &\leq \sup_{z \in J}\{f_i(z) - L|x - z|\} \\ &= \tilde{g}_i(x) = \sup_{z \in J}\{f_i(z) - f_i(x) + f_i(x) - L|x - z|\} \leq f_i(x) \end{aligned}$$

which shows $\tilde{g}_i(x) = f_i(x)$ for all $x \in J$. Then, with (32) and the definition of g, it follows that $g(x) = \mathrm{sat}_M(f(x)) = f(x)$ when $x \in J$. $\square$