Path-Following for Nonlinear Systems with Unstable Zero Dynamics

Dragan B. Dačić, Dragan Nešić and Petar V. Kokotović

***Abstract*— In the path-following problem formulated in this paper, it is required that the error between the system output and the desired geometric path eventually be less than any prespecified constant. If in a nonlinear MIMO system the output derivatives do not enter into its zero dynamics, a condition relating path geometry and stabilizability of the zero dynamics is given under which a solution to this problem exists. The solution is obtained by combining input-to-state stability and hybrid system methodologies.**

I. Introduction

Path-following has recently been formulated to replace the standard reference tracking as more suitable for certain applications [1]-[8]. The primary task in this framework is to steer an object to reach and follow a geometric path, that is, a manifold parameterized by a scalar θ, while properties of the object's motion along the path are of secondary importance. Path-following problems are first solved with respect to the path parameter θ, leaving the choice of a timing law for it as an additional degree of freedom. This additional flexibility of path-following is often a major advantage over reference tracking. For example, stable walking for biped robots [7] is achieved by ensuring that an output converges to a manifold parameterized by the angle of robot's stance leg with respect to the surface. This manifold parametrization led to a dramatic simplification in stability analysis of the underlying system. Another example is the use of an internal model in [8] parameterized by system's output position to reject the disturbance represented by path's varying curvature.

Here, we use the freedom to design a timing law for the path parameter θ to overcome the classical limitations imposed by unstable zero dynamics on tracking accuracy [13]. This idea was introduced in [9]-[11], where a time derivative of the path parameter θ is used to stabilize zero dynamics, while the original control variable is used for steering the system along the path.

We develop a path-following design for nonlinear systems with unstable zero dynamics. Our main assumption is that output derivatives do not enter into zero dynamics. Although restrictive, this assumption is a useful starting point common in the literature, see for example Chapter 2.3 in [12] and Chapter 2.4 in [16]. We design a feedback law for the original control variable to achieve convergence of an auxiliary output to the path. The auxiliary output is to have the same relative degree as the original output and the resulting zero dynamics are to be input-to-state stable (ISS) when the auxiliary output is treated as their input. A feedback law for a time derivative of θ is then designed to ensure closeness of the trajectories of the original and the auxiliary output. We give a sufficient condition on path geometry and stabilizability of the zero dynamics under which difference between the original and the auxiliary output can be made smaller than any prespecified constant. The idea of replacing the original output by an auxiliary one has already been exploited in the flatness approach, see [14]-[15] and references therein. The key difference here is that instead of searching for a flat output which approximates the original output, we construct a feedback law for a derivative of θ to reduce the difference between the two outputs.

In Section II we formulate a path-following problem for a class of nonlinear systems with unstable zero dynamics and give a sufficient condition for its solvability. We first give a design procedure in Section III, and then prove that the designed feedback laws solve the path-following problem in Section IV. An example is given in Section V and concluding remarks are in Section VI.

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II. Problem Statement

In this paper we focus on path-following for nonlinear systems with unstable zero dynamics. We consider systems with vector relative degree $\{r_1, \dots, r_m\}$ which can be transformed by means of a global coordinate and feedback transformation into

$$\dot{z} = f(z, y), \quad z \in I\!R^{n-r}, \tag{1}$$

$$\dot{x}^i = A_{r_i} x^i + B_{r_i} u_i, \ y_i = C_{r_i} x^i, \tag{2}$$

where $x \triangleq [(x^1)^T \dots (x^m)^T]^T$, $x^i \triangleq [x_1^i \dots x_{r_i}^i]^T$, $y \triangleq [y_1 \dots y_m]^T$, $u \triangleq [u_1 \dots u_m]^T$, $r \triangleq \sum_{i=1}^m r_i$, a C^1 map $f : I\!R^{n-r} \times I\!R^m \to I\!R^{n-r}$ satisfies $f(0,0) = 0$, and matrices $A_{r_i} \in I\!R^{r_i \times r_i}$, $B_{r_i}^T, C_{r_i} \in I\!R^{1 \times r_i}$ are given by

$$A_{r_i} = \begin{bmatrix} 0 & I_{r_i-1} \\ 0 & 0 \end{bmatrix}, \quad \begin{array}{l} B_{r_i}^T = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}, \\ C_{r_i} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}. \end{array}$$

For simplicity we assume that this transformation is valid globally, but stress that it needs to exist only in a set containing the path. The class of systems which are globally diffeomorphic to system (1)-(2) is characterized in [16]-[17]. Subsystem (1) represents the unstable zero dynamics driven by the output y but not by any of its derivatives. Subsystem (2) consists of m integrator chains relating the input u with the output y.

Definition 1: Path $\mathcal{Y}_d$ is one-dimensional manifold $\mathcal{Y}_d \triangleq \{y_d(\theta) = [y_{d1}(\theta) \dots y_{dm}(\theta)]^T : \theta \geq 0\}$ where $y_{di} : I\!R \to I\!R$ are smooth bounded functions, $y_{di} \in C^{r_i}$, $i = 1, \dots, m$. □

We augment system (1)-(2) with the following dynamics

$$\dot{\Theta} = A_{r^\star}\Theta + B_{r^\star}\omega, \ \Theta \triangleq [\theta \dots \theta^{(r^\star-1)}]^T, \tag{3}$$

where $r^\star = \max_i r_i$ is the maximal relative degree of an output component y_i in (2), Θ represents $r^\star$ additional states stemming from the path parameter θ, and ω is an additional control input representing the ${r^\star}^{th}$ derivative of θ, $\theta^{(r^\star)} \triangleq \omega$. A key feature of path-following is the possibility to design a feedback law for ω in (3) and determine $\theta(t)$ as a function of system states. Standard reference tracking, in which $\theta(t) = t$, is a special case of path-following where $\omega = 0$ and $\Theta(0) = [0 \ 1 \ 0 \dots 0]^T$.

Given the path-following accuracy $\epsilon > 0$, our goal is to construct feedback laws for the original control input u and for ω, the highest time derivative of the path parameter θ, to ensure that solutions of system (1)-(3) satisfy:

R1) Practical convergence: $\limsup_{t\to\infty}\|y(t)-y_d(\theta(t))\|\le\epsilon$,
R2) Forward motion: $\dot\theta(t)\ge 0$ and $\lim_{t\to\infty}\theta(t)=\infty$,
R3) State boundedness: $\forall t\ge 0$, $\|x(t)\|\le n_x(\|x(0)\|,\|z(0)\|)$, $\|z(t)\|\le n_z(\|x(0)\|,\|z(0)\|)$, $\|[\dot\theta(t)\ldots\theta^{(r^\star-1)}(t)]\|\le M_\Theta$,

where $n_x,n_z : I\!R^+\times I\!R^+\to I\!R^+$ are continuous functions and $M_\Theta>0$. We impose the requirement R2 to mimic reference tracking, that is, to prevent the output y to converge to a point on the path $\mathcal{Y}_d$, $\lim_{t\to\infty}\theta(t)=\infty$, and to prevent its backward motion along the path $\mathcal{Y}_d$, $\dot\theta\ge 0$.

The presence of unstable zero dynamics prevents asymptotic tracking of arbitrary reference signals, and in such situations the bound on tracking error ϵ in $\limsup_{t\to\infty}\|y(t)-y_d(t)\|\le\epsilon$ can not be made arbitrarily small [13]. We exploit the additional freedom of path-following and give conditions on path geometry and stabilizability of zero dynamics such that for arbitrarily small $\epsilon>0$ there exist feedback laws for u and ω guaranteeing R1-R3.

The case of interest is when zero dynamics (1) are unstable but the output y can be used as the control variable to stabilize them. In Assumption 1 we require existence of a feedback law $y=\sigma(z)$ and a Lyapunov function V_z, such that when the feedback law $y=\sigma(z)$ is implemented with a bounded error $d\triangleq y-\sigma(z)$, the derivative of V_z satisfies a particular bound.

Assumption 1: Let $C^{r^\star}$ function $\sigma : I\!R^{n-r}\to I\!R^m$, $\sigma(0)=0$, C^2 Lyapunov function $V_z : I\!R^{n-r}\to I\!R^+$, $V_z(0)=0$, class $\mathcal{K}^\infty$ functions $\alpha_i : I\!R^+\to I\!R^+$, $i=1,2,3$, and C^1 function $\pi : I\!R^{n-r}\to I\!R^m$, $\pi(0)=0$, satisfy $\forall z\in I\!R^{n-r}$ and $\forall d\in I\!R^m$

A1: $\alpha_1(\|z\|)\le V_z(z)\le\alpha_2(\|z\|)$,
A2: $\frac{\partial V_z}{\partial z}f(z,\sigma(z)+d)\le-\alpha_3(\|z\|)+\pi^T(z)d$,
A3: $\lim_{\|z\|\to\infty}\frac{\|\pi(z)\|}{\alpha_3(\|z\|)}=0$. □

In Assumption 2 we relate path geometry with the function π from Assumption 1.

Assumption 2: There exist constants $\theta_M>\theta_m>0$ such that $\forall z\in I\!R^{n-r}$ and $\forall\theta>0$, $\min_{s\in[\theta+\theta_m,\theta+\theta_M]}\pi^T(z)y_d(s)\le 0$. □

Remark 1: Assumption 1 implies that zero dynamics (1) are ISS wrt d. To show that we define a nondecreasing, continuous function

$$\rho(s)\triangleq\inf_{\|z\|\ge s}\frac{\alpha_3(\|z\|)}{\max\{1,\|\pi(z)\|\}},$$

which due to A3 satisfies $\lim_{s\to\infty}\rho(s)=\infty$. Consequently, there exists a function $\varrho\in\mathcal{K}^\infty$ satisfying $\varrho(s)<\rho(s)$, for all $s\neq 0$. From A2, for $\|z\|\ge\varrho^{-1}(\|d\|)$, $\|d\|\neq 0$, we obtain

$$\begin{aligned}\dot V_z &\le -\alpha_3(\|z\|)+\max\{1,\|\pi(z)\|\}\varrho(\|z\|)\\ &= \max\{1,\|\pi(z)\|\}\left\{-\frac{\alpha_3(\|z\|)}{\max\{1,\|\pi(z)\|\}}+\varrho(\|z\|)\right\}\\ &\le \max\{1,\|\pi(z)\|\}\left\{-\rho(\|z\|)+\varrho(\|z\|)\right\}<0,\end{aligned}$$

which implies that zero dynamics (1) are ISS wrt d, [20]. If $d\in\mathcal{L}_\infty$, then there exists a function $\nu\in\mathcal{KL}$ such that solutions of zero dynamics (1) satisfy

$$\|z(t)\|\le\nu(\|z(0)\|,t)+\alpha^\star(\sup_{t\ge 0}\|d(t)\|),\tag{4}$$

where $\alpha^\star\triangleq\alpha_1^{-1}\circ\alpha_2\circ\varrho^{-1}$, and the level set of the Lyapunov function V_z specified by the constant $c=\alpha_2\circ\varrho^{-1}(\limsup_{t\to\infty}\|d(t)\|)$

$$\Omega_z(c)\triangleq\{z\in I\!R^{n-rm} : V_z(z)\le c\},\tag{5}$$

is globally attractive and forward invariant for (1). □

We design a *hybrid* dynamic feedback law of the form

$$\begin{array}{ll} u=\varphi(x_s,x_c), & \omega=\phi(x_s,x_c,q),\\ \left[x_s^T\ x_c^T\right]^T\in\mathcal{S}(q), & \left\{\begin{array}{l}\dot x_c=f_c(x_s,x_c),\\ q=\text{const.}\end{array}\right.\\ \left[x_s^T\ x_c^T\right]^T\in\text{cl}\overline{\mathcal{S}}(q), & \left\{\begin{array}{l}x_c^+=f_d(x_s,x_c),\\ q^+=J_q(x_s,x_c,q)\end{array}\right.\end{array}\tag{6}$$

where x_c and q are its continuous and discrete states, respectively. The discrete state q belongs to a finite set $\mathcal{Q}$, $\mathcal{S}(q)$ is a closed set for all $q\in\mathcal{Q}$, $\text{cl}\overline{\mathcal{S}}(q)$ is the closure of its complement, $x_s\triangleq[x^T\ z^T\ \Theta^T]^T$, $X\triangleq[x_s^T\ x_c^T]^T$, and $x(t^+)\triangleq\lim_{s\downarrow t}x(s)$. We initialize subsystem (3) and feedback law (6) at particular values,

$$\Theta=\Theta_0,\quad x_c=x_{c0},\quad q=q_0.\tag{7}$$

We use the standard definition for the solutions of hybrid system (1)-(3) and (6), [18]-[19].

Definition 2: A function of time $[X^T(t)\ q(t)]^T$, $t\in[0,T_f)$, $T_f\le\infty$, is a solution of hybrid system (1)-(3), (6) if there exists a sequence of instants $\{t_j\}_{j=1}^N$, $t_j\le t_{j+1}$, $N\le\infty$, such that for all $t\in[t_j,t_{j+1})$ and $1\le j\le N$, the following conditions hold

$$\begin{array}{ll} X(t)\in\mathcal{S}(q(t)) & X(t_j)\in\text{cl}\overline{\mathcal{S}}(q(t_j))\\ \dot X(t)=F_s(X(t),q(t))\ , & X(t_j^+)=F_d(X(t_j),q(t_j))\ ,\\ q(t)=q(t_j) & q(t_j^+)=J_q(X(t_j),q(t_j))\end{array}\tag{8}$$

where the functions F_s and F_d are suitably defined by combining (1)-(3), (6). We say that at $t=t_j$ the j^{th} transition occurs. □

Theorem 1: Let zero dynamics (1) and the path $\mathcal{Y}_d$ satisfy Assumptions 1-2. Then for any accuracy $\epsilon>0$ there exists feedback law (6) and initial conditions (7), such that the solutions of system (1)-(3), (6) satisfy the requirements $\text{R}1-\text{R}3$. ■

Remark 2: For controllable linear systems (1)-(2), that is, when $f(z,y)=A_zz+B_zy$ and the pair (A_z,B_z) is controllable, Assumption 1 is automatically satisfied, while Assumption 2 is equivalent to existence of constants $\theta_M>\theta_m>0$ such that for all $w\in I\!R^m$ and $\theta\in I\!R^+$ we have $\min_{s\in[\theta+\theta_m,\theta+\theta_M]}w^Ty_d(s)\le 0$. Thus, for such linear systems the requirements $\text{R}1-\text{R}3$ hinge on Assumption 2, which is then a purely geometric condition. For example, when the output is two dimensional, Assumption 2 requires that for all $\theta\ge 0$ the section of the path $\mathcal{Y}_d$ corresponding to the interval $[\theta+\theta_m,\theta+\theta_M]$ enters into all quadrants. We note that a sufficient condition under which the requirements $\text{R}1-\text{R}3$ can be ensured for $\epsilon=0$ and controllable linear systems (1)-(2) is given in [11] and it implies Assumption 2. □

III. Design

In this Section we design feedback law (6) that ensures the requirements R1-R3. If a quantity in the designed feedback law depends on the path-following accuracy ϵ, we write ϵ in its superscript. By considering dependence of such quantities on ϵ, we study properties of the resulting family of feedback laws with an emphasis on members corresponding to small values of ϵ.

Let zero dynamics (1) and the path $\mathcal{Y}_d$ satisfy Assumptions 1-2, and fix the path-following accuracy $\epsilon>0$. Utilizing the function σ from Assumption 1, the feedback law for u in (6) is given by

$$u_i=\varphi_i(x_s,x_c)\triangleq y_{di}^{(r_i)}(\theta)+\sigma^{(r_i)}(z)-K_i\tilde e^i,\ \ \varphi\triangleq[\varphi_1\ldots\varphi_m]^T,\tag{9}$$

where the gains $K_i\in I\!R^{1\times r_i}$ make the matrix $A_i\triangleq A_{r_i}-B_{r_i}K_i$ Hurwitz, $\tilde e\triangleq[(\tilde e^1)^T\ldots(\tilde e^m)^T]^T$, $\tilde e^i\triangleq[\tilde e_1^i\ldots\tilde e_{r_i}^i]^T$, and

$$\tilde e^i=[x_1^i-\sigma_i-y_d\ \ \ldots\ \ x_{r_i}^i-\sigma_i^{(r_i-1)}-y_d^{(r_i-1)}]^T.\tag{10}$$

The hybrid dynamic feedback law for ω is based on the Lyapunov function V_z and the bound on its derivative from Assumption 1. It has one discrete state $q \in \mathcal{Q} \triangleq \{\text{Start, Wait, Align}\}$ denoting its mode, and three continuous states, $x_c \triangleq [\tau \ z_d^T \ \theta_d]^T$. The state τ measures the duration of the visit to the current mode, while z_d and θ_d respectively represent the values of zero dynamics' states z and path parameter θ at the instant of the most recent transition. The mode Start is used only initially until the zero dynamic states z and the errors $\tilde{e}$ do not become sufficiently small. In the mode Align the path parameter θ is driven to a value at which the bound on derivative of the Lyapunov function V_z is negative. During this mode the Lyapunov function V_z may increase. In the mode Wait the path parameter θ is kept constant and the Lyapunov function V_z is decreased for an amount that is larger than its increase during the previous visit to Align.

The feedback law for ω in (6) is given by

$$\omega = \phi^\epsilon(\tau, z_d, \theta_d, q) \triangleq \begin{cases} 0, & q \neq \text{Align}, \\ \phi_A^\epsilon(\tau, z_d, \theta_d), & q = \text{Align}, \end{cases} \tag{11}$$

where the function $\phi_A^\epsilon : I\!R^+ \times I\!R^{n-r} \times I\!R^+ \to I\!R$ is to be defined. The continuous and jump dynamics for the states $x_c = [\tau \ z_d^T \ \theta_d]^T$ in (6) are respectively given by

$$X \in \mathcal{S} \begin{cases} \dot{\tau} = 1 \\ \dot{z}_d = 0 \\ \dot{\theta}_d = 0 \end{cases}, \ X \in \text{cl}\bar{\mathcal{S}} \begin{cases} \tau^+ = 0 \\ z_d^+ = z \\ \theta_d^+ = \theta \end{cases}, \tag{12}$$

where $\mathcal{S} \triangleq \cup_{q\in\mathcal{Q}} \mathcal{S}(q)$. The sets on which continuous dynamics are valid for a particular mode are defined by

$$\begin{aligned} &\mathcal{S}(\text{Start}) \triangleq \text{cl}\{X : \ \tilde{e} \notin \Omega_{\tilde{e}}(\gamma_1^\epsilon) \vee z \notin \Omega_z(c^\star)\}, \\ &\mathcal{S}(\text{Align}) \triangleq \text{cl}\{X : \ \tau \leq T_a^\epsilon\}, \\ &\mathcal{S}(\text{Wait}) \triangleq \text{cl}\{X : \ z \notin \Omega_z(\tfrac{d^\epsilon}{4}) \wedge \dot{V}_z \leq -\gamma_2^\epsilon\}, \end{aligned} \tag{13}$$

where $\Omega_e(\gamma) \triangleq \{\tilde{e} \in I\!R^r : V_e(\tilde{e}) \leq \gamma\}$ is a level set of the Lyapunov function

$$\begin{aligned} &V_e(\tilde{e}) \triangleq \tilde{e}^T P \tilde{e}, \ P = \text{diag}\{P_1, \dots, P_m\}, \\ &P_i = P_i^T > 0, \ A_i^T P_i + P_i A_i \leq -I, \end{aligned} \tag{14}$$

$\dot{V}_z \triangleq \frac{\partial V_z}{\partial z} f(z, y)$ denotes the derivative of the Lyapunov function V_z from along the solutions of zero dynamics (1), and constants $c^\star$, γ_1^ϵ, γ_2^ϵ, and T_a^ϵ are given in (23) and (25), respectively.

The next mode of feedback law (6) is determined by the function $J_q^\epsilon : I\!R^{n-r} \times I\!R^+ \times \mathcal{Q} \to \mathcal{Q}$

$$J_q^\epsilon(z, \theta, q) = \begin{cases} \text{Align}, & q \neq \text{Align} \vee z \in \Omega_z(\frac{d^\epsilon}{2}), \\ \text{Wait}, & q = \text{Align} \wedge z \notin \Omega_z(\frac{d^\epsilon}{2}), \end{cases} \tag{15}$$

where the constant d^ϵ is defined in (25).

To construct the function ϕ_A^ϵ in (11) that governs evolution of the path parameter θ in the mode Align, we first define the function $J_\theta : I\!R^{n-r} \times I\!R^+ \to I\!R^+$ that determines its target value at the end of the visit in terms of the states at the beginning of the visit

$$J_\theta(z_d, \theta_d) = \arg \min_{s \in [\theta_d+\theta_m, \theta_d+\theta_M]} \pi^T(z_d) y_d(s). \tag{16}$$

Then ϕ_A^ϵ is any continuous function satisfying

$$\phi_A^\epsilon(\tau, z_d, \theta_d) = 0, \forall \tau \geq T_a^\epsilon, \quad \phi_A^\epsilon(0, z_d, \theta_d) = 0 \tag{17}$$

$$[\Phi_{r^\star}(T_a^\epsilon, z_d, \theta_d) \ \Phi_{r^\star-1}(T_a^\epsilon, z_d, \theta_d) \dots \Phi_1(T_a^\epsilon, z_d, \theta_d)]^T = [J_\theta(z_d, \theta_d) - \theta_d \ 0 \dots 0]^T, \tag{18}$$

$$\Phi_{r^\star-1}(\tau, z_d, \theta_d) \geq 0, \ \forall \tau \in [0, T_a^\epsilon), \tag{19}$$

where $\Phi_i(\tau, z_d, \theta_d) \triangleq \int_0^\tau \Phi_{i-1}(\tau_1, z_d, \theta_d) d\tau_1$, $\Phi_0(\tau, z_d, \theta_d) \triangleq \phi_A(\tau, z_d, \theta_d)$ represents the i^{th} integral of the function ϕ_A^ϵ and T_a^ϵ is defined in (25). Condition (17) combined with (11) ensures continuity of the signal $\omega(t)$ with respect to time, (18) requires that states of subsystem (3) reach the value $[J_\theta(z_d, \theta_d) \ 0 \dots 0]^T$ in T_a^ϵ seconds and remain there, while (19) corresponds to the requirement $\dot{\theta}(t) \geq 0$, see R2. We construct the function ϕ_A^ϵ by introducing a polynomial parametrization,

$$\phi_A^\epsilon(\tau, z_d, \theta_d) = \textstyle\sum_{k=1}^{k^\star} p_k^\epsilon(z_d, \theta_d) \tau^k, \ \tau \in [0, T_a^\epsilon), \tag{20}$$

and computing the parameters $p_k^\epsilon(z_d, \theta_d)$, $k = 1, \dots, k^\star$, such that conditions (17)-(19) hold.

We complete our design by setting the initial conditions for subsystem (3) to $\Theta(0) = \Theta_0 = [\theta_0 \ 0 \dots 0]^T$, $\theta_0 \geq 0$, and for (12) to $x_c(0) = x_{c0} \triangleq [0 \ 0 \ 0]^T$, $q(0) = \text{Start}$, that is, we restrict the initial conditions of system (1)-(3), (12) to the set

$$\mathcal{X}_0 \times \mathcal{Q}_0 \triangleq \{X : \Theta = \Theta_0, \ x_c = x_{c0}\} \times \{\text{Start}\}. \tag{21}$$

Remark 3: It can be shown that there exists a sufficiently large integer $k^\star$ such that for all z_d and θ_d there exist coefficients p_k^ϵ, $k = 1, \dots, k^\star$, for which polynomial (20) satisfies conditions (17)-(19). Moreover, using (16) and (18) it follows that the first $r^\star$ integrals of polynomial (20) over the interval $[0, T_a^\epsilon)$ are uniformly bounded for all z_d and θ_d. Then the coefficients p_k^ϵ can be selected to make polynomial (20) uniformly bounded as well,

$$\sup_{z_d \in I\!R^{n-r}, \ \theta_d \in I\!R^+, \ \tau \in (0, T_a^\epsilon]} |\phi_A^\epsilon(\tau, z_d, \theta_d)| \leq M_\omega^\epsilon, \tag{22}$$

where for small ϵ the constant M_ω^ϵ can be written as $M_\omega^\epsilon = M_\omega (1/T_a^\epsilon)^{r^\star}$. We note that bound (22) does not necessarily hold if the function J_θ is unbounded, that is, if $\theta_M = \infty$ in (16). $\square$

We now specify the relevant constants in our design. The path-related constants are given by

$$M_y \triangleq \sup_{\theta \geq 0} \|y_d(\theta)\|, \quad c^\star \triangleq \alpha_2 \circ \varrho^{-1}(M_y) + 1, \tag{23}$$

while constants related to the Lyapunov function V_z and zero dynamics (1) are given by $\pi^\star \triangleq \sup_{z \in \Omega_z(c^\star)} \|\pi(z)\|$,

$$\begin{aligned} &a_1 \triangleq \sup\nolimits_{z \in \Omega_z(c^\star)} | - \alpha_3(\|z\|) + \|\pi(z)\|(M_y + 1)|, \\ &a_2 \triangleq \sup\nolimits_{z \in \Omega_z(c^\star)} \left\|\tfrac{\partial \pi}{\partial z}\right\|, \quad a_3 \triangleq \sup\nolimits_{z \in \Omega_z(c^\star), \ \|y\| \leq 1+M_y} \|f\|, \\ &a_4 \triangleq \sup \ \left(\left\| \tfrac{\partial^2 V_z}{\partial z^2} \right\| a_3^2 + \left\| \tfrac{\partial V_z}{\partial z} \right\| \left(\left\| \tfrac{\partial f}{\partial z} \right\| a_3 + \left\| \tfrac{\partial f}{\partial y} \right\| \right) \right), \end{aligned} \tag{24}$$

where sup for a_4 is taken over $z \in \Omega_z(c^\star)$ and $\|y\| \leq M_y + 1$, and for brevity we write $f \triangleq f(z, \sigma(z)+y)$. Finally, the constants dependent on the path-following accuracy $\epsilon > 0$ are given by $d^\epsilon \triangleq \sup\{c \geq 0 : \sup_{z \in \Omega_z(c)} \|\sigma(z)\| \leq \epsilon\}$, and

$$\begin{aligned} &\gamma^\epsilon \triangleq \alpha_3 \circ \alpha_2^{-1}(d^\epsilon/2), \quad \gamma_1^\epsilon \triangleq p_m \min\left\{1, (\gamma^\epsilon/(4\pi^\star))^2\right\}, \\ &\gamma_2^\epsilon \triangleq \gamma^\epsilon/4, \quad T_w^\epsilon \triangleq \gamma^\epsilon/(4a_4), \quad \Delta^\epsilon \triangleq \min\left\{d^\epsilon/4, \gamma_2^\epsilon T_w^\epsilon\right\}, \\ &\gamma_3^\epsilon \triangleq \min\left\{\Delta^\epsilon, d^\epsilon\right\}/2, \quad T_a^\epsilon \triangleq \min\left\{1, \gamma_3^\epsilon/a_1, \gamma^\epsilon/(4M_\theta a_2 a_3)\right\}. \end{aligned} \tag{25}$$

Note that the constants a_i, $i = 1, 2, 3, 4$, are finite because they represent maxima of continuous functions over compact sets, and the constant d^ϵ is positive, because the function σ is smooth and satisfies $\sigma(0) = 0$.

IV. Proof

In this Section we prove Theorem 1. We first infer properties induced by the feedback law for u in (9), and then with three

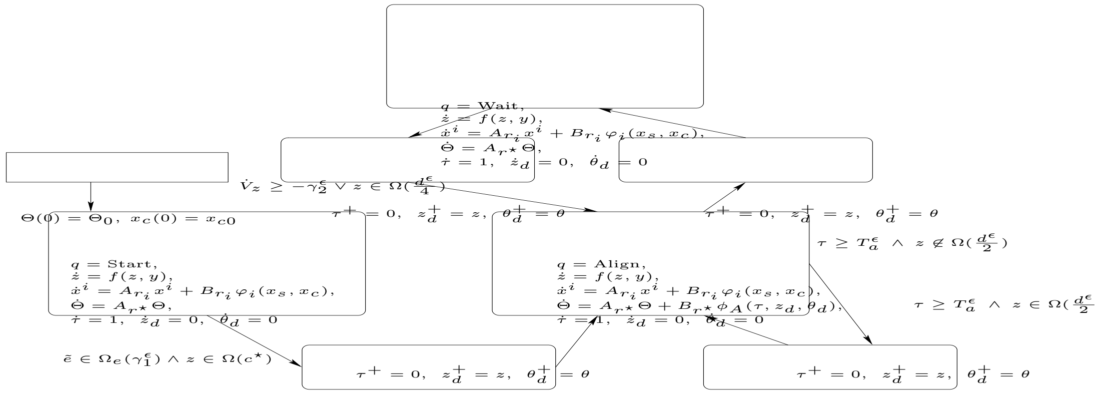

Fig. 1. Hybrid system (1)-(3), (9), (11)-(12), where $i = 1, \ldots, m$.

Lemmata prove that under feedback laws (9), (11) all solutions of system (1)-(3), (12) starting from set (21) satisfy the requirements $\mathrm{R1} - \mathrm{R3}$. Substituting coordinates (10) and feedback law (9) into system (1)-(3), we obtain

$$\begin{aligned} \dot z &= f(z, \sigma(z) + y_d(\theta) + \tilde e_y), && (26) \\ \dot{\tilde e}^i &= A_i \tilde e^i,\ i = 1, \ldots, m, && (27) \\ \dot\Theta &= A_{r^\star}\Theta + B_{r^\star}\omega, && (28) \end{aligned}$$

where $\tilde e_y \triangleq [e_1^1 \ldots e_1^m]^T$. Taking the derivative of Lyapunov function (14) along the solutions of subsystem (27), we get that $\dot V_e \le -\|\tilde e\|^2$, and thus the errors $\tilde e_y$ converge to zero,

$$\|\tilde e_y(t)\| \le \|\tilde e(t)\| \le (p_M/p_m)\|\tilde e(0)\| e^{-t/p_M}. \qquad (29)$$

Using (5) with $d = y_d(\theta) + \tilde e_y$ and (29), we conclude that feedback law (9) renders globally attractive and forward invariant a level set $\Omega_z(c^\star)$ of the Lyapunov function V_z for (26), that is,

$$\limsup_{t\to\infty} V_z(z(t)) < c^\star,\ c^\star = \alpha_2 \circ \varrho^{-1}(M_y) + 1. \qquad (30)$$

Remark 4: Feedback law for u in (9) is designed to achieve asymptotic path-following for the auxiliary output $\tilde y = y - \sigma(z)$, $\lim_{t\to\infty} \|\tilde y(t) - y_d(\theta(t))\| = 0$. This auxiliary output is selected such that the resulting zero dynamics of system (1)-(2) are ISS when the auxiliary output $\tilde y$ is treated as their input. □

Lemma 1: Under feedback laws (9), (11) all solutions of system (1)-(3), (12) starting from set (21) satisfy R2. ■

Proof: Due to (21) the initial mode is Start, $q(0) = \mathrm{Start}$. By using bounds (29)-(30) and set $\mathcal{S}(\mathrm{Start})$ in (13) we conclude that there exits an instant $t_1 \ge 0$ at which the first transition into the mode Align occurs, $q(t_1^+) = \mathrm{Align}$. Combing feedback law (11) and initial condition $\Theta(0) = \Theta_0$, we get $\Theta(t_1) = \Theta_0$. Moreover, $\forall t \ge t_1$ we have that $z(t) \in \Omega_z(c^\star)$ and $\tilde e(t) \in \Omega_e(\gamma_1^\epsilon)$, which with the constant γ_1^ϵ in (25) implies

$$\|\tilde e_y(t)\| \le \sqrt{\frac{p_m}{p_m}\|\tilde e(t)\|^2} \le \sqrt{\frac{1}{p_m} V_e(\tilde e(t))} \le \sqrt{\frac{\gamma_1^\epsilon}{p_m}} \le 1. \qquad (31)$$

Let the j^{th} transition be into the mode Align, $q(t_j^+) = \mathrm{Align}$, and let the states of subsystem (3) satisfy $\Theta(t_j) = [\theta(t_j)\ 0 \ldots 0]^T$. (This hypothesis holds for $j = 1$). We show that there exists an instant $\hat t \ge t_j + T_a^\epsilon$ at which the mode Align is revisited, while the value of the path parameter at $t = \hat t$ has increased by at least $\theta_m > 0$, that is,

$$\begin{aligned} &q(\hat t^+) = \mathrm{Align}, \quad \Theta(\hat t) = [\theta(\hat t)\ 0 \ldots\ 0]^T, \\ &\theta(\hat t) \ge \theta(t_j) + \theta_m, \quad \dot\theta(t) \ge 0, \forall t \in [t_j, \hat t). \end{aligned} \qquad (32)$$

By using induction the claim of Lemma 1 then follows.

Duration of visits to the mode Align is fixed, see (13), thus the $j + 1^{st}$ transition occurs at the instant $t_{j+1} = t_j + T_a^\epsilon$. During the interval $t \in [t_j, t_{j+1})$, combining (11), (18)-(19) and (16), we conclude that $\dot\theta(t) \ge 0$ and $\Theta(t_{j+1}) = [\theta(t_{j+1})\ 0 \ldots 0]^T$, where $\theta(t_{j+1}) - \theta(t_j) \ge J_\theta(z_d(t_{j+1}), \theta_d(t_{j+1})) \ge \theta_m$. Then if $z(t_{j+1}) \in \Omega_z(\frac{d^\epsilon}{2})$, from (15) we get that the $j + 1^{st}$ transition is into the mode Align, hence (32) holds for $\hat t = t_{j+1}$.

If $z(t_{j+1}) \notin \Omega_z(\frac{d^\epsilon}{2})$ the $j + 1^{st}$ transition is into the mode Wait, $q(t_{j+1}^+) = \mathrm{Wait}$. We show that there exists an instant $\bar t > t_{j+1}$ such that

$$z(\bar t) \in \Omega_z\left(d^\epsilon/4\right) \ \vee\ \dot V_z(\bar t) \ge -\gamma_2^\epsilon. \qquad (33)$$

Then from the definition of set $\mathcal{S}(\mathrm{Wait})$ in (13) and (15), we get that $j+2^{nd}$ transition is into the mode Align, $q(\bar t^+) = \mathrm{Align}$, and it occurs at $t_{j+2} = \bar t$. Using (11) we obtain that $\omega(t) = 0$ during the visit to the mode Wait, $t \in [t_{j+1}, t_{j+2})$, hence $\Theta(t_{j+2}) = \Theta(t_{j+1})$. Thus, (32) holds for $\hat t = t_{j+2} > t_j + T_a^\epsilon$.

Taking the derivative of the Lyapunov function V_z along the solutions of (26), using Assumption 1, $\tilde e(t_{j+1}) \in \Omega_e(\gamma_1^\epsilon)$, and $z(t_{j+1}) \in \Omega_z^- \triangleq \Omega_z(c^\star) \setminus \Omega_z(\frac{d^\epsilon}{2})$ we get

$$\begin{aligned} &\dot V_z(t_{j+1}) \le -\alpha_3(\|z(t_{j+1})\|) + \pi^T(t_{j+1})[y_d(t_{j+1}) \\ &+\tilde e_y(t_{j+1})] \le -\inf_{z\notin\Omega_z^-} \alpha_3(\|z\|) + \pi^T(t_{j+1}) y_d(t_{j+1}) \\ &+\pi^\star \sup_{\tilde e\in\Omega_e(\gamma_1^\epsilon)} \|\tilde e\|, \end{aligned} \qquad (34)$$

where, for brevity, we write $\pi(t) \triangleq \pi(z(t))$ and $y_d(\theta(t)) \triangleq y_d(t)$. We compute the bound on the second term in (34) from

$$\begin{aligned} &\pi^T(t_{j+1}) y_d(t_{j+1}) \le \pi^T(t_j) y_d(t_{j+1}) + M_y\|\pi(t_{j+1}) - \pi(t_j)\| \\ &\le \pi^T(t_j) y_d(J_\theta(z(t_j), \theta(t_j))) + M_y\|\pi(t_{j+1}) - \pi(t_j)\| \\ &\le M_y\|\pi(t_{j+1}) - \pi(t_j)\| \\ &\le M_y a_2 \sup_{z\in\Omega_z^-,\ \|\tilde e_y\|\le 1,\ \theta\ge 0} \int_{t_j}^{t_{j+1}} \|\dot z(t)\| dt \\ &\le M_y a_2 a_3 (t_{j+1} - t_j), \end{aligned} \qquad (35)$$

where the third inequality follows from Assumption 2 and (16). Substituting (35) into (34) and utilizing the constants defined in (23)-(25) we obtain

$$\dot V_z(t_{j+1}) \le -\gamma^\epsilon + T_a^\epsilon M_y a_2 a_3 + \pi^\star \sqrt{\frac{\gamma_1^\epsilon}{p_m}} \le -\frac{1}{2}\gamma^\epsilon < -\gamma_2^\epsilon \qquad (36)$$

which ascertains that at the instant of the $j+1^{st}$ transition into the mode Wait the derivative of Lyapunov function V_z is negative. From (36) we deduce existence of an instant $t_{j+2} > t_{j+1}$ at which either V_z becomes smaller than $\frac{d^\epsilon}{4}$, $z(t_{j+2}) \in \Omega_z(\frac{d^\epsilon}{4})$, or its derivative becomes larger than $-\gamma_2^\epsilon$, $\dot{V}_z(t_{j+2}) \geq -\gamma_2^\epsilon$. Thus condition (33) holds and the claim of Lemma 1 follows. ■

The proof of Lemma 1 implies that at most two transitions occur within any interval of duration T_a^ϵ, $t_{j+2} - t_j \geq T_a^\epsilon$, $j > 1$, and that the sequence $\{t_j\}_{j=1}^\infty$ is unbounded, that is, the transitions never stop. Thus for any fixed $\epsilon > 0$ hybrid system (1)-(3), (12) does not have Zeno solutions. Using dependence of the constant T_a^ϵ on the path-following accuracy ϵ in (25), we conclude that smaller ϵ may lead to more frequent transitions.

We next show the trajectory closeness for the original output y and the auxiliary output $\tilde{y}$, that is, $\limsup_{t\to\infty} \|y(t) - \tilde{y}(t)\| = \limsup_{t\to\infty} \|\sigma(z(t))\| \leq \epsilon$. Since feedback law for u in (9) guarantees convergence of the auxiliary output $\tilde{y}$ to the path $\mathcal{Y}_d$, $\lim_{t\to\infty} \|\tilde{y}(t) - y_d(\theta(t))\| = 0$, this implies the requirement R1. We prove that the level set $\Omega_z(d^\epsilon)$ of the Lyapunov function V_z is globally attractive and forward invariant for zero dynamics (26), where the constant $d^\epsilon > 0$ in (25) is chosen such that $z \in \Omega_z(d^\epsilon)$ implies $\|\sigma(z)\| \leq \epsilon$.

Lemma 2: Under feedback laws (9) and (11) all solutions of system (1)-(3), (12) starting from the set (21) satisfy R1. ■

Proof: We compute the minimal decrease of the Lyapunov function V_z during a visit to the mode Wait. Let the j^{th} transition be in the mode Wait, implying that $z(t_j) \notin \Omega_z(\frac{d^\epsilon}{2})$ due to (15), and $\dot{V}_z(t_j) \leq -\frac{1}{2}\gamma^\epsilon$ due to (36). Using the argument below (36) we conclude existence of an instant $t = t_{j+1}$ at which we either have $z(t_{j+1}) \in \Omega_z(\frac{d^\epsilon}{4})$ or $\dot{V}_z(t_{j+1}) \geq -\gamma_2^\epsilon = -\frac{1}{4}\gamma^\epsilon$. If $z(t_{j+1}) \in \Omega_z(\frac{d^\epsilon}{4})$ then $V_z(z(t_{j+1})) - V_z(z(t_j)) \leq -\frac{d^\epsilon}{4}$. If $\dot{V}_z(t_{j+1}) = -\frac{1}{4}\gamma^\epsilon$, the minimal time needed for the derivative of Lyapunov function V_z to reach this value is equal to $T_w^\epsilon \triangleq \frac{1}{4a_4}\gamma^\epsilon \leq t_{j+1} - t_j$, since

$$\frac{1}{4}\gamma^\epsilon \leq \dot{V}_z(t_{j+1}) - \dot{V}_z(t_j) = \int_{t_j}^{t_{j+1}} \ddot{V}_z(t)dt \leq (t_{j+1} - t_j)a_4,$$

where a_4 is given in (25). Hence, we obtain $V_z(t_{j+1}) - V_z(t_j) \leq -\frac{1}{4}\gamma^\epsilon T_w^\epsilon$. Combining the two cases, the minimal decrease of the Lyapunov function V_z during a visit to the mode Wait is

$$V_z(t_{j+1}) - V_z(t_j) \leq -\min\{d^\epsilon/4, \gamma^\epsilon T_w^\epsilon/4\} \triangleq -\Delta^\epsilon. \quad (37)$$

Similarly, we compute the maximal increase of the Lyapunov function V_z during a visit to the mode Align. Since duration of the visit to Align is equal to T_a^ϵ, using (25) we have

$$\begin{aligned} &V_z(t_{j+1}) - V_z(t_j) \\ &\leq \int_{t_j}^{t_{j+1}} \left(-\alpha_3(\|z(t)\|) + \pi^T(t)(y_d(t) + \tilde{e}_y(t))\right) dt \leq T_a^\epsilon a_1 \\ &\leq a_1 \min\left\{\frac{\gamma_3^\epsilon}{a_1}, \frac{\gamma^\epsilon}{4M_\theta a_2 a_3}\right\} \leq \min\left\{\tfrac{1}{2}\Delta^\epsilon, \tfrac{1}{2}d^\epsilon\right\}. \end{aligned} \quad (38)$$

Finally, we show that the level set $\Omega_z(d^\epsilon)$ of the Lyapunov function V_z is forward invariant and globally attractive for zero dynamic (26) by proving the following two claims

$$\exists j^\star,\ q(t_{j^\star}^+) = \text{Align},\ z(t_{j^\star}) \in \Omega_z(d^\epsilon/2), \quad (39)$$

$$\begin{aligned} &q(t_j^+) = \text{Align},\ z(t_j) \in \Omega_z(d^\epsilon/2) \Rightarrow \\ &\forall t > t_j^+,\ z(t) \in \Omega_z(d^\epsilon). \end{aligned} \quad (40)$$

To prove claim (39), let the j^{th} transition be into the mode Align, $q(t_j^+) = \text{Align}$, and $z(t_j) \notin \Omega_z(d^\epsilon/2)$. From (13) and (30), we have that $z(t_j) \in \Omega_z(c^\star)$. If at the end of the current visit, that is, at $t_{j+1} = t_j + T_a^\epsilon$, we have that $z(t_{j+1}) \in \Omega_z(d^\epsilon/2)$, using (15) claim (39) is satisfied for $j^\star = j+1$. If $z(t_{j+1}) \notin \Omega_z(d^\epsilon/2)$ combining (15) and (37)-(38) we conclude that the $j + 2^{nd}$ transition is again into the mode Align, and $z(t_{j+2}) \in \Omega_z(c^\star - \frac{1}{2}\Delta^\epsilon)$. Repeating this argument k times, where $k \geq \frac{2c^\star - d^\epsilon}{\Delta^\epsilon}$, we obtain that $z(t_{j+2k}) \in \Omega_z(c^\star - \frac{k}{2}\Delta^\epsilon) \subseteq \Omega_z(\frac{d^\epsilon}{2})$. This proves claim (39) for $j^\star = j + k$.

To prove claim (40), let the j^{th} transition be into the mode Align, $q(t_j^+) = \text{Align}$, and $z(t_j) \in \Omega_z(d^\epsilon/2)$. If at $t_{j+1} = t_j + T_a^\epsilon$ we have that $z(t_{j+1}) \in \Omega_z(d^\epsilon/2)$, then the next mode is again Align, $q(t_{j+1}^+) = \text{Align}$, and using (38) we obtain that $\forall t \in [t_j, t_{j+1})$, $z(t) \in \Omega_z(d^\epsilon)$. If $z(t_{j+1}) \notin \Omega_z(d^\epsilon/2)$, then from (15) the next two modes are Wait and Align, $q(t_{j+1}^+) = \text{Wait}$ and $q(t_{j+2}^+) = \text{Align}$. Combining (37)-(38) we get that $z(t_{j+2}) \in \Omega_z(d^\epsilon/2 + \min\{d^\epsilon/2, \Delta^\epsilon/2\} - \Delta^\epsilon) \subseteq \Omega_z(d^\epsilon/2)$, and $\forall t \in [t_j, t_{j+2})$, $z(t) \in \Omega_z(d^\epsilon)$. Letting $\hat{t}_j$ be the instant of the first transition into the mode Align after the j^{th} transition, we have

$$\begin{aligned} &q(t_j^+) = \text{Align},\ z(t_j) \in \Omega_z(d^\epsilon/2) \Rightarrow \\ &\exists \hat{t}_j > t_j,\ \forall t \in [t_j, \hat{t}_j),\ z(t) \in \Omega_z(d^\epsilon),\ \text{and} \\ &q(\hat{t}_j^+) = \text{Align},\ z(\hat{t}_j) \in \Omega_z(d^\epsilon/2). \end{aligned}$$

By using induction claim (40) follows. ■

Lemma 3: Under feedback laws (9), (11) all solutions of system (1)-(3), (12) starting from the set (21) satisfy R3. ■

Proof: To prove R3 we construct functions $n_{\tilde{e}}$ and $\tilde{n}_z$ for which $\|\tilde{e}_y(t)\| \leq n_{\tilde{e}}(\|\tilde{e}(0)\|)$, $\|z(t)\| \leq n_z(\|\tilde{e}(0)\|, \|z(0)\|)$ and compute the constant $M_\Theta > 0$ such that $\|\overline{\Theta}(t)\| \leq M_\Theta$, where $\overline{\Theta} \triangleq [\dot{\theta} \ldots \theta^{(r^\star-1)}]^T$. Then using (10) and boundedness of the map y_d and its partial derivatives, existence of the appropriate functions n_x and n_z in R3 follows.

The function $n_{\tilde{e}}$ is obtained from (29) by taking $n_{\tilde{e}}(\|x\|) \triangleq \frac{p_M}{p_m}\|x\|$. The function $\tilde{n}_z$ is obtained by substituting $d(t) = \tilde{e}_y(t) + y_d(\theta(t))$ and (29) into (4), that is, $\|z(t)\| \leq \nu(\|z(0)\|, 0) + \alpha^\star(\frac{p_M}{p_m}\|\tilde{e}(0)\| + M_y) \triangleq \tilde{n}_z(\|\tilde{e}(0)\|, \|z(0)\|)$. Boundedness of the states $\overline{\Theta}$ follows from feedback law (11), conditions (18), and bound (22). In modes Wait and Start we have $\|\overline{\Theta}(t)\| = 0$. In the mode Align we use bound (22) and integrating backwards from conditions (18), we get that $\|\overline{\Theta}(t)\| \leq M_\Theta^\epsilon \triangleq M_\omega^\epsilon \sum_{i=1}^{r^\star-1} (T_a^\epsilon)^i$. Note that the norm of the states $\overline{\Theta}(t)$ can be bounded by a fixed constant because their initial condition $\Theta(0)$ in (21) is fixed. ■

Remark 5: We compute an upper bound on the derivative of the path parameter θ for small path-following accuracies ϵ. Using Remark 2, that is, $M_\omega^\epsilon = M_\omega (1/T_a^\epsilon)^{r^\star}$, it follows that under feedback law (11) we have $\sup_{t\geq 0} |\dot{\theta}(t)| \leq M_{\dot{\theta}}/T_a^\epsilon$, where the constant $M_{\dot{\theta}}$ is independent of ϵ. Thus, the smaller is the path-following accuracy ϵ, the faster the motion along the path $\mathcal{Y}_d$ may be required. □

V. Example

We apply our design to the system

$$\begin{aligned} \dot{z}_1 &= z_1^2 z_2 + z_1^2 y_1, \quad \dot{y}_1 = u_1, \\ \dot{z}_2 &= z_2^2 z_1 + z_2^2 y_2, \quad \dot{y}_2 = u_2, \end{aligned} \quad (41)$$

with output $y = [y_1\ \ y_2]^T$, for which the corresponding zero dynamics $\dot{z}_1 = z_1^2 z_2$, $\dot{z}_2 = z_1 z_2^2$ are unstable. We consider a circular path with radius R, $\mathcal{Y}_d^\star = \{[R\sin\theta\ R\cos\theta]^T : \theta \geq 0\}$.

With the auxiliary output $\tilde{y} \triangleq [\tilde{y}_1\ \tilde{y}_2]^T = y - \sigma(z_1, z_2)$, where $\sigma(z_1, z_2) = -[z_1 + z_2\ \ z_1 + z_2]^T$, and error coordinates $\tilde{e}_y \triangleq$

$[\tilde{e}_{y1}\ \tilde{e}_{y2}]^T = [\tilde{y}_1 - R\sin\theta\ \tilde{y}_2 - R\cos\theta]^T$ system (41) becomes

$$\begin{aligned} \dot{z}_1 &= -z_1^3 + z_1^2(R\sin\theta + \tilde{e}_{y1}), \quad \dot{\tilde{e}}_{y1} = u_1 - \overline{u} - R\omega\cos\theta, \\ \dot{z}_2 &= -z_2^3 + z_2^2(R\cos\theta + \tilde{e}_{y2}), \quad \dot{\tilde{e}}_{y2} = u_2 - \overline{u} + R\omega\sin\theta, \\ \dot{\theta} &= \omega. \end{aligned} \tag{42}$$

where $\overline{u} \triangleq z_1^3 + z_2^3 - z_1^2\tilde{y}_1 - z_2^2\tilde{y}_2$. Taking the derivative of the Lyapunov function $V_z(z) = \frac{1}{2}(z_1^2 + z_2^2)$ along the solutions of system (42) we get $\dot{V}_z = -z_1^4 - z_2^4 + \pi^T(z)(y_d(\theta) + \tilde{e}_y)$, where $\pi(z) \triangleq [z_1^3\ z_2^3]^T$. Hence, Assumption 1 is satisfied. Using (9), we set the feedback law for u to

$$u = [R\omega\cos\theta + \overline{u} - \tilde{e}_{y1}\ \ -R\omega\sin\theta + \overline{u} - \tilde{e}_{y2}]^T, \tag{43}$$

and define the Lyapunov function $V_e(\tilde{e}_y) \triangleq \frac{1}{2}(\tilde{e}_{y1}^2 + \tilde{e}_{y2}^2)$. Since the path $\mathcal{Y}_d^\star$ is circular, Assumption 2 holds with arbitrary $\theta_m > 0$ and $\theta_M = \theta_m + 2\pi$.

We simulate hybrid system system (41), (43), (11)-(12) for $t_S = 100$ seconds. A typical behavior is illustrated in Figs. 2 and 3 for $R = 1$, $\epsilon = 0.05$, and initial conditions $[y_1(0)\ y_2(0)\ z_1(0)\ z_2(0)]^T = [0\ 0\ 1\ 1]^T$. To better illuminate important features we select initial conditions close to the path $\mathcal{Y}_d^\star$ and show only the initial portion, the first 10 seconds, of the simulation in Fig. 3. The mode Wait is visited only once, while afterwards all transitions are from Align to Align. This behavior was observed for all initial conditions for which the simulation was performed.

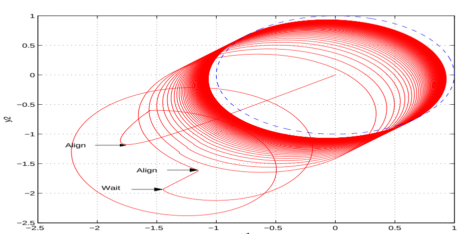

Fig. 2. Output trajectory $y(t)$ in $y_1 - y_2$ plane versus the path $\mathcal{Y}_d^\star$

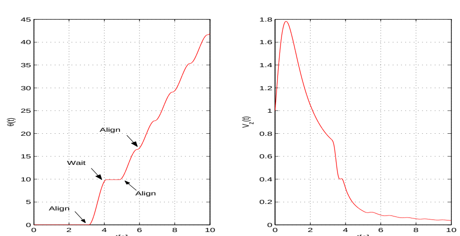

Fig. 3. Left: path parameter $\theta(t)$, Right: Lyapunov function $V_z(t)$.

VI. Conclusion

In the earlier papers [9], [11] it was demonstrated that the path-following problem formulation avoids the well known obstacle to tracking accuracy imposed by unstable zero dynamics. These papers developed control designs for linear systems taking advantage of the system representation [21] in which the only input to the unstable zero dynamics is the system output. While this representation exists for all right invertible linear systems, it exists only for a subclass of feedback linearizable nonlinear systems [16], [17]. For this subclass we provided a constructive solution to the practical path-following problem combining ISS and hybrid system methodologies.

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