Two algorithms arising in analysis of polynomial models

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Abstract

Algorithms for testing observability and forward accessibility of discrete-time polynomial systems are presented. The algorithms are based on symbolic computation packages - the Gröbner basis method and QEPCAD. The observability test checks observability of general polynomial systems in finite time. Forward accessibility test is applicable to a large class of polynomial systems and also stops in finite time.

1 Introduction

Discrete-time polynomial models represent an important and large class of models which have been applied successfully to a range of different processes, such as distillation columns, heat exchangers, cement mills, etc. In [7] a long list of applications of these models can be found. Important subclasses of this class of systems are linear systems, bilinear systems, Wiener-Hammerstein type systems, etc.

Understanding the properties of polynomial models, such as controllability or observability, is important for any in-depth analysis of the systems which allow for these models. Observability for polynomial systems was addressed in [8], different controllability notions for more general discrete-time nonlinear systems in [4], dead-beat controllability in [7, 6], etc. In particular, in [7] an algorithmic approach to dead-beat controllability of polynomial systems, based on symbolic computation ideas, was presented.

In this paper we present two algorithms for testing observability and forward accessibility of polynomial systems. The construction that we use in the observability test can be found in Sontag's work [8]. However, with the newly emerged Gröbner basis and QEPCAD symbolic computation packages, the computations are automated, leading to a straightforward observability test which always stops in finite time. The second algorithm is applicable to a very large class of polynomial models for testing forward accessibility. A very similar algorithm was used for testing output dead-beat controllability of the so called odd polynomial systems [5], and we show here the importance of the same construction for the forward accessibility property. Our forward accessibility test can be used as an alternative approach to the Lie algebraic approaches investigated in [4].

The paper is organized as follows. In Section 2 we present preliminaries and in Sections 3 and 4 respectively we present observability and forward accessibility tests. In the last section we summarize our results.

2 Preliminaries

The ring of polynomials in n variables over a field k is denoted as $k[x_1, x_2, \ldots, x_n]$. Sets of real, rational, natural and complex numbers are respectively denoted as $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{C}$.

We consider the following class of polynomial systems:

\[
\begin{align*}
x(k + 1) &= f(x(k), u(k)) \\
y(k) &= h(x(k))
\end{align*}
\]

(1)

where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}$ and $u(k) \in \mathbb{R}$ are respectively the state, output and input of the system (1) at time $k$. We assume that the vector $f(x, u) = (f_1(x, u) \ldots f_n(x, u))^T$ is such that $f_i(x, u) \in \mathbb{Q}[x, u] = \mathbb{Q}[x_1, x_2, \ldots, x_n, u]$ and $h \in \mathbb{Q}[x_1, \ldots, x_n]$. This assumption is needed since we use the symbolic computation packages (the Gröbner basis method and QEPCAD) which can only be applied to polynomial with rational coefficients.

We denote the composition of function $f$ as:

\[
f_{u(1)} \circ f_{u(0)}(x(0)) = f(f(x(0), u(0)), u(1)).
\]

A sequence of controls is denoted as $U = (u(0), u(1), \ldots)$. The truncation of $U$ to a sequence of length $p$ is denoted as $U_p = (u(0), u(1), \ldots, u(p - 1))$. 

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The output of system (1) that is reached from the initial state \(x(0)\) at time step \(p\) under the action of a control sequence \(\mathcal{U}_p\) is denoted as \(y(p, x(0), \mathcal{U}_p)\). The set of reachable states from an initial state \(x(0)\), at time step \(k\) is denoted as:

\[
V^k(x(0)) = \{z : z = f_{u(k-1)} \circ \ldots \circ f_{u(0)}(x(0)), u(i) \in \mathbb{R}\}
\]

In this paper we investigate the following two notions:

**Definition 1** The system (1) is observable if for each pair of initial states \(\xi \neq \eta\), there exists an integer \(N\) and an input sequence \(U_N\) which yields \(y(N, \xi, U_N) \neq y(N, \eta, U_N)\).

**Definition 2** The system (1) is forward accessible from \(x(0) \in \mathbb{R}^n\) if \(\dim R(x(0)) = n\) \((R(x(0))\) has a non-empty interior), where

\[
R(x(0)) = \{z : z \in \cup_k V^k(x(0))\}
\]

If the system is forward accessible from any \(x(0) \in \mathbb{R}^n\), then we say that it is forward accessible.

Let \(f_1, f_2, \ldots, f_s\) be polynomials in \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). Then we define \(V(f_1, f_2, \ldots, f_s) = \{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n : f_i(a_1, a_2, \ldots, a_n) = 0\text{ for all }1 \leq i \leq s\}\). We call \(V(f_1, f_2, \ldots, f_s)\) the real algebraic set or real variety defined by the polynomials \(f_1, f_2, \ldots, f_s\). A subset \(I \subset \mathbb{R}[x_1, x_2, \ldots, x_n]\) is an ideal if: 0 \(\in I\); \(f, g \in I\), then \(f + g \in I\); \(f \in I\) and \(h \in \mathbb{R}[x_1, x_2, \ldots, x_n]\), then \(hf \in I\). Let \(f_1, f_2, \ldots, f_s\) be polynomials in \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). Then the set \((f_1, \ldots, f_s)\) defined as

\[
(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) : f_i(a_1, \ldots, a_n) = 0\text{ for all }1 \leq i \leq s\}
\]

is called the ideal generated by \(f_1, f_2, \ldots, f_s\). We use the notation \(V(I)\) to denote the variety \(V(f_1, \ldots, f_s)\), where \(f_i\) are the generators of the ideal \(I\). The sum of two ideals \(J_1 = \langle g_1, \ldots, g_r\rangle\) and \(J_2 = \langle p_1, \ldots, p_l\rangle\) is defined as \(J_1 + J_2 = \langle g_1, \ldots, g_r, p_1, \ldots, p_l\rangle\). Product is defined as \(I_1 \cdot I_2 = \langle a_i b_j : 1 \leq i \leq N, 1 \leq j \leq M\rangle\).

Given a set of polynomials, the Gröbner basis algorithm produces a set of “simpler” polynomials (its Gröbner basis) that has the same solutions as the original set. The original algorithm for computing the Gröbner basis, which stops in finite time, is due to B. Buchberger [2]. The Gröbner basis algorithm is implemented in most symbolic computation packages, such as Maple and Mathematica. The Gröbner basis method produces a set of simpler polynomials that depend on the monomial ordering, which is defined by the user [2]. Gröbner bases are not unique. However, given a monomial ordering, there exists a well defined reduced Gröbner basis which is unique. One can then compare whether two ideals are the same by checking whether the reduced Gröbner bases of the ideals are the same. We denote the reduced Gröbner basis of a set of polynomials \(f_1, \ldots, f_s\), for a given ordering as \(GB\{f_1, \ldots, f_s\}\). Due to space limitations it is impossible to present all the theory on Gröbner bases and we refer to [2] for more details on the subject.

QEPCAD is a symbolic computation package for quantifier elimination (QE) in the first order theory of real closed fields. It is based on the cylindrical algebraic decomposition (CAD) algorithm [1]. The input to the algorithm is an expression which consists of polynomial equations and inequalities, Boolean operators and \((\wedge), (\lor)\), or \((\forall)\), implies \((\rightarrow)\) and not \((\neg)\), as well as quantifiers \(\exists\) and \(\forall\). The solution (output formula) is a quantifier free expression. We note that all variables are assumed to be real. For more details on QEPCAD see [1]. The simple examples we present illustrate the operation of the algorithm.

**Example 1** If the input expression is \((\exists t)[a_2 t^2 + a_1 t + a_0 = 0]\), the output expression \((a_1^2 - 4a_2 a_0 \geq 0)\) is obtained.

**Example 2** If the input expression is \((\forall x)(\forall y)[(x^2 + y^2 < 1) \rightarrow (y > x^4 - 2)]\), the output expression \(TRUE\) is obtained.

3 Observability test

The construction given below is presented for SISO systems (1) to simplify presentation but it can be used for general MIMO case. The method that we use below is used in [8] to prove some results. However, in [8] the test for checking observability notion of Definition 1 requires testing if a very general polynomial mapping is bijective, which is not easy and only sufficient conditions for this were referred to in the paper. By using the Gröbner basis method we gain a straightforward algorithmic observability test, which tests necessary and sufficient observability conditions. The algorithm is guaranteed to test this notion in finite time.

We consider all states \(\xi, \eta \in \mathbb{R}^n, \xi \neq \eta\) which produce the same output sequence irrespective of the applied input sequence. A real variety \(V_o \subset \mathbb{R}^n \times \mathbb{R}^n\) is constructed in the following way (see also [8]).

If two states \(\xi, \eta \in \mathbb{R}^n\) can not be distinguished by any input sequence, then necessarily we have that \(h(\xi) = h(\eta)\).
h(\eta). We construct an ideal \( J_1 = (h(\xi) - h(\eta)) \). Suppose that \( h(f(x, u)) \) depends explicitly on \( u \). We can write:

\[
h(f(x, u)) = h_m(x)u^m + \ldots + h_1(x)u + h_0(x)
\]

Then the states \( \xi, \eta \) which produce the same output in first and second time steps irrespective of the applied control satisfy:

\[
\begin{align*}
\xi & = \eta \\
h_i(\xi) & = h_i(\eta), \forall i = 0, 1, \ldots, m
\end{align*}
\]

(2)

We construct an ideal \( J_2 = \langle h(\xi) - h(\eta), h_m(\xi) - h_m(\eta), \ldots, h_0(\xi) - h_0(\eta) \rangle \). Notice, that \( J_1 \subseteq J_2 \) by construction. If \( J_1 = J_2 \), then all states that can not be distinguished by the inputs are contained in the variety \( V(J_1) \subseteq \mathbb{R}^n \times \mathbb{R}^m \). Suppose that \( J_1 \neq J_2 \). Then we have that

\( J_1 \subset J_2 \)

Let us compute the set of states for which the outputs are identical for the first three time steps. These states necessarily satisfy:

\[
h_i(f(\xi, u)) - h_i(f(\eta, u)) = 0, \forall u \in \mathbb{R}, \forall i = 0, 1, \ldots, m_0
\]

If we denote:

\[
h_i(f(x, u)) = h_{p_i, i}(x)u^{p_i} + \ldots + h_{1, i}(x)u + h_{0, i}(x)
\]

we construct the ideal

\[
J_3 = \langle h(\xi) - h(\eta), h_{m, m}(\xi) - h_{m, m}(\eta), \ldots, h_{0, 0}(\xi) - h_{0, 0}(\eta),
\]

\[
h_{p_m, m}(\xi) - h_{p_m, m}(\eta), \ldots, h_{0, 0}(\xi) - h_{0, 0}(\eta) \rangle.
\]

If \( J_3 = J_2 \), the variety \( V(J_2) \) contains all indistinguishable states. Otherwise, we have

\( J_1 \subset J_2 \subset J_3 \)

We continue the same construction to obtain an ascending chain of ideals

\( J_1 \subset J_2 \subset J_3 \subset \ldots \)

which must terminate \([8, 2]\), that is \( J_N = J_{N+1} \) for some \( N \). We let \( V_z = V(J_N) \). All the indistinguishable states belong to the set \( S_z = V_z \cap \{ (\xi, \eta) : \xi \neq \eta \} \). This set is a semi-algebraic subset of \( \mathbb{R}^n \times \mathbb{R}^m \). If we denote the variety of equations obtained when all generators of \( J_N \) are set equal to zero as \( J_N(\xi, \eta) \), using QEPCAD we can check the following decision problem:

\[
(\exists \xi) \, (\exists \eta) \, S_z = J_N(\xi, \eta) \land \xi \neq \eta
\]

(3)

If the answer to (3) is \textit{FALSE} \((S_z = 0)\), the systems are observable and vice versa. We can compare whether the ideals \( J_i \) and \( J_{i+1} \) are equal by comparing their reduced Gröbner bases for the same ordering. The construction which was just described can be formally stated as an algorithmic observability test, which stops in finite time:

\textbf{Observability test:}

1. \( k = 1, f(x, u), h(x), J_1 = \langle h(\xi) - h(\eta) \rangle, G_1 = GB[J_1], \) fix a monomial ordering
2. \( k = k + 1 \)
3. Compute ideal \( J_k \) and its reduced Gröbner basis \( G_k = GB[J_k] \)
4. Compare whether \( G_k = G_{k-1} \). If yes go to 5. If not go to 2.
5. Check using QEPCAD if the decision problem (3) is satisfied. If not, the system is observable and vice versa.

The observability test for polynomial systems terminates always in finite time, since we need to compute finitely many times the Gröbner basis of certain ideals and then use QEPCAD to solve the decision problem (3). The computational cost associated with the observability test is large.

Sometimes we may not have to use the QEPCAD algorithm in testing observability. From construction of \( V_z = V(J_N) \), we see that \( V(\xi - \eta) \subseteq V_z \) and the system is observable if and only if \( V_z = V(\xi - \eta) \). If at one step we obtain that the reduced Gröbner basis of the ideal \( J_k \) is \( \{ \xi - \eta \} \), we know that the system is observable since it immediately follows that \( J_k = J_{k+1} \) and \( V_z = V(\xi - \eta) \). This situation is illustrated by the following example.

\textbf{Example 3} Consider the simple Wiener system:

\[
\begin{align*}
x_1(k + 1) & = x_2(k) \\
x_2(k + 1) & = -x_1(k) - 2x_2(k) + u(k) \\
y(k) & = x_1^2(k)
\end{align*}
\]

(4)

The system consists of a linear dynamical block and quadratic static nonlinearity, which is at the output of the system.

To find the variety \( V_z \subseteq \mathbb{R}^n \times \mathbb{R}^m \), the following ideals are constructed:

\[
\begin{align*}
J_1 & = \langle \eta_1^2 - \xi_1^2 \rangle, \\
J_2 & = \langle \eta_1^2 - \xi_1^2, \eta_2^2 - \xi_2^2 \rangle, \\
J_3 & = \langle \eta_1^2 - \xi_1^2, \eta_2^2 - \xi_2^2, (\eta_1 + 2\eta_2)^2 - (\xi_1 + 2\xi_2)^2, \\
& \quad (\eta_1 + 2\eta_2) - (\xi_1 + 2\xi_2) \rangle, \\
J_4 & = \langle \eta_1^2 - \xi_1^2, \eta_2^2 - \xi_2^2, (\eta_1 + 2\eta_2)^2 - (\xi_1 + 2\xi_2)^2, \\
& \quad (\eta_1 + 2\eta_2) - (\xi_1 + 2\xi_2), (\eta_1 + \eta_2)^2 - (\xi_1 + \xi_2)^2, \\
& \quad 2\eta_1 + \eta_2 - 2\xi_1 - \xi_2, -2\eta_1 - 3\eta_2 + 2\xi_1 + 3\xi_2 \rangle
\end{align*}
\]

(5)

and using the Gröbner basis algorithm with lexicographic ordering \( \xi_1 > \xi_2 > \eta_1 > \eta_2 \) we obtain that

\[
GB[J_4] = \{ \eta_1 - \xi_1, \eta_2 - \xi_2 \}
\]
and therefore $J_4 = J_5$. The system is observable since
$V_s = V(J_s) = V(\eta_1 - \xi_1, \eta_2 - \xi_2)$ and consequently
$S_2 = \emptyset$.

4 Forward accessibility test

Forward accessibility, or reachability, plays a prominent role
for a number of control problems [4]. An algebraic test for testing
the forward accessibility of general discrete-time nonlinear
systems can be found in [4], where the Lie algebraic methods were used to test
the property. We take here another approach, which is
based on ideas of invariant algebraic sets and which is
applicable only to polynomial systems.

Consider the composition:

$$F = f_{u_{n-1}} \circ \ldots \circ f_{u_0}(x)$$

and its Jacobian:

$$J = \frac{\partial F}{\partial u_{n-1}}$$

The Jacobian $J$ is a matrix whose entries are polyno-
mials in $x$ and $u_{n-1} = (u_0 \ldots u_{n-1})^T$. We use the
following assumption:

**Assumption 1** We assume that $\det J \neq 0$.

Assumption 1 does not seem to be restrictive for general
polynomial systems. If for a certain $x$ the determinant
of $J$ is a non-zero polynomial in controls, then the system
(1) is forward accessible from $x$. Hence, if we write
the determinant of $J$ as

$$\det J = \sum_{i_0=0}^{i_0=N_0} \ldots \sum_{i_{n-1}=0}^{i_{n-1}=N_{n-1}} b_{i_0, i_1, \ldots, i_{n-1}}(x)u_{i_0}^i \ldots u_{i_{n-1}}^i$$

we see that if Assumption 1 is satisfied, we define a
critical variety as follows:

$$V_C = V(b_{0,0,\ldots,0}, \ldots, b_{N_0,N_1,\ldots,N_{n-1}})$$

The critical variety $V_C$ contains all states $x \in \mathbb{R}^n$ from
which $\dim V_C(x) < n$. The ideal generated by polyno-
mials $b_{0,0,\ldots,0}, \ldots, b_{N_0,N_1,\ldots,N_{n-1}}$ is denoted as $J_C$ and
we write shortly $V_C = V(J_C)$.

**Definition 3** $S_I \subset V_C$ is an invariant set if $\forall x(0) \in $ $S_I, Vu(\in R$, we have that $f(x(0), u) \in S_I$. $V_I \subset V_C$
is termed the maximal invariant set if it is such that
if $V_I \subset V^*$, where $V^* \subset V_C$ is an invariant set, then
$V_I = V^*$.

We have the following result:

**Theorem 1** Consider a system (1) for which Assumption 1 holds. The system is forward accessible if and only if $V_I = \emptyset$.

**Proof of Theorem 1:** Suppose that $V_I \neq \emptyset$. Consider
any state $x^* \in V_I$. By definition we have that $V_I(x^*) \subset V_I, V_k$ and hence we have that $\forall x \in V_I, V_k(x^*) \subset V_I$. Consequently, we have that $\dim R(x^*) \leq \dim V_I \leq \dim V_C \leq n - 1$. Therefore, the system is not forward accessible from $x^*$ and hence it is not forward accessible.

Suppose $V_I = \emptyset$. Then by definition for any $x \in V_C$
there exists $k^* = k^*(x), k^* \in \mathbb{N}$ such that $\forall x \in V_C, V_k^*(x) \cap (\mathbb{R}^n - V_C) \neq \emptyset$. Therefore, $\forall x \in V_C$ we have that $\dim V^*_k(x) = n$. Q.E.D.

We show below how one can compute the maximal in-
variant set $V_C$. Consider an ideal $I = I(x)$. We use the notation

$$I \circ f_u(x) = I(f(x, u))$$

The following ideals are very important in the sequel.
Suppose that $I = \langle g_1, \ldots, g_s \rangle$. Then we have that $I \circ f_u(x) = \langle g_1(f(x, u)), \ldots, g_s(f(x, u)) \rangle$. Notice that $g_i \circ f_u(x) = g_i(f(x, u)) \in \mathbb{Q}[x_1, \ldots, x_n][u]$. The ideal $I \circ f_u(x)$ is generated by polynomials $g_1 \circ f_u(x), \ldots, g_s \circ f_u(x) \in \mathbb{Q}[x_1, \ldots, x_n][u]$. Regard the polynomials $g_i \circ f_u(x)$ as polynomials in $u$ with coefficients which are polynomials in $x$: $g_j \circ f_u(x) = \sum_{i=0}^{N_j} a_j^i(x)u^i, j = 1, 2, \ldots, n$

Form the ideals $J_I = \langle a_1^1, a_2^1, \ldots, a_{N_J}^1 \rangle$. Finally, we form the ideal $I^{(1)} = J_1 \cdot J_2 \ldots \cdot J_s$. Similarly, we can form the ideals $I^{(k)}$ by taking first composition $I \circ f_u(x) \circ \ldots \circ f_u(0)(x)$, regarding its generators as polynomials in $u(i), i = 0, 1, \ldots, k - 1$ with coefficients polynomials in $x$ and using the same construction as before. Given an arbitrary ideal $J$, we introduce the notation $J(x)$ to denote the set of polynomial equations, such that all the generators of $J$ are set to zero.

The maximal invariant subset of an arbitrary variety can be computed using an algorithm based on the Gröbner basis method and QEPCAD as shown in [5]:

**Forward accessibility test:**

1. Fix a monomial ordering. $k = 0; G_0 = GB[J_C]
2. k = k + 1
3. Compute $G_k = GB[J_C + J^{(1)}_C + \ldots + J^{(k)}_C]
4. Compare whether $G_k = G_{k-1}$. If this is true, go to 5. If this is not true, go to 2.
5. We have computed the maximal invariant set of $V(J)$ which is given by $V(J_C) = V(J_C + J_C^{(1)} + \ldots + J_C^{(k)})$.

6. Check using QEPCAD if the following decision problem:

\[ \exists x [(J_C + J_C^{(1)} + \ldots + J_C^{(k)})(x)] \]

is TRUE. If so, the system is not forward accessible and vice versa.

Note that if at step $k$ we obtain that $G_k = \{1\}$, then we do not need to use QEPCAD since $V_1 = \emptyset$.

**Example 4** In this example we test forward accessibility of the system:

\[
\begin{align*}
x_1(k+1) &= x_2(k) \\
x_2(k+1) &= x_1(k) + x_1(k)u(k) + (x_2(k) + 1)u^3(k)
\end{align*}
\]

Consider the composition:

\[
J = \begin{pmatrix}
\frac{\partial F_1}{F_1} & \frac{\partial F_2}{F_2} \\
\frac{\partial F_1}{F_1} & \frac{\partial F_2}{F_2}
\end{pmatrix}
\]

where $F_1 = x_1 + x_1u_0 + (x_2 + 1)u_0^2$, $F_2 = x_2 + x_2u_1 + (x_1 + x_1u_0 + (x_2 + 1)u_0^2 + 1)u_0^2$. Consider the Jacobian:

\[
\det J = x_1x_2 + 3x_1(x_1 + 1)u_1^2 + 3x_1u_0u_1^2 + 3x_2(x_2 + 1)u_1^2 + 3x_2u_1u_0^2 + 9(x_1 + 1)(x_2 + 1)u_2^3u_1^2 + 9(x_2 + 1)u_0u_1^2
\]

The critical variety consists of states for which all the coefficient polynomials in $x$ in the above given expression vanish simultaneously. By computing the Gröbner basis with lexicographic ordering $x_1 > x_2$ for these polynomials we obtain:

\[ GB[J_C] = GB[x_1x_2, 3x_1(x_1 + 1), 3x_1, 3(x_2 + 1), 3x_2(x_2 + 1), 9(x_1 + 1)(x_2 + 1), 9x_2(x_2 + 1), 9(x_2 + 1)^2u_1^2]
\]

Hence, we have that $G_0 = \{x_2 + 1, x_1\}$. We compute now (the exact expression for $J_C^{(1)}$ is omitted due to space reasons):

\[ G_1 = GB[J_C + J_C^{(1)}] = \{1\} \]

and hence we conclude that $G_1 = G_2 = \{1\}$, which implies that $V_1 = \emptyset$. The system is forward accessible. Note that we obtained the following useful information: $V_2, x \in \mathbb{R}^n - V(x_2 + 1, x_1)$ we have that $dim V_2^*(x) = n$. For the state $x^* = (0 - 1)^T$, on the other hand, we have that $dim V_2^*(x^*) = n$.

5 Conclusions

Two algorithms for testing observability and forward accessibility for discrete-time polynomial systems were presented. The algorithms are based on the Gröbner basis method and QEPCAD. Computational complexity of the algorithmic tests is large and investigation of simpler classes of relevant systems for which simpler conditions may be stated appears to be worthwhile pursuing. For instance, the (simple) examples presented in this paper were solved within seconds using the Gröbner basis package in MAPLE on the standard SPARC station.

References


