A Framework for Observer Design for Sampled-Data Nonlinear Systems *

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Abstract

We present results on observer design for sampled-data nonlinear systems using two approaches: (i) the observer is designed via an approximate discrete-time model of the plant; (ii) the observer is designed based on the continuous-time plant model and then discretized for sampled-data implementation (emulation). Since exact discrete-time models are often unavailable for nonlinear sampled-data systems, a more realistic approach for observer design is to employ approximate discrete-time models. We investigate under what conditions, and in what sense, such an approximate design achieves convergence for the unknown exact discrete-time model. We first present examples which show that designs via approximate discrete-time models may indeed lead to instability when implemented on the exact model. We then present conditions for approximate designs that guarantee robustness for the exact discrete-time model. We finally characterize convergence properties for emulation designs where the discrete-time observer is derived from a continuous-design via an approximate discretization.

1 Introduction

The study of discrete-time observers is important for two reasons: First, continuous-time designs are often implemented digitally and, second, for classes of systems, such as those in [3], observer design may be easier for discrete-time models than for continuous-time models. The main drawback of the existing discrete-time observer theory, however, is that the availability of exact discrete-time models is assumed, which is usually unrealistic. A more practical approach is to employ approximate discrete-time models for design, and to study how robust such approximate designs would be when implemented on the exact model.

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The objective of this paper is to give guidelines for sampled-data observer design when only an approximate model is available. We first present examples which illustrate that observer design via approximate models may fail to produce a stable observer for the exact discrete-time model, regardless of the choice of the sampling period. We then derive conditions under which an approximate observer design guarantees convergence for the exact model in a *semiglobal practical* sense. *Semiglobal* means that the region of attraction of the observer can be rendered as large as desired by reducing the sampling period. *Practical* means that the observer error converges to a a neighborhood of zero, which can be made arbitrarily small either by reducing the sampling period, or by tuning a "modelling parameter" which improves the accuracy of the approximate model for a constant sampling period. Finally, we study *emulation* design of observers, in which the discrete-time observer is obtained from a continuous-time design via an approximate discretization. With appropriate conditions on the approximation, and on the underlying continuous-time observer, we again achieve semiglobal practical convergence. As we shall see, however, in emulation designs we cannot reduce the residual observer error arbitrarily by refining the modelling parameter (integration period of the numerical approximation) only.

The study in this paper for sampled-data observers parallels recent results in [5, 4] for sampled-data control design based on approximate discrete-time models. However, observer design problems pose additional difficulties which cannot be addressed with a direct application of results from control design. We also emphasize that our primary goal in this paper is to study observer convergence properties independently of a feedback design task, because the interest in observers is not restricted to output-feedback control. Results on output-feedback stabilization of nonlinear sampled-data systems include, among others, [1] which studies discrete-time implementation of high-gain observers, and [2] and [4], which address dynamic sampled-data controllers, with observer-based control as a special case.

The paper is organized as follows: Section 2 gives the preliminary definitions and the problem formulation. Section 3 presents examples of non-robust designs, which give clues of why observers based on approximate models may give rise to instability for the exact model. Following these clues, Section 4 derives a set of sufficient conditions for the approximate design, which ensure its robustness when applied to the exact model. Section 5 presents similar conditions for robustness of the emulation design. Section 6 presents a robust design example which conforms to the conditions derived in Section 5, and ensures practical convergence. Throughout the paper we make use of the following classes of functions, which are now standard in nonlinear control literature: $\mathcal{K}$ is the class of functions $I\!R_{\geq 0} \to I\!R_{\geq 0}$ which are zero at zero, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subset of class-$\mathcal{K}$ functions that are unbounded. $\mathcal{L}$ is the set of functions $I\!R_{\geq 0} \to I\!R_{\geq 0}$ which are continuous, decreasing and converging to zero as their argument tends to $+\infty$. $\mathcal{KL}$ is the class of functions $I\!R_{\geq 0} \times I\!R_{\geq 0} \to I\!R_{\geq 0}$ which are class-$\mathcal{K}$ on the first argument and class-$\mathcal{L}$ on the second argument. Given $c > 0$ we define $\mathcal{B}_c := \{x : |x| \leq c\}$.

2 Preliminaries and Problem Statement

We consider the system

$$\begin{aligned} \dot{x} &= f(x,u) && (1)\\ y &= h(x), && (2) \end{aligned}$$

where $x \in I\!R^n, u \in I\!R^m, y \in I\!R^p$, and $f(x,u)$ is locally Lipschitz. Given a sampling period $T > 0$, we assume that the control u is constant during sampling intervals $[kT, (k+1)T)$ and that the output y is measured at sampling instants kT; that is, $y(k) := y(kT)$. The family of exact discrete-time models of (1)-(2) is:

$$\begin{aligned} x(k+1) &= F_T^e(x(k), u(k)) \\ y(k) &= h(x(k)), \end{aligned} \tag{3}$$

where $F_T^e(x, u)$ is the solution of (1) at time T starting at x, with the constant input u. This model is well-defined when the continuous model (1) does not exhibit finite-escape time. Even when there is finite escape time, (3) is valid on compact sets which can be rendered arbitrarily large by reducing T.

To compute (3) we need a closed-form solution for the initial value problem

$$\dot{x} = f(x, u(k)), \qquad x_0 = x(k) \tag{4}$$

over one sampling interval $[kT, (k+1)T)$, which is impossible to obtain in general. It is realistic, however, to assume that a family of approximate discrete-time models is available:

$$\begin{aligned} x(k+1) &= F_{T,\delta}^a(x(k), u(k)) \\ y(k) &= h(x(k)). \end{aligned} \tag{5}$$

This family is parameterized by the sampling period T, and a "modelling parameter" δ which will be used to refine the approximate model when T is fixed. It can be interpreted as the *integration period* in numerical schemes for solving differential equations. The case where $\delta = T$ is of separate interest because several approximations of this type (such as Euler approximation) preserve the structure and types of nonlinearities of the continuous-time system and, hence, may be preferable to the designer. When $\delta = T$ we use the short-hand notation

$$F_T^a(x, u) := F_{T,T}^a(x, u). \tag{6}$$

As an example, for the linear system $\dot{x} = Ax$, the forward Euler numerical scheme $x(t+\delta) = (I + \delta A)x(t)$ can be used to generate an approximate model by dividing the sampling period T into N integration periods $\delta = T/N$, and by applying the Euler approximation $(I + \delta A)$ for each integration period; that is, $F_{T,\delta}^a = (I + \delta A)^{T/\delta} x$. As $\delta \to 0$, this $F_{T,\delta}^a$ converges to the exact model $F_T^e = \exp(AT)x$. Another way to use the Euler approximation is to assume that $\delta = T$ and then we have $F_T^a = (I + TA)x$.

Throughout the paper we assume that the approximate model (5) is *consistent* with the exact model, as defined in [5, 4] with motivation from numerical analysis literature [6]:

Definition 1
a) When $\delta = T$ the family $F_T^a(x, u)$ is said to be consistent with $F_T^e(x, u)$ if for each compact set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ there exists $T_0 > 0$ such that, for all $(x, u) \in \Omega$ and all $T \in (0, T_0]$,

$$|F_T^e(x, u) - F_T^a(x, u)| \leq T\rho(T). \tag{7}$$

b) When δ is independent of T, $F_{T,\delta}^a(x, u)$ is said to be *consistent* with $F_T^e(x, u)$ if, for each compact set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists a class-$\mathcal{K}$ function $\rho(\cdot)$ and a constant $T_0 > 0$, and for each fixed $T \in (0, T_0]$ there exists $\delta_0 \in (0, T]$ such that, for all $(x, u) \in \Omega$ and $\delta \in (0, \delta_0]$,

$$|F_T^e(x, u) - F_{T,\delta}^a(x, u)| \leq T\rho(\delta). \tag{8}$$

□

When $\delta = T$ inequality (7) means that the difference between F_T^a and F_T^e is to converge to zero faster than linearly in T as $T \to 0$. It is not necessary to know the exact model $F_T^e(x, u)$ to verify these consistency properties. Verifiable conditions to check (7) and (8) are given in [5, 4].

For the approximate model (5), we design a family of observers (depending on T and δ) of the form

$$\hat{x}(k+1) = F_{T,\delta}^a(\hat{x}(k), u(k)) + \ell_{T,\delta}(\hat{x}(k), u(k), y(k)), \tag{9}$$

and analyze under what conditions, and in what sense, this design guarantees convergence when applied to the exact model (3). Due to the mismatch of the exact and approximate models, the observer error system is now driven by the plant trajectories $x(t)$ and controls $u(t)$, which act as disturbance inputs. When these inputs are bounded, we want the observer to guarantee *semiglobal practical* convergence. "Practical" means that the observer error is to converge to a small set, the size of which can be assigned to be arbitrarily small by tuning δ or T. Likewise, "semiglobal" means that its region of attraction can be rendered as large as desired, by reducing T:

Definition 2
a) When $\delta = T$ we say that the observer (9) is *semiglobal practical in* T, if there exists a class-$\mathcal{KL}$ function $\beta(\cdot, \cdot)$ such that for any $D > d > 0$ and compact sets $\mathcal{X} \subset I\!R^n, \mathcal{U} \subset I\!R^m$, we can find a $T^* > 0$ with the property that for all $T \in (0, T^*]$,

$$|\hat{x}(0) - x(0)| \leq D, \quad \text{and} \quad x(k) \in \mathcal{X}, u(k) \in \mathcal{U}, \ \forall k \geq 0 \tag{10}$$

imply

$$|\hat{x}(k) - x(k)| \leq \beta(|\hat{x}(0) - x(0)|, kT) + d. \tag{11}$$

b) When δ is independent of T we say that the observer (9) is *semiglobal in* T and *practical in* δ, if there exists a class-$\mathcal{KL}$ function $\beta(\cdot, \cdot)$ such that for any given real number $D > 0$, and compact sets $\mathcal{X} \subset I\!R^n, \mathcal{U} \subset I\!R^m$, we can find and a $T^* > 0$, and for any $T \in (0, T^*]$ and $d \in (0, D)$, we can find $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$, (10) implies (11).

c) We say that the observer (9) is *semiglobal in* T and *practical in* T *and* δ, if there exists a class-$\mathcal{KL}$ function $\beta(\cdot, \cdot)$ such that for any $D > d_1 > 0$, and compact sets $\mathcal{X} \subset I\!R^n, \mathcal{U} \subset I\!R^m$, we can find and a $T^* > 0$, and for any $T \in (0, T^*]$ and $d_2 \in (0, D - d_1)$, we can find $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$, (10) implies

$$|\hat{x}(k) - x(k)| \leq \beta(|\hat{x}(0) - x(0)|, kT) + d_1 + d_2. \tag{12}$$

□

Unlike Definition 2(b) where we can arbitrarily reduce the residual observer error d in (11) by decreasing δ, in Definition 2(c) we can only reduce d_2 with δ, while d_1 is dictated by the sampling period T. As we shall see in Section 5, this situation arises in emulation design where, decreasing δ can reduce the residual observer error, but cannot eliminate it completely if T is held constant.

3 Examples of Non-Robust Designs

We first illustrate with examples that, unless the observer design for the approximate model meets several criteria, it may not ensure semiglobal practical convergence for the exact model. In the following examples, the observer designs are based on approximate models with $\delta = T$:

Example 1 Consider the quadruple chain of integrators

$$\dot{x}_1 = x_2; \qquad \dot{x}_2 = x_3; \qquad \dot{x}_3 = x_4; \qquad \dot{x}_4 = u, \tag{13}$$

where the output

$$y = x_1 \tag{14}$$

is sampled at times $t = kT, k \in \mathbb{Z}$. For the Euler approximate model

$$\begin{aligned} x(k+1) &= A_T^a x(k) + B_T^a u(k) \\ y(k) &= Cx(k), \end{aligned}$$

where

$$A_T^a = \begin{pmatrix} 1 & T & 0 & 0 \\ 0 & 1 & T & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B_T^a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ T \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},$$

we design a family of *dead-beat* observers

$$\hat{x}(k+1) = A_T^a \hat{x}(k) + B_T^a u(k) - L_T[y(k) - C\hat{x}(k)], \tag{15}$$

in which the injection matrix

$$L_T = \begin{pmatrix} -4 \\ -\frac{6}{T} \\ -\frac{4}{T^2} \\ -\frac{1}{T^3} \end{pmatrix}$$

places the eigenvalues of $A_T^a + L_T C$ at the origin for all $T > 0$. However, for the exact model,

$$A_T^e = \begin{pmatrix} 1 & T & \frac{T^2}{2!} & \frac{T^3}{3!} \\ 0 & 1 & T & \frac{T^2}{2!} \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_T^e = \begin{pmatrix} \frac{T^4}{4!} \\ \frac{T^3}{3!} \\ \frac{T^2}{2!} \\ T \end{pmatrix}$$

the eigenvalues of $A_T^e + L_T C$ are, for all T,

$$-0.5897 \pm 1.6865i; \ 0.5897 \pm 0.1334i,$$

in which the first two are outside the unit circle. This means that, in the observer error $e := \hat{x} - x$ dynamics

$$e(k+1) = (A_T^a + L_T C)e(k) + (A_T^a - A_T^e)x(k) + (B_T^a - B_T^e)u(k), \tag{16}$$

the ℓ_∞-gain from x to e is $\gamma_{xe} \geq 1$ for all values of T. (Otherwise, substituting $x = -e$ ($\gamma_{ex} = 1$) and $u = 0$ in (16), we would infer from the small-gain property $\gamma_{ex}\gamma_{xe} < 1$ that the resulting system $e(k+1) = (A_T^e + L_T C)e(k)$ is asymptotically stable, which contradicts our computation of eigenvalues above.) Because the ℓ_∞-gain cannot be reduced arbitrarily by reducing T, we cannot assign d arbitrarily small in (11) even when $x(k)$ is bounded. Thus, the approximate design (15) does not guarantee practical convergence.

The reason why this dead-beat design is non-robust for the exact model is because, when T is reduced, it attempts to achieve faster convergence at the cost of larger overshoots. The combination of this "peaking" in the transients, and the mismatch between the exact and approximate discrete-time models, leads to instability of $A^e_T + L_T C$. In the next example, the approximate design is non-peaking, but the convergence rate is slower for smaller T, which again leads to instability of $A^e_T + L_T C$.

Example 2 We consider the system:

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= x_2 + u \\ y &= x_1 \end{aligned}$$

and, again, design a Luenberger observer based on the Euler approximation

$$A^a_T = \begin{pmatrix} 1+T & T \\ 0 & 1+T \end{pmatrix}.$$

The injection matrix

$$L_T = \begin{pmatrix} -2T - 2T^3 \\ -T(1+T^2)^2 \end{pmatrix}$$

places both eigenvalues of $A^a_T + L_T C$ at $1 - T^3$, which is inside the unit circle. However, for the exact model

$$A^e_T = \begin{pmatrix} e^T & Te^T \\ 0 & e^T \end{pmatrix},$$

the eigenvalues of $A^e_T + L_T C$ are complex and located at

$$\{e^T - T(1+T^2)\} \pm \{T(1+T^2)\sqrt{e^T - 1}\}i,$$

which are outside the unit circle for all $T > 0$. Then, arguments similar to those at the end of Example 1 show that this design does not ensure practical convergence for the exact model.

4 Observer Design via Approximate Discrete-Time Models

In the examples of Section 3, either the overshoot or the convergence rate of the approximate design is not uniform in the sampling period T. We now derive conditions which exclude such designs, and guarantee semiglobal practical convergence for the exact model. For our analysis we first note from (3) and (9) that the observer error

$$e := \hat{x} - x \tag{17}$$

satisfies

$$e(k+1) = F^a_{T,\delta}(\hat{x}(k), u(k)) + \ell_{T,\delta}(\hat{x}(k), u(k), y(k)) - F^e_T(x(k), u(k)). \tag{18}$$

Adding and subtracting the approximate model $F^a_{T,\delta}(x(k), u(k))$, we rewrite (18) as

$$e(k+1) = E_{T,\delta}(e(k), x(k), u(k)) + F^a_{T,\delta}(x(k), u(k)) - F^e_T(x(k), u(k)), \tag{19}$$

where

$$E_{T,\delta}(e, x, u) := F^a_{T,\delta}(\hat{x}, u) + \ell_{T,\delta}(\hat{x}, u, y) - F^a_{T,\delta}(x, u) \tag{20}$$

represents the nominal observer error dynamics of the approximate design, and $F^a_{T,\delta}(x(k),u(k)) - F^e_T(x(k),u(k))$ is the mismatch between the approximate and exact plant models.

In Theorem 1 below, we study the case $\delta = T$, and prove semiglobal practical convergence in T under conditions (i)-(iii). In particular, condition (iii) guarantees that we can find T-independent estimates for the overshoot and convergence rate in the approximate design and, thus, rules out the non-robust designs of the previous section:

Theorem 1 **($\delta = T$)** The observer (9) is semiglobal practical in T as in Definition 2(a) if the following conditions hold:

(i) $\delta = T$.

(ii) F^e_T is consistent with F^a_T as in Definition 1(a).

(iii) There exists a family of Lyapunov functions $V_T(x,\hat{x})$, class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, $\rho_0(\cdot)$, and nondecreasing functions $\gamma_0(\cdot)$, $\gamma_1(\cdot)$, $\gamma_2(\cdot)$, with the following property:

For any compact sets $\mathcal{X} \subset I\!R^n$, $\hat{\mathcal{X}} \subset I\!R^n$, $\mathcal{U} \subset I\!R^m$, there exist constants $T^* > 0$ and $M > 0$, such that, for all x, x_1, $x_2 \in \mathcal{X}$, $\hat{x} \in \hat{\mathcal{X}}$, $u \in \mathcal{U}$, and $T \in (0, T^*]$,

$$|V_T(x_1,\hat{x}) - V_T(x_2,\hat{x})| \le M|x_1 - x_2| \tag{21}$$

$$\alpha_1(|e|) \le V_T(x,\hat{x}) \le \alpha_2(|e|) \tag{22}$$

$$\frac{V_T(F^a_T(x,u), F^a_T(\hat{x},u) + \ell_T(\hat{x},y,u)) - V_T(x,\hat{x})}{T} \le -\alpha_3(|e|) + \rho_0(T)[\gamma_0(|e|) + \gamma_1(|x|) + \gamma_2(|u|)]. \tag{23}$$

□

The proof of Theorem 1 relies on the following Proposition, which is proved separately in the Appendix:

Proposition 1 Suppose all conditions of Theorem 1 hold. Then, for any quadruple of strictly positive numbers $(\Delta_x, \Delta_{\hat{x}}, \Delta_u, \nu)$, there exists $T^* > 0$ such that for all x, $\hat{x}$, u and T satisfying $|x| \le \Delta_x$, $|\hat{x}| \le \Delta_{\hat{x}}$, $|u| \le \Delta_u$, $T \in (0, T^*]$,

$$\frac{V_T(F^e_T(x,u), F^a_T(\hat{x},u) + \ell_T(\hat{x},y,u)) - V_T(x,\hat{x})}{T} \le -\alpha_3(|e|) + \nu. \tag{24}$$

Proof of Theorem 1: We let $\mathcal{X}$ and $\mathcal{U}$ be as in Definition 2(a) and claim that, given any pair of numbers $0 < r < R$, we can find $T^* > 0$ such that, for all $T \in (0, T^*]$,

$$r \le V_T(x(k),\hat{x}(k)) \le R \quad \Rightarrow \quad \frac{V_T(x(k+1),\hat{x}(k+1)) - V_T(x(k),\hat{x}(k))}{T} \le -\frac{1}{2}\alpha_3(|e(k)|). \tag{25}$$

To see this, let $(\Delta_x, \Delta_{\hat{x}}, \Delta_u, \nu)$ be numbers such that

$$\Delta_x \ge \sup_{x\in\mathcal{X}} |x| \quad \Delta_u \ge \sup_{u\in\mathcal{U}} |u| \quad \Delta_{\hat{x}} \ge \sup_{x\in\mathcal{X}} |x| + \alpha_1^{-1}(R) \quad \nu \le \frac{1}{2}\alpha_3(\alpha_2^{-1}(r)). \tag{26}$$

It then follows that $|x(k)| \le \Delta_x$, $|u(k)| \le \Delta_u$, and, from (40) and $V_T(x(k),\hat{x}(k)) \le R$,

$$|\hat{x}(k)| = |x(k) + e(k)| \le |x(k)| + |e(k)| \le |x(k)| + \alpha_1^{-1}(V_T) \le \sup_{x\in\mathcal{X}} |x| + \alpha_1^{-1}(R) \le \Delta_{\hat{x}}. \tag{27}$$

Thus, if we choose T^* as in Proposition 1, then we guarantee, for all $T \in (0, T^*]$,

$$\frac{V_T(x(k+1), \hat{x}(k+1)) - V_T(x(k), \hat{x}(k))}{T} \leq -\alpha_3(|e(k)|) + \nu. \tag{28}$$

Moreover, $V_T(x(k), \hat{x}(k)) \geq r$ and (22) together imply $|e(k)| \geq \alpha_2^{-1}(r)$ and, hence, (25) follows from the choice of ν in (26). Having proven (25) we next note from (28) that

$$V_T(x(k), \hat{x}(k)) \leq r \quad \Rightarrow \quad V_T(x(k+1), \hat{x}(k+1)) \leq r + \nu T. \tag{29}$$

This means that, if we choose T such that $r + \nu T < R$, then, from (25) and (29),

$$V_T(x(0), \hat{x}(0)) \leq R \quad \Rightarrow \quad V_T(x(k), \hat{x}(k)) \leq \max\{\beta_1(V_T(x(0), \hat{x}(0)), kT), r + \nu T\} \quad \forall k \geq 0 \tag{30}$$

for some class-$\mathcal{KL}$ function $\beta_1(\cdot, \cdot)$, which can be calculated from (25) as in the proof of [5, Theorem 2]. We thus conclude from (22) that

$$|e(0)| \leq \alpha_2^{-1}(R) \quad \Rightarrow \quad |e(k)| \leq \beta(|e(0)|, kT) + \alpha_1^{-1}(r + \nu T), \tag{31}$$

where

$$\beta(s, \tau) := \alpha_1^{-1}(\beta_1(\alpha_2(s), \tau)). \tag{32}$$

Thus, given any real numbers $D > d > 0$ as in Definition 2(a), we can select $R = \alpha_2(D)$, $r = \frac{1}{2}\alpha_1(d)$, and T^* small enough that $\nu T^* \leq \frac{1}{2}\alpha_1(d)$, and verify from (31) that the observer (9) is semiglobal practical in T. $\square$

Theorem 1 established semiglobal practical convergence by reducing the sampling period T. When T is fixed and cannot be reduced, it is still possible to achieve practical convergence by, instead, refining the accuracy of the approximate models with the parameter δ:

Theorem 2 (δ independent of T) The observer (9) is semiglobal in T and practical in δ as in Definition 2(b) if the following conditions hold:

(i) δ can be adjusted independently of T.

(ii) $F^a_{T,\delta}(x, u)$ is consistent with the exact model $F^e_T(x, u)$ as in Definition 1(b).

(iii) There exists a family of Lyapunov functions $V_{T,\delta}(x, \hat{x})$, class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, $\rho_0(\cdot)$, and nondecreasing functions $\gamma_0(\cdot)$, $\gamma_1(\cdot)$, $\gamma_2(\cdot)$, with the following property:

For any compact sets $\mathcal{X} \subset \mathbb{R}^n$, $\hat{\mathcal{X}} \subset \mathbb{R}^n$, $\mathcal{U} \subset \mathbb{R}^m$, there exists a constant $T^* > 0$, and for any fixed $T \in (0, T^*]$ there exists $\delta^* > 0$, and for any $\varepsilon_1 > 0$ there exists $c > 0$, such that, for all x, x_1, $x_2 \in \mathcal{X}$, $\hat{x} \in \hat{\mathcal{X}}$, $u \in \mathcal{U}$, and $\delta \in (0, \delta^*]$,

$$|x_1 - x_2| \leq c \quad \Rightarrow \quad |V_{T,\delta}(x_1, \hat{x}) - V_{T,\delta}(x_2, \hat{x})| \leq \varepsilon_1 \tag{33}$$

$$\alpha_1(|e|) \leq V_{T,\delta}(x, \hat{x}) \leq \alpha_2(|e|) \tag{34}$$

$$\begin{aligned} \frac{V_{T,\delta}(F^a_{T,\delta}(x, u), F^a_{T,\delta}(\hat{x}, u) + \ell_{T,\delta}(\hat{x}, y, u)) - V_{T,\delta}(x, \hat{x})}{T} \leq\ & -\alpha_3(|e|) \\ & + \rho_0(\delta)[\gamma_0(|e|) + \gamma_1(|x|) + \gamma_2(|u|)]. \end{aligned} \tag{35}$$

$\square$

We will use the following Proposition, proved in the Appendix:

Proposition 2 Suppose that all conditions of Theorem 2 hold. Then, for any triple of strictly positive real numbers $(\Delta_x, \Delta_{\hat{x}}, \Delta_u)$ there exists $T^* > 0$ such that, for any fixed $T \in (0, T^*]$ and $\nu > 0$, there exists $\delta^* > 0$ such that $|x| \le \Delta_x$, $|\hat{x}| \le \Delta_{\hat{x}}$, $|u| \le \Delta_u$ and $\delta \in (0, \delta^*]$ imply:

$$\frac{V_{T,\delta}(F^e_{T,\delta}(x,u), F^a_{T,\delta}(\hat{x},u) + \ell_{T,\delta}(\hat{x},y,u)) - V_{T,\delta}(x,\hat{x})}{T} \le -\alpha_3(|e|) + \nu \ . \tag{36}$$

□

Proof of Theorem 2: We first let Δ_x, $\Delta_{\hat{x}}$ and Δ_u be as in (26), and determine T^* from Proposition 2. Next, we fix $T \in (0, T^*]$, and choose $\nu > 0$ to satisfy both (26) and

$$\nu T \le \frac{1}{2}\alpha_1(d). \tag{37}$$

Finally, using Proposition 2 and arguments similar to those in the proof of Theorem 1, we can find $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$ the estimate (31) holds. Note that, unlike Theorem 1 where we tune T to reduce $r + \nu T$ in (31), here we ensure $r + \nu T \le \alpha_1(d)$ by further restricting the choice of ν by (37). Using the resulting δ^* obtained from Proposition 2, we conclude from (31) that the observer (9) is semiglobal in T and practical in δ as in Definition 2(b). □

5 Observer Design via Emulation

A common method for digital implementation of controllers and observers, known as "emulation", is to discretize continuous-time designs using approximate techniques. In this section we assume that a continuous-time observer of the form

$$\dot{\hat{x}} = g(\hat{x}, y, u) \tag{38}$$

is available, and implement it with the approximate discrete-time equation:

$$\hat{x}(k+1) = G^a_{T,\delta}(\hat{x}(k), y(k), u(k)). \tag{39}$$

We assume in this section that functions $f(\cdot,\cdot)$ and $g(\cdot, h(\cdot), \cdot)$ are locally Lipschitz in all their arguments.

When $\delta = T$ we establish semiglobal practical convergence in T under a Lyapunov condition on the continuous-time observer (38), and a consistency property of the approximate discretization in (39):

Theorem 3 **($\delta = T$)** The observer (39) is semiglobal practical in T as in Definition 2(a) if the following conditions hold:
(i) $\delta = T$.
(ii) G^a_T is consistent with G^e_T as in Definition 1(a), with (y, u) interpreted as constant inputs during sampling intervals.
(iii) The continuous-time observer (38) ensures convergence with a C^1 function $V(x, \hat{x})$ satisfying, for all $x, \hat{x} \in I\!R^n$ and for all $u \in I\!R^m$,

$$\alpha_1(|e|) \le V(x, \hat{x}) \le \alpha_2(|e|) \tag{40}$$

$$\frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial \hat{x}} g(\hat{x}, y, u) \le -\alpha_3(|e|). \tag{41}$$

□

The proof follows the same steps as the proof of Theorem 1, and is not repeated here. Instead of $V_T(x, \hat{x})$ in Theorem 1, in this proof we use the Lyapunov function $V(x, \hat{x})$ in (40)-(41) above. Likewise, Proposition 1 is to be replaced by the following result adapted from [2, Theorem 3.1]:

Proposition 3 Let all conditions of Theorem 3 hold. Then, given any strictly positive numbers $(\Delta_x, \Delta_{\hat{x}}, \Delta_u, \nu)$, there exists $T^* > 0$ such that, for all $|x| \leq \Delta_x$, $|\hat{x}| \leq \Delta_{\hat{x}}$, $|u| \leq \Delta_u$, and $T \in (0, T^*]$, the following inequality holds:

$$\frac{V(F_T^e(x,u), G_T^a(\hat{x}, y, u)) - V(x, \hat{x})}{T} \leq -\alpha_3(|e|) + \nu. \tag{42}$$

□

Finally we study the situation where δ can be tuned independently of T. Unlike the design in Theorem 2 where practical convergence is achieved by reducing δ when T is fixed, in emulation design δ cannot reduce the parameter d in (11) arbitrarily. Even if the exact zero-order-hold equivalent $G_T^e(\hat{x}, y, u)$ of the observer (38) is available (with y and u considered as inputs), perfect observer convergence would not be achievable with fixed T because the sampling of y is ignored in the emulation design. The following theorem shows that by tuning δ we can reduce d in (11) not arbitrarily, but to a level dictated by T:

Theorem 4 (δ independent of T) The observer (39) is semiglobal in T, and practical in T and δ as in Definition 2(c), if the following conditions hold:

(i) δ can be adjusted independently of T.

(ii) $G_{T,\delta}^a$ is consistent with G_T^e as in Definition 1(b), with (y, u) interpreted as constant inputs during sampling intervals.

(iii) The continuous-time observer (38) satisfies condition (iii) of Theorem 3 with a C^1 function $V(x, \hat{x})$. □

Rather than giving a separate proof for Theorem 4, we refer to the proof of Theorem 2, and use Proposition 4 below instead of Proposition 2 used in Theorem 2. We note that the term $r + \nu T$ in (31) is now $r + \nu_1 T + \nu_2 T$ in which the first component $r + \nu_1 T$ determines d_1 in Definition 2(c). The second component, $\nu_2 T$, determines d_2 because, for fixed T, we can assign ν_2 in Proposition 4 arbitrarily small by tuning δ.

Proposition 4 Suppose all conditions of Theorem 4 hold. Then, given any strictly positive numbers $(\Delta_x, \Delta_{\hat{x}}, \Delta_u, \nu_1)$, there exists $T^* > 0$, and for any fixed $T \in (0, T^*]$ and $\nu_2 > 0$, there exists $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$, $|x| \leq \Delta_x$, $|\hat{x}| \leq \Delta_{\hat{x}}$, $|u| \leq \Delta_u$, we have

$$\frac{V(F_T^e(x,u), G_{T,\delta}^a(\hat{x}, y, u)) - V(x, \hat{x})}{T} \leq -\alpha_3(|e|) + \nu_1 + \nu_2. \tag{43}$$

6 Design Example

Theorems 1-4 are also applicable to reduced-order observers, when e is interpreted as the difference between the unmeasured components of x, and their observer estimates. For the

Duffing oscillator

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_1^3 \\ y &= x_1\,,\end{aligned} \tag{44}$$

a reduced-order continuous-time observer is

$$\begin{aligned}\hat{x}_2 &= \chi + y \\ \dot{\chi} &= -\chi - 2y - y^3.\end{aligned} \tag{45}$$

Because the observer error $e := \hat{x}_2 - x_2$ satisfies $\dot{e} = -e$, condition (iii) of Theorems 3 and 4 holds with the Lyapunov function

$$V(x, \hat{x}) = \frac{1}{2}(x_2 - \hat{x}_2)^2 = \frac{1}{2}e^2. \tag{46}$$

To discretize (45) we first use an Euler approximation with $\delta = T$; that is,

$$\chi(k+1) = \chi(k) + T[-\chi(k) - 2y(k) - y(k)^3]. \tag{47}$$

As predicted by Theorem 3, simulation results in Figure 1 show that the residual error between the solid trajectories of (44) and the dashed observer estimates of (47) becomes smaller as the sampling period T is decreased.

We next study the situation where T is fixed as in Theorem 4. Instead, we refine the discretization for (45) by dividing the sampling period into N steps of size $\delta = T/N$, and by applying an Euler approximation for each step. As $N \to \infty$, this approximation converges to the exact zero-order-hold equivalent of the continuous-time observer (45):

$$\chi(k+1) = \exp(-T)\chi(k) + (1 - \exp(-T))[-2y(k) - y^3(k)]\,, \tag{48}$$

which is computable in this example because the only nonlinearity in (45) is in the output-injection term $[-2y - y^3]$. Simulation results in Figure 2 show that the residual observer error is smaller for $N = 3$ in the middle plot, than for $N = 1$ ($\delta = T$) in the top plot. However, as predicted by Theorem 4, increasing N (that is, decreasing δ) does not reduce this residual error arbitrarily. Even with the exact zero-order-hold equivalent (48), we note in the bottom plot of Figure 2 that an observer error remains because the sampling period T is fixed.

7 Conclusions

We have given a framework for semiglobal practical asymptotic observer design for sampled-data nonlinear systems using two methods: (i) the observer design is carried out via an approximate discrete-time plant model; (ii) the observer is designed for the continuous-time plant model and then discretized (emulation). We specified conditions on the approximate model, continuous-time model, and the observer, guaranteeing that the observer that performs well on an approximate discrete-time model will also perform well on the exact discrete-time model. We have further discussed the effect of refining the approximate models with a modelling parameter δ, independently of the sampling period T. For emulation design, however, we have shown that this approach does not guarantee practical convergence if T is fixed, and leads to a residual observer error. Because emulation designs using pre-specified sampling periods are commonly practiced in engineering applications, an important research direction would be to pursue re-design methods that remove this residual error without decreasing T.

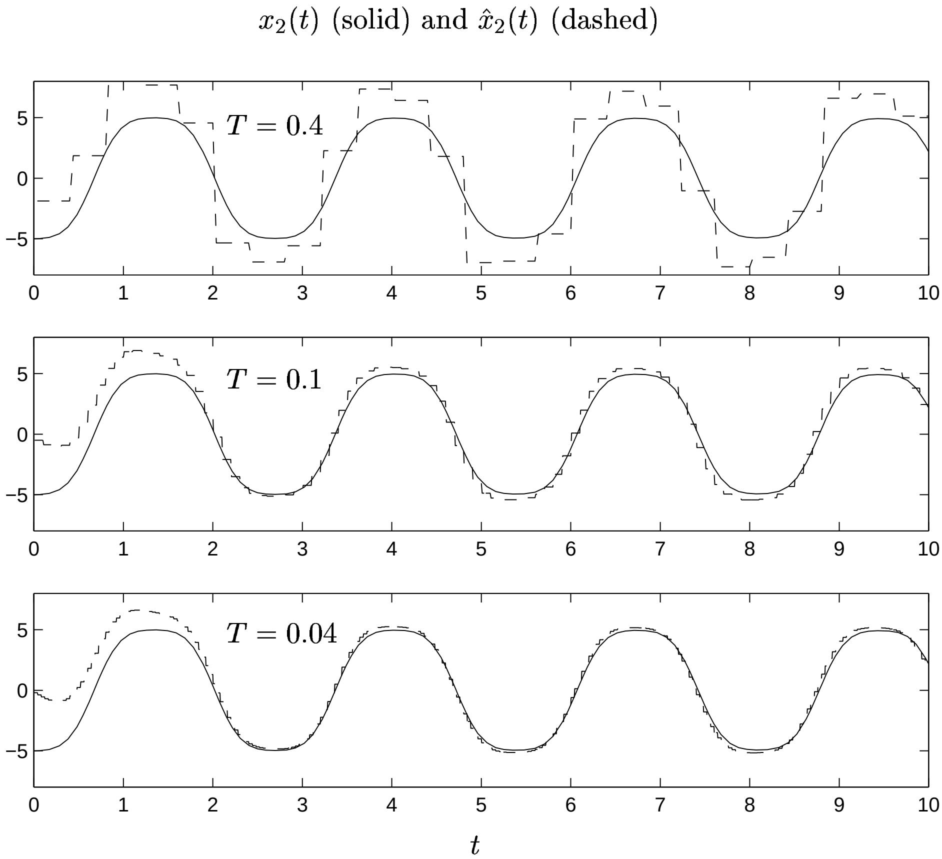

Figure 1: Simulation results for $x_2(t)$ (solid) from the Duffing oscillator (44), and $\hat{x}_2(t)$ (dashed) from the observer (47). The sampling period is $T = 0.4$ sec. for the top plot, $T = 0.1$ sec. for the middle plot, and $T = 0.04$ sec. for the bottom plot. As predicted by Theorem 3, the residual observer error diminishes as T is decreased.

Appendix

Proof of Proposition 1

Let $(\Delta_x, \Delta_{\hat{x}}, \Delta_u, \nu)$ be given. Let $T_1^* > 0$ be such that $F_T^e(x,u)$ is well-defined for all x, u and T such that $|x| \le \Delta_x$, $|u| \le \Delta_u$ and $T \in (0, T_1^*]$. Let $\Omega := \mathcal{B}_{\Delta_x} \times \mathcal{B}_{\Delta_u}$ generate $T_2^* > 0$ and $\rho \in \mathcal{K}$ via consistency (item (ii) of Theorem 1) of F_T^e and F_T^a. Let

$$M_1 := \sup_{x \in \Delta_x,\, u \in \Delta_u,\, \hat{x} \in \Delta_{\hat{x}}} \max\{|F_T^e|, |F_T^a| + |\ell_T|, \Delta_x, \Delta_{\hat{x}}\}. \tag{49}$$

Let $\mathcal{X} = \hat{\mathcal{X}} = \mathcal{B}_{M_1}$ and $\mathcal{U} = \mathcal{B}_{\Delta_u}$ generate T_3^* and $M > 0$ via item (iii) of Theorem 1, and let

$$\begin{aligned} T_4^* &= \rho_0^{-1}\left(\frac{\nu}{2[\gamma_0(2\Delta_{M_1}) + \gamma_1(\Delta_{M_1}) + \gamma_2(\Delta_u)]}\right) & (50)\\ T_5^* &= \rho^{-1}\left(\frac{\nu}{2M}\right). & (51)\end{aligned}$$

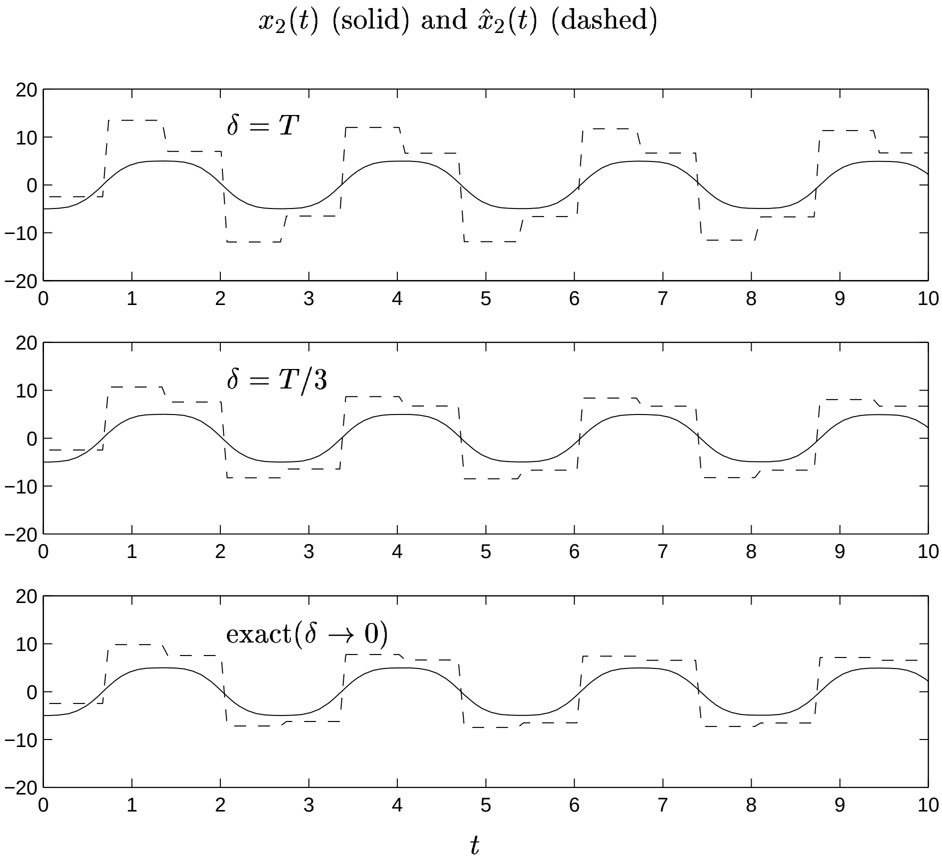

Figure 2: Simulation results for $x_2(t)$ (solid) from the Duffing oscillator (44), and $\hat{x}_2(t)$ (dashed) from the Euler approximation of the continuous-time observer (47), when the sampling period is held constant at $T = 0.67$ sec. and the integration period δ is reduced. For $\delta = T/3$ (middle plot) the residual observer error is smaller than for $\delta = T$ (top plot). As predicted by Theorem 4, however, the residual error does not vanish (bottom plot) even with the exact zero-order-hold equivalent (48), which would be recovered in the limit as $\delta \to 0$.

Defining $T^* = \min\{T_1^*, T_2^*, T_3^*, T_4^*, T_5^*\}$ and letting $x \in \mathcal{B}_{\Delta_x}$, $\hat{x} \in \mathcal{B}_{\Delta_{\hat{x}}}$, $u \in \mathcal{B}_{\Delta_u}$ we can write from item (iii),

$$\begin{aligned}
\frac{V_T(F_T^e, F_T^a + \ell_T) - V_T(x, \hat{x})}{T} &\leq -\alpha_3(|e|) + \rho_0(T)[\gamma_0(|e|) + \gamma_1(|x|) + \gamma_2(|u|)] \\
&\qquad + \frac{V_T(F_T^e, F_T^a + \ell_T) - V_T(F_T^a, F_T^a + \ell_T)}{T} \\
&\leq -\alpha_3(|e|) + \rho_0(T^*)[\gamma_0(2\Delta_{M_1}) + \gamma_1(\Delta_{M_1}) + \gamma_2(\Delta_u)] + \rho(T^*)M \\
&\leq -\alpha_3(|e|) + \frac{\nu}{2} + \frac{\nu}{2}. \qquad (52)
\end{aligned}$$

Proof of Proposition 2

Let $(\Delta_x, \Delta_{\hat{x}}, \Delta_u, \nu)$ be given. Let $T_1^* > 0$ be such that $F_T^e(x, u)$ is well defined for all $|x| \leq \Delta_x$, $|u| \leq \Delta_u$ and $T \in (0, T_1^*]$. Let $\Omega := \mathcal{B}_{\Delta_x} \times \mathcal{B}_{\Delta_u}$ generate $\rho \in \mathcal{K}$ and $T_2^* > 0$ using the consistency between F_T^e and $F_{T,\delta}^a$. Let $T^* := \min\{T_1^*, T_2^*\}$ and consider arbitrary fixed $T \in (0, T^*]$. Let

$$M := \sup_{x \in \mathcal{B}_{\Delta_x}, \hat{x} \in \mathcal{B}_{\Delta_{\hat{x}}}, u \in \mathcal{B}_{\Delta_u}, \delta \in (0,T]} \max\{|F_T^e|, |F_{T,\delta}^a| + |\ell_{T,\delta}|, \Delta_x, \Delta_{\hat{x}}\}. \tag{53}$$

Let $\mathcal{X} = \hat{\mathcal{X}} = \mathcal{B}_M$ and $\mathcal{U} = \mathcal{B}_{\Delta_u}$ generate $\delta_1^* > 0$, and let $\varepsilon_1 := \frac{\nu}{2}T$ generate $c > 0$ via item (iii) of Theorem 2. Thus, for all $x_1, x_2 \in \mathcal{X}$, $\hat{x} \in \hat{\mathcal{X}}$, $u \in \mathcal{U}$, and $\delta \in (0, \delta_1^*]$, we have

$$|V_{T,\delta}(x_1, \hat{x}) - V_{T,\delta}(x_2, \hat{x})| \leq \varepsilon_1 \ .$$

Let T and Ω generate $\delta_2^* > 0$ using the consistency assumption, so that for all $(x,u) \in \Omega$ and $\delta \in (0, \delta_2^*]$ we have:

$$|F_{T,\delta}^a(x,u) - F_T^e(x,u)| \leq T\rho(\delta) \ .$$

Let $\delta_3^* := \rho^{-1}\left(\frac{c}{T}\right)$ and

$$\delta_4^* := \rho_0^{-1}\left(\frac{\nu}{2(\gamma_0(2\Delta_M) + \gamma_1(\Delta_M) + \gamma(\Delta_u))}\right) \ .$$

Defining $\delta^* := \min\{\delta_1^*, \delta_2^*, \delta_3^*, \delta_4^*\}$, and using item (iii) of Theorem 2 and the definition of δ^* we can write for all $x \in B_{\Delta_x}$, $u \in B_{\Delta_u}$, $\hat{x} \in B_{\Delta_{\hat{x}}}$ and $\delta \in (0, \delta^*]$ that the following holds:

$$\begin{aligned}
\frac{V_{T,\delta}(F_T^e, F_{T,\delta}^a + \ell_{T,\delta}) - V_{T,\delta}(x,\hat{x})}{T} &\leq -\alpha_3(|e|) + \rho_0(\delta)(\gamma_0(|e|) + \gamma_1(|x|) + \gamma_2(|u|)) \\
&\quad + \frac{V_{T,\delta}(F_T^e, F_{T,\delta}^a + \ell_{T,\delta}) - V_{T,\delta}(F_T^a, F_{T,\delta}^a + \ell_{T,\delta})}{T} \\
&\leq -\alpha_3(|e|) + \rho_0(h^*)(\gamma_0(2\Delta_M) + \gamma_1(\Delta_M) + \gamma(\Delta_u)) + \frac{\varepsilon_1}{T} \\
&\leq -\alpha_3(|e|) + \frac{\nu}{2} + \frac{\nu}{2} \ ,
\end{aligned}$$

which completes the proof.

Proof of Proposition 3: See [2, Theorem 3.1].

Proof of Proposition 4: We first state and prove a fact that is instrumental in proving Proposition 4. For this purpose we consider two different initial value problems:

$$\dot{x}(t) = f(x(t), u(0)) \qquad x(0) = x \tag{54}$$
$$\dot{\hat{x}}(t) = g(\hat{x}(t), h(x(t)), u(0)) \qquad \hat{x}(0) = \hat{x} \ , \tag{55}$$

and

$$\dot{x}_1(t) = f(x_1(t), u(0)) \qquad x_1(0) = x \tag{56}$$
$$\dot{\hat{x}}_1(t) = g(\hat{x}_1(t), h(x(0)), u(0)) \qquad \hat{x}_1(0) = \hat{x} \ , \tag{57}$$

where $u := u(0)$ and $y := y(0) = h(x(0))$. We denote solutions of the initial value problem (54), (55) at time t as $x(t)$ and $\hat{x}(t)$. Solutions of the initial value problem (56), (57) at time t are denoted as $x_1(t)$ and $\hat{x}_1(t)$. Note that by definition we have that $x_1(T) = F_T^e(x,u)$ and $\hat{x}_1(T) = G_T^e(x,y,u)$. Now we can state the following fact:

Fact 1: Let conditions of Proposition 4 hold. Then, for any strictly positive numbers $(\Delta_x, \Delta_{\hat{x}}, \Delta_u)$ there exist $T^* > 0$ and $B > 0$ such that for all $T \in [0, T^*]$, $|x| \leq \Delta_x$, $|\hat{x}| \leq \Delta_{\hat{x}}$ and $|u| \leq \Delta_u$ the following holds:

$$
\begin{aligned}
x(T) &= F_T^e(x, u) && (58)\\
|\hat{x}(T) - G_T^e(x, y, u)| &\leq BT^2 \ . && (59)
\end{aligned}
$$

Proof of Fact 1: Let $(\Delta_x, \Delta_{\hat{x}}, \Delta_u)$ be given. Let $T^* > 0$ be such that for all $|x| \leq \Delta_x$, $|\hat{x}| \leq \Delta_{\hat{x}}$ and $|u| \leq \Delta_u$ the solutions of the initial value problems (54), (55) and (56), (57) exist for all $t \in [0, T^*]$. The proof of (58) is trivial since $x(\cdot)$ and $x_1(\cdot)$ are solutions of the same system with the same input and from the same initial condition. Introduce the following numbers:

$$
\begin{aligned}
M_1 &:= \max\{\sup_{t\in[0,T^*]} |x(t)|\,, \sup_{t\in[0,T^*]} |\hat{x}(t)|\,, \sup_{t\in[0,T^*]} |x_1(t)|\,, \sup_{t\in[0,T^*]} |\hat{x}_1(t)|\} && (60)\\
M_2 &= \max_{|x|\leq M_1, |\hat{x}|\leq M_1, |u|\leq \Delta_u} \max\{|f(x,u)|, |g(\hat{x}, h(x), u)|\} \ . && (61)
\end{aligned}
$$

Let $L > 0$ be the Lipschitz constant for functions $f(\cdot,\cdot)$ and $g(\cdot, h(\cdot), \cdot)$ on the set $(x, \hat{x}, u) \in \mathcal{B}_{M_1} \times \mathcal{B}_{M_1} \times \mathcal{B}_{\Delta_u}$. Introduce

$$
B := \frac{3M_2 L}{2} \ . \qquad (62)
$$

Note first that for any $s \in [0, T^*]$ we can write the following:

$$
\begin{aligned}
|\hat{x}(s) - \hat{x}_1(s)| &\leq \int_0^s |g(\hat{x}(\tau), h(x(\tau)), u(0)) - g(\hat{x}_1(\tau), h(x(0)), u(0))|\, d\tau && (63)\\
&\leq \int_0^s 2M_2 d\tau \ = \ 2M_2 s\\
|x(s) - x(0)| &\leq \int_0^s |f(x(\tau), u(0))|\, d\tau \ = \ \int_0^s M_2 d\tau \ = \ M_2 s \ .
\end{aligned}
$$

Then, using the inequalities (63), (64), the Lipschitz property of $f(\cdot,\cdot)$ and $g(\cdot, h(\cdot), \cdot)$ and differentiability of $u(\cdot)$ we can write:

$$
\begin{aligned}
|\hat{x}(T) - G_T^e(x, y, u)| &\leq \int_0^T |g(\hat{x}(s), h(x(s)), u(0)) - g(\hat{x}_1(s), h(x(0)), u(0))|\, ds\\
&\leq \int_0^T L\left(|\hat{x}(s) - \hat{x}_1(s)| + |x(s) - x(0)|\right) ds\\
&\leq \int_0^T L(2M_2 + M_2) s ds\\
&= \frac{3LM_2}{2} T^2 \ = \ BT^2 \ ,
\end{aligned}
$$

which completes the proof of the fact.

We complete now the proof of Proposition 4. Let $(\Delta_x, \Delta_{\hat{x}}, \Delta_u, \nu_1)$ be given. Let $T_1^* > 0$, M_1, M_2 and B come from the proof of Fact 1. Recall that $e := x - \hat{x}$. From continuity of solutions of the initial value problems (54), (55) and (56), (57) and continuity of $\alpha_3(\cdot)$ it follows that there exists $T_2^* > 0$ such that for all $|x| = |x(0)| \leq \Delta_x$, $|\hat{x}| = |\hat{x}(0)| \leq \Delta_{\hat{x}}$, $|u| = |u(0)| \leq \Delta_u$ we can write:

$$|\alpha_3(|e(s)|) - \alpha_3(|e(0)|)| \leq \frac{\nu_1}{2} \qquad \forall s \in [0, T_2^*] .$$

We define

$$\begin{aligned} M_3 &:= \max\{M_2, \Delta_x\} \\ M_4 &:= \max\{M_2, \max_{\substack{|x| \leq \Delta_x, |\hat{x}| \leq \Delta_{\hat{x}}, |u| \leq \Delta_u \\ T \in [0,1], \delta \in [0,T]}} |G^a_{T,\delta}(\hat{x}, h(x), u)|, \Delta_{\hat{x}}\} \end{aligned}$$

Let $L_V > 0$ denote the Lipschitz constant of $V(x, \hat{x})$ on the set $(x, \hat{x}) \in \mathcal{B}_{M_3} \times \mathcal{B}_{M_4}$. Introduce $T_3^* := \frac{\nu_1}{2BL_V}$ and $T^* := \min\{T_1^*, T_2^*, T_3^*, 1\}$. Let $T \in (0, T^*]$ be arbitrary but fixed and let $\nu_2 > 0$ be given. Let $\rho(\cdot)$ come from consistency. Introduce $h_1^* := \rho^{-1}\left(\frac{\nu_2}{L_V}\right)$ and let $h^* := \min\{T, h_1^*\}$.

By integrating the second inequality in (41) along the solutions of the continuous time system with the constant input $u(t) = u(0) = u$ (the notation is the same as in the proof of Fact 1, with $e := e(0) = x(0) - \hat{x}(0)$), we can write using our choice of T_2^* that:

$$\begin{aligned} \frac{V(x(T), \hat{x}(T)) - V(x, \hat{x})}{T} &\leq -\alpha_3(|e|) + \frac{1}{T}\int_0^T |\alpha_3(|e(s)|) - \alpha_3(|e|)|\, ds \\ &\leq -\alpha_3(|e|) + \frac{\nu_1}{2} . \end{aligned} \tag{64}$$

By adding and subtracting $\frac{V(F^e_T, G^e_T)}{T}$ and $\frac{V(F^e_T, G^a_{T,\delta})}{T}$ to (64), we can write using Fact 1, Lipschitzness of V, consistency of G^e_T and $G^a_{T,\delta}$ and our choice of T^* and h^* (we omit arguments of all functions for simplicity):

$$\begin{aligned} \frac{V(F^e_T, G^a_{T,\delta}) - V(x, \hat{x})}{T} &\leq -\alpha_3(|e|) + \frac{\nu_1}{2} + \frac{V(F^e_T, G^e_T) - V(x(T), \hat{x}(T))}{T} + \frac{V(F^e_T, G^a_{T,\delta}) - V(F^e_T, G^e_T)}{T} \\ &\leq -\alpha_3(|e|) + \frac{\nu_1}{2} + \frac{L_V(|x(T) - F^e_T| + |\hat{x}(T) - G^e_T|)}{T} + \frac{L_V |G^e_T - G^a_{T,\delta}|}{T} \\ &\leq -\alpha_3(|e|) + \frac{\nu_1}{2} + \frac{L_V B T^2}{T} + \frac{L_V T \rho(h)}{T} \\ &\leq -\alpha_3(|e|) + \frac{\nu_1}{2} + \frac{\nu_1}{2} + \nu_2 , \end{aligned}$$

which completes the proof of Proposition 4.

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