A trajectory based approach for stability robustness of nonlinear systems with inputs

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Abstract

We show, for two different definitions of semiglobal practical external stability, that the stability property holds on semi-infinite time intervals if and only if it holds on arbitrarily long but finite time intervals. These results have immediate applications in analysis of stability properties of highly oscillatory systems with inputs using averaging or for systems with inputs that are slowly varying. Results are stated for general flows and the stability is given with respect to arbitrary (not necessarily compact) sets.

1 Introduction

An important part of the classical stability theory is robustness of uniform asymptotic stability of an equilibrium to regular perturbations (see [4]), singular perturbations [1, 4, 2, 5], highly oscillatory signals [11, 22], slowly varying parameters [7, 3, 2] and other types of perturbations. These classical results originated from the dynamical systems theory and are usually given for systems without inputs. In control theory, on the other hand, systems with inputs are prevalent and understanding their stability properties is still attracting a lot of interest, especially in the area of nonlinear control systems.

Among the different stability definitions for systems with inputs, L_∞ and L_2 stability play a major role in control theory for their practical importance and intuitive appeal. The former captures the property "bounded disturbances imply bounded outputs", whereas the latter implies that "bounded energy disturbances imply bounded energy outputs". A particularly useful tool for L_∞ stability analysis of nonlinear control systems is the notion of input-to-state stability (ISS) introduced by Sontag in [12] (see also [13, 14, 16, 17] and references therein). ISS has proved to be an invaluable tool in solving a large number of important nonlinear control problems and continues to attract large interest for its different interpretations and applications (see for instance [6]). Moreover, ISS has recently been proved to be equivalent in an appropriate sense to a notion of L_2 stability (from inputs to states). In other words, ISS systems are both L_∞ and L_2 stable in an appropriate sense (see [17] for more details) and more importantly there exist Lyapunov function tools to check both of these important properties.

Recently, an approach has been proposed in [8, 20] for a unified investigation of robustness of the ISS property to different types of perturbations such as, slowly varying parameters, singular perturbations, highly oscillatory signals, etc. The novelty of the approach in [8] and [20] is that it gives a trajectory based proof of robustness of ISS, that is, it does not resort to converse ISS Lyapunov theorems which is the usual proof technique to address the robustness investigations. Moreover, the approach in [8, 20] is given for the L_∞ formulation of the ISS property, as in [12].

In this paper, we extend the trajectory based approach of [8] to investigate robustness of two new stability properties:
Property 1: the L_2 formulation of the ISS property as defined in [17];
Property 2: L_∞ input-output stability (IOS) in combination with an appropriate detectability condition (IOSS).

Our results provide a unified approach to investigation of robustness for both stability definitions. We show that our results have immediate applications, for example, to the study of systems with inputs that are either slowly varying or highly oscillatory (via averaging). Moreover, our results can be used to investigate robustness of the Property 1 and 2 in a range of seemingly different situations, such as in singularly perturbed systems. However, due to space limitations, we do not present all consequences of our results here and we refer to [20] for such examples.

The paper is organized as follows. In Section 2 we present preliminaries. The trajectory based proofs for robustness of the two stability properties are given in Section 3. Applications of our results to averaging of highly oscillatory systems with inputs and slowly varying systems with inputs is given in the last section.

2 Preliminaries

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{G}$ if it is zero at zero, continuous and nondecreasing. It is of class $\mathcal{K}$ if it is of class-$\mathcal{G}$ and strictly increasing. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if for each fixed value of $t \geq 0$ the function $\beta(\cdot, t)$ is a $\mathcal{K}$ function and for each fixed value of $s > 0$ it is decreasing to zero. Given a measurable function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$, we define its infinity norm $\|w\|_\infty := \text{ess sup}_{t \geq 0} |w(t)|$. If we have $\|w\|_\infty < \infty$ ($\|w\|_\infty < r$), then we write $w \in \mathcal{L}_\infty$ ($w \in \mathcal{L}_\infty(r)$). If $w(\cdot)$ is absolutely continuous, its derivative is defined almost everywhere and we can write $w(t) - w(t_\circ) = \int_{t_\circ}^t \dot{w}(\tau) d\tau$. Given an arbitrary nonempty set $\mathcal{A} \subset \mathbb{R}^n$, we define the distance of a point $x \in \mathbb{R}^n$ to the set $\mathcal{A}$ as:

$$|x|_\mathcal{A} := \inf_{s \in \mathcal{A}} |x - s| \ ,$$

where $|x - s|$ is the Euclidean norm of $x - s$.

For the sake of generality, the results in the next section are for general flows parameterized by the initial state $x_\circ \in \mathbb{R}^n$, disturbance $w \in \mathcal{F}$ (a subset of measurable functions taking values in $\mathbb{R}^d$) and true parameter $\theta \in \Theta$ (a subset of $\mathbb{R}^p$ such that the origin either belongs to the boundary of the set or to the set itself). Such a flow, which is a subset of functions defined on intervals of the form $[0, T)$ where $T \in [0, \infty]$ (T does not need to be the same for each element of the flow) taking values in $\mathbb{R}^n$, is denoted

$$\mathcal{S}_1(x_\circ, w(\cdot), \theta) \ . \tag{1}$$

The properties that define a flow are two of the main properties associated with the solution set of differential equations or inclusions:

Definition 1 *A parameterized set $\mathcal{S}_1$ of functions defined on intervals of the form $[0, T)$ with $T \in [0, \infty]$ is said to be a* flow *if*

1. $x(\cdot) \in \mathcal{S}_1(x_\circ, w(\cdot), \theta) \implies x(0) = x_\circ$

2 $$\left. \begin{array}{l} x(\cdot) \in \mathcal{S}_1(x_\circ, w(\cdot), \theta) \\ t_1 \in \text{domain}(x(\cdot)) \end{array} \right\} \implies x(t_1 + \cdot) \in \mathcal{S}_1(x(t_1), w(t_1 + \cdot), \theta) \ .$$

It often turns out that the analysis of properties of the parameterized flow $\mathcal{S}_1$ is too complicated and one way to overcome this is to introduce a simpler auxiliary flow that is denoted

$$\mathcal{S}_2(y_\circ, w(\cdot)) \tag{2}$$

where $y_\circ \in \mathbb{R}^n$ and $w \in \mathcal{L}_\infty$ and then to perform analysis of $\mathcal{S}_2$. If the flow $\mathcal{S}_2$ is a good approximation of $\mathcal{S}_1$ for small values of the parameter θ, then this approach yields good results. This is a common approach in a variety of methods, such as averaging, singular perturbations, slowly varying systems, etc. For instance, in averaging (see [11, 22]) one investigates the properties of the parameterized time-varying differential equation $\dot{x} = f\left(\frac{t}{\epsilon}, x\right)$, where $\epsilon > 0$ is a small parameter, via an auxiliary time invariant differential equation $\dot{x} = f_{av}(x)$, called the average of the original system. The average system is usually simpler to deal with than the original system (it is not parameterized with ϵ and it is time invariant) and under appropriate conditions it can be shown that the flow $\mathcal{S}_2$ that is generated by the solutions of the average system is a good approximation of the flow $\mathcal{S}_1$ generated by the solutions of the actual system, for sufficiently small values of the parameter ϵ. By presenting results for general flows, we will be able to apply our main results in a unified manner to investigation of a range of problems, such as highly oscillatory systems with inputs via averaging, slowly varying systems with inputs and singularly perturbed systems. For space reasons, however, we only present results on averaging and slowly varying systems.

3 Main Results

In this section we state and prove our main results on robustness of the stability properties mentioned in the introduction.

The results, stated for general flows with inputs, are applied in the next section to problems of averaging of highly oscillatory and slowly varying systems with inputs.

First, we consider a nonlinear version of the L_2 stability property from inputs to states that was introduced in [17] and shown to be equivalent to ISS. We introduce a generalization of the original definitions presented in [17] since the definition in [17] are global whereas we need a semiglobal and practical definition of the same property. A special case of this stability property is the finite gain L_2 stability, which is ubiquitous in robust control [21]. Roughly speaking, we show that if the auxiliary flow $\mathcal{S}_2$ is L_2-stable and if it approximates well the flow $\mathcal{S}_1$ for small values of θ, then the flow $\mathcal{S}_1$ is semiglobally practically L_2 stable in the parameter θ. Precise definitions and statements are given later.

Second, we prove similar results for L_∞ Input-Output stability (IOS, as introduced in [12]) when combined with an appropriate detectability condition, the so called input-to-output-to-state stability (IOSS) first considered in [15]. This type of results is important, for example, if we can prove that the average system $\dot{x} = f_{av}(x)$ with the output $y = h(x)$ is IOS from input u to output y and moreover the system is detectable, that is IOSS. This is typical in cases when a LaSalle type argument is used to prove asymptotic stability of an equilibrium. We emphasize, however, that we do not use Lyapunov arguments anywhere in the proofs and the proofs are based entirely on investigations of appropriate bounds on trajectories.

3.1 Integral-input to integral-state stability

Let us define semiglobal practical integral-Input to integral-State stability (iIiSS) for flows; ignoring expressions in square brackets we obtain the definitions for the semi-infinite horizon case, reading them yields the same property on arbitrary finite time intervals:

Definition 2 *Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty closed set, $\gamma_1, \gamma_2, \gamma_3$ in $\mathcal{K}_\infty$. The flow $\mathcal{S}$ is said to be semiglobally practically $(\gamma_1, \gamma_2, \gamma_3)$-iIiSS with respect to $\mathcal{A}$ [*on finite time intervals *] if for each pair [triple] of strictly positive real numbers $\delta, r[, T]$, there exists a strictly positive real number $\theta^\star$ such that for all $\theta \in \Theta$ with $|\theta| \leq \theta^\star$, for all $w \in \mathcal{L}_\infty(r)$ and $x_\circ \in \mathbb{R}^n$ with $|x_\circ|_\mathcal{A} \leq r$, each element $x(\cdot)$ of the flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ exists and satisfies for all $0 \leq t$ [$\leq T$]*

$$\int_0^t \gamma_1(|x(s)|_\mathcal{A})\, ds \leq \max\left\{\gamma_2(|x(0)|_\mathcal{A}), \int_0^t \gamma_3(|w(s)|)\, ds\right\} + \delta t. \tag{3}$$

If the inequality (3) holds with $\delta = 0$ and for all $t \geq 0$ uniformly over the set of initial conditions and input disturbances, we simply say that the system is $(\gamma_1, \gamma_2, \gamma_3)$-iIiSS with respect to $\mathcal{A}$. Notice how dividing both sides of (3) by $t > 0$ the iIiSS property can be thought of as an averaged-Input to averaged-State stability notion. In this way also the "practical" aspect involved in Definition 2 becomes apparent.

Theorem 1 *The flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ is semiglobally practically $(\gamma_1, \gamma_2, \gamma_3)$-iIiSS with respect to $\mathcal{A}$ on finite time intervals if and only if it is semiglobally practically $(\gamma_1, \gamma_2, \gamma_3)$-iIiSS with respect to $\mathcal{A}$.*

Proof. The sufficiency is clear. To establish necessity, take arbitrary strictly positive (δ, r) and let $T \geq 1$ be large enough so that

$$\gamma_2(\max\{r, \gamma_1^{-1}(2\gamma_3(r) + \delta)\}) \leq \frac{\delta T}{2}. \tag{4}$$

Let $\theta^\star$ come from the assumption of semiglobal practical $(\gamma_1,\gamma_2,\gamma_3)$-iIiSS on finite time intervals for the triple $(\delta/2, \max\{r, \gamma_1^{-1}(2\gamma_3(r)+\delta)\}, 4T)$.
Claim: for all $x_\circ$ with $|x_\circ|_{\mathcal{A}} \le r$, all $w \in \mathcal{L}_\infty(r)$ and all $\theta \in \Theta$ with $|\theta| \le \theta^\star$ and given any $x(\cdot) \in \mathcal{S}(x_\circ, w(\cdot), \theta)$ there exists a sequence of times $\tau_k \to +\infty$ such that $\tau_0 = 0$, $T \le \tau_k - \tau_{k-1} \le 2T$ and

$$|x(\tau_k)|_{\mathcal{A}} \le \max\{r, \gamma_1^{-1}(2\gamma_3(r)+\delta)\} \qquad \forall\, k \in \mathbb{N}.$$

We prove the claim by induction. For $k=0$ the induction hypothesis is trivially satisfied:

$$|x(\tau_0)|_{\mathcal{A}} = |x_\circ|_{\mathcal{A}} \le r \le \max\{r, \gamma_1^{-1}(2\gamma_3(r)+\delta)\}.$$

Assume by induction that $|x(\tau_{k-1})|_{\mathcal{A}} \le \max\{r, \gamma_1^{-1}(2\gamma_3(r)+\delta)\}$. It is straightforward by contradiction that there exists $\bar\tau \in [\tau_{k-1}+T, \tau_{k-1}+2T]$ such that

$$\gamma_1(|x(\bar\tau)|_{\mathcal{A}}) \le \frac{\int_{\tau_{k-1}+T}^{\tau_{k-1}+2T} \gamma_1(|x(s)|_{\mathcal{A}})\, ds}{T}.$$

In other words, there always exists a point of an interval (in this case $[\tau_{k-1}+T, \tau_{k-1}+2T]$) where the value of a function is less or equal than its average over the same interval. As a flow is by definition time-invariant (see Property 2), the assumption of semiglobal practical iIiSS (over finite intervals) is uniform with respect to the initial time. Therefore, exploiting this last inequality in combination with semiglobal practical iIiSS over finite intervals initialized at τ_{k-1} yields :

$$\begin{aligned}
\gamma_1(|x(\bar\tau)|_{\mathcal{A}}) &\le \frac{\int_{\tau_{k-1}}^{\tau_{k-1}+2T} \gamma_1(|x(s)|_{\mathcal{A}})\, ds}{T} \le \frac{\max\left\{\gamma_2(|x(\tau_{k-1})|_{\mathcal{A}}), \int_{\tau_{k-1}}^{\tau_{k-1}+2T} \gamma_3(|w(s)|)\, ds\right\} + \delta/2}{T} \\
&\le \max\left\{\frac{\gamma_2(\max\{r, \gamma_1^{-1}(2\gamma_3(r)+\delta)\})}{T}, 2\gamma_3(r)\right\} + \delta/(2T) \\
&\le \max\{\delta/2, 2\gamma_3(r)\} + \delta/2 \le 2\gamma_3(r) + \delta, \qquad (5)
\end{aligned}$$

where, for the last inequality, we used the fact that $T \ge 1$. Hence, $|x(\bar\tau)|_{\mathcal{A}} \le \gamma_1^{-1}(2\gamma_3(r)+\delta) \le \max\{r, \gamma_1^{-1}(2\gamma_3(r)+\delta)\}$ and the claim follows just by letting $\tau_k = \bar\tau$.

Let us look now at stability over a semi-infinite horizon. From (4) and the fact that $T \le \tau_k - \tau_{k-1}$ we obtain the following estimate

$$\begin{aligned}
\int_{\tau_{k-1}}^{\tau_k} \gamma_1(|x(s)|_{\mathcal{A}})\, ds &\le \max\left\{\gamma_2(|x(\tau_{k-1})|_{\mathcal{A}}), \int_{\tau_{k-1}}^{\tau_k} \gamma_3(|w(s)|)\, ds\right\} + \frac{\delta}{2}(\tau_k - \tau_{k-1}) \\
&\le \frac{\delta}{2}T + \int_{\tau_{k-1}}^{\tau_k} \gamma_3(|w(s)|)\, ds + \frac{\delta}{2}(\tau_k - \tau_{k-1}) \\
&\le \delta(\tau_k - \tau_{k-1}) + \int_{\tau_{k-1}}^{\tau_k} \gamma_3(|w(s)|)\, ds. \qquad (6)
\end{aligned}$$

Pick an arbitrary $t \ge 0$. We consider separately the two cases $t > \tau_1$ and $t \le \tau_1$. In the first case, there exists $k \in \mathbb{N}$ such that $0 \le t - \tau_k \le 2T$. In particular then $T \le t - \tau_{k-1} \le 4T$. In order to get an estimate for the integral of $\gamma_1(|x|_{\mathcal{A}})$ we proceed as follows

$$\begin{aligned}
\int_0^t \gamma_1(|x(s)|_{\mathcal{A}})\, ds &= \left(\sum_{i=1}^{k-1} \int_{\tau_{i-1}}^{\tau_i} \gamma_1(|x(s)|_{\mathcal{A}})\, ds)\right) + \int_{\tau_{k-1}}^t \gamma_1(|x(s)|_{\mathcal{A}})\, ds \\
&\le \left(\sum_{i=1}^{k-1} \delta(\tau_i - \tau_{i-1}) + \int_{\tau_{i-1}}^{\tau_i} \gamma_3(|w(s)|)\, ds\right) + \int_{\tau_{k-1}}^t \gamma_1(|x(s)|_{\mathcal{A}})\, ds \\
&= \delta\tau_{k-1} + \int_0^{\tau_{k-1}} \gamma_3(|w(s)|)\, ds + \int_{\tau_{k-1}}^t \gamma_1(|x(s)|_{\mathcal{A}})\, ds \qquad (7)
\end{aligned}$$

The last term of (7) can be bounded as follows

$$\begin{aligned}\int_{\tau_{k-1}}^{t} \gamma_1(|x(s)|_{\mathcal{A}})\, ds &\le \max\left\{\gamma_2(|x(\tau_{k-1})|_{\mathcal{A}}), \int_{\tau_{k-1}}^{t} \gamma_3(|w(s)|)\, ds\right\} + \frac{\delta}{2}(t-\tau_{k-1}) \\ &\le \frac{\delta}{2}T + \int_{\tau_{k-1}}^{t} \gamma_3(|w(s)|)\, ds + \frac{\delta}{2}(t-\tau_{k-1}) \le \delta(t-\tau_{k-1}) + \int_{\tau_{k-1}}^{t} \gamma_3(|w(s)|)\, ds. \quad (8)\end{aligned}$$

Combining (7) and (8) yields

$$\int_0^t \gamma_1(|x(s)|_{\mathcal{A}})\, ds \le \delta t + \int_0^t \gamma_3(|w(s)|)\, ds. \tag{9}$$

The case $t \le \tau_1$ is treated separately:

$$\int_0^t \gamma_1(|x(s)|_{\mathcal{A}})\, ds \le \max\left\{\gamma_2(|x(0)|_{\mathcal{A}}), \int_0^t \gamma_3(|w(s)|)\, ds\right\} + \delta t/2. \tag{10}$$

The theorem follows taking the maximum of (9) and (10):

$$\int_0^t \gamma_1(|x(s)|_{\mathcal{A}})\, ds \le \max\left\{\gamma_2(|x(0)|_{\mathcal{A}}), \int_0^t \gamma_3(|w(s)|)\, ds\right\} + \delta t.$$

■

We now state the corollary that if the elements of the parameterized flow can be made arbitrarily close on arbitrary compact time intervals to the elements of another flow that is iIiSS, then the parameterized flow is semiglobally practically iIiSS. The utility of this result is that the closeness of solutions result may be evident from the regularity hypotheses of the system generating the flow and the iIiSS property for the auxiliary flow may be easy to check while the (semiglobal practical) iIiSS property for the parameterized flow may be difficult to check directly. We need the following definition:

Definition 3 *Let $\mathcal{A} \subset \mathbb{R}^n$ be nonempty and closed. The flow $\mathcal{S}_2(x_\circ, w)$ is said to be forward complete with respect to $\mathcal{A}$ if for each $r > 0$ and $T > 0$ there exists $R \ge r$ such that for all $x_\circ \in \mathbb{R}^n$ with $|x_\circ|_{\mathcal{A}} \le r$ and all $w \in \mathcal{L}_\infty(r)$, each element of the flow is defined on $[0, T]$ and satisfies $|x(t)|_{\mathcal{A}} \le R$ for all $t \in [0, T]$.* ■

Corollary 1 *Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty closed set. Assume that the following conditions hold:*

1. *The flow $\mathcal{S}_2$ is $(\gamma_1, \gamma_2, \gamma_3)$-iIiSS with respect to $\mathcal{A}$,*
2. *The flow $\mathcal{S}_2$ is forward complete with respect to $\mathcal{A}$;*
3. *For each triple (T, δ, r) of strictly positive real numbers there exists a strictly positive real number θ^* such that, for each $\theta \in \Theta$ with $|\theta| < \theta^*$, $x_\circ \in \mathbb{R}^n$ with $|x_\circ|_{\mathcal{A}} \le r$, $w \in \mathcal{L}_\infty(r)$, each element $x(\cdot)$ of the flow $\mathcal{S}_1(x_\circ, w(\cdot), \theta)$ is defined for all $t \in [0, T]$ and there exists an an element $y(\cdot)$ of the flow $\mathcal{S}_2(x_\circ, w(\cdot))$ such that*

$$|x(t) - y(t)| \le \delta \qquad \forall t \in [0, T]\ . \tag{11}$$

Then the flow $\mathcal{S}_1$ is semiglobally practically $(\gamma_1, \gamma_2, \gamma_3)$-iIiSS with respect to $\mathcal{A}$. ■

Proof: Using the result of Theorem 1, we just need to show that the flow $\mathcal{S}_1$ is semiglobally practically iIiSS with respect to $\mathcal{A}$ on finite time intervals. Let (T, δ, r) be given. Let (r, T) generate R from the second condition of the corollary. Let $\tilde{\delta} > 0$ satisfy:

$$\sup_{s \in [0,R]} \left[\gamma_1(s + \tilde{\delta}) - \gamma_1(s)\right] \le \delta\ .$$

Let the data $(T, \tilde{\delta}, r)$ generate $\theta^* > 0$ from the third condition of Corollary 1. Consider arbitrary $\theta \in \Theta$ with $|\theta| \le \theta^*$, $x_\circ \in \mathbb{R}^n$ with $|x_\circ| \le r$ and $w \in \mathcal{L}_\infty(r)$. Condition 2 of the corollary guarantees that each element $y(\cdot)$ of the flow $\mathcal{S}_2(x_\circ, w)$ is defined for all $t \in [0, T]$ and satisfies $|y(t)|_{\mathcal{A}} \le R$. Condition 3 guarantees that each element $x(\cdot)$ of the flow $\mathcal{S}_1$ is defined on $[0.T]$ and there exists an element of the flow $\mathcal{S}_2(x_\circ, w)$ such that:

$$|x(t) - y(t)| \le \tilde{\delta}, \ \forall t \in [0, T] \ . \tag{12}$$

Condition 1 of the corollary guarantees that the flow $\mathcal{S}_2$ is $(\gamma_1, \gamma_2, \gamma_3)$-iIiSS with respect to $\mathcal{A}$ and hence for all $y_\circ \in \mathbb{R}^n, w \in \mathcal{L}_{\gamma_3}$ we have that all solutions of the flow $\mathcal{S}_2$ exist for all $t \in [0, \infty)$ and satisfy:

$$\int_0^t \gamma_1\left(|y(s)|_{\mathcal{A}}\right) ds \le \max\left\{\gamma_2\left(|y_\circ|_{\mathcal{A}}\right), \int_0^t \gamma_3(|w(s)|) ds\right\} \qquad \forall t \ge 0 \ . \tag{13}$$

Using the definition of $\tilde{\delta}$, (12) and (13) and the fact that $|\cdot|_{\mathcal{A}}$ is globally Lipschitz with Lipschitz constant one, we have that each element $x(\cdot)$ of the flow $\mathcal{S}_1$ satisfies for all $t \in [0, T]$:

$$\begin{aligned}
\int_0^t \gamma_1\left(|x(s)|_{\mathcal{A}}\right) ds &\le \int_0^t \gamma_1\left(|y(s)|_{\mathcal{A}} + |x(s) - y(s)|\right) ds \\
&\le \int_0^t \gamma_1\left(|y(s)|_{\mathcal{A}}\right) ds + \int_0^t \left[\gamma_1\left(|y(s)|_{\mathcal{A}} + \tilde{\delta}\right) - \gamma_1\left(|y(s)|_{\mathcal{A}}\right)\right] ds \\
&\le \max\left\{\gamma_2\left(|y_\circ|_{\mathcal{A}}\right), \int_0^t \gamma_3(|w(s)|) ds\right\} + \int_0^t \delta ds \\
&= \max\left\{\gamma_2\left(|x_\circ|_{\mathcal{A}}\right), \int_0^t \gamma_3(|w(s)|) ds\right\} + \delta t \ .
\end{aligned} \tag{14}$$

This establishes the result. ■

Remark 1 *The third condition in Corollary 1 (see also the second condition in Corollary 2) states that the trajectories of the parameterized flow and the unparameterized auxiliary flow can be made arbitrarily close on arbitrarily large compact time intervals by reducing the parameter θ. That type of results stems from classical results on continuous dependence of solutions of ordinary differential equations on initial conditions and parameters (see Theorems 2.5 and 2.6 in [4]) and it has been proved in a range of different situations for systems without inputs, such as averaging (Theorem 8.4 in [4]), singular perturbations (Theorem 9.1 in [4]), etc. Similar results for systems with inputs are straightforward generalization of the classical results and can be found, for instance, in [18, 8] for averaging of systems with inputs, in [8] for slowly varying systems with inputs and in [20] for singularly perturbed systems with inputs. We emphasize that our results can be applied in any of these situations but for space reasons we present applications only to averaging of systems with inputs and slowly varying systems with inputs.* ■

3.2 Robustness of input-output stability and detectability

We now define semiglobal practical input-output stability and input-output-to-state stability:

Definition 4 *Let $\mathcal{A}$,$\mathcal{D} \subset \mathbb{R}^n$ be nonempty closed sets, $\beta_0 \in \mathcal{KL}$ and $\gamma_0 \in \mathcal{K}_\infty$. The flow $\mathcal{S}$ is said to be semiglobally practically (β_0, γ_0)-IOS with respect to $\mathcal{A}$ and $\mathcal{D}$ [on finite time intervals] if for each pair [triple] of strictly positive real numbers $\delta, r[, T]$, there exists a strictly positive real number $\theta^\star$ such that for all $\theta \in \Theta$ with $|\theta| < \theta^\star$, for all $w \in \mathcal{L}_\infty(r)$ and $x_\circ$ in $\mathbb{R}^n$ with $|x_\circ|_{\mathcal{A}} \le r$, each element $x(\cdot)$ of the flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ exists and satisfies for all $0 \le t$ [$\le T$]*

$$|x(t)|_{\mathcal{D}} \le \max\{\beta_0(|x_\circ|_{\mathcal{A}}, t), \gamma_0(\|w\|_\infty)\} + \delta. \tag{15}$$

If the inequality (15) holds with $\delta = 0$ and for all $t \ge 0$ uniformly over the set of initial conditions and input disturbances, we simply say that the system is (β_0, γ_0)-IOS with respect to $\mathcal{A}$ and $\mathcal{D}$. Notice that the point-set distance $|\cdot|_{\mathcal{D}}$ can be thought of as the norm of a system output (the set $\mathcal{D}$ being the kernel of the output map); this motivates in the following the use of IOS and IOSS acronyms which usually refer to input-to-*output* and input-*output*-to-state stability notions.

Definition 5 *Let $\mathcal{A}$,$\mathcal{D} \subset \mathbb{R}^n$ be nonempty closed sets, $\beta_1 \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$. The flow $\mathcal{S}$ is said to be semiglobally practically $(\beta_1, \gamma_1, \gamma_2)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$ [on finite time intervals] if for each pair [triple] of strictly positive real numbers $\delta, r[, T]$, there exists a strictly positive real number $\theta^\star$ such that* *for all $\theta \in \Theta$ with $|\theta| < \theta^\star$, for all $w \in \mathcal{L}_\infty(r)$ and $x_\circ$ in $\mathbb{R}^n$ with $|x_\circ|_\mathcal{A} \leq r$, each element $x(\cdot)$ of the flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ exists and satisfies for all $0 \leq t$ [$\leq T$]*

$$|x(t)|_\mathcal{A} \leq \max\{\beta_1(|x_\circ|_\mathcal{A}, t), \gamma_1(\|w\|_\infty), \gamma_2(\||x_{[0,t]}|_\mathcal{D}\|_\infty)\} + \delta. \tag{16}$$

If the inequality (16) holds with $\delta = 0$ and for all $t \geq 0$ uniformly over the set of initial conditions and input disturbances, we simply say that the system is $(\beta_1, \gamma_1, \gamma_2)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$.

Lemma 1 *Let a flow $\mathcal{S}$ be semiglobally practically IOS and IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$ on finite time intervals; then there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ so that the flow $\mathcal{S}$ is semiglobally practically (β, γ)-ISS with respect to $\mathcal{A}$ on finite time intervals.*

Proof. Let us fix a triple of strictly positive real numbers $(\tilde{\delta}, \tilde{r}, T)$. Let $\theta^\star = \min\{\theta_1^\star, \theta_2^\star\}$ where $\theta_1^\star$ and $\theta_2^\star$ are generated by the semiglobal practical IOS and IOSS assumptions for (δ, r, T) where δ and r are suitable positive constants to be defined later on. Let $t \in [0, T]$ be arbitrary; by the IOSS assumption, initialized at time $t/2$ we have

$$|x(t)|_\mathcal{A} \leq \max\{\beta_1(|x(t/2)|_\mathcal{A}, t/2), \gamma_1(\|w_{[t/2,t]}\|_\infty), \gamma_2(\||x_{[t/2,t]}|_\mathcal{D}\|_\infty)\} + \delta. \tag{17}$$

By virtue of semiglobal practical IOS the term inside γ_2 in equation (17) can be upper bounded as follows

$$\||x_{[t/2,t]}|_\mathcal{D}\|_\infty \leq \max\{\beta_0(|x_\circ|_\mathcal{A}, t/2), \gamma_0(\|w\|_\infty)\} + \delta. \tag{18}$$

We also need to estimate $|x(t/2)|_\mathcal{A}$ which appears inside the $\mathcal{KL}$ function β_1 in (17):

$$\begin{aligned} |x(t/2)|_\mathcal{A} &\leq \max\{\beta_1(|x_\circ|_\mathcal{A}, t/2), \gamma_1(\|w\|_\infty), \gamma_2(\||x_{[0,t/2]}|_\mathcal{D}\|)\} + \delta \\ &\leq \max\{\beta_1(|x_\circ|_\mathcal{A}, t/2), \gamma_1(\|w\|_\infty), \gamma_2(\max\{\beta_0(|x_\circ|_\mathcal{A}, 0), \gamma_0(\|w\|_\infty)\} + \delta)\} + \delta \\ &= \max\{\beta_1(|x_\circ|_\mathcal{A}, t/2), \gamma_1(\|w\|_\infty), \gamma_2(\beta_0(|x_\circ|_\mathcal{A}, 0) + \delta), \gamma_2(\gamma_0(\|w\|_\infty) + \delta)\} + \delta. \end{aligned} \tag{19}$$

Combining (17), (18) and (19) and using standard techniques in the treatment of comparison functions yields the desired inequality. The following result was proved in [8]:

Lemma 2 *The flow $\mathcal{S}_1$ in (1) is semiglobally practically (β, γ)-ISS with respect to $\mathcal{A}$ on finite time intervals if and only if it is semiglobally practically (β, γ)-ISS with respect to $\mathcal{A}$.*

The main result of this section is stated next.

Theorem 2 *The flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ is semiglobally practically (β_0, γ_0)-IOS with respect to $\mathcal{A}$ and semiglobally practically $(\beta_1, \gamma_1, \gamma_2)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$ if and only if it is semiglobally practically (β_0, γ_0)-IOS with respect to $\mathcal{A}$ on finite time intervals and semiglobally practically $(\beta_1, \gamma_1, \gamma_2)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$ on finite time intervals.*

Proof. By Lemma 1 the system is semiglobally practically (β, γ)-ISS with respect to the set $\mathcal{A}$ over finite time intervals. Hence, for each triple of strictly positive real numbers (T, δ, r), there exists a strictly positive real number $\theta^\star$ such that for all $\theta \in \Theta$ with $|\theta| < \theta^\star$, for all $w \in \mathcal{L}_\infty(r)$ and $x_\circ \in \mathbb{R}^n$ with $|x_\circ|_\mathcal{A} \leq r$, each element $x(\cdot)$ of the flow $\mathcal{S}(x_\circ, w(\cdot), \theta)$ exists for all $t \in [0, T]$ and satisfies

$$|x(t)|_\mathcal{A} \leq \max\{\beta(|x_\circ|_\mathcal{A}, t), \gamma(\|w\|)\} + \delta \qquad \forall\, t \in [0, T].$$

Pick arbitrary positive real numbers (δ, r). Let T be sufficiently large so that:

$$\begin{array}{c} \beta(\max\{r, \gamma(r) + \delta\}, T) \leq \delta/2 \\ \beta_0(\max\{r, \gamma(r) + \delta\}, T) \leq \delta/2 \\ \beta_1(\max\{r, \gamma(r) + \delta\}, T) \leq \delta/2 \end{array}\ . \tag{20}$$

Let $\theta^\star$ be the minimum of the $\theta^\star$s generated from the IOS, IOSS and ISS assumptions with the triple $(2T, \delta/2, \max\{r, \gamma(r) + \delta\})$. Consider a fixed but arbitrary $\theta \in \Theta$, with $|\theta| < \theta^\star$, and $w \in \mathcal{L}_\infty(r)$. For all $x_\circ \in \mathbb{R}^n$ with $|x_\circ|_\mathcal{A} \leq \max\{r, \gamma(r) + \delta\}$, each element of the flow exists for all $t \in [0, 2T]$ and satisfies

$$\begin{aligned} |x(t)|_\mathcal{D} &\leq \max\{\beta_0(|x_\circ|_\mathcal{A}, t), \gamma_0(\|w\|_\infty)\} + \delta/2 \qquad \forall\, t \in [0, 2T], \\ |x(t)|_\mathcal{A} &\leq \max\{\beta_1(|x_\circ|_\mathcal{A}, t), \gamma_1(\|w\|_\infty), \gamma_2(\||x|_\mathcal{D}\|_\infty)\} + \delta/2 \qquad \forall\, t \in [0, 2T]. \end{aligned} \tag{21}$$

By virtue of inequalities (20) we have

$$\begin{array}{rcll} |x(t)|_{\mathcal{D}} & \leq & \gamma_0(\|w\|_\infty)+\delta, & \forall\, t\in[T,2T], \\ |x(t)|_{\mathcal{A}} & \leq & \max\{\gamma_1(\|w\|_\infty),\gamma_2(\||x|_{\mathcal{D}}\|_\infty)\}+\delta, & \forall\, t\in[T,2T] \\ |x(T)|_{\mathcal{A}} & \leq & \gamma(\|w\|_\infty)+\delta \leq \max\{r,\gamma(r)+\delta\}. & \end{array} \tag{22}$$

Now, exploiting time-invariance of the flow $\mathcal{S}$ (property 2 of Definition 1) we may apply the above argument again with initial state $\tilde{x}_\circ = x(T)$ and disturbance $\tilde{w}(\cdot) = w(\cdot + T)$. In fact a repeated application of the argument implies that for all $x_\circ \in \mathbb{R}^n$ with $|x_\circ|_{\mathcal{A}} \leq \max\{r, \gamma(r)+\delta\}$, each element of the flow exists for all $t \in [0,+\infty)$ and satisfies

$$\begin{array}{rcll} |x(t)|_{\mathcal{D}} & \leq & \gamma_0(\|w\|_\infty)+\delta, & \forall\, t\in[T,+\infty), \\ |x(t)|_{\mathcal{A}} & \leq & \max\{\gamma_1(\|w\|_\infty),\gamma_2(\||x|_{\mathcal{D}}\|_\infty)\}+\delta, & \forall\, t\in[T,+\infty) \\ |x(kT)|_{\mathcal{A}} & \leq & \gamma(\|w\|_\infty)+\delta \leq \max\{r,\gamma(r)+\delta\} & \forall\, k\in\mathbb{N}. \end{array} \tag{23}$$

The conclusion of the theorem follows by (23) and (21). ■

We now state the corollary that if the elements of the parameterized flow can be made arbitrarily close on arbitrary compact time intervals to the elements of another flow that is IOS and IOSS then the parameterized flow is semiglobally practically ISS. The utility of this result is similar to Corollary 1: the closeness of solutions result may be evident from the regularity hypotheses of the system generating the flow and the IOS and IOSS properties for the auxiliary flow may be easy to check while the (semiglobal practical) ISS property for the parameterized flow may be difficult to check directly. We again consider the auxiliary flow (2) to state the next corollary.

Corollary 2 *Let $\mathcal{A},\mathcal{D} \subset \mathbb{R}^n$ be nonempty closed sets. Assume that the following conditions hold:*

1. *The flow $\mathcal{S}_2$ in (2) is:*
   (a) *(β_0,γ_0)-IOS with respect to $\mathcal{A}$ and $\mathcal{D}$; and*
   (b) *$(\beta_1,\gamma_1,\gamma_2)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$.*
2. *For each triple (T,δ,r) of strictly positive real numbers there exists a strictly positive real number θ^* such that, for each $\theta \in \Theta$ with $|\theta| < \theta^*$, $x_\circ \in \mathbb{R}^n$ with $|x_\circ|_{\mathcal{D}} \leq r$, $w \in \mathcal{L}_\infty(r)$, each element $x(\cdot)$ of the flow $\mathcal{S}_1(x_\circ, w(\cdot), \theta)$ is defined for all $t \in [0,T]$ and there exists an $y_\circ \in \Re^n$ and an element $y(\cdot)$ of the flow $\mathcal{S}_2(y_\circ, w(\cdot))$ such that*
$$|x(t)-y(t)| \leq \delta \qquad \forall t \in [0,T]\ . \tag{24}$$

Then the following statements are true:

1. *The flow $\mathcal{S}_1$ in (1) is semiglobally practically (β_0,γ_0)-IOS with respect to $\mathcal{A}$ and $\mathcal{D}$.*
2. *The flow $\mathcal{S}_1$ in (1) is semiglobally practically $(\beta_1,\gamma_1,\gamma_2)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$.*
3. *There exist $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that the flow $\mathcal{S}_1$ in (1) is semiglobally practically (β,γ)-ISS with respect to $\mathcal{D}$.*

■

Proof Using the result of Lemmas 1, 2 and 2, we just need to show that the flow $\mathcal{S}_1$ is semiglobally practically (β_0,γ_0)-IOS with respect to $\mathcal{A}$ and $\mathcal{D}$ on finite time intervals and semiglobally practically $(\beta_1,\gamma_1,\gamma_2)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$ on finite time intervals. Let (T,δ,r) be given. Without loss of generality we assume that $\delta \leq 1$ and that $\mathcal{KL}$ functions be of the following kind: $\beta_i(s,t) = \lambda_1(s)\lambda_2(e^{-t})$ for some $\mathcal{K}$ functions λ_1 and λ_2 (this is always possible by virtue of the $\mathcal{KL}$ lemma in [17]) . Thus we can find $\tilde{\delta} > 0$ satisfying all of the inequalities:

$$\sup_{s\in[0,r],t\in[0,\infty)} [\beta_0(s+\tilde{\delta},t)-\beta_0(s,t)]+\tilde{\delta} \ \leq \ \delta \qquad (25)$$

$$\sup_{s\in[0,r],t\in[0,\infty)} [\beta_1(s+\tilde{\delta},t)-\beta_1(s,t)]+\tilde{\delta} \ \leq \ \frac{\delta}{2} \qquad (26)$$

$$\sup_{s\in[0,\max\{\beta_0(r,0),\gamma_0(r)\}+1],t\in[0,\infty)} [\gamma_2(s+\tilde{\delta})-\gamma_2(s)] \ \leq \ \frac{\delta}{2} \qquad (27)$$

Let the data $(T,\tilde{\delta},r)$ generate $\theta^*>0$ from the second condition of Corollary 1. Consider arbitrary $\theta\in\Theta$ with $|\theta|<\theta^*$, $x_\circ\in\mathbb{R}^n$ with $|x_\circ|_\mathcal{D}\leq r$ and $w\in\mathcal{L}_\infty(r)$. Condition 2 of the corollary's statement guarantees that for each element $x(\cdot)$ of the flow $\mathcal{S}_1$ is defined for all $t\in[0,T]$ and there exists an $y_\circ\in\mathbb{R}^n$ and an element $y(\cdot)$ of the flow $\mathcal{S}_2$ such that

$$|x(t)-y(t)|\leq\tilde{\delta} \qquad \forall t\in[0,T]\ . \qquad (28)$$

Using the first property of flows, this implies that $|x_\circ-y_\circ|\leq\tilde{\delta}$. Condition 1a of the corollary's statement guarantees that the flow $\mathcal{S}_2$ is (β_0,γ_0)-IOS with respect to $\mathcal{A}$ and $\mathcal{D}$ and hence for all $y_\circ\in\mathbb{R}^n, w\in\mathcal{L}_\infty$ we have that

$$|y(t)|_\mathcal{D}\leq\max\{\beta(|y_\circ|_\mathcal{A},t),\gamma(\|w\|_\infty)\} \qquad \forall t\geq 0\ . \qquad (29)$$

Using (28) and (29), (25) and the fact that $|\cdot|_\mathcal{A}$ is globally Lipschitz with Lipschitz constant one, we can see that each element $x(\cdot)$ of the flow $\mathcal{S}_1$ satisfies, for all $t\in[0,T]$,

$$\begin{aligned}
|x(t)|_\mathcal{D} &\leq |y(t)|_\mathcal{D}+|x(t)-y(t)| \\
&\leq \max\{\beta_0(|y_\circ|_\mathcal{A},t),\gamma_0(\|w\|_\infty)\}+\tilde{\delta} \\
&\leq \max\{\beta_0(|x_\circ|_\mathcal{A}+\tilde{\delta},t),\gamma_0(\|w\|_\infty)+\tilde{\delta} \\
&\leq \max\{\beta_0(|x_\circ|_\mathcal{A},t),\gamma_0(\|w\|_\infty)\}+\delta\ . \qquad (30)
\end{aligned}$$

Hence, the flow S_1 is semiglobally practically (β_0,γ_0)-IOS with respect to $\mathcal{A}$ and $\mathcal{D}$ on finite time intervals. Condition 1b of the corollary's statement guarantees that the flow $\mathcal{S}_2$ is $(\beta_1,\gamma_1,\gamma_2)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$ and hence for all $y_\circ\in\mathbb{R}^n, w\in\mathcal{L}_\infty$ we have that

$$|y(t)|_\mathcal{A}\leq\max\{\beta_1(|y_\circ|_\mathcal{A},t),\gamma_1(\|w\|_\infty),\gamma_2(\||y_{[0,t]}|_\mathcal{D}\|)\} \qquad \forall t\geq 0\ . \qquad (31)$$

Using (28) and (31), (26), (27) and the fact that $|\cdot|_\mathcal{A}$ is globally Lipschitz with Lipschitz constant one, we can see that each element $x(\cdot)$ of the flow $\mathcal{S}_1$ satisfies, for all $t\in[0,T]$,

$$\begin{aligned}
|x(t)|_\mathcal{A} &\leq |y(t)|_\mathcal{A}+|x(t)-y(t)| \\
&\leq \max\left\{\beta_1(|y_\circ|_\mathcal{A},t),\gamma_1(\|w\|_\infty),\gamma_2\left(\||y_{[0,t]}|_\mathcal{D}\|\right)\right\}+\tilde{\delta} \\
&\leq \max\left\{\beta_1(|x_\circ|_\mathcal{A}+\tilde{\delta},t),\gamma_1(\|w\|_\infty),\gamma_2\left(\||x_{[0,t]}|_\mathcal{D}\|+\tilde{\delta}\right)\right\}+\tilde{\delta} \\
&\leq \max\left\{\beta_1(|x_\circ|_\mathcal{A},t),\gamma_1(\|w\|_\infty),\gamma_2\left(\||x_{[0,t]}|_\mathcal{D}\|\right)\right\}+\delta\ . \qquad (32)
\end{aligned}$$

Hence, the flow S_1 is semiglobally practically $(\beta_1,\gamma_1,\gamma_2)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$ on finite time intervals, which completes the proof. ■

Remark 2 *It is clear from proofs that the results in this section also hold for more general notions of stability than the one considered so far; for instance we may easily restrict all definitions to positively-invariant subsets of $\mathbb{R}^n$ and forward shift-invariant classes of inputs; recall that $\mathcal{F}$ is forward shift invariant, if $w(\cdot)\in\mathcal{F}$, $t_1\geq 0$ implies $w(t_1+\cdot)\in\mathcal{F}$.*

4 Applications of main results

In this section we apply results of the previous section to investigation of stability properties of systems with inputs that are either highly oscillatory or slowly time varying. First, we consider averaging of systems with inputs following closely the results in [10, 19]. Then, we present similar results for slowly varying systems with inputs using results of [8]. Note that the second condition in Corollaries 1 and 2 is a closeness of solutions result for the two flows on finite time intervals. This condition together with appropriate stability/detectability condition for flow $\mathcal{S}_2$ guarantees that the same stability/detectability property holds in a semiglobal practical sense for the flow $\mathcal{S}_1$. We emphasize that the appropriate closeness of solutions results are classical in averaging and slowly varying systems and for systems with disturbances such results were stated and proved in [8]. Hence, by combining the closeness of solutions results of [8] and our Corollaries 1 and 2, we can state several interesting new stability results in the context of averaging and slowly varying systems.

4.1 Averaging of highly oscillatory systems

In this section, we consider systems of the form (for motivation and more details see [8])

$$\begin{aligned} \dot{x} &= f(q,p,x,w,\nu) \\ \dot{p} &= 1 \\ \epsilon\dot{q} &= 1 \end{aligned} \tag{33}$$

where $x \in \mathbb{R}^n$, $p \in \mathbb{R}_{\geq 0}$ and $q \in \mathbb{R}_{\geq 0}$, and where $\epsilon > 0$ and ν belongs to a small neighborhood of the origin in $\mathbb{R}^{n_\nu}$. We will consider the situation where the function f has a well-defined average when $\nu = 0$, p and x are held constant, and the function is integrated with respect to q. Whether w is held constant or is allowed to vary will depend on the type of average we are considering. We will consider two types of averages: *weak* and *strong* averages. When working with weak averages we will assume that our disturbances belong to a set of equi-uniformly continuous functions. This is defined precisely as follows:

Definition 6 *Let $\mathcal{F}$ be a set of locally essentially bounded functions.* The set $\mathcal{F}$ is equi-uniformly continuous *if for each $\rho > 0$ there exists $\delta > 0$ such that, for all $w(\cdot) \in \mathcal{F}$ and all $t \geq 0$,*

$$\tau \in [0,\delta] \qquad \Longrightarrow \qquad |w(t+\tau) - w(t)| \leq \rho \ .$$

■

Whether the function f has a well-defined average or not will depend partly on the specification of a closed set $\mathcal{A} \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$. This set is related to the set with respect to which we want to study the stability properties for the system (33). In particular, we are ultimately interested in studying the (semiglobal practical) stability property with respect to the set $\mathcal{A}_e := \{(x,p,q) : |(x,p)|_{\mathcal{A}} = 0\}$ from the initial conditions $\mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. While making the p dependence explicit should make it clear to the reader that we are considering partial averages, the only real mathematical advantage is to allow weaker regularity properties for f with respect to p compared to its regularity properties with respect to x. Without this motivation, we could just as easily consider p to be a part of x since we will be considering the ISS property with respect to sets that are not necessarily compact. The standing regularity hypotheses on the function f in this subsection are given by the following:

Assumption 1 *The function f is:*

1. *measurable in (q,p) for each (x,w,ν),*
2. *for each $R > 0$ and $\sigma > 0$ there exist $\rho > 0$ and $B > 0$ such that,*
   (a) *for all $(x,p,q,w,\nu) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^d \times \mathbb{R}^{n_\nu}$ with $|(x,p)|_{\mathcal{A}} \leq R$, $|w| \leq R$, and $|\nu| < \rho$,*

   $$\begin{aligned} |f(q,p,x,w,\nu) - f(q,p,x,w,0)| &\leq \sigma \\ |f(q,p,x,w,0)| &\leq B \end{aligned}$$

   (b) *for $(i = 1,2)$ all $(x_i,p,q,w_i) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^d$ with $|(x_i,p)|_{\mathcal{A}} \leq R$, $|w_i| \leq R$, $|x_1 - x_2| \leq \rho$ and $|w_1 - w_2| \leq \rho$,*

   $$|f(q,p,x_1,w_1,0) - f(q,p,x_2,w_2,0)| \leq \sigma \ .$$

■

The (partial) average systems we consider will have the form:

$$\begin{aligned} \dot{x} &= f_{\rm av}^{\mathcal{A}}(p,x,w) \\ \dot{p} &= 1 \; . \end{aligned} \tag{34}$$

The standing regularity hypotheses imposed on this system throughout this section are the following:

Assumption 2 *The function $f_{\rm av}^{\mathcal{A}}$ has the property that for each $R>0$ there exist $L>0$ and $B>0$ such that for $(i=1,2)$ all $(x_i,p,w)\in\mathbb{R}^n\times\mathbb{R}_{\geq 0}\times\mathbb{R}^d$ with $|(x_i,p)|_{\mathcal{A}}\leq R$ and $|w|\leq R$,*

$$\begin{aligned} |f_{\rm av}^{\mathcal{A}}(p,x_1,w)-f_{\rm av}^{\mathcal{A}}(p,x_2,w)| &\leq L|x_1-x_2| \\ |f_{\rm av}^{\mathcal{A}}(p,x_1,w)| &\leq B \; . \end{aligned}$$

■

The following two definitions of $\mathcal{A}$-strong and $\mathcal{A}$-weak average for a time-varying system with exogenous disturbances generalize the definitions that were introduced in [10] (see also [19]):

Definition 7 ($\mathcal{A}$-weak average) *Let $\mathcal{A}\subset\mathbb{R}^n\times\mathbb{R}_{\geq 0}$ be an arbitrary nonempty closed set. A function $f_{\rm wa}^{\mathcal{A}}:\mathbb{R}_{\geq 0}\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is said to be* the $\mathcal{A}$-weak average of *$f(\cdot,p,x,w,\nu)$ if there exist $\beta_{\rm av}\in\mathcal{KL}$ and $T^*>0$ such that $\forall T\geq T^*$ and $\forall t\geq 0$ we have*[1]

$$\begin{aligned} &\left| f_{\rm wa}^{\mathcal{A}}(p,x,w)-\frac{1}{T}\int_t^{t+T} f(s,p,x,w,0)ds\right| \\ &\qquad \leq \beta_{\rm av}\left(\max\{|(x,p)|_{\mathcal{A}},|w|,1\},T\right) \;, \end{aligned} \tag{35}$$

for all $(x,p)\in\mathbb{R}^n\times\mathbb{R}_{\geq 0}$ and $w\in\mathbb{R}^m$. The $\mathcal{A}$-weak average of system (33) is then defined as in (34) with $f_{\rm av}^{\mathcal{A}}:=f_{\rm wa}^{\mathcal{A}}$. ■

Definition 8 ($\mathcal{A}$-strong average) *Let $\mathcal{A}\subset\mathbb{R}^n\times\mathbb{R}_{\geq 0}$ be an arbitrary nonempty closed set. An $\mathcal{A}$-locally Lipschitz function $f_{\rm sa}^{\mathcal{A}}:\mathbb{R}_{\geq 0}\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is said to be* the $\mathcal{A}$-strong average of *$f(\cdot,p,x,w,\nu)$ if there exist $\beta_{\rm av}\in\mathcal{KL}$ and $T^*>0$ such that $\forall T\geq T^*$ and $\forall t\geq 0$ the following holds:*

$$\begin{aligned} &\left|\frac{1}{T}\int_t^{t+T}\left[f_{\rm sa}^{\mathcal{A}}(p,x,w(s))-f(s,p,x,w(s),0)\right]ds\right| \\ &\qquad \leq \beta_{\rm av}\left(\max\{|x|_{\mathcal{A}},\|w\|_\infty,1\},T\right) \;, \end{aligned} \tag{36}$$

for all $(x,p)\in\mathbb{R}^n\times\mathbb{R}_{\geq 0}$ and $w\in\mathcal{L}_\infty$. The $\mathcal{A}$-strong average of system (33) is then defined as in (34) with $f_{\rm av}^{\mathcal{A}}:=f_{\rm sa}^{\mathcal{A}}$. ■

Remark 3 *It has been shown in [10] that functions f, which are periodic in t, that have a strong average are, in essence, functions of the form $f(\cdot,p,x,w,0)=\widetilde{f}(\cdot,p,x)+g(p,x,w)$ where $\widetilde{f}(\cdot,p,x)$ has a well-defined (weak) average.* ■

The results that are given below were first presented in [19] and they give conditions under which the solutions of (33) are close to the solutions of (33)'s $\mathcal{A}$-weak or $\mathcal{A}$-strong average, when these averages exist. Proofs of these results are not presented and they can be found in [19].

Theorem 3 (Closeness to $\mathcal{A}$-weak average) *Let the nonempty closed set $\mathcal{A}\subset\mathbb{R}^n\times\mathbb{R}_{\geq 0}$ and the set of functions $\mathcal{F}$ be given. Suppose:*

1. *Assumption 1 holds,*

[1] Note that w in the integral is a constant vector.

2. *the $\mathcal{A}$-weak average (34) of the system (33) exists, is $(\mathbb{R}^n \times \mathbb{R}_{\geq 0}, \mathcal{F}, \mathcal{A})$-forward complete and satisfies Assumption 2,*

3. *$\mathcal{F}$ equi-uniformly continuous;*

Then, for each triple (T, δ, r) of strictly positive real numbers there exists a triple (ϵ^*, ν^*, μ) of strictly positive real numbers such that, for each $\epsilon \in (0, \epsilon^*)$, $\nu \in \mathbb{R}^{n_\nu}$ with $|\nu| < \nu^*$, each $(y_\circ, p_\circ) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ (an initial condition of (34)) with $|(y_\circ, p_\circ)|_{\mathcal{A}} \leq r$, each $w \in \mathcal{F} \bigcap \mathcal{L}(r)$ and each $(x_\circ, q_\circ) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ (together with $p_\circ$, an initial condition for (33)) such that $|x_\circ - y_\circ| \leq \mu$, each solution of (33) (with x component denoted $x_{\epsilon,\nu}(\cdot)$) and the solution of (34) (with x component denoted $y(\cdot)$) satisfy

$$|x_{\epsilon,\nu}(t) - y(t)| \leq \delta \qquad \forall t \in [0, T] \ . \tag{37}$$

■

Identifying $\theta := (\epsilon, \nu)$ and $\Theta := \mathbb{R}_{>0} \times \mathbb{R}^{n_\nu}$, and relying on Remark 2, a consequence of Theorem 3, Corollary 1 is the following:

Corollary 3 *Let a set $\mathcal{F}$ and an arbitrary nonempty closed set $\mathcal{A}$ be given. Suppose:*

1. *Assumption 1 holds,*

2. *the $\mathcal{A}$-weak average (34) of the system (33) exists, satisfies Assumption 2, and it is $(\mathbb{R}^n \times \mathbb{R}_{\geq 0}, \mathcal{F}, \beta, \gamma)$-iIiSS with respect to $\mathcal{A}$.*

3. *$\mathcal{F}$ is equi-uniformly continuous and forward shift invariant.*

Then, the system (33) is semiglobally practically $(\mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}, \mathcal{F}, \beta, \gamma)$-iIiSS with respect to $\mathcal{A}_e := \{(x, p, q) : |(x, p)|_{\mathcal{A}} = 0\}$. ■

Corollary 4 *Let a set $\mathcal{F}$ and an arbitrary nonempty closed sets $\mathcal{A}$ and $\mathcal{D}$ be given. Suppose:*

1. *Assumption 1 holds,*

2. *the $\mathcal{A}$-weak average (34) of the system (33) exists, satisfies Assumption 2, and it is $(\mathbb{R}^n \times \mathbb{R}_{\geq 0}, \mathcal{F}, \beta, \gamma)$-IOS with respect to $\mathcal{A}$ and $\mathcal{D}$ and $(\mathbb{R}^n \times \mathbb{R}_{\geq 0}, \mathcal{F}, \beta, \gamma)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$.*

3. *$\mathcal{F}$ is equi-uniformly continuous and forward shift invariant.*

Then, the system (33) is semiglobally practically $(\mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}, \mathcal{F}, \beta, \gamma)$-ISS with respect to $\mathcal{A}_e := \{(x, p, q) : |(x, p)|_{\mathcal{A}} = 0\}$. ■

The assumption that $\mathcal{F}$ is equi-uniformly continuous can be removed when the strong average exists and is $\mathcal{F}$-forward complete (see [8]):

Theorem 4 (Closeness to $\mathcal{A}$-strong average) *Let a set $\mathcal{F}$ and a nonempty closed set $\mathcal{A}$ be given. Suppose:*

1. *Assumption 1 holds,*

2. *the $\mathcal{A}$-strong average (34) of the system (33) exists, satisfies Assumption 2, and is $(\mathbb{R}^n \times \mathbb{R}_{\geq 0}, \mathcal{F}, \mathcal{A})$-forward complete.*

Then, for each triple (T, δ, r) of strictly positive real numbers there exists a triple (ϵ^*, ν^*, μ) of strictly positive real numbers such that, for each $\epsilon \in (0, \epsilon^*)$, $\nu \in \mathbb{R}^{n_\nu}$ with $|\nu| < \nu^*$, each $(y_\circ, p_\circ) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ (an initial condition of (34)) with $|(y_\circ, p_\circ)|_{\mathcal{A}} \leq r$, each $w \in \mathcal{F} \bigcap \mathcal{L}_\infty(r)$ and each $(x_\circ, q_\circ) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ (together with $p_\circ$, an initial condition for (33)) such that $|x_\circ - y_\circ| \leq \mu$, each solution of (33) (with x component denoted $x_{\epsilon,\nu}(\cdot)$) and each solution of (34) (with x component denoted $y(\cdot)$) satisfy

$$|x_{\epsilon,\nu}(t) - y(t)| \leq \delta \qquad \forall t \in [0, T] \ . \tag{38}$$

■

A consequence of Theorem 4, Corollary 1 and Remark 2 is the following:

Corollary 5 *Let a nonempty closed set $\mathcal{A}$ be given. Suppose:*

1. *Assumption 1 holds,*
2. *the $\mathcal{A}$-strong average (34) of the system (33) exists, satisfies Assumption 2, and it is $(\mathbb{R}^n \times \mathbb{R}_{\geq 0}, \beta, \gamma)$-iIiSS with respect to $\mathcal{A}$.*

Then the system (33) is semiglobally practically $(\mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}, \beta, \gamma)$-iIiSS with respect to $\mathcal{A}_e := \{(x,p,q) : |(x,p)|_{\mathcal{A}} = 0\}$. ■

Corollary 6 *Let a nonempty closed sets $\mathcal{A}$ and $\mathcal{D}$ be given. Suppose:*

1. *Assumption 1 holds,*
2. *the $\mathcal{A}$-strong average (34) of the system (33) exists, satisfies Assumption 2, and it is $(\mathbb{R}^n \times \mathbb{R}_{\geq 0}, \beta, \gamma)$-IOS with respect to $\mathcal{A}$ and $\mathcal{D}$ and $(\mathbb{R}^n \times \mathbb{R}_{\geq 0}, \beta, \gamma)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$.*

Then the system (33) is semiglobally practically $(\mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}, \beta, \gamma)$-ISS with respect to $\mathcal{A}_e := \{(x,p,q) : |(x,p)|_{\mathcal{A}} = 0\}$. ■

Remark 4 *Corollaries 3 and 5 generalize results in [9] since:*

1. *The iIiSS property considered here is defined with respect to an arbitrary (possibly non-compact) set $\mathcal{A}$ as opposed to the case considered in [9] where $\mathcal{A} = \{0\}$. Consequently, we can use our results for the case of partially averaged systems (for more details see [19]) that are not covered by results in [9].*
2. *The trajectory based proof that we presented here gives the same disturbance gain in the iIiSS property for the actual and averaged systems. This was not possible to prove in [9] since the proofs in [9] were based on converse Lyapunov ISS theorems that introduce certain increase (conservatism) in the disturbance gain.*

On the other hand, Corollaries 4 and 6 appear to be new results in the context of averaging of systems with disturbances. ■

4.2 Slowly varying systems

In this section, we will consider the system (for motivation and more details see [8])

$$\begin{aligned} \dot{x} &= f(x,u,v,w) \\ \dot{u} &= v \end{aligned} \tag{39}$$

where v and the initial value for u are assumed to be such that $u(t) \in \Gamma$ for all $t \geq 0$. Note that, for the sake of generality, we are allowing explicit dependence on $\dot{u}$, i.e., v in the $\dot{x}$ equation. Another interesting special case of this form is when f is independent of u and we want to study robustness of the stability properties with respect to disturbances v that are small in the (averaged) sense we will consider here. We will also consider the following auxiliary system:

$$\begin{aligned} \dot{x} &= f(x,u,0,w) \\ \dot{u} &= 0\ . \end{aligned} \tag{40}$$

Regarding our assumption on $v(\cdot)$ that enables comparing the solutions of (39) to the solutions of (40), we assume $v(\cdot)$ is small on average. In particular, we assume that

$$\frac{1}{\Delta}\int_t^{t+\Delta} |v(\tau)| d\tau$$

is sufficiently small, where Δ is a parameter that can be chosen arbitrarily. Notice that this condition does not necessarily imply that $v \in \mathcal{L}_\infty$. We may, however, impose this extra condition if the regularity conditions for f are not strong enough. A special case where the quantity given above is sufficiently small is when $||v||_\infty$ is sufficiently small. Below we state the regularity conditions for f that we will impose. The first assumption is compatible with the assumption that $v \in \mathcal{L}_\infty$ while the second, stronger, assumption is used when this assumption is not in place.

Assumption 3 *For each $C > 0$ there exists $L > 0$ such that for all $x, y, \alpha_1, \alpha_2, w, v$ satisfying*

$$\max\{|(x,\alpha_1)|_{\mathcal{A}}, |(y,\alpha_2)|_{\mathcal{A}}, |w|, |v|\} \le C \ , \qquad \alpha_1, \alpha_2 \in \Gamma$$

we have that:

$$|f(x,\alpha_1,v,w) - f(y,\alpha_2,0,w)| \le L\{|x-y| + |\alpha_1 - \alpha_2| + |v|\} \ . \tag{41}$$

■

Assumption 4 *For each $C > 0$ there exists $L > 0$ such that for all $x, y, \alpha_1, \alpha_2, w, v$ satisfying*

$$\max\{|(x,\alpha_1)|_{\mathcal{A}}, |(y,\alpha_2)|_{\mathcal{A}}, |w|\} \le C \ , \qquad \alpha_1, \alpha_2 \in \Gamma \ , \qquad v \in \mathbb{R}^p$$

we have that (41) holds. ■

Remark 5 *In the situation where the state x includes a component corresponding to time, that dependence can be made explicit in f and the regularity of f with respect to this variable can be relaxed to measurability, as was done in the results on averaging.* ■

In what follows, we let $\Delta > 0$ and $\Omega_v > 0$ be arbitrary but fixed and then, for each $\epsilon > 0$, we use $\mathcal{S}_1^{\Delta}(x_\circ, u_\circ, w, \epsilon)$ (respectively $\mathcal{S}_1^{(\Delta,\Omega_v)}(x_\circ, u_\circ, w, \epsilon)$) to denote all solutions of (39) resulting from all locally integrable functions $v(\cdot)$ that yield $u(t) \in \Gamma$ for all $t \ge 0$ and that satisfying

$$\frac{1}{\Delta}\int_t^{t+\Delta} |v(\tau)| d\tau \le \epsilon \qquad \forall t \in \mathbb{R}_{\ge 0} \tag{42}$$

(and satisfy $||v||_\infty \le \Omega_v$). The notation $x(\cdot)$ represents the x component, and $u(\cdot)$ the u component, of an element of either of these flows. We use $\mathcal{S}_2(x_\circ, u_\circ, w)$ to denote the solution of (40) starting from the initial condition $(x_\circ, u_\circ)$, under the action of the disturbance w and $y(\cdot)$ to represent the x component of the solution to (40). The following result was proved in [8]:

Theorem 5 *Let an arbitrary nonempty closed set $\mathcal{A} \subset \mathbb{R}^n \times \mathbb{R}^p$ be given and suppose that Assumption 4 holds (respectively, Assumption 3 holds). Let Δ (and Ω_v) be arbitrary strictly positive real number(s). Suppose that the following conditions hold:*

1. *The set $\mathcal{F}$ is equi-essentially bounded, i.e., each $w \in \mathcal{F}$ is locally essentially bounded and there exists $\Omega_w > 0$ such that for each $w \in \mathcal{F}$, $||w||_\infty \le \Omega_w$.*
2. *The system (40) is $(\mathbb{R}^n \times \Gamma, \mathcal{F}, \mathcal{A})$-forward complete;*

Then, for each triple (T, δ, r) of strictly positive real numbers there exists a pair (ϵ^*, μ) of strictly positive real numbers such that for each $\epsilon \in (0, \epsilon^*)$, each $(y_\circ, u_\circ) \in \mathbb{R}^n \times \Gamma$ with $|(y_\circ, u_\circ)|_{\mathcal{A}} \le r$, each $w \in \mathcal{F}$, and each $x_\circ \in \mathbb{R}^n$ such that $|x_\circ - y_\circ| \le \delta$, each element $(x(\cdot), u(\cdot))$ of $\mathcal{S}_1^{\Delta}(x_\circ, u_\circ, w, \epsilon)$ (respectively, each element of $\mathcal{S}_1^{(\Delta,\Omega_v)}(x_\circ, u_\circ, w, \epsilon)$) and the solution $(y(\cdot), u_\circ) = \mathcal{S}_2(y_\circ, u_\circ, w)$ satisfy:

$$\left| \begin{pmatrix} x(t) - y(t) \\ u(t) - u_\circ \end{pmatrix} \right| \le \delta \tag{43}$$

for all $t \in [0, T]$. ■

Identifying $\theta := \epsilon$ and $\Theta = \mathbb{R}_{>0}$, and relying on Remark 2, consequences of Theorem 5 and Corollaries 1 and 2 are:

Corollary 7 *Suppose that $\mathcal{A} \subset \mathbb{R}^n \times \mathbb{R}^p$ is an arbitrary nonempty closed set, Assumption 4 (resp., Assumption 3) holds and the system (40) is $(\mathbb{R}^n \times \Gamma, \beta, \gamma)$-iIiSS with respect to $\mathcal{A}$. Then the flow $\mathcal{S}_1^{\Delta}$ (resp., the flow $\mathcal{S}_1^{(\Delta,\Omega_v)}$) is semiglobally practically $(\mathbb{R}^n \times \Gamma, \beta, \gamma)$-iIiSS with respect to $\mathcal{A}$.* ■

Corollary 8 *Suppose that $\mathcal{A}, \mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^p$ are arbitrary nonempty closed sets, Assumption 4 (resp., Assumption 3) holds and the system (40) is $(\mathbb{R}^n \times \Gamma, \beta, \gamma)$-IOS with respect to $\mathcal{A}$ and $\mathcal{D}$ and $(\mathbb{R}^n \times \Gamma, \beta, \gamma)$-IOSS with respect to $\mathcal{A}$ and $\mathcal{D}$. Then the flow $\mathcal{S}_1^{\Delta}$ (resp., the flow $\mathcal{S}_1^{(\Delta,\Omega_v)}$) is semiglobally practically $(\mathbb{R}^n \times \Gamma, \beta, \gamma)$-ISS with respect to $\mathcal{A}$.* ■

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