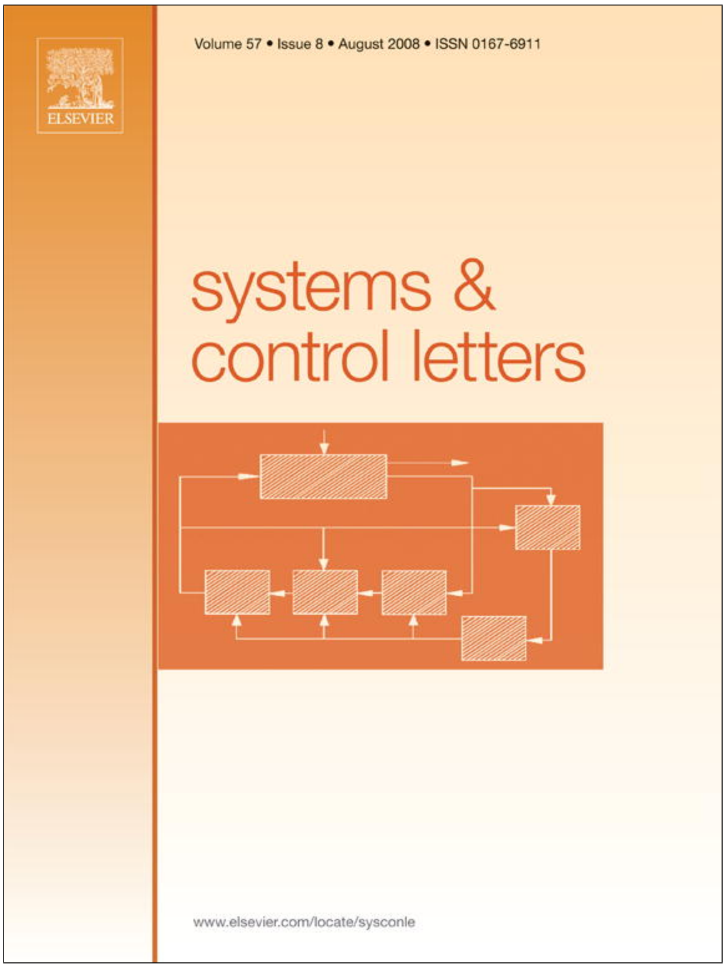
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Practical encoders for controlling nonlinear systems under communication constraints

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Abstract

We introduce a new class of dynamic encoders for continuous-time nonlinear control systems which update their parameters only at discrete times. We prove that the information reconstructed from the encoded feedback can be used to deliver a piece-wise constant control law which yields semi-global practical stability.

Keywords: Nonlinear control; Sampled-data systems; Observers; Encoders; Measurement noise

1. Introduction

Controlling (nonlinear) systems via encoded feedback is of paramount importance in distributed control systems. For systems of the form

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \tag{1}$$

it has been very recently illustrated in the literature ([24,15,6] — for other contributions on control via encoded or quantized feedback, the reader is referred to the papers e.g. [17,23,2,8,21,18,14,16,13,24,10,9,15]) how to design encoders to achieve (semi-) global asymptotic stabilization via encoded feedback by assuming standard stabilizability. The encoders are impulsive systems of the form

$$\begin{array}{ll} \dot{\xi}(t) = f(\xi(t), u(t)) & \xi(t) = \varphi(x(t^-), \xi(t^-), \ell(t^-)) \\ \dot{\ell}(t) = 0, \ t \neq kT & \ell(t) = \lambda \ell(t^-), \ t = kT \end{array} \tag{2}$$

where the symbol $r(\bar{t}^-)$ denotes the limit $\lim_{t\to\bar{t}^-} r(t)$, $(\xi, \ell) \in \mathbb{R}^n \times \mathbb{R}_+$, $0 < \lambda < 1$ is a design parameter, $T > 0$ is the sampling period, and φ is the nonlinear function defined as

$$\varphi(x, \xi, \ell) = \xi + 2\left(\left\lceil \frac{B}{2\ell}(x - \xi) + \frac{1}{2}\mathrm{sgn}(x - \xi) \right\rceil\right) \frac{\ell}{B}$$

with $B > 0$ the integer equal to the quantization levels available for each state component (see [24,15,6]), here taken odd for the sake of simplicity, and $\lceil\cdot\rceil$ the ceiling function defined in the last paragraph of this Section. By the definition of φ, it is not difficult to see that $|x-\varphi(x, \xi, \ell)| \leq \ell/B$ whenever $|x-\xi| \leq \ell$. Further, observe that, as long as $|x - \xi| \leq \ell$, each component of the function

$$\left\lceil \frac{B}{2\ell}(x - \xi) + \frac{1}{2}\mathrm{sgn}(x - \xi) \right\rceil,$$

takes values in the finite and discrete set $\{0, \pm 1, \ldots, \pm (B-1)/2\}$. As a consequence, this vector can be transmitted through the finite data-rate communication channel, and received at the other end of the (noiseless, delay free) channel by a device called decoder. This works synchronously with the encoder and is composed of exactly the same Eq. (2), with identical initial conditions. This implies [24,15,6] that the encoder state variables $\xi(\cdot)$ and $\ell(\cdot)$ are known to the decoder as well. Moreover, it can be proved that $\xi(\cdot)$ is actually an asymptotic estimate of the state $x(\cdot)$, and as such it can be employed to deliver the control action which stabilizes the system despite the channel.

The actual adoption of devices such as (2) in networked control systems very much depends on the possibility of easing the computational burden involved in the solution of (2). In this note, we aim to address such issues by proposing an

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encoder which does not require a continuous update of its state and which is able to reconstruct an asymptotically practically correct estimate of the state starting from encoded information. We also illustrate the possibility of using this estimate for the purpose of stabilizing the system. Other approaches are also possible [19]. We also mention dwell-time switching control laws to cope with the stabilization problem under limited data-rate constraints [10,9]. They represent a durable solution to the problem, due to the simplicity of their implementation. The approach discussed in this paper can also be viewed as a general framework in which many of the results available for quantized discrete-time or sampled continuous-time systems can be rephrased or interpreted, although this is not discussed here in detail. A preliminary version of the paper has appeared in [7].

In the next section, we consider an *approximate* discrete-time version of the system (1) and design a simplified version of the encoder (2). Then we prove that its estimate of the state $x(\cdot)$ is asymptotically practically correct at the sampling times. In Section 3, we extend this result to the case in which only partial-state measurements are available. In Section 4, under asymptotic controllability assumption, we study the evolution of system (1) in closed-loop with a piece-wise constant control law designed on the basis of the feedback generated by the encoders examined in Sections 2 and 3. Before ending this section we summarize in the following paragraph the notation in use throughout the paper.

Notation. $\mathbb{R}$ ($\mathbb{R}_+$) is the set of (nonnegative) real numbers. For $r > 0$, the ceiling function $\lceil r \rceil$ equals the smallest integer larger than r if $r > 0$, is equal to 0 if $r = 0$, and moreover $\lceil -r \rceil = -\lceil r \rceil$ if $r < 0$. If r is a vector, then $\lceil r \rceil$ is a vector whose ith entry is $\lceil r_i \rceil$. Similarly, $\mathrm{sgn}(r)$ represents the sign function, namely $\mathrm{sgn}(r) = +1$ if $r > 0$, $\mathrm{sgn}(r) = -1$ if $r < 0$, and $\mathrm{sgn}(r) = 0$ for $r = 0$. Again, if r is a vector, then $\mathrm{sgn}(r)$ is a vector whose component i coincides with $\mathrm{sgn}(r_i)$. In this paper, the infinity norm of a vector x, $\max_{1\le i\le n} |x_i|$, is simply indicated as $|x|$. The symbol $r(\bar{t}^-)$ denotes the limit $\lim_{t\to\bar{t}^-} r(t)$, while the symbol C^r_s, with $r > 0$ an integer and $s > 0$ a real number, denotes the cube in $\mathbb{R}^r$ centered around the origin and with edges of length $2s$. $\mathcal{V}(S)$ is the level set $\{x : V(x) \le S\}$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a class-$\mathcal{K}$ function if it is continuous, zero at zero, and strictly increasing. A class-$\mathcal{K}$ function which is also unbounded is a class-$\mathcal{K}_\infty$ function. The function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a class-$\mathcal{KL}$ function if, for each fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s)$ is a class-$\mathcal{K}$ function, and, for each fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is a continuous decreasing function which converges to zero as $s \to \infty$.

2. A consistent discretized encoder

Under piece-wise constant control laws, let us introduce as in [1] the exact discrete-time model of (1), that is

$$x((k+1)T) = f^e_T(x(kT), u(kT)) := x(kT) + \int_{kT}^{(k+1)T} f(x(s), u(kT))\mathrm{d}s. \quad (3)$$

This model is in general not available, and an approximate discrete-time model[1]

$$x((k+1)T) = f^a_T(x(kT), u(kT)), \quad (4)$$

must instead be taken into account. Following [20,1], we consider approximate models (4) which are consistent with the exact model (3):

Assumption 1. The model $f^a_T(x, u)$ is consistent with $f^e_T(x, u)$, that is for each compact set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, there exist a class-$\mathcal{K}$ function $\varrho(\cdot)$ and a constant $T_0 > 0$ such that for all $(x, u) \in \Omega$ and all $T \in (0, T_0]$,

$$|f^e_T(x, u) - f^a_T(x, u)| \le T\varrho(T).$$

Inspired by [24,15,3], we propose the following discrete-time implementation of the modified encoder (2):

$$\begin{aligned} \xi((k+1)T) &= f^a_T(\hat{\xi}(kT), u(kT)) \\ \ell((k+1)T) &= \lambda\ell(kT) + \eta, \end{aligned} \quad (5)$$

with $\hat{\xi}(kT) = \varphi(x(kT), \xi(kT), \ell(kT))$, and $0 < \lambda < 1$, $\eta > 0$ parameters to design. Under the hypothesis that the process initial condition satisfies $x(0) \in C^n_X$, the encoder initialization is chosen as

$$\xi(0) = 0, \qquad \ell(0) = X.$$

Assume the following [3]:

Assumption 2. For any pair of constants $X > 0$ and $U > 0$, there exists a number $Y > 0$ such that, if $x(0) \in C^n_X$ and $u(kT) \in C^m_U$ for each $k \in \mathbb{Z}_+$, then $x(kT) \in C^n_Y$ for each $k \in \mathbb{Z}_+$, where $x(kT)$ is the solution of (1) at time kT.

The result below shows a clear relation between the degree of accuracy achievable by the "asymptotic estimate" $\xi(\cdot)$ of $x(\cdot)$, the parameter B and the sampling period T. It relies on the concept of consistency [20,1] recalled above and employs arguments inspired by those in [3], proof of Proposition 1.

Proposition 1. *Let Assumptions 1 and 2 hold. Then for any $X > 0$ and for any $U > 0$ there exists $T_0 > 0$ with the property that, for all $T \in (0, T_0]$, $x(0) \in C^n_X$ implies*

$$|x(kT) - \xi(kT)| \le \lambda^k X + \frac{1}{1-\lambda}\eta$$

provided that $\eta \ge T\varrho(T)$, $\lambda \triangleq F/B < 1$, and $B > F + 1$, with $F > 0$ the Lipschitz constant for which

$$|f^a_T(x, u) - f^a_T(\hat{x}, u)| \le F|x - \hat{x}|$$

for all (x, u), $(\hat{x}, u) \in \mathcal{C}^n_{X+Y+\eta} \times \mathcal{C}^U_m$.

Proof. In Assumption 1, let Ω be $\mathcal{C}^Y_n \times \mathcal{C}^U_m$ and fix $\varrho(\cdot)$ and T_0 accordingly. Let the quantity T_0 of the statement be the same as the latter one. Fix $T \in (0, T_0]$. Note that $x(0) \in C^n_X$, $\xi(0) = 0$ and $\ell(0) = X$ imply $|\hat{\xi}(0)| \le X$ and $|x(0) - \hat{\xi}(0)| \le X/B$. Assume that, for some $k \in \mathbb{Z}_+$, $|x(jT) - \xi(jT)| \le \ell(jT)$

[1] For the sake of conciseness, we do not consider in this note the presence of the "modelling parameter" δ [1], but the conclusions we draw hold analogously for the case in which δ is present.

and $|x(jT) - \hat{\xi}(jT)| \le \ell(jT)/B$ for each $j = 0, 1, \ldots, k$. In particular,

$$|x(kT) - \xi(kT)| \le \ell(kT) \tag{6}$$

and

$$|x(kT) - \hat{\xi}(kT)| \le \ell(kT)/B. \tag{7}$$

The evolution of $\ell(\cdot)$ as given by the second equation in (5) is described by the relation

$$\begin{aligned}\ell(kT) &= \lambda^k \ell(0) + \sum_{j=0}^{k-1} \lambda^{k-1-j}\eta \\ &= \lambda^k X + \frac{1-\lambda^k}{1-\lambda}\eta.\end{aligned}$$

Hence,

$$\frac{\ell(kT)}{B} \le \lambda^k \frac{X}{B} + \frac{1}{B-F}\eta.$$

As $|x(kT)| \le Y$ for all $k \in \mathbb{Z}_+$, relation (7) guarantees that $|\hat{\xi}(kT)| \le \ell(kT)/B + Y \le X + Y + \eta$. Consider now the following chain of relations:

$$\begin{aligned}&|x((k+1)T) - \xi((k+1)T)| \\ &\quad = |f_T^e(x(kT), u(kT)) - f_T^a(\hat{\xi}(kT), u(kT))| \\ &\quad = |f_T^e(x(kT), u(kT)) - f_T^a(x(kT), u(kT)) \\ &\qquad + f_T^a(x(kT), u(kT)) - f_T^a(\hat{\xi}(kT), u(kT))| \\ &\quad \le T\varrho(T) + F|x(kT) - \hat{\xi}(kT)| \le T\varrho(T) + F\ell(kT)/B \\ &\quad \le \lambda\ell(kT) + \eta = \ell((k+1)T).\end{aligned}$$

Hence $|x((k+1)T) - x((k+1)T)| \le \ell((k+1)T)/B$. By induction we conclude that both (6) and (7) hold for each $k \in \mathbb{Z}_+$. Bearing in mind that $\lambda = F/B < 1$, the thesis descends straightforwardly. ■

Notice that the encoder (5), endowed with the output function

$$\hat{\xi}(kT) = \varphi(x(kT), \xi(kT), \ell(kT))$$

can be viewed as an asymptotically practically observer of (1):

Corollary 1. *Let Assumptions 1 and 2 hold. Then there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that for any $X > \chi > 0$ and for any $U > 0$, there exist $T_0 > 0$ and a $B_0 > 0$ with the property that, for all $T \in (0, T_0]$, for all $B \ge B_0$, for all $x(0) \in C_X^n$, we have*

$$|x(kT) - \hat{\xi}(kT)| \le \beta(X, kT) + \chi \quad k \in \mathbb{Z}_+.$$

Remark. Analogous results as the one above can be given for the other encoders to be presented below. These results are however omitted for the sake of conciseness.

Remark. It is worth stressing that $T_0 > 0$ and $B_0 > 0$ are not independent. Consider, for instance, the case when the approximate model is given by the Euler approximation. Then, it is immediate to see that $F = 1 + KT$, with K a suitable constant. As $B > F + 1$, B_0 cannot be smaller than $2 + KT_0$.

Proof. In Assumption 1, let Ω be $\mathcal{C}_n^Y \times \mathcal{C}_m^U$ and fix $\varrho(\cdot)$ and T_0 accordingly. Let η, F and λ be defined as in Proposition 1. Set $B_0 = \max\{\eta/\chi, 1\} + F$. From the proof of Proposition 1, we recall that

$$|x(kT) - \hat{\xi}(kT)| \le \lambda^k \frac{X}{B} + \frac{1}{B-F}\eta,$$

and the thesis then trivially follows. ■

The term η in (5) must satisfy $\eta \ge T\varrho(T)$ (cf. Proposition 1), where in turn the term $T\varrho(T)$ measures how close the approximate model is to the actual one (cf. Assumption 1). As such, in some cases it may be difficult to find an appropriate value for the parameter. For those cases, it may be preferable to turn to a different kind of encoder presented below. This can be done under the following assumption [20]:

Assumption 3. For each compact set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, there exist $K > 0$ and $T_0 > 0$ such that, for all (x, u) and (z, u) in Ω, and all $T \in (0, T_0]$,

$$|f_T(x, u) - f_T(z, u)| \le (1 + KT)|x - z|,$$

where f_T is either f_T^a or f_T^e.

We recall now the following result (cf. Lemma 2, Lemma 3 and Remark 2 in [20]):

Lemma 1. *Let Assumptions 1–3 hold. Then, for any $X > 0$, $U > 0$, $L > 0$ and $\eta > 0$, there exists $T_0 > 0$ such that $T \in (0, T_0]$ and $x(0) = x^a(0)$ imply*

$$|x(kT) - x^a(kT)| \le \eta, \quad \forall k : kT \in [0, L],$$

where $x^a(kT)$ is obtained from the recursive equation $x^a((k+1)T) = f_T^a(x^a(kT), u(kT))$.

The proposed modified encoder is as follows:

$$\begin{aligned}x^a((k+1)T) &= f_T^a(x^a(kT), u(kT)) \\ \xi((k+1)T) &= f_T^a(\hat{\xi}(kT), u(kT)) \\ \ell((k+1)T) &= \lambda\ell(kT),\end{aligned} \tag{8}$$

which incorporates the equations of the approximate model. In the present case, $\hat{\xi} = \varphi(x^a, \xi, \ell)$, and therefore $\hat{\xi}(kT)$ is a vector for which $|x^a(kT) - \hat{\xi}(kT)| \le \ell(kT)/B$, provided that $|x^a(kT) - \xi(kT)| \le \ell(kT)$. Moreover, $x^a(0) = x(0)$, $\xi(0) = 0$, $\ell(0) = X$. We stress that the decoder equations coincide with the last two of the encoder, namely those in ξ and ℓ. Then the following result holds:

Proposition 2. *Let Assumptions 1–3 hold. Then for any $X > 0$, $U > 0$, $L > 0$ and $\eta > 0$ there exists $T_0 > 0$ with the property that, for all $T \in (0, T_0]$, $x(0) \in C_X^n$ implies*

$$|x(kT) - \xi(kT)| \le \lambda^k X + \eta \quad \forall k : kT \in [0, L],$$

provided that $\lambda \triangleq F/B$, $B > F + 1$, with $F > 0$ the Lipschitz constant for which

$$|f_T^a(x, u) - f_T^a(\hat{x}, u)| \le F|x - \hat{x}|$$

for all (x, u), $(\hat{x}, u) \in \mathcal{C}_{X+Y+\eta}^n \times \mathcal{C}_u^n$.

Proof. Take T_0 as in Lemma 1. As before, one first shows that

$$|x^a(0) - \xi(0)| \le \ell(0), \qquad |x^a(0) - \hat{\xi}(0)| \le \frac{\ell(0)}{B}.$$

Then, assume that for some $k \in \mathbb{Z}_+$ for which $kT \le L$,

$$|x^a(jT) - \xi(jT)| \le \ell(jT),$$
$$|x^a(jT) - \hat{\xi}(jT)| \le \frac{\ell(jT)}{B} \qquad \forall j = 0, 1, \ldots, k-1.$$

By Assumption 2 and Lemma 1, the following chain of implications is shown promptly:

$$|x((k-1)T)| \le Y \Rightarrow |x^a((k-1)T)| \le Y + \eta$$
$$\Rightarrow |\hat{\xi}((k-1)T)| \le X + Y + \eta.$$

Hence,

$$|x^a(kT) - \xi(kT)| = |f_T^a(x^a((k-1)T), u((k-1)T))$$
$$- f_T^a(\hat{\xi}((k-1)T), u((k-1)T))|$$
$$\le \lambda\ell((k-1)T) = \ell(kT).$$

Then, for each k for which $kT \in [0, L]$,

$$|x(kT) - \xi(kT)| \le |x(kT) - x^a(kT)| + |x^a(kT) - \xi(kT)|$$
$$\le \lambda^k X + \eta. \quad \blacksquare$$

Remark. A modification of the structure of the practical encoders described in this section may lead to encoders which employ lower data rates. See [24,16,4] for details.

Remark. In Section 4, we shall comment on how the estimate generated by the encoder (8) is used to control the process. It will be shown that, thanks to the action of the controller, the result of Proposition 2 can be propagated for all the times, provided that, at times $kT = L$ and its multiples, the encoder is re-initialized with the actual value of the state of the process at that time. In this regard, the encoder (and therefore the decoder) is genuinely operating in *closed-loop*.

3. Observer-based practical encoders

The previous section has focused on the case in which full state was available for measurements. Here we consider the case in which the system (1) is endowed with a readout map which is different from the identity, namely

$$y = h(x) \in \mathbb{R}^p. \tag{9}$$

In this scenario, the design of the encoders is based on observers [1]. A common approach to the design of sampled-data observer lies on a suitable discretization of a continuous-time observer. This is examined in the next subsection. Another approach, which typically exhibits a better performance in simulations, consists of designing the discrete-time observer directly. This approach is studied in Section 3.2. Observe that in the emulation-based design, the observer (and hence the encoder) is designed without relying on any consistency hypothesis (in fact, a continuous-time observer is supposed to be known, and the sampled-data observer is obtained by discretization). On the other hand, relying on a discrete-time observer, we assume that a consistent approximate model of the system is available. This difference reflects on the design of the encoders and on the required data rates.

3.1. Encoder design by emulation

In this subsection, we assume that a continuous-time observer

$$\dot{\sigma}(t) = g(\sigma(t), y(t), u(t)) \tag{10}$$

is actually available, and consider its zero order hold equivalent [12]:

$$\sigma((k+1)T) = g_T^a(\sigma(kT), y(kT), u(kT)). \tag{11}$$

Namely, we assume the following [1]:

Assumption 4. System (11) is a semi-global practical observer. i.e. there exists a class-$\mathcal{KL}$ function $\omega(\cdot,\cdot)$ such that, for any $X > \chi > 0$ and any $Y > 0$ and $U > 0$, we can find T_0 such that, for all $T \in (0, T_0]$,

$$|x(0)| \le X, \qquad |\sigma(0) - x(0)| \le 2X,$$

and

$$|x(kT)| \le Y, \qquad |u(kT)| \le U,$$

for each $k \in \mathbb{Z}_+$, imply

$$|\sigma(kT) - x(kT)| \le \omega(|\sigma(0) - x(0)|, kT) + \chi, \tag{12}$$

for each $k \in \mathbb{Z}_+$.

Remark. There are precise conditions under which the assumption above is fulfilled, and these are investigated in [1]. Suppose the following hypotheses hold true:

(i) The model $g_T^a(\cdot,\cdot,\cdot)$ is consistent with the model $g(\cdot,\cdot,\cdot)$.
(ii) There exist a continuously differentiable function $V(x,\sigma)$ and class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)$ such that

$$\alpha_1(|x-\sigma|) \le V(x,\sigma) \le \alpha_2(|x-\sigma|)$$
$$\frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial \sigma} g(\sigma, y, u) \le -\alpha_3(|x-\sigma|).$$

Then, by [1], Theorem 3, Assumption 2 implies Assumption 4.

In the following, it will be useful to single out a part of the observer (11) not affected by the output:

Assumption 5. Map $g_T^a(\sigma(kT), y(kT), u(kT))$ can be decomposed as

$$g_T^a(\sigma(kT), y(kT), u(kT)) = g_{T1}^a(\sigma(kT), u(kT))$$
$$+ g_{T2}^a(\sigma(kT), y(kT), u(kT)).$$

Furthermore, there exist class-$\mathcal{KL}$ function $\omega(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ and positive constants χ, η such that, $\omega(r,s) \le \beta(r,s)$ for all $r, s \in \mathbb{R}_+$, $\chi \le \eta$, and $|\sigma(kT) - x(kT)| \le \omega(|\sigma(0) - x(0)|, kT) + \chi$ implies

$$|g_{T2}^a(\sigma(kT), y(kT), u(kT))| \le \beta(|\sigma(0) - x(0)|, kT) + \eta.$$

Remark. There is a loss of generality associated with the Assumption. Nevertheless a meaningful class of observers which satisfy it, and which include e.g. those in [22] and references therein, are discussed below. If $g(\cdot,\cdot,\cdot)$ is a continuously differentiable function, and the Lyapunov function in the Remark following Assumption 4 holds with $V(x,\sigma) = V(x-\sigma) =: V(e)$, then it implies that $x = \sigma$ must be an invariant manifold for systems (1), (9) and (10) and therefore $g(x, h(x), u) = f(x, u)$ which in turn implies [22]

$$\dot\sigma(t) = f(\sigma(t), u(t)) + \tilde g(\sigma(t), y(t), u(t))(y(t) - h(\sigma(t))),$$

with $\tilde g(\cdot,\cdot,\cdot)$ a suitable continuous function. Using e.g. Euler discretization for the latter system and letting $h(\cdot)$ be Lipschitz continuous, then Assumption 4 implies:

$$\begin{aligned}&|g^a_{T2}(\sigma(kT), y(kT), u(kT))|\\ &\quad = |\tilde g(\sigma(kT), y(kT), u(kT))(y(kT) - h(\sigma(kT)))|\\ &\quad \le \tilde G(\omega(|\sigma(0) - x(0)|, kT) + \chi),\end{aligned}$$

for some constant $\tilde G > 0$, which can be taken not smaller than 1 without loss of generality.

Following [24] and [5], we propose the following observer-based encoder:

$$\begin{aligned}\sigma((k+1)T) &= g^a_T(\sigma(kT), y(kT), u(kT))\\ \varsigma((k+1)T) &= g^a_{T1}(\hat\varsigma(kT), u(kT))\\ \ell((k+1)T) &= \lambda\ell(kT) + \beta(2X, kT) + \eta\end{aligned} \tag{13}$$

with $\hat\varsigma(kT) = g(\sigma(kT), \varsigma(kT), \ell(kT))$, $0 < \lambda < 1$ to design and $\beta(\cdot,\cdot)$ and $\eta > 0$ as in Assumption 5. Notice that, differently from the state feedback case, here the signal $\hat\varsigma(\cdot)$ represents not the encoded state of the process $x(\cdot)$ but the encoded state of the observer $\sigma(\cdot)$. Furthermore $\varsigma(\cdot)$ represents the center of the quantization region. Hence, analogously to the state feedback case, $|\sigma(kT) - \varsigma(kT)| \le \ell(kT)$ implies $|\sigma(kT) - \hat\varsigma(kT)| \le \ell(kT)/B$. It is worth stressing that the decoder at the other end of the channel will implement only the last two equations in (13) ($y(\cdot)$ is not available to the decoder). Finally the encoder initial conditions are chosen as $\varsigma(0) = 0$, $\sigma(0) \in C^n_X$, $\ell(0) = X$.

Now, we introduce the constant:

$$Z \triangleq X + Y + 2\beta(2X, 0) + 2\eta.$$

The main result of this subsection is as follows:

Proposition 3. *Let Assumptions 2, 4 and 5 hold. Then for any $X > 0$ and for any $U > 0$, here exists $T_0 > 0$ with the property that, for all $T \in (0, T_0]$, $x(0) \in C^n_X$ implies*

$$|x(kT) - \varsigma(kT)| \le \beta(|\sigma(0) - x(0)|, kT) + \eta + \ell(kT) \quad \forall k \in \mathbb{Z}_+$$

provided that $\lambda \triangleq G/B$, and $B > G + 1$, with $G > 0$ the Lipschitz constant for which

$$|g^a_{T1}(\sigma, u) - g^a_{T1}(\hat\sigma, u)| \le G|\sigma - \hat\sigma|$$

for all (σ, u), $(\hat\sigma, u) \in \mathcal{C}^Z_n \times \mathcal{C}^U_m$.

Proof. Take T_0 as in Assumption 4. A consequence of Assumptions 2 and 4 is that

$$\begin{aligned}|\sigma(kT)| &\le Y + \omega(2X, 0) + \chi \le Y\\ &\quad + \beta(2X, 0) + \eta \le Z, \quad k \in \mathbb{Z}_+.\end{aligned}$$

Note that $\varsigma(0) = 0$ and $\sigma(0) \in C^n_X$ imply $|\sigma(0) - \varsigma(0)| \le X = \ell(0)$. Assume now that $|\sigma(kT) - \varsigma(kT)| \le \ell(kT)$ for some $k \in \mathbb{Z}_+$. We prove that the latter inequality holds true at time $(k+1)T$. Note first that

$$\frac{\ell(kT)}{B} \le X + \beta(2X, 0) + \eta.$$

Hence

$$\begin{aligned}|\hat\varsigma(kT)| &\le |\sigma(kT)| + \frac{\ell(kT)}{B}\\ &\le X + Y + 2\beta(2X, 0) + 2\eta = Z.\end{aligned}$$

We have

$$\begin{aligned}&|\sigma((k+1)T) - \varsigma((k+1)T)|\\ &\quad \le |g^a_{T1}(\sigma(kT), u(kT)) - g^a_{T1}(\hat\varsigma(kT), u(kT))|\\ &\qquad + |g^a_{T2}(\sigma(kT), y(kT), u(kT))|\\ &\quad \le G|\sigma(kT) - \hat\varsigma(kT)| + \beta(2X, kT) + \eta\\ &\quad \le \lambda\ell(kT) + \beta(2X, kT) + \eta = \ell((k+1)T).\end{aligned}$$

Hence, $|\sigma(kT) - \varsigma(kT)| \le \ell(kT)$ for all $k \in \mathbb{Z}_+$. We conclude

$$\begin{aligned}|x(kT) - \varsigma(kT)| &\le |x(kT) - \sigma(kT)| + |\sigma(kT) - \hat\varsigma(kT)|\\ &\le \omega(|\sigma(0) - x(0)|, kT) + \chi + \ell(kT)\\ &\le \beta(|\sigma(0) - x(0)|, kT) + \eta + \ell(kT),\end{aligned}$$

which proves the thesis at time $(k+1)T$. By induction, we conclude the thesis for each $k \in \mathbb{Z}_+$. ■

3.2. *Encoder design by approximate discrete-time models*

In order to design practical encoders for system (1) with output map (9), in this subsection we pursue another approach, namely we assume the existence of a discrete-time observer for the approximate model (4), where $f^a_T(x, u)$ is required to be consistent with the exact model, as specified in Assumption 1. We have [1]:

Assumption 6. System

$$\begin{aligned}\sigma((k+1)T) &= f^a_T(\sigma(kT), u(kT))\\ &\quad + g_T(\sigma(kT), y(kT), u(kT))\end{aligned}$$

is a practical observer for systems (1) and (9), i.e. there exists a class-$\mathcal{KL}$ function $\omega(\cdot,\cdot)$ such that, for any $0 < \chi < X$ and any $Y > 0$, $U > 0$, there exists $T_0 > 0$ such that, for all $T \in (0, T_0]$, $x(0) \in C^n_X$, $x(0) - \sigma(0) \in C^n_{2X}$, and $x(kT) \in C^n_Y$, $u(kT) \in C^m_U$ for each $k \in \mathbb{Z}_+$, imply

$$|x(kT) - \sigma(kT)| \le \omega(|x(0) - \sigma(0)|, kT) + \chi, \quad \forall k \in \mathbb{Z}_+.$$

The encoder is formally the same as in the previous subsection:

$$\sigma((k+1)T) = f_T^a(\sigma(kT), u(kT)) + g_T(\sigma(kT), y(kT), u(kT))$$
$$\varsigma((k+1)T) = f_T^a(\hat{\varsigma}(kT), u(kT))$$
$$\ell((k+1)T) = \lambda\ell(kT) + \beta(2X, kT) + \eta,$$

with $\hat{\varsigma}(kT) = g(\sigma(kT), \varsigma(kT), \ell(kT))$ where, besides λ, also the class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and the positive constant η are to be chosen.

As in the previous subsection, the decoder will implement only the last two equations above. To correctly encode the observer state σ, an estimate of the term $g_T(\sigma(kT), y(kT), u(kT))$ is needed. This will be provided in the result below.

Proposition 4. *Let Assumptions 1, 2 and 6 hold. Then for any $X > 0$ and for any $U > 0$, there exists $T_0 > 0$ with the property that, for all $T \in (0, T_0]$, $x(0) \in C_X^n$ implies the existence of $Z > 0$, a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$, and $\eta > 0$, such that*

$$|x(kT) - \varsigma(kT)| \le \beta(|x(0) - \varsigma(0)|, kT) + \eta + \ell(kT) \quad \forall k \in \mathbb{Z}_+$$

provided that $\varsigma(0) = 0$, $\sigma(0) \in C_X^n$, $\ell(0) = X$ and $\lambda \triangleq F/B$, with $B > 2F + 1$ and F the Lipschitz constant for which

$$|f_T^a(x,u) - f_T^a(\hat{x},u)| \le F|x - \hat{x}|$$

for all (x,u), $(\hat{x},u) \in \mathcal{C}_n^Z \times \mathcal{C}_m^U$.

Proof. Take Ω as the compact set which includes $C_Y^n \times C_U^m$, and fix T_0 as in Assumption 1. Take further $\varpi(r,s) \ge \omega(r,s)$ for all positive pairs (r,s) and $\theta \ge \chi$. Then set

$$Z \triangleq X + Y + 2\varpi(2X,0) + 2\theta + T\varrho(T).$$

Let F be defined as in the statement, and $\beta(r,s) \triangleq (F+1)\varpi(r,s)$ and $\eta \triangleq (F+1)\theta + T\varrho(T)$. By Assumption 6, and bearing in mind that $x(kT) \in C_Y^n$, we have

$$|\sigma(kT)| \le Y + \omega(2X,0) + \chi \le Y + \varpi(2X,0) + \theta \le Z.$$

Proceeding by induction, assume that $|\sigma(kT) - \varsigma(kT)| \le \ell(kT)$ (this is true for $k = 0$), which implies

$$\begin{aligned}|\sigma(kT) - \hat{\varsigma}(kT)| &\le \ell(kT)/B \\ &\le X + (B-F)^{-1}((F+1)(\varpi(2X,0)+\theta) + T\varrho(T)) \\ &\le X + \varpi(2X,0) + \theta + T\varrho(T).\end{aligned}$$

Hence,

$$|\hat{\varsigma}(kT)| \le X + \varpi(2X,0) + \theta + T\varrho(T) + Y + \varpi(2X,0) + \theta = Z.$$

We provide now an estimate of $g_T(\sigma(kT), y(kT), u(kT))$. First we have:

$$\begin{aligned}&g_T(\sigma(kT), y(kT), u(kT)) \\ &= \sigma((k+1)T) - f_T^a(\sigma(kT), u(kT)) \\ &= \sigma((k+1)T) - x((k+1)T) + f_T^e(x(kT), u(kT)) - f_T^a(\sigma(kT), u(kT)) \\ &= \sigma((k+1)T) - x((k+1)T) + f_T^e(x(kT), u(kT)) \\ &\quad - f_T^a(x(kT), u(kT)) + f_T^a(x(kT), u(kT)) \\ &\quad - f_T^a(\sigma(kT), u(kT)).\end{aligned}$$

By Assumption 6, and the choice of Ω and F above, we have

$$|g_T(\sigma(kT), y(kT), u(kT))| \le \omega(2X, (k+1)T) + \chi + T\varrho(T) + F(\omega(2X,kT) + \chi).$$

Hence,

$$\begin{aligned}&|\sigma((k+1)T) - \varsigma((k+1)T)| \\ &\le |f_T^a(\sigma(kT), u(kT)) - f_T^a(\hat{\varsigma}(kT), u(kT))| \\ &\quad + |g_T(\sigma(kT), y(kT), u(kT))| \\ &\le \lambda\ell(kT) + \omega(2X,(k+1)T) + \chi + T\varrho(T) \\ &\quad + F(\omega(2X,(k+1)T) + \chi) \\ &\le \lambda\ell(kT) + \beta(2X,kT) + \eta = \ell((k+1)T).\end{aligned}$$

Having proved that $|\sigma(kT) - \varsigma(kT)| \le \ell(kT)$ for all k, we easily infer the thesis. ■

4. Practical stabilization

The attention is now turned to the design of the controller, for which we follow very closely [11]. We shall refer to Proposition 1 for the state feedback case, and to Proposition 3, for the output-feedback case. Analogous results can be given using Proposition 2 and, respectively, Proposition 4. A number of notions from that paper are now introduced. The positive numbers $r < R$ and $r_m < R_m$ are given.

Assumption 7. There exists a continuous Lyapunov function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$ for which:

- There exist class-$\mathcal{K}_\infty$ functions $\nu(\cdot)$, $\rho(\cdot)$ such that $|V(x_1) - V(x_2)| \le \rho(|x_1 - x_2|)$. Moreover, $V(x) \ge \nu(|x|)$.
- There exist a feedback function $\kappa(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ and constants $T > 0$, $c > 0$ such that the solution of $\dot{x}(t) = f(x(t), \kappa(x(0)))$ with $x(0) \in \mathcal{V}(R)$ satisfies
  (S1) $V(x(T)) \le \max\{V(x(0)) - c, r\}$
  (S2) $V(x(t)) \le \max\{V(x(0)), r\} + r_m$, $\forall t \in [0,T]$.

We also introduce the following:

Assumption 8. Set $\mathcal{U} := \{u \in \mathbb{R}^m : u = \kappa(x), x \in \mathcal{V}(R + R_m)\}$.

(1) There exists $M \ge 0$ such that $|f(x,u)| \le M$ for all $x \in \mathcal{V}(R + R_m)$, for all $u \in \mathcal{U}$.
(2) There exists $L_{fx}, L_{fu} > 0$ such that $|f(x_1,u_1) - f(x_2,u_2)| \le L_{fx}|x_1 - x_2| + L_{fu}|u_1 - u_2|$, for all $x_1, x_2 \in \mathcal{V}(R + R_m)$, for all $u_1, u_2 \in \mathcal{U}$.

We recall the following statement from [11]:

Theorem 1. *Under Assumptions 7 and 8, consider*

$$\dot{x}(t) = f(x(t), \kappa(x(0))) + d(t)$$

where $x(0) \in \mathcal{V}(R)$. Let $\sigma \in [0, R_m - r_m)$. If the disturbance $d(\cdot)$ satisfies

$$\max_{t\in[0,T]} \left| \int_0^t d(s)\mathrm{d}s \right| \le \rho^{-1}(\sigma)\mathrm{e}^{-L_{fx}T}$$

then for all $t \in [0, T]$, *the solution* $x(\cdot)$ *exists and satisfies*

$$V(x(T)) \leq \max\{V(x(0)) - c, r\} + \sigma$$
$$V(x(t)) \leq \max\{V(x(0)), r\} + (r_m + \sigma).$$

We now apply this theorem and the results in the previous section to show practical stabilization when using the control law

$$u(t) = \kappa(\hat{\xi}(kT)), \quad t \in [kT, (k+1)T),$$

where the samples $\hat{\xi}(\cdot)$ are generated by the decoder (5). To proceed, fix any $R > 0$, set $R + R_m \triangleq \rho \circ \nu^{-1}(R) \geq R$, and choose $r = r_m < R, R_m$. Set

$$\nu(X) = R, \qquad \nu(Y) = R + R_m, \qquad U = \max_{x \in C_Y^n} |\kappa(x)|,$$

so that $\mathcal{V}(R) \subseteq C_X^n$ and $\mathcal{V}(R + R_m) \subseteq C_Y^n$. The result below is concerned with proving that under the assumptions just stated, and despite the quantization error, a control law exists which keeps the state confined in $\mathcal{V}(R + R_m)$, and hence in C_Y^n, where, applying the results established in the previous sections, an increasingly improved estimation of the state is possible. Practical stability of the resulting closed-loop system is then concluded.

We specify some quantities needed in the statement below. First, one can choose $T \in (0, T_0]$, with T_0 and T as in Assumptions 1 and 7, respectively. Choose the constants η and F in Proposition 1 accordingly. Then define the quantities

$$E_k \triangleq \lambda^k \frac{X}{B} + \frac{1}{B - F}\eta, \quad k \in \mathbb{Z}_+,$$

with B to determine, and where in particular

$$E_0 \triangleq \frac{X}{B} + \frac{1}{B - F}\eta.$$

We have:

Proposition 5. *Let Assumptions* 1, 7 *and* 8 *hold*, $T \in (0, T_0]$, *with* T_0 *and* T *as in Assumptions* 1 *and* 7, *respectively. Let* $\sigma \in [0, \min\{R_m - r_m, R - r, c/4\})$ *and choose* $B > F + 1$ *so that:*

$$E_0 \in \left[0, \min\left\{\frac{\rho^{-1}(\sigma)}{2 + L_{fx}T} e^{-L_{fx}T}, \rho^{-1}(R - r - \sigma)\right\}\right). \tag{14}$$

Then, the solution of the closed-loop system

$$\dot{x}(t) = f(x(t), \kappa(\hat{\xi}(kT))), \quad t \in [kT, (k+1)T), \ k \in \mathbb{Z}_+$$

from the initial condition $x(0) \in \mathcal{V}(R - 2\rho(E_0))$ *exists for all* $t \geq 0$ *and satisfies*

$$\begin{aligned} V(x(kT)) \ &\leq \ \max\left\{V(x(0)) - \frac{(3k-1)c}{4}, r\right\} \\ &\quad + \rho(E_k) + \sigma, \ \forall k \in \mathbb{Z}_+ \\ V(x(t)) \ &\leq \ \max\{V(x(kT)) + \rho(E_k), r\} + (r + \sigma + \rho(E_k)), \\ &\quad \forall t \in [kT, (k+1)T), \forall k \in \mathbb{Z}_+. \end{aligned}$$

Remark. From the first one of the inequalities, we see that the state of the closed-loop system at the sampling times asymptotically converges to the level set

$$\left\{x \in \mathbb{R}^n : V(x) \leq r + \sigma + \rho\left(\eta/(B - F)\right)\right\}.$$

Proof. The result is an application of [11], Proposition 1, and its proof is basically the same. The differences are as follows: We explicitly take into account the fact that the "measurement noise" (i.e. the quantization error) is vanishing and we take care of the fact that the measurement noise itself is not known a priori unless we guarantee that the state is confined within a region of interest. To make the paper as self-contained as possible, the proof is reported in the Appendix. ■

Remark. Under the addition of Assumption 3 in Proposition 5, one can infer similar results employing the encoder (8) in place of (5) to generate $\hat{\xi}(\cdot)$. Then, in view of Lemma 1, the same conclusions of the proposition can be drawn over a finite horizon of time, i.e. for each k such that $kT \in [0, L]$. Nevertheless, the result can be propagated over an infinite time horizon. As a matter of fact, from the arguments in [20], it is easily seen that in Lemma 1, as $L \to +\infty$, $T_0 \to 0$ and therefore, denoted by k_L the largest integer such that $k_L T \leq L$, $k_L \to +\infty$ as well. As a consequence, given any arbitrarily small v greater than $r + \sigma$, there always exists a choice of L and η in Proposition 2 so that

$$V(x(k_L T)) \leq r + \sigma + \rho(E_{K_L}) \leq v,$$

where

$$E_{k_L} \triangleq \lambda^{k_L} \frac{X}{B} + \frac{\eta}{B}.$$

At time $k_L T$ one can always re-initialize the encoder and the decoder and set

$$x^a(k_L T) = x(k_L T), \qquad \xi(k_L T) = 0, \qquad \ell(k_L T) = \hat{X},$$

where $\hat{X} \triangleq \nu^{-1}(v)$ (of course the decoder re-initializes the state variables ξ and ℓ only). Mimicking the previous arguments, set

$$\hat{E}_0 \triangleq \frac{\hat{X}}{B} + \frac{\eta}{B},$$

and

$$\begin{aligned} &\hat{R} \triangleq v + 2\rho(\hat{E}_0), \qquad &&\hat{R} + \hat{R}_m \triangleq \rho \circ \nu^{-1}(\hat{R}), \\ &\nu(\hat{Y}) = \hat{R} + \hat{R}_m, \qquad &&\hat{U} = \max_{x \in C_{\hat{Y}}^n} |\kappa(x)|. \end{aligned}$$

As a consequence of this choice, $x(k_L T) \in \mathcal{V}(\hat{R} - 2\rho(\hat{E}_0))$ and a Lipschitz constant $\hat{F}$ for $f(x, u)$ remains defined over the set $C_{X+Y+\eta}^n \times C_U^n$. Then, an appropriate choice of $\hat{B} > B$ and $\hat{\lambda} \triangleq \hat{F}/\hat{B}$ shows that

$$V(x((k_L + 1)T)) \leq r + \sigma + \rho\left(\hat{\lambda}\frac{\hat{X}}{B} + \frac{\eta}{B}\right) \leq V(x(k_L T)),$$

and the same is true replacing $k_L + 1$ with any $k > k_L + 1$.

Mutatis mutandi, an output-feedback version of the previous result can also be stated. As observer-based encoder, we can choose either the one designed by emulation or by approximate discrete-time models. In the following, we proceed with the former, but analogous results can be given using the latter.

In the following proposition, the solution $x(\cdot)$ we refer to is the solution of the process (1) in closed-loop with the "output-feedback" control law:

$$u(t) = \kappa(\hat{\varsigma}(kT)), \quad t \in [kT, (k+1)T),$$

where the samples $\hat{\varsigma}(\cdot)$ are generated by the encoder (13). Fix the constants r, R, r_m, R_m and X, Y as before, Z as in Section 3.1, and $U \triangleq \max_{|x|\le Z} |\kappa(x)|$. Again, having fixed T_0 and T according to Assumptions 4 and 7, respectively, let $G > 0$ be as in Proposition 3. Finally set

$$E_k \triangleq \lambda^k \frac{X}{B} + \frac{1}{B-G}\eta + \sum_{j=0}^{k-1} \lambda^{k-1-j} \frac{\beta(2X, jT)}{B}, \quad k \in \mathbb{Z}_+, \tag{15}$$

where η and $\beta(\cdot,\cdot)$ are those introduced in Section 3.1, while B is to be specified below. Then we can state:

Proposition 6. *Let Assumptions 4, 5, 7 and 8 hold. Let $T \in (0, T_0]$, with T_0 and T as in Assumptions 4 and 7, respectively. Choose $B > G + 1$ so that E_0 defined in (15) satisfies (14). Then, the solution of the closed-loop system*

$$\dot{x}(t) = f(x(t), \kappa(\hat{\varsigma}(kT))), \quad t \in [kT, (k+1)T),\ k \in \mathbb{Z}_+$$

with $\hat{\varsigma}(kT) = g(\sigma(kT), \varsigma(kT), \ell(kT))$ generated by (13), from the initial condition $x(0) \in \mathcal{V}(R - 2\rho(E_0))$ exists for all $t \ge 0$ and satisfies

$$V(x(kT)) \le \max\left\{V(x(0)) - \frac{(3k-1)c}{4}, r\right\} + \rho(E_k) + \sigma, \quad \forall k \in \mathbb{Z}_+$$

$$V(x(t)) \le \max\{V(x(kT)) + \rho(E_k), r\} + (r + \sigma + \rho(E_k)), \quad \forall t \in [kT, (k+1)T), \forall k \in \mathbb{Z}_+.$$

Proof. The proof proceeds similarly as the previous one and is therefore omitted. ∎

5. Conclusion

The paper deals with the design of encoders for continuous-time nonlinear systems via their approximate discrete-time models. This approach has several advantages. With respect to previous dynamic encoding schemes presented in the literature, the encoder designed in this way allows to achieve (semi-global practical) stability with less computational effort. Moreover, the methods presented in the paper allow us to extend the results available for the quantized control of discrete-time systems to continuous-time systems, and to overcome the drawbacks in connection with some existing methods. In the results established in the paper, decrease in sampling time and increase in bandwidth improve the performance of the system. The introduction of another parameter in addition to the sampling period T may allow us to refine the model independently of T, and thus to achieve the same results while relieving the communication constraints. The paper has also shown how to apply the results of [11] to the study of the robustness of nonlinear systems with respect to quantization errors.

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Appendix. Proof of Proposition 5

Rewrite the closed-loop system as

$$\dot{x}(t) = f(x(t), \kappa(\hat{\xi}(kT))) = f(x(t), \kappa(x(kT) + \eta(kT))),$$

where the "measurement noise" $\eta(kT)$ coincides in this case with the quantization error $\hat{\xi}(kT) - x(kT)$ (see also Section III.A in [11]). As $x(0) \in \mathcal{V}(R - 2\rho(E_0))$, we have $\hat{\xi}(0) \in \mathcal{V}(R + R_m)$ as well. In fact, we can be more precise: As $\rho(E_0) \ge \rho(|x(0) - \hat{\xi}(0)|)$ and $x(0), \hat{\xi}(0) \in \mathcal{V}(R + R_m)$, we can exploit the second item in Assumption 7 (also recall the remark after that assumption) to write:

$$\rho(E_0) \ge |V(\hat{\xi}(0)) - V(x(0))| \ge V(\hat{\xi}(0)) - V(x(0)).$$

Recalling that $V(x(0)) \le R - 2\rho(E_0)$, we obtain $\hat{\xi}(0) \in \mathcal{V}(R - \rho(E_0))$.

As in [11], we would like to define a fictitious signal $\eta_L(\cdot)$, which is globally Lipschitz and matches $\eta(\cdot)$ at the sampling times. One difficulty here is that the samples $\eta(kT) = \hat{\xi}(kT) - x(kT)$ are not known a priori for $k > 0$ because they depend on variables $(x(kT), \hat{\xi}(kT))$ which have not been guaranteed to exist yet — at least until we do guarantee that $x(kT) \in \mathcal{V}(R + R_m)$; if this in fact is the case, then $\hat{\xi}(kT)$ would be well defined and $|x(kT) - \hat{\xi}(kT)| \le E_k$. Nevertheless the same reasoning as in [11] works. Indeed, pick any value for $\eta_L(T)$ satisfying $|\eta_L(T)| \le E_1$, and construct the signal $\tilde{\eta}_L(\cdot)$ over $[0, T]$ as the segment connecting $\eta(0)$ with $\eta_L(T)$ chosen before. Note that

$$|\eta_L(t)| \le E_0 \quad \text{and} \quad |\dot{\eta}_L(t)| \le (E_0 + E_1)/T < 2E_0/T$$

for all $t \in [0, T]$. Now consider the system

$$\dot{x}(t) = f(x(t), \kappa(x(kT) + \eta_L(kT))). \tag{16}$$

Of course, for $k = 0$, the solutions of the two systems (the one with $\eta(\cdot)$ and the one with $\eta_L(\cdot)$) coincide wherever such solutions exist over $[kT, (k+1)T]$. Following [11], define the change of coordinates $\hat{\xi}(t) = x(t) + \eta_L(t)$ for all t in the subinterval of $[0, T]$ where $x(\cdot)$ exists. We have:

$$\dot{\hat{\xi}}(t) = f(\hat{\xi}(t), \kappa(\hat{\xi}(0))) + d(t),$$

with

$$d(t) \triangleq f(\hat{\xi}(t) - \eta_L(t), \kappa(\hat{\xi}(0))) - f(\hat{\xi}(t), \kappa(\hat{\xi}(0))) + \dot{\eta}_L(t).$$

Recall that $\hat{\xi}(0) - \eta_L(0) = \hat{\xi}(0) - \eta_L(0) = x(0) \in \mathcal{V}(R - 2\rho(E_0)) \subseteq \mathcal{V}(R + R_m)$ and that $\hat{\xi}(0) \in \mathcal{V}(R - \rho(E_0)) \subseteq \mathcal{V}(R + R_m - \rho(E_0))$. We aim to prove that actually

$$x(t) \in \mathcal{V}(R + R_m) \quad \text{and} \quad \hat{\xi}(t) \in \mathcal{V}(R + R_m - \rho(E_0)) \tag{17}$$

for all $t \in [0, T]$. In fact suppose this is not true, and let $\bar{t} \in (0, T)$ be the largest time for which (17) holds. Then, for all $t \in [0, \bar{t}\,]$,

$$|d(t)| \leq L_{fx} E_0 + (E_0 + E_1)/T \leq E_0(L_{fx} + 2/T)$$

which shows, together with (14), that

$$\int_0^t |d(s)|\,\mathrm{d}s \leq \rho^{-1}(\sigma)\mathrm{e}^{-L_{fx}T}.$$

But then, applying Theorem 1, one concludes that

$$V(\hat{\xi}(t)) \leq \max\{V(\hat{\xi}(0)), r\} + (r + \sigma)$$

for all $t \in [0, \bar{t}\,]$. As in [11], this yields the contradictory statement that $V(\hat{\xi}(t)) < R + R_m - \rho(E_0)$ for all $t \in [0, \bar{t}\,]$. In fact, $\sigma < R_m - r_m$ implies $V(\hat{x}(0)) + (r + \sigma) < R + R_m - \rho(E_0)$ on the one hand, and $E_0 < \rho^{-1}(R - r - \sigma)$ implies $r + r + \sigma < R + R_m - \rho(E_0)$, on the other. Also, an application of the second item in Assumption 7 again shows that $V(x(t)) \leq V(\hat{\xi}(t)) + \rho(E_0) < R + R_m$ for all $t \in [0, \bar{t}\,]$. Hence we have a contradiction and (17) holds for all $t \in [0, T]$. We also have

$$\begin{aligned} &V(\hat{\xi}(T)) \leq \max\{V(\hat{\xi}(0)) - c, r\} + \sigma \\ &V(\hat{\xi}(t)) \leq \max\{V(\hat{\xi}(0)), r\} + (r_m + \sigma), \quad \forall t \in [0, T]. \end{aligned} \tag{18}$$

As, in particular, the solution $x(t)$ exists for all $t \in [0, T]$, it is possible to determine the value of $\eta(T)$, which by definition satisfies $|\eta(T)| \leq E_1 \leq E_0$. Notice now that the first inequality in (18) implies $V(\hat{\xi}(T)) < R - \rho(E_0)$, and, in turn, $V(x(T)) < R$. As the conclusion in the previous paragraph, regarding the $\hat{\xi}(\cdot)$-subsystem over the interval $[0, T]$, was drawn under the hypothesis that $|\eta(0)| \leq E_0$, $V(\hat{\xi}(0)) < R - \rho(E_0)$ and $V(x(0)) < R + R_m$, time-invariance and iteration of the arguments above yield

$$\begin{aligned} &V(\hat{\xi}(kT)) \leq \max\{V(\hat{\xi}(0)) - k(c - \sigma), r\} + \sigma, \quad k \in \mathbb{Z}_+ \\ &V(\hat{\xi}(t)) \leq \max\{V(\hat{\xi}(kT)), r\} + (r + \sigma), \\ &\quad t \in [kT, (k+1)T),\ k \in \mathbb{Z}_+. \end{aligned}$$

Furthermore, $\eta_L(t)$ can be designed over each interval $[kT, (k+1)T]$, by connecting $\eta_L(kT) = \eta(kT)$, hence such that $|\eta_L(kT)| \leq E_k$, with any point $\eta_L((k+1)T)$ satisfying $|\eta_L((k+1)T)| \leq E_{k+1}$.

We have already noticed that $V(\hat{\xi}(0)) \leq V(x(0)) + \rho(E_0)$. Again, time-invariance and iterative arguments also prove that $x(t) \in \mathcal{V}(R + R_m)$, $\hat{\xi}(t) \in \mathcal{V}(R + R_m - \rho(E_0))$ for all $t \geq 0$. Hence,

$$\rho(E_k) \geq \rho(|x(kT) - \hat{\xi}(kT)|) \geq |V(x(kT)) - V(\hat{\xi}(kT))|,$$

and hence $V(x(kT)) \leq V(\hat{\xi}(kT)) + \rho(E_k)$ for each $k \in \mathbb{Z}_+$. Similarly, the same arguments and $|\hat{\xi}(t) - x(t)| \leq E_k$, yield $V(x(t)) \leq V(\hat{\xi}(t)) + \rho(E_k)$ for all $t \in [kT, (k+1)T)$, for each $k \in \mathbb{Z}_+$. The thesis follows easily. ∎

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