On Stochastic Stability of Packetized Predictive Control of Non-linear Systems over Erasure Channels ⋆

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Abstract: We study a predictive control formulation for discrete-time non-linear plant models where controller output data is transmitted over an unreliable communication channel. The channel is affected by random data-loss and does not provide acknowledgments of receipt. To achieve robustness with respect to dropouts, at every sampling instant the controller transmits packets of data. These contain possible control inputs for a finite number of future time instants, and minimize a finite horizon cost function. At the plant actuator side, received packets are buffered, providing the plant inputs. Within this context, we adopt a stochastic Lyapunov function approach to establish stability results of this networked control system.

Keywords: Control over networks, predictive control, packet dropouts, non-linear systems, mean-square stability.

1. INTRODUCTION

Motivated by both practical and also theoretical aspects, over the last decade significant research has concentrated on Networked Control Systems (NCSs), as documented, e.g., in Baillieul and Antsaklis (2007); Matveev and Savkin (2009) and the many references therein. In a NCS, plant and controller communicate via a network which may be shared with other applications. The sharing of a network simplifies the cabling (especially if the network is wireless) and, thus, increases overall system reliability. However, since general purpose network platforms were not originally designed for applications with critical timing requirements, their use for closed-loop control presents some serious challenges. The network itself is a dynamical system that exhibits characteristics which traditionally have not been taken into account in control system design. In addition to being quantized, transmitted data may be affected by time delays and data-dropouts. Thus, in a NCS, links are not transparent, often constituting a significant bottleneck in the achievable performance.

One important feature of modern communication protocols, such as Ethernet, is that data is sent in large and time-stamped packets. This alleviates quantization issues and opens the possibility to conceive control algorithms in which *packets of data*, rather than individual values, are sent through the network. In particular, one can formulate schemes where entire signal predictions stemming from model predictive controllers are transmitted. Through buffering and appropriate selection logic at the receiver node, time delays and packet dropouts can be compensated for; see, e.g., Tang and de Silva (2007); Liu et al. (2006); Zhao et al. (2008); Pin and Parisini (2009); Quevedo et al. (2007); Findeisen and Varutti (2009).

In the present work, we study such a *packetized predictive control* method for discrete-time non-linear plant models. The controller uses model predictive control principles, where at each sampling instant a finite-horizon cost function is minimized. The resulting optimizing sequences are then transmitted over an unreliable communication channel. This channel is affected by random data-loss and does not provide acknowledgments of receipt. To be amenable to practical situations where dropout-rates are unknown, the control algorithm is designed without requiring knowledge of the packet dropout distribution.

The main purpose of this work is to investigate stability issues of the NCS described above. To be more specific, we establish sufficient conditions for the optimal value function to constitute a stochastic Lyapunov function of the NCS *at the successful transmission instants* and show how this property ensures stochastic stability of the NCS. A distinguishing aspect of our approach, when compared to existing literature, such as, e.g., Quevedo and Nešić (2010); Xiong and Lam (2007), is that the stability results presented in the present work apply to channels where packet dropouts are independent and identically distributed (i.i.d.), in which case the maximum number of consecutive dropouts becomes unbounded.

The remainder of this manuscript is organized as follows: In Section 2, we present the NCS architecture to be studied. Section 3 describes the control law. Stochastic stability results are established in Section 4. Section 5 draws conclusions.

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2. PACKETIZED CONTROL OVER ERASURE CHANNELS

We consider discrete-time nonlinear MIMO plant models with state $x(k) \in \mathbb{R}^n$ and input $u(k) \in \mathbb{U} \subseteq \mathbb{R}^p$, described via:

$$x(k+1) = f(x(k), u(k)), \quad k \in \mathbb{N}_0 \triangleq \{0, 1, \dots\} \tag{1}$$

where $f(0,0) = 0$ and where the initial state $x(0) = x_0$ is arbitrarily distributed.

2.1 Network effects

Our interest lies in clock-driven Ethernet-like networks situated between controller output and plant input. Thus, all data to be transmitted is sent in large time-stamped packets. Due to transmission errors and congestion, the network introduces packet-dropouts. This motivates us to model the network as an erasure channel, which operates at the same sampling rate as the plant model (1). More precisely, we characterize transmission effects via the following discrete Bernoulli process $\{d(k)\}_{k \in \mathbb{N}_0}$:

$$d(k) \triangleq \begin{cases} 1 \text{ if packet-dropout occurs at instant } k, \\ 0 \text{ if packet-dropout does not occur at instant } k. \end{cases}$$

Each variable $d(k)$ is i.i.d. with distribution

$$\text{Prob}(d(k) = 1) = p, \quad \text{Prob}(d(k) = 0) = 1 - p, \tag{2}$$

where $p \in (0,1)$ is the *dropout-rate*. In practice, p is not known exactly. Accordingly, in the present work our focus is on situations where the controller does not have knowledge about p. (Of course, closed loop stability will depend upon the dropout-rate, see Section 4.)

As foreshadowed in the introduction, at each time instant k, the packetized predictive controller sends a control packet, say $\boldsymbol{u}(k)$ to the plant input node. To achieve good performance despite unreliable communication, $\boldsymbol{u}(k)$ contains possible control inputs for a finite number of N future time instants, i.e., we have

$$\boldsymbol{u}(k) = \begin{bmatrix} u_0(k) \\ u_1(k) \\ \vdots \\ u_{N-1}(k) \end{bmatrix} \in \mathbb{U}^N. \tag{3}$$

At the plant input side, the received packets are buffered, providing the plant inputs, see Fig. 1.

In what follows, we will first describe the buffering procedure. In Section 3 we present the control packet design.

2.2 Buffering

The buffering mechanism amounts to a parallel-in serial-out shift register, which acts as a safeguard against dropouts. For that purpose, the buffer state, denoted via $b(k) \in \mathbb{R}^{pN}$, is overwritten whenever a valid (i.e., uncorrupted and undelayed) control packet arrives. Actuator values are passed on to the plant sequentially until the next valid control packet is received. More formally, we have:

$$\begin{aligned} b(k) &= d(k) S b(k-1) + (1 - d(k)) \boldsymbol{u}(k), \\ u(k) &= e_1^T b(k) \end{aligned} \tag{4}$$

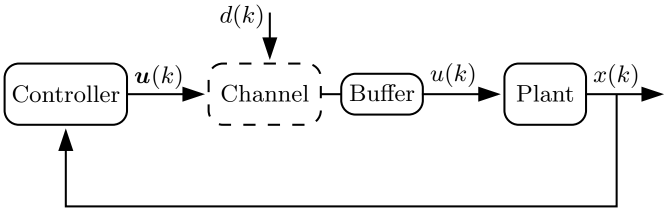

Fig. 1. Closed-loop control with an erasure channel

where the initial buffer state $b(0) = 0$ and

$$\begin{aligned} S &\triangleq \begin{bmatrix} 0_p & I_p & 0_p & \dots & 0_p \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_p & \dots & 0_p & I_p & 0_p \\ 0_p & \dots & \dots & 0_p & I_p \\ 0_p & \dots & \dots & \dots & 0_p \end{bmatrix} \in \mathbb{R}^{pN \times pN}, \\ e_1^T &\triangleq [I_p \ 0_p \ \dots \ 0_p] \in \mathbb{R}^{p \times pN}, \end{aligned} \tag{5}$$

where I_p is the $p \times p$ identity matrix and where $0_p \triangleq 0 \cdot I_p$; see, e.g., Quevedo and Nešić (2010).

Remark 1. (Holding the control input). The choice of S in (5) corresponds to setting the buffer state to zero if no data is received over N consecutive instants. Alternatively, if one wished to hold the latest value, one could set the "last" element of S equal to I_p. □

3. PACKETIZED PREDICTIVE CONTROL

The control packets $\boldsymbol{u}(k)$ in (3) are formed by adapting the ideas underpinning model predictive controllers. More precisely, at each time instant k and for a given plant state $x(k)$, the following cost function is minimized:

$$J(\boldsymbol{u}', x(k)) \triangleq F(x'(N)) + \sum_{\ell=0}^{N-1} L(x'(\ell), u'(\ell)). \tag{6}$$

The cost function in (6) examines predictions of the nominal system (1) over a finite horizon of length N, which is taken equal to the buffer size. The predicted state trajectories are generated by the model:

$$x'(\ell+1) = f(x'(\ell), u'(\ell)), \quad \ell \in \{0, 1, \dots, N-1\}$$

starting from $x'(0) = x(k)$ and where the entries in

$$\boldsymbol{u}' = \left[(u'(0))^T \ \dots \ (u'(N-1))^T \right]^T \in \mathbb{U}^N$$

are the associated plant inputs. Predicted plant states and inputs are penalized via the per-stage weighting function $L(\cdot,\cdot)$ and the terminal weighting $F(\cdot)$. These design variables allow one to trade-off control performance versus control effort. As in situations without dropouts, see, e.g., Mayne et al. (2000), the choices made for $L(\cdot,\cdot)$, $F(\cdot)$ and N influence closed loop stability. This issue will be further examined in Section 4.

The control packet $\boldsymbol{u}(k)$, see (3), is set equal to the constrained optimizer,

$$\boldsymbol{u}(k) \triangleq \arg \min_{\boldsymbol{u}' \in \mathbb{U}^N} J(\boldsymbol{u}', x(k)) \tag{7}$$

and is sent through the network to the buffer.

Following the receding horizon optimization idea, at the next sampling step and given $x(k+1)$, the horizon is shifted by one and another optimization is carried out, providing

$$\boldsymbol{u}(k+1) = \arg \min_{\boldsymbol{u}' \in \mathbb{U}^N} J(\boldsymbol{u}', x(k+1)),$$

sequence, which is transmitted to the buffer. This procedure is repeated *ad infinitum.*

Note that $\boldsymbol{u}(k)$ in (7) contains possible plant input values for instants $\{k, \dots, k+N-1\}$. If $\boldsymbol{u}(k)$ is received at time k, then these values are written into the buffer and implemented sequentially until some future (valid) control packet arrives.

In the NCS architecture under study, the plant input design is done dynamically such as to optimize performance. It is important to note that whilst $\boldsymbol{u}(k)$ is found by evaluating open-loop predictions (and not closed-loop policies, see also Quevedo et al. (2008)), the resultant control policy is a closed-loop one. Indeed, the loop is closed at all successful transmission instants, i.e., at all instants where $d(k) = 0$.

Since the plant model and cost function adopted here are time-invariant, the optimization in (7) gives rise to a time-invariant mapping, say $\kappa_N \colon \mathbb{R}^n \to \mathbb{U}^N$, which characterizes the sequence $\boldsymbol{u}(k)$ via:

$$\boldsymbol{u}(k) = \kappa_N(x(k)), \quad \forall k \in \mathbb{N}_0. \tag{8}$$

If we now introduce the augmented state

$$\theta(k) \triangleq \begin{bmatrix} x(k) \\ b(k-1) \end{bmatrix} \in \mathbb{R}^{n+pN},$$

then expressions (1), (4) and (8) allow us to describe the NCS via:

$$\theta(k+1) = \mathcal{F}_{d(k)}(\theta(k)), \quad k \in \mathbb{N}_0,$$

where

$$\begin{aligned} \mathcal{F}_0(\theta(k)) &= \begin{bmatrix} f(x(k), e_1^T \kappa_N(x(k))) \\ \kappa_N(x(k)) \end{bmatrix}, \\ \mathcal{F}_1(\theta(k)) &= \begin{bmatrix} f(x(k), e_1^T S b(k-1)) \\ S b(k-1) \end{bmatrix}. \end{aligned} \tag{9}$$

We see that the NCS which results from using packetized predictive control over an erasure channels amounts to a jump non-linear system where the jump variable is i.i.d. The *linear* case has been extensively studied within the context of Markov Jump Linear Systems; see, e.g., Ji et al. (1991). To treat the non-linear case (9), we will use an alternative model, which is presented in the following section.

4. STABILITY OF THE NCS

Various stability notions for stochastic systems have been studied in the literature; see, e.g., Kushner (1971); Ji et al. (1991). In the present work, we will adopt the following definitions:

Definition 2. (Stochastic Stability). For system (1), the equilibrium point $x = 0$

(1) is *stochastically stable*, if for every initial state x_0:[1]

$$\mathbb{E}_{x_0}\left\{\sum_{k=0}^{\infty} |x(k)|\right\} < \infty,$$

(2) is *mean square stable*, if for every initial state x_0:

$$\lim_{k\to\infty} \mathbb{E}_{x_0}|x(k)| = 0.$$

(Clearly, stochastic stability implies mean-square stability.) □

[1] Here and in the sequel, we denote expectation given y via $\mathbb{E}_y(\cdot)$.

4.1 The NCS at successful transmission instants

The approach taken in the present work is based upon Quevedo et al. (2007), where the instants of successful transmission are examined to conclude about stability properties of the system at all instants $k \in \mathbb{N}_0$.[2] For that purpose, we denote the time instants where there are no packet-dropouts ($d(k) = 0$) via

$$K = \{k_i\}_{i\in\mathbb{N}_0} \subseteq \mathbb{N}_0, \quad k_{i+1} > k_i, \ \forall i \in \mathbb{N}_0. \tag{10}$$

and also define:

$$\Delta_i \triangleq k_{i+1} - k_i, \quad i \in \mathbb{N}_0.$$

Note that $\Delta_i \geq 1$, with equality if and only if no dropouts occur between instants k_i and k_{i+1}. Furthermore, it follows directly from (2) and (10), that Δ_i is i.i.d. with geometric distribution:

$$\text{Prob}(\Delta_i = j) = (1-p)p^{j-1}, \quad \forall j \in \{1, 2, \dots\}. \tag{11}$$

For our subsequent analysis, it is convenient to introduce the mappings:

$$\bar{f}^j(x(k)) \triangleq \begin{cases} x(k), & \text{if } j = 0, \\ f(\bar{f}^{j-1}(x(k)), u_{j-1}(k)), & \text{if } j \in \{1, \dots, N\}, \end{cases} \tag{12}$$

see (7) and (3), and:

$$f_{ol}^j(\xi) \triangleq \begin{cases} \xi, & \text{if } j = 0, \\ f(f_{ol}^{j-1}(\xi), 0), & \text{if } j \in \{1, 2, \dots\}, \end{cases}$$

where $\xi \in \mathbb{R}^n$.

Given the buffering mechanism, see (4), and in terms of the above definitions, it is easy to see that the NCS *at the successful transmission instants* $k_i \in K$ is characterized via:

$$x(k_{i+1}) = \begin{cases} \bar{f}^{\Delta_i}(x(k_i)), & \text{if } \Delta_i \leq N, \\ f_{ol}^{\Delta_i - N}(\bar{f}^N(x(k_i))) & \text{if } \Delta_i \geq N. \end{cases} \tag{13}$$

Thus, $\{x(k_i)\}_{k_i\in K}$ is a Markov chain.[3]

Remark 3. (Relationship to previous works). A key difference between the current situation and that studied in Quevedo et al. (2007) and in Quevedo and Nešić (2010) is that the results in the latter works require that $\Delta_i \leq N$, for all $i \in \mathbb{N}_0$. In the present work, we remove this assumption by allowing the maximum number of consecutive packet dropouts to be unbounded, see (11). For that purpose, we extend the approach of Quevedo et al. (2010) to encompass non-linear plant models. □

4.2 Assumptions

For ease of exposition (and to keep the presentation reasonably brief), in the sequel, we will assume that the first successful transmission instant occurs at $k = 0$, i.e, we have $k_0 = 0$. We will furthermore assume that the plant and cost function satisfy the following assumptions:

Assumption 4. (Tuning Parameters). The terms $F(\cdot)$ and $L(\cdot,\cdot)$ in (6) are chosen such that:

$$F(x) \geq 0, \quad \forall x \in \mathbb{R}^n, \qquad F(0) = 0,$$
$$L(0,0) = 0, \ L(x,u) \geq \alpha|x|, \ \forall x \in \mathbb{R}^n, \ \forall u \in \mathbb{U},$$

where $\alpha > 0$ and $|\cdot|$ denotes the Euclidean norm.

[2] This idea is related to methods used to study randomly sampled systems; see, e.g., Kushner and Tobias (1969); Xie and Xie (2009).

[3] $\{x(k)\}_{k\in\mathbb{N}_0}$ will in general not be a Markov chain.

There exists a *terminal control law* $\kappa_f : \mathbb{R}^n \to \mathbb{U}$ such that:

$$F(f(\xi,\kappa_f(\xi))) - F(\xi) + L(\xi,\kappa_f(\xi)) \leq 0, \quad \forall \xi \in \mathbb{R}^n. \quad (14)$$

Assumption 5. (Bound on p). There exists $1 \leq \gamma < 1/p$, such that:

$$F(f(x,0)) \leq \gamma F(x), \quad \forall x \in \mathbb{R}^n. \quad (15)$$

Assumption 4 (and variations thereof) has been widely used for establishing stability of predictive control loops (without dropouts); see, e.g., Mayne et al. (2000); Raimondo et al. (2009).

Assumption 5 amounts to an upper bound of the dropout probability for a given plant model, or, conversely, to an upper bound of the rate of growth of $f(x,0)$ for a given p. Interestingly, (15) is an extension of a *necessary* condition for stability when the plant is linear; see Ishii (2009); Gupta et al. (2009) and the references therein.

Motivated by our analysis in Quevedo et al. (2007), in the remainder of this section we will establish stability of the NCS via study of the optimal costs $V(x(k_i))$, where:

$$V(x(k)) \triangleq J(\boldsymbol{u}(k),x(k)), \quad k \in \mathbb{N}_0,$$

Before proceeding, we note that:

Lemma 6. Suppose that Assumption 4 holds. Then

$$L(x(k),u_0(k)) \leq V(x(k)) \leq F(x(k))$$

for all $x(k) \in \mathbb{R}^n$.

Proof. By (6) and (12), we have

$$V(x(k)) = F(\bar{f}^N(x(k))) + \sum_{\ell=0}^{N-1} L(\bar{f}^\ell(x(k)),u_\ell(k)), \quad (16)$$

so that

$$V(x(k)) = F(x(k)) + \sum_{\ell=0}^{N-1} \big(L(\bar{f}^\ell(x(k)),u_\ell(k)) - F(\bar{f}^\ell(x(k))) + F(\bar{f}^{\ell+1}(x(k)))\big).$$

Since $V(x(k))$ is optimal, (14) allows us to conclude that $V(x(k)) \leq F(x(k))$. The other inequality follows directly from (16). □

4.3 Main Results

Our first result establishes that the parameters of the cost function in (6) can be chosen to ensure monotonicity of the expected value of the optimal cost at the successful transmission instants $k_i \in K$. This property is then used, in Theorem 8, for characterizing sufficient conditions for stochastic stability and mean-square stability of the closed loop *at all time instants* $k \in \mathbb{N}_0$.

Lemma 7. (Monotonicity of $V(x(k_i))$). Suppose that Assumptions 4 and 5 hold and define, for all $k_i \in K$:

$$\phi(x(k_i)) \triangleq L(x(k_i),u_0(k_i)) - p^N \left(\frac{\gamma-1}{1-p\gamma}\right) F(\bar{f}^N(x(k_i))).$$

We then have that:

$$\mathbb{E}_{x(k_0)} V(x(k_1)) - V(x(k_0)) \leq -\phi(x(k_0)), \quad (17)$$

where $\mathbb{E}_{x(k_0)} V(x(k_1))$ denotes the conditional expectation of $V(x(k_1))$ given $x(k_0)$.

Proof. See Appendix A. □

Theorem 8. (Stability). Suppose that Assumptions 4 and 5 hold, that $F(x(k_0)) < \infty$, and that there exists $\epsilon > 0$ such that

$$\phi(x(k_i)) \geq \epsilon F(x(k_i)), \quad \forall k_i \in K. \quad (18)$$

Then the NCS described by (1)–(7) is stochastically stable and mean-square stable.

Proof. See Appendix B. □

We note that Theorem 8 establishes stability of the NCS at all time instants $k \in \mathbb{N}_0$. The result suggests that one incorporate (18) as an additional constraint in the minimization of the cost function. Alternatively, the following corollary shows how to design the cost function parameters in (6) such that the NCS is stochastically stable (without requiring additional constraints in the optimization):

Corollary 9. Suppose that Assumptions 4 and 5 hold, that $F(x(k_0)) < \infty$, and that

$$F(x(k_i)) \leq \left(1 + \frac{1-p\gamma}{p^N(\gamma-1)}\right) L(x(k_i),u_0(k_i)), \quad (19)$$

for all $k_i \in K$. Then the NCS is stochastically stable and mean-square stable.

Proof. If (19) holds, then

$$F(x(k_i)) \leq \left(\frac{1-p\gamma+p^N(\gamma-1)}{\epsilon(1-p\gamma)+p^N(\gamma-1)}\right) L(x(k_i),u_0(k_i)),$$

for all $k_i \in K$ and all $\epsilon > 0$. On the other hand, Lemma 6 implies that

$$F(\bar{f}^N(x(k_i))) \leq F(x(k_i)) - L(x(k_i),u_0(k_i)) - \sum_{\ell=1}^{N-1} \big(L(\bar{f}^\ell(x(k_i)),u_\ell(k_i)) \leq F(x(k_i)) - L(x(k_i),u_0(k_i)).$$

Stability of the NCS now follows from Theorem 8. □

Corollary 9 allows us to conclude that the NCS will be stable if $F(\cdot)$ is chosen small enough, when compared to $L(\cdot,\cdot)$. To further elucidate this result, we note that

$$\psi(N) \triangleq 1 + \frac{1-p\gamma}{p^N(\gamma-1)},$$

is monotonically increasing, with

$$\lim_{N\to\infty} \psi(N) = \infty.$$

Thus, choosing larger horizons N in (6) is beneficial for fulfilling (19) and hence guaranteeing stochastic stability. Moreover, for any given weighting functions $F(\cdot)$ and $L(\cdot,\cdot)$, the closed loop will be stochastically stable, if N is chosen large enough.

Remark 10. It is worth emphasizing that (14) is a *global* condition. It would be convenient to replace it by a *local* condition, which needs to hold only in some bounded set $\mathbb{X}_f$, see also, e.g., Cannon et al. (2003). To prove monotonicity of $\{V(x(k_i))\}_{k_i \in K}$ by proceeding as in Lemma 7, one would then need that

$$\bar{f}^{N+\ell}(x(k_i)) \in \mathbb{X}_f, \quad \forall \ell \geq 0, \forall k_i \in K. \quad (20)$$

Unfortunately, since the maximum number of consecutive dropouts is unbounded, for open-loop unstable plants, (20) can, in general, not be satisfied if $\mathbb{X}_f$ is bounded. We conclude that the issue of formulating *local conditions* certainly deserves further study. □

5. CONCLUSIONS

This work has studied a NCS architecture where a packetized predictive controller uses an unreliable network affected by packet-dropouts to control a nonlinear plant. It has been shown that, provided that the plant and network satisfy suitable conditions, stochastic stability can be ensured by appropriate choice of tuning parameters.

Future work could include the study of more general situations, including where the plant is affected by random disturbances, where dropout distributions do not satisfy the model (2), and where the controller does not have direct access to the plant state.

Appendix A. PROOF OF LEMMA 7

For notational convenience, throughout this proof, we will write x instead of $x(k_0)$, u_ℓ for $u_\ell(k_0)$, and Δ for Δ_1. By the law of total expectation and (11), we have that

$$\begin{aligned}\mathbb{E}_x V(x(k_1)) &= (1-p)\sum_{i=1}^{\infty} p^{i-1}\mathbb{E}_x\{V(x(k_1))\,|\,\Delta=i\}\\ &= (1-p)\sum_{i=1}^{N} p^{i-1}\mathbb{E}_x\{V(x(k_1))\,|\,\Delta=i\}\\ &\quad + (1-p)\sum_{i=N+1}^{\infty} p^{i-1}\mathbb{E}_x\{V(x(k_1))\,|\,\Delta=i\}.\end{aligned} \tag{A.1}$$

In what follows, we will separately bound the two sums in the last expression.

(1) For $\Delta \leq N$, and since Assumption 4 holds, we can adapt (Quevedo and Nešić, 2010, Lemma 1). More precisely, we consider the sequence
$$\boldsymbol{u}^\sharp = \{u_i, u_{i+1}, \dots, u_{N-1}, u^\sharp_N, u^\sharp_{N+1}\dots, u^\sharp_{N+i-1}\},$$
whose first $N-i$ elements are taken from $\boldsymbol{u}(k_0)$. The remaining i elements of $\boldsymbol{u}^\sharp$ are provided by:
$$u^\sharp_{N+j} = \kappa_f(x^\sharp_{N+j}) \in \mathbb{U}, \quad j\in\{0,1,\dots,i-1\} \tag{A.2}$$
where $\kappa_f(\cdot)$ is such that (14) holds and where:
$$x^\sharp_{j+1} = f(x^\sharp_j, u^\sharp_j), \quad j\in\{N, N+1,\dots,N+i-1\}$$
with $x^\sharp_N = \bar{f}^N(x)$, see (12).
It follows from (6) that the associated cost satisfies:
$$\begin{aligned}J(\boldsymbol{u}^\sharp, \bar{f}^i(x)) &= F(x^\sharp_{N+i}) + \sum_{\ell=i}^{N-1} L(\bar{f}^\ell(x), u_\ell)\\ &\quad + \sum_{\ell=N}^{N+i-1} L(x^\sharp_\ell, u^\sharp_\ell)\\ &= V(x) - \sum_{\ell=0}^{i-1} L(\bar{f}^\ell(x), u_\ell) + F(x^\sharp_{N+i})\\ &\quad - F(\bar{f}^N(x)) + \sum_{\ell=N}^{N+i-1} L(x^\sharp_\ell, u^\sharp_\ell)\\ &= V(x) - \sum_{\ell=0}^{i-1} L(\bar{f}^\ell(x), u_\ell)\\ &\quad + \sum_{\ell=N}^{N+i-1}\big(F(x^\sharp_{\ell+1}) - F(x^\sharp_\ell) + L(x^\sharp_\ell, u^\sharp_\ell)\big).\end{aligned}$$
Since, due to optimality, it holds that
$$V(\bar{f}^i(x)) \leq J(\boldsymbol{u}^\sharp, \bar{f}^i(x)),$$
we can use (14) to obtain:
$$\begin{aligned}&\mathbb{E}_x\{V(x(k_1))\,|\,\Delta=i\leq N\}\\ &\qquad\leq V(x) - \sum_{\ell=0}^{i-1} L(\bar{f}^\ell(x), u_\ell).\end{aligned} \tag{A.3}$$
For the cases $\Delta = N$, we consider the sequence
$$\boldsymbol{u}^\sharp = \{u^\sharp_N, u^\sharp_{N+1}\dots, u^\sharp_{2N-1}\},$$
where now all N elements of $\boldsymbol{u}^\sharp$ are as in (A.2). By using the definition $\sum_{\ell=N}^{N-1} = 0$, expression (A.3) follows as in the case $i \leq N-1$ studied above.

(2) To study the events where $\Delta > N$, we recall (13) and Lemma 6, which give:
$$\mathbb{E}_x\{V(x(k_1))\,|\,\Delta=i>N\} \leq F(f_{ol}^{i-N}(\bar{f}^N(x))). \tag{A.4}$$

Substitution of (A.4) and (A.3) into (A.1) and use of (16) and (15) provide that:

$$\begin{aligned}\mathbb{E}_x V(x(k_1)) - V(x) &\leq -(1-p)\sum_{i=1}^{\infty} p^{i-1}\sum_{\ell=0}^{\min(i,N)-1} L(\bar{f}^\ell(x), u_\ell)\\ &\quad + (1-p)\sum_{i=N+1}^{\infty} p^{i-1}\big(F(f_{ol}^{i-N}(\bar{f}^N(x))) - F(\bar{f}^N(x))\big)\\ &\leq -L(x,u_0) - p^N\sum_{\ell=1}^{N-1} L(\bar{f}^\ell(x), u_\ell)\\ &\quad + (1-p)p^N\sum_{j=0}^{\infty} p^j\big(F(f_{ol}^{j+1}(\bar{f}^N(x))) - F(\bar{f}^N(x))\big)\\ &\leq -L(x,u_0(0)) - p^N\sum_{\ell=1}^{N-1} L(\bar{f}^\ell(x), u_\ell)\\ &\quad - p^N F(\bar{f}^N(x)) + (1-p)p^N\gamma\sum_{j=0}^{\infty}(p\gamma)^j F(\bar{f}^N(x)).\end{aligned}$$

By Assumption 5, we have that $|p\gamma| < 1$. Hence, $\sum_{j=0}^{\infty}(p\gamma)^j = (1-p\gamma)^{-1}$, proving (17). $\square$

Appendix B. PROOF OF THEOREM 8

Throughout this proof we will denote $x(k_0)$ by x, Δ_1 by Δ, and $\bar{f}^N(x(k_0))$ by x_N. Lemma 7 gives that $V(x(k_i))$ is a stochastic Lyapunov function for the closed loop at the time instants $k_i \in K$. In fact, Lemma 6 and Equation (18) ensure that there exists $\epsilon > 0$ such that
$$\phi(x(k_i)) \geq \epsilon V(x(k_i)), \quad \forall k_i \in K.$$
Thus, (17) provides:
$$\mathbb{E}_x V(x(k_1)) - V(x) \leq -\epsilon V(x).$$
Since $\{x(k_i)\}_{k_i\in K}$ is a Markov chain, Theorem 2 in (Kushner, 1971, Ch. 8.4.2) then allows us to conclude that $\epsilon \leq 1$ and that *at instants* $k_i \in K$ we have exponential stability:
$$\mathbb{E}_x V(x(k_i)) \leq (1-\epsilon)^i V(x), \quad \forall i \geq 1. \tag{B.1}$$

We next examine instants $k \notin K$, $k > k_0$. For that purpose, we proceed as in (A.1) and condition upon Δ to obtain:

$$
\begin{aligned}
\mathbb{E}_x \sum_{\ell=k_0}^{k_1-1} |x(\ell)| &\leq (1-p)\sum_{i=1}^{\infty} p^{i-1}\mathbb{E}_x\left\{\sum_{\ell=k_0}^{k_0+N-1}|x(\ell)| \,\Big|\, \Delta \geq N\right\} \\
&\quad + (1-p)\sum_{i=N+1}^{\infty} p^{i-1}\mathbb{E}_x\left\{\sum_{\ell=k_0+N}^{k_1-1}|x(\ell)| \,\Big|\, \Delta = i\right\} \\
&= \mathbb{E}_x \sum_{\ell=0}^{N-1} |\bar{f}^{\ell}(x)| + (1-p)p^N \sum_{\ell=0}^{\infty} p^{\ell}\mathbb{E}_x \sum_{j=0}^{\ell} |f_{ol}^{j}(x_N)| \\
&\leq \mathbb{E}_x \sum_{\ell=0}^{N-1} |\bar{f}^{\ell}(x)| + \frac{(1-p)p^N}{\alpha} \sum_{\ell=0}^{\infty} p^{\ell}\mathbb{E}_x \sum_{j=0}^{\ell} F(f_{ol}^{j}(x_N)) \\
&\leq \mathbb{E}_x \left\{\sum_{\ell=0}^{N-1} |\bar{f}^{\ell}(x)| + \frac{(1-p)p^N}{\alpha} \sum_{\ell=0}^{\infty} p^{\ell} \sum_{j=0}^{\ell} \gamma^j F(x_N)\right\} \\
&= \mathbb{E}_x \left\{\sum_{\ell=0}^{N-1} |\bar{f}^{\ell}(x)| + \frac{p^N}{\alpha(1-p\gamma)} F(x_N)\right\}
\end{aligned}
$$

where we have used (15), Assumption 4 and Lemma 6. Assumption 4 furthermore ensures that there exists a constant $0 < \beta < \infty$, such that

$$
\mathbb{E}_x \sum_{\ell=k_0}^{k_1-1} |x(\ell)| \leq \beta \mathbb{E}_x V(x).
$$

In a similar manner, it can be shown that

$$
\mathbb{E}_{x(k_i)} \left\{\sum_{\ell=k_i}^{k_{i+1}-1} |x(\ell)|\right\} \leq \beta\, \mathbb{E}_{x(k_i)}\big\{V(x(k_i))\big\},\ \forall k_i \in K,
$$

so that (B.1) gives:

$$
\mathbb{E}_{x(k_0)} \left\{\sum_{\ell=k_i}^{k_{i+1}-1} |x(\ell)|\right\} \leq \beta\,(1-\epsilon)^i V(x(k_0)), \quad \forall k_i \in K.
$$

Since $\{x(k_i)\}_{k_i \in K}$ is a Markov chain, we obtain:

$$
\begin{aligned}
\mathbb{E}_{x(k_0)} \left\{\sum_{\ell=k_0}^{k_{m+1}-1} |x(\ell)|\right\} &\leq \beta \mathbb{E}_{x(k_0)} \left\{\sum_{i=0}^{m} (1-\epsilon)^i V(x(k_0))\right\} \\
&= \beta \frac{1-(1-\epsilon)^m}{\epsilon} V(x(k_0))
\end{aligned}
$$

If we now let $m \to \infty$, and use Lemma 6, then:

$$
\mathbb{E}_{x(k_0)} \left\{\sum_{\ell=k_0}^{\infty} |x(\ell)|\right\} \leq \frac{\beta}{\epsilon} V(x(k_0)) \leq \frac{\beta}{\epsilon} F(x(k_0)) < \infty,
$$

thus, proving the result. □

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