Networked Control Systems with Communication Constraints: Tradeoffs between Transmission Intervals, Delays and Performance

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***Abstract*—There are many communication imperfections in networked control systems (NCS) such as varying transmission delays, varying sampling/transmission intervals, packet loss, communication constraints and quantization effects. Most of the available literature on NCS focuses on only some of these aspects, while ignoring the others. In this paper we present a general framework that incorporates both communication constraints, varying transmission intervals and varying delays. Based on a newly developed NCS model including all these network phenomena, we will provide an explicit construction of a continuum of Lyapunov functions. Based on this continuum of Lyapunov functions we will derive bounds on the maximally allowable transmission interval (MATI) and the maximally allowable delay (MAD) that guarantee stability of the NCS in the presence of communication constraints. The developed theory includes recently improved results for delay-free NCS as a special case. After considering stability, we also study semi-global practical stability (under weaker conditions) and performance of the NCS in terms of $\mathcal{L}_p$ gains from disturbance inputs to controlled outputs. The developed results lead to tradeoff curves between MATI, MAD and performance gains that depend on the used protocol. These tradeoff curves provide quantitative information that supports the network designer when selecting appropriate networks and protocols guaranteeing stability and a desirable level of performance, while being robust to specified variations in delays and tranmission intervals. The complete design procedure will be illustrated using a benchmark example.**

***Index Terms*—Networked control systems, Lyapunov functions, stability, delays, communication constraints, time scheduling, protocols, $\mathcal{L}_p$ gains.**

I. INTRODUCTION

Networked control systems (NCS) have received considerable attention in recent years. The interest for NCS is motivated by many benefits they offer such as the ease of maintenance and installation, the large flexibility and the low cost. However, still many issues need to be resolved before all the advantages of wired and wireless networked control systems can be harvested. Next to improvements in the communication infrastructure itself, there is a need for control algorithms that can deal with communication imperfections and constraints. This latter aspect is recognized widely in the control community, as evidenced by the many publications appearing recently, see e.g. the overview papers [23], [47], [54], [56].

Roughly speaking, the networked-induced imperfections and constraints can be categorized in five types:

(i) Quantization errors in the signals transmitted over the network due to the finite word length of the packets;
(ii) Packet dropouts caused by the unreliability of the network;
(iii) Variable sampling/transmission intervals;
(iv) Variable communication delays;
(v) Communication constraints caused by the sharing of the network by multiple nodes and the fact that only one node is allowed to transmit its packet per transmission.

It is well known that the presence of these network phenomena can degrade the performance of the control loop significantly and can even lead to instability, see e.g. [10] for an illustrative example. Therefore, it is of importance to understand how these phenomena influence the closed-loop stability and performance properties, preferably in a quantitative manner. Unfortunately, much of the available literature on NCS considers only some of above mentioned types of network phenomena, while ignoring the other types. There are, for instance, systematic approaches that analyse stability of NCSs subject to only one of these network-induced imperfections. Indeed, the effects of quantization are studied in [3], [12], [20], [22], [28], [36], [45], of packet dropouts in [41], [42], of time-varying transmission intervals and delays in [14], [32], and [10], [16], [24], [27], [35], [55], respectively, and of communication constraints in [2], [11], [26], [40].

Since in any practical communication network all aforementioned network-induced imperfections are present, there is a need for analysis and synthesis methods including all these imperfections. This is especially of importance, because the design of a NCS often requires tradeoffs between the different types. For instance, reducing quantization errors (and thus transmitting larger or more packets) typically results in larger transmission delays. To support the designers in making these tradeoffs, tools are needed that provide quantitative information on the consequences of each of the possible choices. However, less results are available that study combinations of these imperfections. References that simultaneously consider two types of network-induced limitations are given in Table I. Moreover, [37] consider imperfections of type (i), (iii), (v), [8], [33], [34] study simultaneously type (ii), (iii), (iv), [38] focusses on type (ii), (iii), (v) and [15] incorporates type (i), (ii) and (iv). In addition some of the approaches mentioned in Table I that study varying transmission intervals and/or varying communication delays can be extended to

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include type (ii) phenomena as well by modeling dropouts as prolongations of the maximal transmission interval or delay (cf. also Remark II.4 below).

TABLE I
REFERENCES THAT STUDY MULTIPLE NETWORKED INDUCED IMPERFECTIONS SIMULTANEOUSLY.

&	(iv)	(v)
(i)	[29]	
(ii)	[9], [18], [31]	
(iii)	[25], [48]	[5], [13], [39], [44], [51], [52]

Another paper that studies three different types of network imperfections is written by Chaillet and Bicchi [6]. This paper studies NCS involving both variable delays, variable transmission intervals and communication constraints, and uses a method for delay compensation. The delay compensation is based on sending a larger control packet to the plant containing not just one control value at one particular time instant, but containing a control signal valid for a given future time horizon. For this particular control scheme, [6] provides bounds on the tolerable delays and transmission intervals such that stability of the NCS is guaranteed. Also in this paper we will study NCS corrupted by varying delays, varying transmission intervals and communication constraints, while packet dropouts can be included as well (in the way explained in Remark II.4 below). In other words, this paper considers networked-induced imperfections of type (iii), (iv) and (v). After developing a novel NCS model incorporating all these types of network phenomena, we will present allowable bounds on delays and transmission intervals guaranteeing both stability and performance of the NCS. However, in contrast with [6], we consider the more basic *emulation approach* in the spirit of [5], [11], [38], [39], [51], [52], which encompasses no specific delay compensation schemes. The work in [6] is of interest, as it aims at allowing larger delays by including specific delay compensation schemes, at the cost of sending larger control-packets and requiring time-stamping of messages. The features of compensation and time-stamping of messages are not needed in our framework. Another distinction with [6] is related to the admissible protocols that schedule which node is allowed to transmit its packet at a transmission time. Our work applies for all protocols satisfying the UGES property (see below for an exact definition) and not only for so-called *invariably* UGES protocols (cf. [6]), which exclude the commonly used Round-Robin (RR) protocol.

One of the main contributions of this paper is that we explicitly construct a *continuum* of Lyapunov functions based on the standard delay-free conditions as adopted in [5], [11], [38], [39], [51], [52]. This continuum of Lyapunov functions leads to tradeoff curves between the maximally allowable transmission interval (MATI) and the maximally allowable delay (MAD) guaranteeing stability of the NCS. These tradeoff curves will depend on the specific communication protocol used, so that they even allow for the comparison of different protocols. In addition to stability, which is only a basic property that has to be satisfied by the control loop, there are often additional requirements with respect to the performance of the NCS. This paper also studies the performance in terms of $\mathcal{L}_p$ gains between specific exogenous inputs (e.g. disturbances) and controlled outputs of the system. We will show how $\mathcal{L}_p$ performance of the NCS depends on the MATI, the MAD and the protocol used, leading to tradeoff curves as well. This design methodology and the method to compute the tradeoff curves will be demonstrated on the case study of the batch reactor that has developed over the years as a benchmark system for NCS, see e.g. [5], [38], [39]. Next to stability and $\mathcal{L}_p$ performance, also semiglobal practical stability results will be presented that can be obtained under weaker conditions.

The paper is organized as follows. The current section will end with introducing some notational conventions and concepts. Next, in Section II we present a general NCS modeling framework that extends the NCS models in [5], [11], [38], [39], [51], [52] to include both communication constraints as well as varying transmission delays and transmission intervals. In Section III we will transform this new NCS model into the hybrid system framework as introduced in [17] as this will facilitate further analysis. Also the stability and performance concepts as used in this paper are defined in this section. In Section IV we will derive the Lyapunov-based conditions that determine both the maximally allowable transmission interval (MATI) and the maximally allowable delay (MAD) guaranteeing global asymptotic stability and $\mathcal{L}_p$ performance. We also present the results on semiglobal practical stability in this section. In section V we show how the Lyapunov functions can be constructed on the basis of the widely adopted non-delay conditions in [5], [11], [38], [39], [51], [52] and show that the non-delay case is a particular case of general framework. To demonstrate how the developed methods can be used for explicitly computing MATI and MAD guaranteeing stability or certain $\mathcal{L}_p$ performance, we apply the framework to the benchmark problem of the batch reactor [5], [38], [39]. Finally, we state the conclusions and our ideas for future work.

The following notational conventions will be used in this paper. $\mathbb{N}$ will denote all nonnegative integers, $\mathbb{R}$ denotes the field of all real numbers and $\mathbb{R}_{\geq 0}$ denotes all nonnegative reals. By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the Euclidean norm and the usual inner product of real vectors, respectively. For a number of real vectors $(a_1, \ldots, a_M)$ with $a_i \in \mathbb{R}^{n_i}$, we denote the column vector $(a_1^\top, \ldots, a_M^\top)^\top$ obtained by stacking the vectors a_i, $i = 1, \ldots, M$ on top of each other by $(a_1, \ldots, a_M)$. For a symmetric matrix A, $\lambda_{max}(A)$ denotes the largest eigenvalue of A. By $\vee$ and $\wedge$ we denote the logical 'or' and 'and,' respectively. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is nonincreasing and satisfies $\lim_{t\to\infty} \beta(s, t) = 0$ for each $s \geq 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KLL}$ if, for each $r \geq 0$, $\beta(\cdot, r, \cdot)$ and $\beta(\cdot, \cdot, r)$ belong to class $\mathcal{KL}$. We write $\exp(\cdot)$ for the standard exponential function.

We recall now some definitions given in [17] that will be used for developing a hybrid model of a NCS later. For the motivation and more details on these definitions, one can consult [17].

Definition I.1 A *compact hybrid time domain* is a set $\mathcal{D} = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j) \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ with $J \in \mathbb{N}_{>0}$ and $0 = t_0 \leq t_1 \cdots \leq t_J$. A *hybrid time domain* is a set $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0}$ such that $\mathcal{D} \cap ([0, T] \times \{0, \ldots, J\})$ is a compact hybrid time domain for each $(T, J) \in \mathcal{D}$. ■

Definition I.2 A *hybrid trajectory* is a pair (dom ξ, ξ) consisting of hybrid time domain dom ξ and a function ξ defined on dom ξ that is absolutely continuous in t on (dom $\xi) \cap (\mathbb{R}_{\geq 0} \times \{j\})$ for each $j \in \mathbb{N}$. ■

Definition I.3 For the hybrid system $\mathcal{H}$ given by the open state space $\mathbb{R}^n$, an input space $\mathbb{R}^{n_w}$ and the data (F, G, C, D), where $F : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ is continuous, $G : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded, and C and D are subsets of $\mathbb{R}^n$, a hybrid trajectory (dom ξ, ξ) with ξ : dom $\xi \to \mathbb{R}^n$ is a *solution to* $\mathcal{H}$ for a locally integrable input function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_w}$ if

1) For all $j \in \mathbb{N}$ and for almost all $t \in I_j := \{t \mid (t, j) \in \text{dom } \xi\}$, we have $\xi(t, j) \in C$ and $\dot{\xi}(t, j) = F(\xi(t, j), w(t))$.
2) For all $(t, j) \in$ dom ξ such that $(t, j+1) \in$ dom ξ, we have $\xi(t, j) \in D$ and $\xi(t, j+1) = G(\xi(t, j))$.

Hence, the hybrid systems that we consider are of the form:

$$\begin{array}{rcll} \dot{\xi}(t, j) & = & F(\xi(t, j), w(t)) & \xi(t, j) \in C \\ \xi(t_{j+1}, j+1) & = & G(\xi(t_{j+1}, j)) & \xi(t_{j+1}, j) \in D . \end{array}$$

We sometimes omit the time arguments and write:

$$\dot{\xi} = F(\xi, w), \text{ when } \xi \in C, \qquad \xi^+ = G(\xi), \text{ when } \xi \in D, \tag{1}$$

where we denoted $\xi(t_{j+1}, j+1)$ as ξ^+. We also note that typically $C \cap D \neq \emptyset$ and, in this case, if $\xi(0, 0) \in C \cap D$ we have that either a jump or flow is possible, the latter only if flowing keeps the state in C. Hence, the hybrid model (1) may have non-unique solutions.

In addition, for $p \in \mathbb{N}$, $p \geq 1$, we introduce the $\mathcal{L}_p$ norm of a function ξ defined on a hybrid time domain dom$\xi = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ with J possibly ∞ and/or $t_J = \infty$, by

$$\|\xi\|_p = \left(\sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} |\xi(t, j)|^p dt \right)^{\frac{1}{p}} \tag{2}$$

provided the right-hand side exists and is finite. In case the $\|\xi\|_p$ norm exists and is finite, we say that $\xi \in \mathcal{L}_p$. Note that this definition is essentially identical to the usual $\mathcal{L}_p$ norm in case a function is defined on a subset of $\mathbb{R}_{\geq 0}$.

II. NCS MODEL AND PROBLEM STATEMENT

In this section, we introduce the model that will be used to describe NCS including both communication constraints as well as varying transmission intervals and transmission delays. This model will form an extension of the NCS models used before in [38], [39] that were motivated by the work in [52]. All these previous models did not include transmission delays. We consider the continuous-time plant

$$\dot{x}_p = f_p(x_p, \hat{u}, w), \qquad y = g_p(x_p) \tag{3}$$

that is sampled. Here, $x_p \in \mathbb{R}^{n_p}$ denotes the state of the plant, $\hat{u} \in \mathbb{R}^{n_u}$ denotes the most recent control values available at the plant, $w \in \mathbb{R}^{n_w}$ is a disturbance input and $y \in \mathbb{R}^{n_y}$ is the output of the plant. The controller[1] is given by

$$\dot{x}_c = f_c(x_c, \hat{y}, w), \qquad u = g_c(x_c), \tag{4}$$

where the variable $x_c \in \mathbb{R}^{n_c}$ is the state of the controller, $\hat{y} \in \mathbb{R}^{n_y}$ is the most recent output measurement of the plant that is available at the controller and $u \in \mathbb{R}^{n_u}$ denotes the control input. The functions f_p, f_c are assumed to be continuous and g_p and g_c are assumed to be continuously differentiable. At times t_{s_i}, $i \in \mathbb{N}$, (parts of) the input u at the controller and/or the output y at the plant are sampled and sent over the network. The tranmission/sampling times satisfy $0 \leq t_{s_0} < t_{s_1} < t_{s_2} < \ldots$ and there exists a $\delta > 0$ such that the tranmission intervals $t_{s_{i+1}} - t_{s_i}$ satisfy $\delta \leq t_{s_{i+1}} - t_{s_i} \leq \tau_{mati}$ for all $i \in \mathbb{N}$, where τ_{mati} denotes the maximally allowable transmission interval (MATI). At each transmission time t_{s_i}, $i \in \mathbb{N}$, the protocol determines which of the nodes $j \in \{1, 2, \ldots, l\}$ is granted access to the network. Each node corresponds to a collection of sensors or actuators. The sensors/actuators corresponding to the node that is granted access collects their values of the entries in $y(t_{s_i})$ or $u(t_{s_i})$ that will be sent over the communication network. They will arrive after a transmission delay of τ_i time units at the controller or actuator. This results in updates of the corresponding entries in $\hat{y}$ or $\hat{u}$ at times $t_{s_i} + \tau_i$, $i \in \mathbb{N}$. The situation described above is illustrated for y and $\hat{y}$ in Fig. 1.

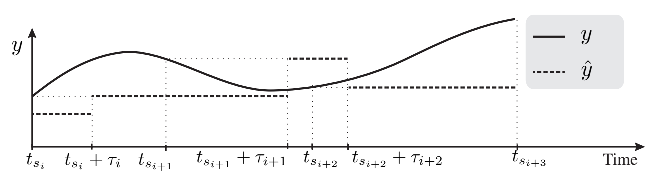

Fig. 1. Illustration of a typical evolution of y and $\hat{y}$.

It is assumed that there are bounds on the maximal delay in the sense that $\tau_i \in [0, \tau_{mad}]$, $i \in \mathbb{N}$, where $0 \leq \tau_{mad} \leq \tau_{mati}$ is the maximally allowable delay (MAD). To be more precise, we adopt the following standing assumption.

Standing Assumption II.1 The transmission times satisfy $\delta \leq t_{s_{i+1}} - t_{s_i} < \tau_{mati}$, $i \in \mathbb{N}$ and the delays satisfy $0 \leq \tau_i \leq \min\{\tau_{mad}, t_{s_{i+1}} - t_{s_i}\}$, $i \in \mathbb{N}$, where $\delta \in (0, \tau_{mati}]$ is arbitrary.

The latter condition implies that each transmitted packet arrives before the next sample is taken. This assumption indicates that we are considering the so-called small delay case as opposed to the large delay case, where delays can be larger than the transmission interval. The inequalities $\tau_i \leq t_{s_{i+1}} - t_{s_i}$ and $\tau_{mad} \leq \tau_{mati}$ can be taken non-strict with the understanding that in case the update instant $t_{s_i} + \tau_i$ coincides with the next transmission instant $t_{s_{i+1}}$, the update

[1] Extensions of the theory presented below to the case of time-dependent systems (3) and time-dependent controllers (4) are straightforward.

is performed before the next sample is taken. The updates at $t_{s_i}+\tau_i$ satisfy

$$\begin{aligned}\hat{y}((t_{s_i}+\tau_i)^+) &= y(t_{s_i})+h_y(i,e(t_{s_i}))\\ \hat{u}((t_{s_i}+\tau_i)^+) &= u(t_{s_i})+h_u(i,e(t_{s_i}))\end{aligned} \tag{5}$$

where e denotes the vector (e_y,e_u) with $e_y:=\hat{y}-y$ and $e_u:=\hat{u}-u$. Hence, $e\in\mathbb{R}^{n_e}$ with $n_e=n_y+n_u$. If the NCS has l links, then the error vector e can be partitioned as $e=(e_1,e_2,\dots,e_l)$. The functions h_y and h_u are now update functions that are related to the protocol that determines on the basis of i and the networked error $e(t_{s_i})$ which node is granted access to the network. Typically when the j-th node gets access to the network at some transmission time t_{s_i} we have that the corresponding part in the error vector has a jump at $t_{s_i}+\tau_i$. In most situations, the jump will actually be to zero, since we assume that the quantization effects are negligible. For instance, when y_j is sampled at time t_{s_i}, then we have that $h_{y,j}(i,e(t_{s_i}))=0$. However, we allow for more freedom in the protocols by allowing general functions h. See [38], [39] for more details. We will refer to $h=(h_y,h_u)$ as the protocol.

In between the updates of the values of $\hat{y}$ and $\hat{u}$, the network is assumed to operate in a zero order hold (ZOH) fashion, meaning that the values of $\hat{y}$ and $\hat{u}$ remain constant in between the updating times $t_{s_i}+\tau_i$ and $t_{s_{i+1}}+\tau_{i+1}$ for all $i\in\mathbb{N}$:

$$\dot{\hat{y}}=0,\qquad \dot{\hat{u}}=0. \tag{6}$$

To compute the resets of e at the update times $\{t_{s_i}+\tau_i\}_{i\in\mathbb{N}}$, we proceed as follows:

$$\begin{aligned}&e_y((t_{s_i}+\tau_i)^+)=\hat{y}((t_{s_i}+\tau_i)^+)-y(t_{s_i}+\tau_i)\\ &=y(t_{s_i})+h_y(i,e(t_{s_i}))-y(t_{s_i}+\tau_i)\\ &=h_y(i,e(t_{s_i}))+\underbrace{y(t_{s_i})-\hat{y}(t_{s_i})}_{-e(t_{s_i})}+\underbrace{\hat{y}(t_{s_i}+\tau_i)-y(t_{s_i}+\tau_i)}_{e(t_{s_i}+\tau_i)}\\ &=h_y(i,e(t_{s_i}))-e(t_{s_i})+e(t_{s_i}+\tau_i).\end{aligned}$$

In the third equality we used that $\hat{y}(t_{s_i})=\hat{y}(t_{s_i}+\tau_i)$ due to the zero order hold character of the network. We also implicitly employed our Standing Assumption II.1 as we used that there always occurs an update before the next sample is taken ($t_{s_i}+\tau_i\le t_{s_{i+1}}$).

A similar derivation holds for e_u, leading to the following model for the NCS:

$$\left.\begin{aligned}\dot{x}(t) &= f(x(t),e(t),w(t))\\ \dot{e}(t) &= g(x(t),e(t),w(t))\end{aligned}\right\}\ t\in[t_{s_i},t_{s_{i+1}}]\backslash\{t_{s_i}+\tau_i\} \tag{7a}$$

$$e((t_{s_i}+\tau_i)^+)=h(i,e(t_{s_i}))-e(t_{s_i})+e(t_{s_i}+\tau_i), \tag{7b}$$

where $x=(x_p,x_c)\in\mathbb{R}^{n_x}$ with $n_x=n_p+n_c$, f, g are appropriately defined functions depending on f_p, g_p, f_c and g_c and $h=(h_y,h_u)$. See [38] for the explicit expressions of f and g, which also reveal how we use the differentiability conditions on g_c and g_p imposed earlier.

Standing Assumption II.2 f and g are continuous and h is locally bounded. ■

Observe that the system

$$\dot{x}=f(x,0,w) \tag{8}$$

is the closed-loop system (3)-(4) without the network (i.e. $y(t)=\hat{y}(t)$ and $u(t)=\hat{u}(t)$ in (3)-(4)).

The problem that we consider in this paper is formulated as follows.

Problem II.3 Suppose that the controller (4) was designed for the plant (3) rendering the closed-loop (3)-(4) (or equivalently, (8)) stable in some sense. Determine the value of τ_{mati} and τ_{mad} so that the NCS given by (7) is stable as well when the transmission intervals and delays satisfy Standing Assumption II.1. ■

Remark II.4 Of course, there are certain extensions that can be made to the above setup. The inclusion of packet dropouts is relatively easy, if one models them as prolongations of the transmission interval. Indeed, if we assume that there is a bound $\bar{\delta}\in\mathbb{N}$ on the maximum number of successive dropouts, the stability bounds derived below are still valid for the MATI given by $\tau'_{mati}:=\frac{\tau_{mati}}{\bar{\delta}+1}$, where τ_{mati} is the obtained value for the dropout-free case. Another extension of the framework in this paper could be the inclusion of quantization effects. This step is more involved. It might be based on recent work in [37] that unifies the areas of networked and quantized control systems *without* communication delays. It can be envisioned that the results presented here can be combined with the framework in [37] leading to an overall methodology capable of handling all types of networked phenomena that were mentioned in the introduction. The specific conditions and types of quantizers for which this methodology is effective are subject of future research and some preliminary results are report in [21]. Another possible extension of interest is the consideration of the large-delay case (in which the delays can be larger than the transmission interval). This would require a more involved NCS model that does not have the periodicity between transmission and update events as implied by Standing Assumption II.1. This is a hard problem, which will be considered in future research. ■

III. REFORMULATION IN A HYBRID SYSTEM FRAMEWORK

To facilitate the stability analysis, we transform the above NCS model into the hybrid system framework as developed in [17]. This hybrid systems framework was also employed in [5], where a similar model was obtained *without* the incorporation of delays. To do so, we introduce the auxiliary variables $s\in\mathbb{R}^n$, $\kappa\in\mathbb{N}$, $\tau\in\mathbb{R}_{\ge 0}$ and $\ell\in\{0,1\}$ to reformulate the model in terms of flow equations and reset equations. The variable s is an auxiliary variable containing the memory in (7b) storing the value $h(i,e(t_{s_i}))-e(t_{s_i})$ for the update of e at the update instant $t_{s_i}+\tau_i$, κ is a counter keeping track of the transmission, τ is a timer to constrain both the transmission interval as well as the transmission delay[2] and ℓ is a Boolean keeping track whether the next event is a transmission event or an update event. To be precise, when $\ell=0$ the next event will be related to transmission and when $\ell=1$ the next event

[2]We could also have introduced two timers, one corresponding to the transmission interval and one to the transmission delay. However, it turns out that the NCS can be described using only one timer, which has the advantage of resulting in a more compact hybrid model.

will be an update. The Boolean ℓ will be used to guarantee in the model below that the transmission and update events are alternating in the sense that before a next sample is taking the previous update is implemented in the NCS.

The hybrid system $\mathcal{H}_{NCS}$ is now given by the flow equations

$$\left.\begin{array}{rcl} \dot{x} &=& f(x,e,w) \\ \dot{e} &=& g(x,e,w) \\ \dot{s} &=& 0 \\ \dot{\tau} &=& 1 \\ \dot{\kappa} &=& 0 \\ \dot{\ell} &=& 0 \end{array}\right\} \begin{array}{l} (\ell = 0 \wedge \tau \in [0, \tau_{mati}])\vee \\ \vee(\ell = 1 \wedge \tau \in [0, \tau_{mad}]) \end{array} \tag{9}$$

and the reset equations are obtained by combining the "transmission reset relations," active at the transmission instants $\{t_{s_i}\}_{i\in\mathbb{N}}$, and the "update reset relations", active at the update instants $\{t_{s_i} + \tau_i\}_{i\in\mathbb{N}}$, given by

$$(x^+, e^+, s^+, \tau^+, \kappa^+, \ell^+) = G(x, e, s, \tau, \kappa, \ell), \text{ when } (\ell = 0 \wedge \tau \in [\delta, \tau_{mati}]) \vee (\ell = 1 \wedge \tau \in [0, \tau_{mad}]) \tag{10}$$

with the mapping G given by the transmission resets (when $\ell = 0$)

$$G(x, e, s, \tau, \kappa, 0) = (x, e, h(\kappa, e) - e, 0, \kappa + 1, 1) \tag{11}$$

and the update resets (when $\ell = 1$)

$$G(x, e, s, \tau, \kappa, 1) = (x, s + e, -s - e, \tau, \kappa, 0). \tag{12}$$

Two comments on this model are in order. First of all, the role of $\delta > 0$ is to exclude (instantaneous Zeno) solutions to $\mathcal{H}_{NCS}$ satisfying $x(0, j) = x(0, 0)$, $\tau(0, j) = \tau(0, 0)$ and $\kappa(0, j) = \kappa(0, 0) + p$ for $j = 2p$ or $j = 2p - 1$ with $p \in \mathbb{N}$, when $\ell(0, 0) = 0$. Also when $\ell(0, 0) = 1$ similar solutions exist that are only resetting (sometimes called 'livelock' in hybrid systems theory [50]). However, $\delta > 0$ can be taken arbitrarily small to still allow for small transmission intervals.

Secondly, the choice for s^+ when $\ell = 1$ is irrelevant from a modeling point of view. However, it was selected here as $s^+ = -s - e$, because it will simplify the analysis later. By taking $\xi = (x, e, s, \tau, \kappa, \ell)$ the hybrid system $\mathcal{H}_{NCS}$ above is in the form (1).

Definition III.1 For the hybrid system $\mathcal{H}_{NCS}$ with $w = 0$, the set given by
$\mathcal{E} := \{(x, e, s, \tau, \kappa, \ell) \mid x = 0,\ e = s = 0\}$ is said to be *uniformly globally asymptotically stable* (UGAS) if there exists a function $\beta \in \mathcal{KLL}$ such that, for each $0 < \delta \leq \tau_{mati}$, and any initial condition $x(0, 0) \in \mathbb{R}^{n_x}$, $e(0, 0) \in \mathbb{R}^{n_e}$, $s(0, 0) \in \mathbb{R}^{n_e}$, $\tau(0, 0) \in \mathbb{R}_{\geq 0}$, $\kappa(0, 0) \in \mathbb{N}$, $\ell(0, 0) \in \{0, 1\}$ with[3] $(\ell(0, 0) = 0 \wedge \tau(0, 0) \in [0, \tau_{mati}]) \vee (\ell(0, 0) = 1 \wedge \tau(0, 0) \in [0, \tau_{mad}])$, all corresponding solutions satisfy

$$|(x(t, j), e(t, j), s(t, j))| \leq \beta\left(|(x(0, 0), e(0, 0), s(0, 0))|, t, \delta j\right) \tag{13}$$

for all (t, j) in the solution's domain. The set $\mathcal{E}$ is *uniformly globally exponentially stable* (UGES) if β can be taken of the form $\beta(r, t, k) = Mr \exp(-\varrho(t + k))$ for some $M \geq 0$ and $\varrho > 0$. ■

[3]Note that the next condition is just saying that $\xi(0, 0) \in C \cup D$ in the terminology of (1).

Remark III.2 The factor δ multiplying j in the right-hand side of (13) is motivated by the fact that for small values of $\delta > 0$ solutions of $\mathcal{H}_{NCS}$ exist that have many resets without t progressing too fast. Actually, the limit case $\delta = 0$ has 'livelock' solutions that are only resetting (with t remaining 0 and $j \longrightarrow \infty$) as discussed above. The δ scales the right-hand side of (13) for this effect. ■

We also introduce the concept of uniform semiglobal practical asymptotical stability.

Definition III.3 For the hybrid system $\mathcal{H}_{NCS}$ with $w = 0$, the set $\mathcal{E}$ is said to be *uniformly semiglobally practically asymptotically stable* (USPAS) with respect to τ_{mati} and τ_{mad}, if there exists $\beta \in \mathcal{KLL}$ and for any pair of positive numbers (ϵ, Δ) such that there exist $\tau_{mati} > 0$ and $0 < \tau_{mad} \leq \tau_{mati}$ such that for each $0 < \delta \leq \tau_{mati}$, each initial condition $x(0, 0) \in \mathbb{R}^{n_x}$, $e(0, 0) \in \mathbb{R}^{n_e}$, $s(0, 0) \in \mathbb{R}^{n_e}$, $\tau(0, 0) \in \mathbb{R}_{\geq 0}$, $\kappa(0, 0) \in \mathbb{N}$, $\ell(0, 0) \in \{0, 1\}$ with $(\ell(0, 0) = 0 \wedge \tau(0, 0) \in [0, \tau_{mati}]) \vee (\ell(0, 0) = 1 \wedge \tau(0, 0) \in [0, \tau_{mad}])$, $|x(0, 0)| \leq \Delta$, $|e(0, 0)| \leq \Delta$, $|s(0, 0)| \leq \Delta$ and each corresponding solution we have

$$|(x(t, j), e(t, j), s(t, j))| \leq \max\left\{\beta\left(|(x(0, 0), e(0, 0), s(0, 0))|, t, \delta j\right)\ ,\ \epsilon\right\}, \tag{14}$$

for all (t, j) in the solution's domain. ■

In the presence of disturbance inputs w in $\mathcal{H}_{NCS}$ we might be interested in reducing its influence on a particular controlled output variable

$$z = q(x, w) \tag{15}$$

in terms of the induced $\mathcal{L}_p$ gain, as formally defined below. The hybrid model $\mathcal{H}_{NCS}$ expanded with the output equation (15) is denoted by $\mathcal{H}^z_{NCS}$.

Definition III.4 Consider $p \in \mathbb{N}$ with $p \geq 1$ and let $\theta \geq 0$ be given. The hybrid system $\mathcal{H}^z_{NCS}$ is said to be $\mathcal{L}_p$ stable with gain θ, if there is a $\mathcal{K}_\infty$-function S such that for any $0 < \delta \leq \tau_{mati}$, any input $w \in \mathcal{L}_p$ and any initial condition $x(0, 0) \in \mathbb{R}^{n_x}$, $e(0, 0) \in \mathbb{R}^{n_e}$, $s(0, 0) \in \mathbb{R}^{n_e}$, $\tau(0, 0) \in \mathbb{R}_{\geq 0}$, $\kappa(0, 0) \in \mathbb{N}$, $\ell(0, 0) \in \{0, 1\}$ with $(\ell(0, 0) = 0 \wedge \tau(0, 0) \in [0, \tau_{mati}]) \vee (\ell(0, 0) = 1 \wedge \tau(0, 0) \in [0, \tau_{mad}])$, each corresponding solution to $\mathcal{H}^z_{NCS}$ satisfies

$$\|z\|_p \leq S\left(|(x(0, 0), e(0, 0), s(0, 0))|\right) + \theta\|w\|_p. \tag{16}$$

IV. Stability and performance analysis

In this section we focus on the analysis of UGAS and UGES, USPAS, and $\mathcal{L}_p$ stability.

A. Stability analysis

In order to guarantee UGAS or UGES, we assume the existence of a Lyapunov function $\widetilde{W}(\kappa,\ell,e,s)$ for the reset equations (11) and (12) satisfying

$$\begin{aligned}\widetilde{W}(\kappa+1,1,e,h(\kappa,e)-e) &\leq \lambda\widetilde{W}(\kappa,0,e,s) && \text{(17a)}\\ \widetilde{W}(\kappa,0,s+e,-s-e) &\leq \widetilde{W}(\kappa,1,e,s) && \text{(17b)}\end{aligned}$$

for all $\kappa\in\mathbb{N}$ and all $s,e\in\mathbb{R}^{n_e}$ and the bounds

$$\underline{\beta}_W(|(e,s)|)\leq\widetilde{W}(\kappa,\ell,e,s)\leq\overline{\beta}_W(|(e,s)|) \tag{18}$$

for all $\kappa\in\mathbb{N}$, $\ell\in\{0,1\}$ and $s,e\in\mathbb{R}^{n_e}$ for some functions $\underline{\beta}_W$ and $\overline{\beta}_W\in\mathcal{K}_\infty$ and $0\leq\lambda<1$.

In Section V we will show how a function $\widetilde{W}$ satisfying (17)-(18) can be derived from the generally accepted conditions on the protocol h as used for the delay-free case in [5], [38]. To solve Problem II.3, we extend (17) and (18) to the following condition.

Condition IV.1 There exist a function $\widetilde{W}:\mathbb{N}\times\{0,1\}\times\mathbb{R}^{n_e}\times\mathbb{R}^{n_e}\to\mathbb{R}_{\geq 0}$ with $\widetilde{W}(\kappa,\ell,\cdot,\cdot)$ locally Lipschitz for all $\kappa\in\mathbb{N}$ and $\ell\in\{0,1\}$, a locally Lipschitz function $\widetilde{V}:\mathbb{R}^{n_x}\to\mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$-functions $\underline{\beta}_V$, $\overline{\beta}_V$, $\underline{\beta}_W$ and $\overline{\beta}_W$, continuous functions $H_i:\mathbb{R}^{n_x}\to\mathbb{R}_{\geq 0}$, positive definite functions ρ_i and σ_i and constants $L_i\geq 0$, $\gamma_i>0$, for $i=0,1$, and $0\leq\lambda<1$ such that:

- for all $\kappa\in\mathbb{N}$ and all $s,e\in\mathbb{R}^{n_e}$ (17) holds and (18) holds for all $\ell\in\{0,1\}$;
- for all $\kappa\in\mathbb{N}$, $\ell\in\{0,1\}$, $s\in\mathbb{R}^{n_e}$, $x\in\mathbb{R}^{n_x}$ and almost all $e\in\mathbb{R}^{n_e}$ it holds that
$$\left\langle\frac{\partial\widetilde{W}(\kappa,\ell,e,s)}{\partial e},g(x,e,0)\right\rangle\leq L_\ell\widetilde{W}(\kappa,\ell,e,s)+H_\ell(x); \tag{19}$$
- for all $\kappa\in\mathbb{N}$, $\ell\in\{0,1\}$, $s,e\in\mathbb{R}^{n_e}$ and almost all $x\in\mathbb{R}^{n_x}$
$$\begin{aligned}\langle\nabla\widetilde{V}(x),f(x,e,0)\rangle\leq{}&-\rho_\ell(|x|)-H_\ell^2(x)\\&-\sigma_\ell(\widetilde{W}(\kappa,\ell,e,s))+\gamma_\ell^2\widetilde{W}^2(\kappa,\ell,e,s)\end{aligned} \tag{20}$$
and
$$\underline{\beta}_V(|x|)\leq\widetilde{V}(x)\leq\overline{\beta}_V(|x|). \tag{21}$$

The inequalities (19) and (20) are similar in nature to the delay-free situation as studied in [5] and are directly related to the $\mathcal{L}_2$ gain conditions from $\widetilde{W}$ to H_ℓ as adopted in [38]. Although these conditions may seem difficult to obtain at first sight, this is not the case. We will demonstrate this in Section VI, where the complete computational set-up for determining the parameters in the above conditions is provided. See also Remark V.2 for a more detailed discussion.

Consider now the differential equations

$$\begin{aligned}\dot{\phi}_0 &= -2L_0\phi_0-\gamma_0(\phi_0^2+1) && \text{(22a)}\\ \dot{\phi}_1 &= -2L_1\phi_1-\gamma_0\Big(\phi_1^2+\frac{\gamma_1^2}{\gamma_0^2}\Big), && \text{(22b)}\end{aligned}$$

where $L_\ell\geq 0$ and $\gamma_\ell>0$, $\ell=0,1$ are the real constants as given in Condition IV.1. Observe that the solutions to these differential equations are strictly decreasing as long as $\phi_\ell(\tau)\geq 0$, $\ell=0,1$.

Theorem IV.2 *Consider the system $\mathcal{H}_{NCS}$ that satisfies Condition IV.1. Suppose $\tau_{mati}\geq\tau_{mad}\geq 0$ satisfy*

$$\begin{aligned}\phi_0(\tau) &\geq \lambda^2\phi_1(0) \text{ for all } 0\leq\tau\leq\tau_{mati} && \text{(23a)}\\ \phi_1(\tau) &\geq \phi_0(\tau) \text{ for all } 0\leq\tau\leq\tau_{mad} && \text{(23b)}\end{aligned}$$

for solutions ϕ_0 and ϕ_1 of (22) corresponding to certain chosen initial conditions $\phi_\ell(0)>0$, $\ell=0,1$, with $\phi_1(0)\geq\phi_0(0)\geq\lambda^2\phi_1(0)\geq 0$, $\phi_0(\tau_{mati})>0$ and λ as in Condition IV.1. Then for the system $\mathcal{H}_{NCS}$ with $w=0$ the set $\mathcal{E}$ is UGAS. If in addition, there exist strictly positive real numbers b_1, b_2, c_1, c_2 and c_3 such that $\underline{\beta}_W(r)=b_1r$, $\overline{\beta}_W(r)=b_2r$, $\underline{\beta}_V(r)=c_1r^2$, $\overline{\beta}_V(r)=c_2r^2$, $\rho_i(r)\geq c_3r^2$ and $\sigma_i(r)\geq c_3r^2$, $i=0,1$, then this set is UGES. ■

Proof: The solutions ϕ_ℓ to the differential equations (22) are scaled as $\phi_0=\widetilde{\phi}_0$ and $\phi_1=\frac{\gamma_1}{\gamma_0}\widetilde{\phi}_1$ for new functions $\widetilde{\phi}_\ell$, $\ell=0,1$. The differential equations (22) and the conditions (23) transform into

$$\dot{\widetilde{\phi}}_\ell=-2L_\ell\widetilde{\phi}_\ell-\gamma_\ell(\widetilde{\phi}_\ell^2+1),\ \ell=0,1, \tag{24}$$

and

$$\begin{aligned}\gamma_0\widetilde{\phi}_0(\tau) &\geq \lambda^2\gamma_1\widetilde{\phi}_1(0) \text{ for all } \tau\in[0,\tau_{mati}] && \text{(25a)}\\ \gamma_1\widetilde{\phi}_1(\tau) &\geq \gamma_0\widetilde{\phi}_0(\tau) \text{ for all } \tau\in[0,\tau_{mad}]. && \text{(25b)}\end{aligned}$$

We consider now the function

$$U(\xi)=\widetilde{V}(x)+\gamma_\ell\widetilde{\phi}_\ell(\tau)\widetilde{W}^2(\kappa,\ell,e,s) \tag{26}$$

and show that this constitutes a suitable Lyapunov function for the system $\mathcal{H}_{NCS}$, which can be used to conclude UGAS and UGES under the stated conditions.

Below, by abuse of notation, we consider the quantity $\langle\nabla U(\xi),F(\xi,w)\rangle$ with $F(\xi,w):=(f(x,e,w),g(x,e,w),0,0,1,0)$ as in (9) even though $\widetilde{W}$ is not differentiable with respect to κ and ℓ. This is justified since the components in $F(\xi,w)$ corresponding to κ and ℓ are zero. We will first show that $U(\xi^+)\leq U(\xi)$ whenever the system $\mathcal{H}_{NCS}$ with $w=0$ resets.

When $\ell=0$ and a jump occurs, we have that $\tau\in[0,\tau_{mati}]$ and obtain, using (11), that

$$\begin{aligned}U(\xi^+)&=\widetilde{V}(x)+\gamma_1\widetilde{\phi}_1(0)\widetilde{W}^2(\kappa+1,1,e,h(\kappa,e)-e)\\&\overset{\text{(17a),(25a)}}{\leq}\widetilde{V}(x)+\gamma_0\widetilde{\phi}_0(\tau)\widetilde{W}^2(\kappa,0,e,s)=U(\xi).\end{aligned}$$

Similarly, when $\ell=1$ we have that $\tau\in[0,\tau_{mad}]$ and obtain, using (12),

$$\begin{aligned}U(\xi^+)=&\widetilde{V}(x)+\gamma_0\widetilde{\phi}_0(\tau)\widetilde{W}^2(\kappa,0,s+e,-s-e)\\&\overset{\text{(17b),(25b)}}{\leq}\widetilde{V}(x)+\gamma_1\widetilde{\phi}_1(\tau)\widetilde{W}^2(\kappa,1,e,s)=U(\xi).\end{aligned}$$

We also have, for all (τ,κ,ℓ) and almost all (x,e,s), that

$$\begin{aligned}&\langle \nabla U(\xi), F(\xi,0)\,\rangle \leq -\rho_\ell(|x|) - \sigma_\ell(\widetilde{W}(\kappa,\ell,e,s)) - H_\ell^2(x)\\ &\quad +\gamma_\ell^2 \widetilde{W}^2(\kappa,\ell,e,s)\\ &\quad +2\gamma_\ell\widetilde{\phi}_\ell(\tau)\widetilde{W}(\kappa,\ell,e,s)[L_\ell\widetilde{W}(\kappa,\ell,e,s)+H_\ell(x)]\\ &\quad -\gamma_\ell\widetilde{W}^2(\kappa,\ell,e,s)[2L_\ell\widetilde{\phi}_\ell(\tau)+\gamma_\ell(\widetilde{\phi}_\ell^2(\tau)+1)]\\ &\leq -\rho_\ell(|x|) - \sigma_\ell(\widetilde{W}(\kappa,\ell,e,s)) - H_\ell^2(x)\\ &\quad +2\gamma_\ell\widetilde{\phi}_\ell(\tau)\widetilde{W}(\kappa,\ell,e,s)H_\ell(x) - \gamma_\ell^2\widetilde{W}^2(\kappa,\ell,e,s)\widetilde{\phi}_\ell^2(\tau)\\ &\leq -\rho_\ell(|x|) - \sigma_\ell(\widetilde{W}(\kappa,\ell,e,s))\ .\end{aligned}$$

The above proves that U forms a Lyapunov function for the system $\mathcal{H}_{NCS}$ with $w=0$ and UGAS and UGES follows now using standard Lyapunov arguments as are provided in [5] for the delay-free case. This completes the proof. ∎

From the above theorem quantitative numbers for τ_{mati} and τ_{mad} can be obtained by constructing the solutions to (22) for certain initial conditions. By computing the τ value of the intersection of ϕ_0 and the constant line $\lambda^2\phi_1(0)$ provides τ_{mati} according to (23a), while the intersection of ϕ_0 and ϕ_1 gives a value for τ_{mad} due to (23b). Different values of the initial conditions $\phi_0(0)$ and $\phi_1(0)$ lead, of course, to different solutions ϕ_0 and ϕ_1 of the differential equations (22) and thus also to different Lyapunov functions in (26). Hence, a *continuum* of Lyapunov functions is obtained by varying the initial conditions $\phi_0(0)$ and $\phi_1(0)$. Moreover, each different choice of $\phi_0(0)$ and $\phi_1(0)$ provides different τ_{mati} and τ_{mad}. As a result, tradeoff curves between τ_{mati} and τ_{mad} can be obtained that indicate when stability of the NCS is still guaranteed. This will be illustrated in Section VI, where the complete analysis framework will be illustrated on a benchmark example.

Remark IV.3 The existence of strictly positive τ_{mati} and τ_{mad} such that (23) holds follows from $0\leq\lambda<1$ in (17) as this implies the existence of $\phi_\ell(0)>0$, $\ell=0,1$ with $\phi_1(0)>\phi_0(0)>\lambda^2\phi_1(0)$. Moreover, when using $\phi_0(0)=\phi_1(0)=\lambda^{-1}$ in case $\lambda\neq 0$, we recover the explicit formula for the MATI obtained in [5] (which improved earlier results in [38], [39]) in the sense that we have

$$\tau_{mati}=\begin{cases}\frac{1}{L_0 r}\arctan(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}(\frac{\gamma_0}{L_0})+1+\lambda}), & \gamma_0>L_0\\ \frac{1-\lambda}{L_0(1+\lambda)}, & \gamma_0=L_0\\ \frac{1}{L_0 r}\text{arctanh}(\frac{r(1-\lambda)}{2\frac{\lambda}{1+\lambda}(\frac{\gamma_0}{L_0})+1+\lambda}), & \gamma_0<L_0,\end{cases} \tag{27}$$

where $r=\sqrt{|(\frac{\gamma_0}{L_0})^2-1|}$. Hence, for the delay-free case ($\tau_{mad}=0$) we recover the results in [5] as a special case.

B. Uniform semiglobal practical stability

Under a version of Condition IV.1, which is weaker at various points, we can obtain semi-global practical asymptotical stability results with respect to τ_{mati} and τ_{mad} for the zero-input system of $\mathcal{H}_{NCS}$ ($w=0$).

Condition IV.4 There are a function $\widetilde{W}:\mathbb{N}\times\{0,1\}\times\mathbb{R}^{n_e}\times\mathbb{R}^{n_e}\to\mathbb{R}_{\geq 0}$ with $\widetilde{W}(\kappa,\ell,\cdot,\cdot)$ locally Lipschitz for all $\kappa\in\mathbb{N}$ and $\ell\in\{0,1\}$, $\mathcal{K}_\infty$-functions $\underline{\beta}_W$, $\overline{\beta}_W$, $\widetilde{\alpha}$, and $0\leq\lambda<1$ s.t.

- for all $\kappa\in\mathbb{N}$ and all $s,e\in\mathbb{R}^{n_e}$ (17) holds and (18) holds for all $\ell\in\{0,1\}$;
- for all $\kappa\in\mathbb{N}$, $\ell\in\{0,1\}$, all $s\in\mathbb{R}^{n_e}$ and almost all $e\in\mathbb{R}^{n_e}$ it holds that

$$\left|\frac{\partial\widetilde{W}(\kappa,\ell,e,s)}{\partial e}\right|\leq\widetilde{\alpha}(|(e,s)|). \tag{28}$$

- The origin of the networked-free and zero-input system $\dot{x}=f(x,0,0)$ is globally asymptotically stable.

Theorem IV.5 *Consider the system $\mathcal{H}_{NCS}$ with $w=0$ that satisfies Condition IV.4. The set $\mathcal{E}$ is USPAS with respect to τ_{mati} and τ_{mad}.* ∎

Proof: We will prove the USPAS property with respect to τ_{mad} and τ_{mati} considering $\tau_{mad}=\tau_{mati}$. Since $\dot{x}=f(x,0,0)$ is globally asymptotically stable and f is continuous, we can apply a converse Lyapunov theorem [7] that yields the existence of a continuously differentiable function $V:\mathbb{R}^{n_x}\to\mathbb{R}_{\geq 0}$ and a $\varrho_V\in\mathcal{K}_\infty$ such that

$$\langle\nabla V(x),f(x,0,0)\rangle\leq-\varrho_V(|x|). \tag{29}$$

We consider now the Lyapunov function

$$U(\xi)=V(x)+\exp(-\frac{\sigma\tau}{\tau_{mati}})\widetilde{W}(\kappa,\ell,e,s) \tag{30}$$

using ξ and $F(\xi,w)$ as in the proof of Theorem IV.2 and taking the constant $\sigma>0$ such that $e^{-\sigma}=\lambda$ with λ as in (17). Now exploiting (17), yields that in case of $\ell=0$ and $0\leq\tau\leq\tau_{mati}$

$$\begin{aligned}U(\xi^+)&=V(x)+\widetilde{W}(\kappa+1,1,e,h(\kappa,e)-e)\\ &\leq V(x)+\exp(-\sigma)\widetilde{W}(\kappa,0,e,s)\\ &\leq V(x)+\exp(\frac{-\sigma\tau}{\tau_{mati}})\widetilde{W}(\kappa,0,e,s)=U(\xi).\end{aligned}$$

In case of $\ell=1$ and $0\leq\tau\leq\tau_{mad}=\tau_{mati}$, we have

$$\begin{aligned}U(\xi^+)&=V(x)+\exp(\frac{-\sigma\tau}{\tau_{mati}})\widetilde{W}(\kappa+1,0,s+e,-s-e)\\ &\leq V(x)+\exp(\frac{-\sigma\tau}{\tau_{mati}})\widetilde{W}(\kappa,1,e,s)=U(\xi).\end{aligned}$$

Finally, considering the evolution of U along the flow of $\mathcal{H}_{NCS}$ with $w=0$ gives using (18) and (28) for $0\leq\tau\leq\tau_{mati}=\tau_{mad}$, all x,s,κ,ℓ and almost all e

$$\begin{aligned}&\langle\nabla U(\xi),F(\xi,0)\rangle=\\ &\quad\langle\nabla V(x),f(x,0,0)\rangle+\langle\nabla V(x),f(x,e,0)-f(x,0,0)\rangle\\ &\quad+\exp(\tfrac{-\sigma\tau}{\tau_{mati}})\left\langle\tfrac{\partial\widetilde{W}(\kappa,\ell,e,s)}{\partial e},g(x,e,0)\right\rangle\\ &\quad-\tfrac{\sigma}{\tau_{mati}}\exp(\tfrac{-\sigma\tau}{\tau_{mati}})\widetilde{W}(\kappa,\ell,e,s)\\ &\leq-\varrho_V(|x|)-\tfrac{\sigma}{\tau_{mati}}\exp(-\sigma)\underline{\beta}_W(|(e,s)|)\varphi(x,e,s),\end{aligned}$$

where

$$\begin{aligned}&\varphi(x,e,s):=\\ &\quad\langle\nabla V(x),f(x,e,0)-f(x,0,0)\rangle+\widetilde{\alpha}(|(e,s)|)|g(x,e,0)|.\end{aligned}$$

Note that φ is a continuous function due to the continuous differentiability of V and the continuity of f and g. Moreover, $\varphi(x,0,0)=0$ for all $x\in\mathbb{R}^{n_x}$. Using now Lemma 2.1 in [46] guarantees, for each pair of strictly positive numbers $0 < \bar{\delta} < \bar{\Delta}$, the existence of $\tau_{mati} = \tau_{mad} > 0$ such that for almost all ξ in $\{\xi \mid \bar{\delta} \leq |(x,e,s)| \leq \bar{\Delta},\ \tau \in [0,\tau_{mati}],\ \kappa \in \mathbb{N},\ \ell \in \{0,1\}\}$it holds that $\langle \nabla U(\xi), F(\xi,0)\rangle \leq -0.5\varrho_V(|x|) - 0.5\underline{\beta}_W(|(e,s)|)$. In a similar way as in the proof of Theorem IV.2, the USPAS property can now be derived by straightforward reasoning. ■

C. $\mathcal{L}_p$ stability analysis

For the $\mathcal{L}_p$ stability analysis, we replace Condition IV.1 by the following.

Condition IV.6 There exist a function $\widetilde{W} : \mathbb{N}\times\{0,1\}\times\mathbb{R}^{n_e}\times\mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ with $\widetilde{W}(\kappa,\ell,\cdot,\cdot)$ locally Lipschitz for all $\kappa \in \mathbb{N}$ and $\ell \in \{0,1\}$, a locally Lipschitz function $\widetilde{V} : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$-functions $\underline{\beta}_V$, $\overline{\beta}_V$, $\underline{\beta}_W$ and $\overline{\beta}_W$, continuous functions $H_i : \mathbb{R}^{n_x}\times\mathbb{R}^{n_w} \to \mathbb{R}_{\geq 0}$, and constants $L_i \geq 0$, $\gamma_i > 0$, for $i = 0,1$, and $0 \leq \lambda < 1$ such that:

- for all $\kappa \in \mathbb{N}$ and all $s, e \in \mathbb{R}^{n_e}$ (17) holds and (18) holds for all $\ell \in \{0,1\}$;
- for all $\kappa \in \mathbb{N}$, $\ell \in \{0,1\}$, $s \in \mathbb{R}^{n_e}$, $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$ and almost all $e \in \mathbb{R}^{n_e}$ it holds that
$$\left\langle \frac{\partial \widetilde{W}(\kappa,\ell,e,s)}{\partial e}, g(x,e,w)\right\rangle \leq L_\ell \widetilde{W}(\kappa,\ell,e,s) + H_\ell(x,w); \tag{31}$$
- for all $\kappa \in \mathbb{N}$, $\ell \in \{0,1\}$, $s, e \in \mathbb{R}^{n_e}$, $w \in W$ and almost all $x \in \mathbb{R}^{n_x}$
$$\begin{aligned}&\langle \nabla \widetilde{V}(x), f(x,e,w)\rangle \leq \\ &-H_\ell^2(x,w)\gamma_\ell^2 \widetilde{W}^2(\kappa,\ell,e,s) + \tilde{\mu}(\theta^p|w|^p - |q(x,w)|^p)\end{aligned} \tag{32}$$
for some $\tilde{\mu} > 0$ and $\theta \geq 0$, and
$$\underline{\beta}_V(|x|) \leq \widetilde{V}(x) \leq \overline{\beta}_V(|x|). \tag{33}$$

The main difference between Condition IV.6 and Condition IV.1 is the presence of the disturbance input w and the dependence of H_ℓ on both w and x instead of on x only. Moreover, comparing (20) and (32), we observe the additional term $\tilde{\mu}(\theta^p|w|^p - |q(x,w)|^p)$ in the right-hand side of (32), which is needed to obtain a bound θ on the $\mathcal{L}_p$ gain between w and z.

Theorem IV.7 *Consider the system $\mathcal{H}^z_{NCS}$ that satisfies Condition IV.6. Suppose $\tau_{mati} \geq \tau_{mad} \geq 0$ satisfy* (23) *for solutions ϕ_0 and ϕ_1 of* (22) *corresponding to certain initial conditions $\phi_\ell(0) > 0$, $\ell = 0,1$, with $\phi_1(0) \geq \phi_0(0) \geq \lambda^2\phi_1(0) \geq 0$, $\phi_0(\tau_{mati}) > 0$ and λ as in Condition IV.6. Then the system $\mathcal{H}^z_{NCS}$ is $\mathcal{L}_p$ stable with gain θ.* ■

Proof: In a similar manner as in the proof of Theorem IV.2, we obtain that the function U in (26) satisfies
$$U(\xi^+) \leq U(\xi) \tag{34}$$
whenever there is a reset and
$$\langle \nabla U(\xi), F(\xi,w)\rangle \leq \tilde{\mu}(\theta^p|w|^p - |q(x,w)|^p) \tag{35}$$
during the flow of the hybrid system $\mathcal{H}^z_{NCS}$. Since $\tilde{\mu} > 0$ we can without loss of generality assume that $\tilde{\mu} = 1$ by scaling $U(\xi)$ to $\frac{1}{\tilde{\mu}}U(\xi)$. Let ξ be a solution to $\mathcal{H}^z_{NCS}$ with corresponding output z for initial condition $\xi(0,0)$ and input $w \in \mathcal{L}_p$. Denote the hybrid time domain of ξ by dom $\xi = \bigcup_{j=0}^{J-1}([t_j,t_{j+1}],j)$ with $t = t_J$ and J possibly ∞ and/or $t_J = \infty$. Let z be the corresponding output also considered on dom ξ. Reformulating (34) by using the hybrid time domain dom ξ gives that for each $j = 0,\ldots,J-1$ we have
$$U(\xi(t_{j+1},j+1)) \leq U(\xi(t_{j+1},j)) \tag{36}$$
and integrating (35) yields for each $(t',j) \in$ dom ξ and $(t'',j) \in$ dom ξ with $t' \leq t''$ that
$$\int_{t'}^{t''} |z(t)|^p dt \leq -U(\xi(t'',j)) + U(\xi(t',j)) + \theta^p \int_{t'}^{t''} |w(t)|^p dt. \tag{37}$$
Computing now the $\mathcal{L}_p$ norm of z gives
$$\begin{aligned}\|z\|_p^p &= \sum_{j=0}^{J-1}\int_{t_j}^{t_{j+1}} |z(t,j)|^p dt \\ &\overset{(37)}{\leq} \sum_{j=0}^{J-1}\Big[-U(\xi(t_{j+1},j)) + U(\xi(t_j,j)) \\ &\qquad\qquad + \theta^p\int_{t_j}^{t_{j+1}} |w(t)|^p dt\Big] \\ &= U(\xi(0,0)) - U(\xi(t_J,J-1)) + \theta^p\|w\|_p^p \\ &+ \sum_{j=0}^{J-2}[U(\xi(t_{j+1},j+1)) - U(\xi(t_{j+1},j))] \\ &\overset{(36)}{\leq} U(\xi(0,0)) + \theta^p\|w\|_p^p \leq (U(\xi(0,0))^{\frac{1}{p}} + \theta\|w\|_p)^p.\end{aligned}$$
As a consequence, we have that $\|z\|_p \leq U(\xi(0,0))^{\frac{1}{p}} + \theta\|w\|_p$. Due to the bounds (33) and (18) on $\widetilde{V}$ and $\widetilde{W}$, respectively, we can bound U as $U(\xi) \leq \alpha_U(|(x,e,s)|)$ for a suitable $\alpha_U \in \mathcal{K}_\infty$. This proves that the system $\mathcal{H}^z_{NCS}$ is $\mathcal{L}_p$ stable with gain θ, where the $\mathcal{K}_\infty$ function S in (16) can be taken as $S(r) = (\alpha_U(r))^{\frac{1}{p}}$. ■

Remark IV.8 Essentially, in the above proof we constructed a so-called *storage function* [53] given by U as in (26) for the system $\mathcal{H}^z_{NCS}$ with supply rate $\tilde{\mu}(\theta^p|w|^p - |z|^p)$ during the flow phases and supply equal to 0 during the resets. In the analysis of passivity and $\mathcal{L}_p$ stability these concepts are exploited for various classes of systems in, for instance, [4], [19], [49], [53].

V. Constructing Lyapunov and storage functions

In this section we will construct Lyapunov and storage functions $\widetilde{V}$ and $\widetilde{W}$ as in Condition IV.1, Condition IV.6 and Condition IV.4 from the commonly adopted assumptions in [5], [38], [39], [51], [52] for the delay-free case. We will start with constructing Lyapunov and storage functions as in Condition IV.1 and Condition IV.6, respectively. For the delay-free case, one considers in [5], [38] protocols satisfying the following condition:

Condition V.1 The protocol given by h is UGES (uniformly globally exponentially stable), meaning that there exists a function $W : \mathbb{N} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument such that

$$\underline{\alpha}_W |e| \leq W(\kappa, e) \leq \overline{\alpha}_W |e| \tag{38a}$$
$$W(\kappa+1, h(\kappa, e)) \leq \lambda W(\kappa, e) \tag{38b}$$

for constants $0 < \underline{\alpha}_W \leq \overline{\alpha}_W$ and $0 < \lambda < 1$. ■

Additionally we assume here that

$$W(\kappa+1, e) \leq \lambda_W W(\kappa, e) \tag{39}$$

for some constant[4] $\lambda_W \geq 1$ and that for almost all $e \in \mathbb{R}^{n_e}$ and all $\kappa \in \mathbb{N}$

$$\left| \frac{\partial W}{\partial e}(\kappa, e) \right| \leq M_1 \tag{40}$$

for some constant $M_1 > 0$. For all protocols discussed in [5], [38], [39], [51], [52] such constants exist. In Lemma V.4 below, we specify appropriate values for these constants in case of the often used Round Robin (RR) and the Try-Once-Discard (TOD) protocols (see [38], [52] for their definitions). We also assume the growth condition on the NCS model (7)

$$|g(x, e, 0)| \leq m_x(x) + M_e |e|, \tag{41}$$

where $m_x : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, if we are looking for a *Lyapunov function* establishing UGAS, and

$$|g(x, e, w)| \leq m(x, w) + M_e |e|, \tag{42}$$

where $m : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}_{\geq 0}$, if we are looking for a *storage function* establishing $\mathcal{L}_p$ stability. In both cases $M_e \geq 0$ is a constant. Building upon slightly modified conditions as used for the *delay-free case* in [5] given by the existence of a locally Lipschitz continuous function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ satisfying the bounds

$$\underline{\alpha}_V(|x|) \leq V(x) \leq \overline{\alpha}_V(|x|) \tag{43}$$

for some $\mathcal{K}_\infty$-functions $\underline{\alpha}_V$ and $\overline{\alpha}_V$, and, in case of constructing a Lyapunov function, the condition

$$\langle \nabla V(x), f(x, e, 0) \rangle \leq -m_x^2(x) - \rho(|x|) + (\gamma^2 - \varepsilon) W^2(\kappa, e) \tag{44}$$

for almost all $x \in \mathbb{R}^{n_x}$ and all $e \in \mathbb{R}^{n_e}$ with $\rho \in \mathcal{K}_\infty$, and, in case of constructing a storage function, the condition

$$\begin{aligned}\langle \nabla V(x), f(x, e, w) \rangle \leq{}& -m^2(x, w) + \\ &+ \gamma^2 W^2(\kappa, e) + \mu(\theta^p |w|^p - |q(x, w)|^p)\end{aligned} \tag{45}$$

for almost all $x \in \mathbb{R}^{n_x}$ and all $e \in \mathbb{R}^{n_e}$ and all $w \in \mathbb{R}^{n_w}$, we can derive functions $\widetilde{V}$ and $\widetilde{W}$ satisfying Condition IV.1 and Condition IV.6, respectively. The constants in (44) satisfy $0 < \varepsilon < \max\{\gamma^2, 1\}$, where $\varepsilon > 0$ is sufficiently small.

Remark V.2 Condition V.1 and inequality (44) are essentially the same as in [5] with $H(x) = m_x(x)$. The constant $\varepsilon > 0$ is selected small to sacrifice only a little of the $\mathcal{L}_2$ gain from W to m_x. In [38, Thm. 4] the inequality (44) was actually formulated in terms of an $\mathcal{L}_2$ gain, while we use a Lyapunov-based formulation here. As $\mathcal{L}_2$ gains are established often using Lyapunov functions, this seems to be a natural reformulation (see also [5, Rem. 2]). The only additional condition we add here is (39), which holds for all protocols considered in [5], [11], [38], [39], [51], [52] as is demonstrated in Lemma V.4 below for the RR and TOD protocols. ■

[4] In principle this constant can be taken non-negative. However, as all protocols available in the literature satisfy $\lambda_W \geq 1$, we take $\lambda_W \geq 1$ to reduce some notational burden later.

Theorem V.3 *Consider the systems $\mathcal{H}_{NCS}$ and $\mathcal{H}^z_{NCS}$, respectively, such that*

- *Condition V.1,* (39) *with $\lambda_W \geq 1$ and* (40) *with constant $M_1 > 0$ hold;*
- (41) *is satisfied for some function $m_x : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ (and* (42) *for some function $m : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}_{\geq 0}$, respectively) and $M_e \geq 0$.*
- *there exists a locally Lipschitz continuous function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ satisfying the bounds* (43) *for some $\mathcal{K}_\infty$-functions $\underline{\alpha}_V$, $\overline{\alpha}_V$, and* (44) *with $\gamma > 0$ and $0 < \varepsilon <$* $\max\{\gamma^2, 1\}$ *(and* (45) *with $\mu > 0$, $\theta \geq 0$ and $\gamma > 0$, respectively).*

Then, the functions given by

$$\widetilde{W}(\kappa, \ell, e, s) := \begin{cases} \max\{W(\kappa, e), W(\kappa, e+s)\}, & \ell = 0, \\ \max\{\frac{\lambda}{\lambda_W} W(\kappa, e), W(\kappa, e+s)\}, & \ell = 1, \end{cases} \tag{46}$$

$$\widetilde{V}(x) = M_1^2 V(x) \tag{47}$$

satisfy Condition IV.1 (and Condition IV.6, respectively) with $\underline{\beta}_W(r) = \underline{\beta}_W r$, $\overline{\beta}_W(r) = \overline{\beta}_W r$, $\underline{\beta}_V = M_1^2 \underline{\alpha}_V$, $\overline{\beta}_V = M_1^2 \overline{\alpha}_V$, $\sigma_0(r) = \varepsilon M_1^2 r^2$, $\sigma_1(r) = \varepsilon M_1^2 \frac{\lambda_W^2}{\lambda^2}$ and $\rho_\ell(r) = M_1^2 \rho(r)$, $H_\ell(x) = M_1 m_x(x)$ (and $H_\ell(x, w) = M_1 m(x, w)$, resp.), $\ell = 0, 1$, with λ as in Condition V.1,

$$L_0 = \frac{M_1 M_e}{\underline{\alpha}_W};\ L_1 = \frac{M_1 M_e \lambda_W}{\lambda \underline{\alpha}_W};\ \gamma_0 = M_1 \gamma;\ \gamma_1 = \frac{M_1 \gamma \lambda_W}{\lambda}, \tag{48}$$

and some positive constants $\underline{\beta}_W$, $\overline{\beta}_W$ (and $\tilde{\mu} = M_1 \mu$, respectively). ■

Proof: We only prove the theorem for establishing Condition IV.1, as the case for Condition IV.6 is analogous. The condition (17a) with $\widetilde{W}$ of the form (46) is equivalent to

$$\max\left\{ \frac{\lambda}{\lambda_W} W(\kappa+1, e), W(\kappa+1, h(\kappa, e)) \right\} \leq \lambda \max\{W(\kappa, e), W(\kappa, e+s)\}.$$

Using (38b) and (39), this follows trivially. The condition (17b) is identical to

$$\max\{W(\kappa, s+e), W(\kappa, 0)\} \leq \max\left\{ \frac{\lambda}{\lambda_W} W(\kappa, e), W(\kappa, e+s) \right\},$$

which is true as $W(\kappa, 0) = 0$. Based on (46) we obtain four cases for (19):

- Case 1: $\ell = 0$ and $W(\kappa, e) \geq W(\kappa, e+s)$

$$\begin{aligned} &\left\langle \tfrac{\partial \widetilde{W}(\kappa,0,e,s)}{\partial e}, g(x,e,0) \right\rangle = \left\langle \tfrac{\partial W(\kappa,e)}{\partial e}, g(x,e,0) \right\rangle \\ &\overset{(40),(41)}{\leq} M_1(m_x(x) + M_e|e|) \\ &\overset{(38a)}{\leq} M_1 m_x(x) + M_1 \tfrac{M_e}{\underline{\alpha}_W} W(\kappa, e) \\ &= M_1 m_x(x) + M_1 \tfrac{M_e}{\underline{\alpha}_W} \widetilde{W}(\kappa, 0, e, s). \end{aligned} \tag{49}$$

- Case 2: $\ell = 0$ and $W(\kappa, e) \leq W(\kappa, e+s)$

$$\begin{aligned} &\left\langle \tfrac{\partial \widetilde{W}(\kappa,0,e,s)}{\partial e}, g(x,e,0) \right\rangle = \left\langle \tfrac{\partial W(\kappa,e+s)}{\partial e}, g(x,e,0) \right\rangle \\ &\overset{(40),(41),(38a)}{\leq} M_1 m_x(x) + M_1 \tfrac{M_e}{\underline{\alpha}_W} W(\kappa, e) \\ &\leq M_1 m_x(x) + M_1 \tfrac{M_e}{\underline{\alpha}_W} \widetilde{W}(\kappa, 0, e, s). \end{aligned} \tag{50}$$

- Case 3: $\ell = 1$ and $\frac{\lambda}{\lambda_W} W(\kappa, e) \geq W(\kappa, e+s)$

$$\begin{aligned} &\left\langle \tfrac{\partial \widetilde{W}(\kappa,1,e,s)}{\partial e}, g(x,e,0) \right\rangle = \tfrac{\lambda}{\lambda_W} \left\langle \tfrac{\partial W(\kappa,e)}{\partial e}, g(x,e,0) \right\rangle \\ &\overset{(40),(41),(38a)}{\leq} \tfrac{\lambda}{\lambda_W} M_1 m_x(x) + M_1 \tfrac{M_e}{\underline{\alpha}_W} \tfrac{\lambda}{\lambda_W} W(\kappa, e) \\ &= \tfrac{\lambda}{\lambda_W} M_1 m_x(x) + M_1 \tfrac{M_e}{\underline{\alpha}_W} \widetilde{W}(\kappa, 1, e, s). \end{aligned} \tag{51}$$

- Case 4: $\ell = 1$ and $\frac{\lambda}{\lambda_W} W(\kappa, e) \leq W(\kappa, e+s)$

$$\begin{aligned} &\left\langle \tfrac{\partial \widetilde{W}(\kappa,1,e,s)}{\partial e}, g(x,e,0) \right\rangle = \left\langle \tfrac{\partial W(\kappa,e+s)}{\partial e}, g(x,e,0) \right\rangle \\ &\overset{(40),(41),(38a)}{\leq} M_1 m_x(x) + M_1 \tfrac{M_e}{\underline{\alpha}_W} W(\kappa, e) \\ &\leq M_1 m_x(x) + M_1 \tfrac{M_e \lambda_W}{\underline{\alpha}_W \lambda} \widetilde{W}(\kappa, 1, e, s). \end{aligned} \tag{52}$$

Hence, (19) holds for $\ell = 0, 1$ with $H_0(x) = H_1(x) = M_1 m_x(x)$ and L_0, L_1 as in (48). Here we used that $\frac{\lambda_W}{\lambda} \geq 1$ as $0 < \lambda < 1 \leq \lambda_W$.

Finally, to obtain (20) observe that for the case $\ell = 0$, we have due to (44)

$$\begin{aligned} &\langle \nabla V(x), f(x,e,0) \rangle \leq -\rho(|x|) - m_x^2(x) + (\gamma^2 - \varepsilon) W^2(\kappa, e) \\ &\qquad \leq -\rho(|x|) - m_x^2(x) + (\gamma^2 - \varepsilon) \widetilde{W}^2(\kappa, 0, e, s) \\ &= -\rho(|x|) - \varepsilon \widetilde{W}^2(\kappa, 0, e, s) - M_1^{-2} H_0^2(x) + \gamma^2 \widetilde{W}^2(\kappa, 0, e, s). \end{aligned} \tag{53}$$

Similarly, for $\ell = 1$ we obtain

$$\begin{aligned} &\langle \nabla V(x), f(x,e,0) \rangle \leq -\rho(|x|) - m_x^2(x) + (\gamma^2 - \varepsilon) W^2(\kappa, e) \\ &\leq -\rho(|x|) - m_x^2(x) + (\gamma^2 - \varepsilon) \tfrac{\lambda_W^2}{\lambda^2} \widetilde{W}^2(\kappa, 1, e, s) \\ &= -\rho(|x|) - \varepsilon \tfrac{\lambda_W^2}{\lambda^2} \widetilde{W}^2(\kappa, 1, e, s) \\ &\qquad - M_1^{-2} H_1^2(x) + \gamma^2 \tfrac{\lambda_W^2}{\lambda^2} \widetilde{W}^2(\kappa, 1, e, s). \end{aligned} \tag{54}$$

Take $\widetilde{V}(x) = M_1^2 V(x)$ and multiply the inequalities (53) and (54) by M_1^2, which give for $\ell = 0$

$$\begin{aligned} \langle \nabla \widetilde{V}(x), f(x,e,0) \rangle &\leq -M_1^2 \rho(|x|) - \varepsilon M_1^2 \widetilde{W}^2(\kappa, 0, e, s) \\ &\quad - H_0^2(x) + \gamma^2 M_1^2 \widetilde{W}^2(\kappa, 0, e, s) \end{aligned}$$

and for $\ell = 1$

$$\begin{aligned} \langle \nabla \widetilde{V}(x), f(x,e,0) \rangle &\leq -M_1^2 \rho(|x|) - \varepsilon \frac{\lambda_W^2}{\lambda^2} M_1^2 \widetilde{W}^2(\kappa, 1, e, s) \\ &\quad - H_1^2(x) + \gamma^2 \frac{\lambda_W^2}{\lambda^2} M_1^2 \widetilde{W}^2(\kappa, 1, e, s). \end{aligned}$$

Note that the bounds on $\widetilde{W}$ as in (18) with linear bounding functions $\underline{\beta}_W$ and $\overline{\beta}_W$ can be easily obtained from the fact that W satisfies (38a) with linear functions. This completes the proof. ■

To apply the above theorem for a given protocol we need to establish the values λ, M_1, λ_W, $\underline{\alpha}_W$ and $\overline{\alpha}_W$. The following lemma determines these constants for the well-known RR and TOD protocols. See [38], [52] for the exact definitions of these protocols.

Lemma V.4 *Let l denote the number of nodes in the network. For the RR protocol there is a $W_{RR} : \mathbb{N} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument satisfying* (38), (39) *and* (40) *with $\lambda_{RR} = \sqrt{\frac{l-1}{l}}$, $\underline{\alpha}_{W,RR} = 1$, $\overline{\alpha}_{W,RR} = \sqrt{l}$, $\lambda_{W,RR} = \sqrt{l}$ and $M_{1,RR} = \sqrt{l}$. For the TOD protocol there is a $W_{TOD} : \mathbb{N} \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz in its second argument satisfying* (38), (39) *and* (40) *with $\lambda_{TOD} = \sqrt{\frac{l-1}{l}}$, $\underline{\alpha}_{W,TOD} = \overline{\alpha}_{W,TOD} = 1$, $\lambda_{W,TOD} = 1$ and $M_{1,TOD} = 1$.* ■

Proof: The constants λ, $\underline{\alpha}_W$, $\overline{\alpha}_W$ are derived for both protocols in [38] and corresponding Lyapunov functions W. For the RR protocol $W_{RR}^2(i, e)$ can be taken $\sum_{j=1}^{n_e} a_i(j) e_j^2$ with $a_i(j) \in \{1, \ldots, l\}$, $j = 1, \ldots, n$, $i = 1, 2, \ldots$ and l is the number of nodes in the network, see Example 3 in [38]. This implies that

$$\begin{aligned} W_{RR}^2(i+1, e) &= \sum_{j=1}^{n_e} a_{i+1}(j) e_j^2 \leq \sum_{j=1}^{n_e} \frac{a_{i+1}(j)}{a_i(j)} a_i^2(j) e_j^2 \\ &\leq l \sum_{j=1}^{n_e} a_i(j) e_j^2 = \underbrace{l}_{=(\lambda_{W,RR})^2} W^2(i, e). \end{aligned}$$

Using $\frac{\partial W_{RR}^2(i,e)}{\partial e} = 2 W_{RR}(i,e) \frac{\partial W_{RR}(i,e)}{\partial e}$ and the identity $\frac{\partial W_{RR}^2(i,e)}{\partial e} = 2(a_i^2(1) e_1, \ldots, a_i^2(n_e) e_{n_e})^\top$, we obtain that $\left| \frac{\partial W_{RR}(i,e)}{\partial e} \right| = \frac{\sqrt{\sum_j a_i^2(j) e_j^2}}{\sqrt{\sum_j a_i(j) e_j^2}} \leq \sqrt{\max_j a_i(j)} = \sqrt{l} = M_{1,RR}$. Since $W_{TOD}(i, e) = |e|$ for the TOD protocol, $\lambda_{W,TOD} = M_{1,TOD} = 1$ follows from the above by taking $a_i(j) = 1$ for all i, j. ■

In the next theorem, we show how to obtain Condition IV.4, used for establishing USPAS in Theorem IV.5, from the commonly adopted conditions in the delay-free case. As the proof is similar in nature as the proof of Theorem V.3, it is omitted.

Theorem V.5 *Consider the system $\mathcal{H}_{NCS}$. Assume that Condition V.1 and* (39) *with $\lambda_W \geq 1$ hold. Moreover, assume that there exists a $\mathcal{K}_\infty$-function α such that*

$$\left| \frac{\partial W(\kappa, e)}{\partial e} \right| \leq \alpha(|e|) \tag{55}$$

for almost all $e \in \mathbb{R}^{n_e}$ and all $\kappa \in \mathbb{N}$. Assume that the origin of the networked-free and zero-input system $\dot{x} = f(x, 0, 0)$ is globally asymptotically stable. Then, the function $\widetilde{W}$ given by (46) *satisfies Condition IV.4 with $\widetilde{\alpha} = \alpha$, $\underline{\beta}_W(r) = \underline{\beta}_W r$, $\overline{\beta}_W(r) = \overline{\beta}_W r$ for some positive constants $\underline{\beta}_W$ and $\overline{\beta}_W$ and λ as in Condition V.1.* ■

The conditions in Theorem V.5 are precisely those used in the delay-free case for obtaining USPAS as adopted in [5, Thm. 2]. To obtain the constants λ, λ_W, $\underline{\alpha}_W$ and $\overline{\alpha}_W$ for the RR or TOD protocol again Lemma V.4 above can be employed.

VI. CASE STUDY OF THE BATCH REACTOR

In this section we will illustrate how the derived conditions can be verified and how this leads to quantitative tradeoff curves between τ_{mati}, τ_{mad} and $\mathcal{L}_2$ performance for the case study of the batch reactor. This case study has developed over the years as a benchmark example in NCS, see e.g. [5], [11], [38], [52]. The functions in the NCS (7) for the batch reactor are given by the linear functions $f(x,e,w) = A_{11}x + A_{12}e + A_{13}w$ and $g(x,e,w) = A_{21}x + A_{22}e + A_{23}w$. The batch reactor, which is open-loop unstable, has $n_u = 2$ inputs, $n_y = 2$ outputs, $n_p = 4$ plant states and $n_c = 2$ controller states and $l = 2$ nodes (only the outputs are assumed to be sent over the network). See [38], [52] for more details on this example. We included here also a disturbance input w that takes values in $\mathbb{R}^2$ and a controlled output z, taking values in $\mathbb{R}^2$, given by $z = q(x,w) = Cx + Dw$. The numerical values for A_{ij}, $i,j = 1,2$, as provided in [38], [52], are given in (56) In [38], [52] z and w were absent, as they considered the disturbance-free case. We selected for the remaining matrices A_{13}, A_{23}, C and D in this case study the numerical values as in (56).

This means that the disturbance w is such that w_1 affects the first and third state of the reactor and, w_2 affects the second and fourth state. The controlled output z is chosen to be equal to the measured output y, see [38], [52].

A. Stability analysis

To apply the developed framework for stability analysis, we first ignore the controlled output z and set $w = 0$. Moreover, we take $M_e = |A_{22}| := \sqrt{\lambda_{max}(A_{22}^\top A_{22})}$ and $m_x(x) = |A_{21}x|$ in (41). To verify (44) we take $\rho(r) = \varepsilon r^2$ and consider a quadratic Lyapunov function $V(x) = x^\top Px$ to compute the $\mathcal{L}_2$ gain from $|e| = W_{TOD}(\kappa, e)$ to $m_x(x)$ (or actually a value close to the $\mathcal{L}_2$ gain by selecting $\varepsilon > 0$ small) by minimizing γ subject to the linear matrix inequalities (LMIs)

$$\begin{pmatrix} A_{11}^\top P + PA_{11} + \varepsilon I + A_{21}^\top A_{21} & PA_{12} \\ A_{12}^\top P & (\varepsilon - \gamma^2)I \end{pmatrix} \preceq 0, \quad (57a)$$

$$P = P^\top \succ 0. \quad (57b)$$

Minimizing γ subject to the LMI (57) with $\varepsilon = 0.01$ using the SEDUMI solver [43] with the YALMIP interface [30] provides the minimal value of $\gamma = 15.9165$. This value of γ applies for both the TOD as well as the RR protocol in (44) since $W_{RR}(k,e) \geq |e|$ and $W_{TOD}(k,e) = |e|$ due to Lemma V.4. From Lemma V.4 also the values for the constants λ, $\underline{\alpha}_W$, $\overline{\alpha}_W$ and λ_W can be obtained. Then Theorem V.3 can be applied to construct suitable Lyapunov functions for the closed-loop NCS system. This results for the TOD protocol in the values $L_0 = 15.7300$, $L_1 = 22.2456$, $\gamma_0 = 15.9165$ and $\gamma_1 = 22.5093$. Note that L_0 and γ_0 are the same as found in [5] and [38] for the delay-free case (up to some small numerical differences). In case we now take $\phi_0(0) = \phi_1(0) = \lambda_{TOD}^{-1} = \sqrt{2}$ as $\lambda_{TOD} = \sqrt{\frac{l-1}{l}} = \sqrt{\frac{1}{2}}$, we recover exactly the results in [5], see Fig. 2. Indeed, checking the conditions (23) gives $\tau_{mad} = 0$ and $\tau_{mati} = 0.0108$, as also found in [5].

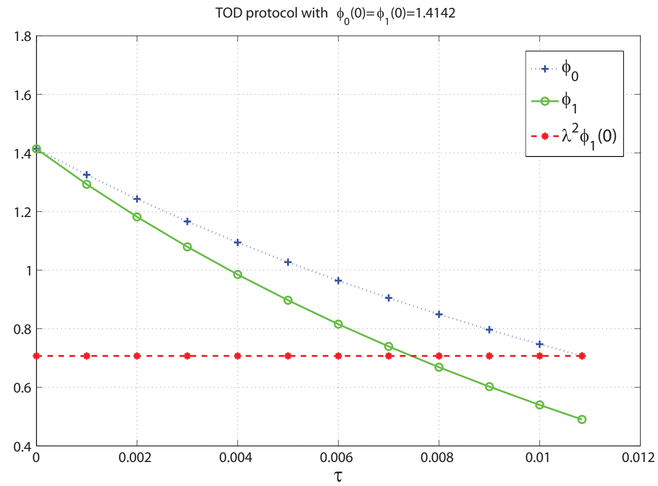

Fig. 2. Batch reactor functions ϕ_ℓ, $\ell = 0, 1$ with $\phi_0(0) = \phi_1(0) = \frac{1}{\lambda_{TOD}}$.

Besides this delay-free limit case, the above numerical values provide much more combinations of $(\tau_{mati}, \tau_{mad})$ that yield stability of the NCS by varying the initial conditions $\phi_0(0)$ and $\phi_1(0)$. Actually, each pair of initial conditions provides a different Lyapunov function $U(\xi)$ as in (26) and different values for $(\tau_{mati}, \tau_{mad})$ as discussed after the proof of Theorem IV.2. To illustrate this, consider Fig. 3, which displays the solutions ϕ_ℓ, $\ell = 0, 1$, to (22) for initial conditions $\phi_0(0) = 1.4142$ and $\phi_1(0) = 1.6142$. The solutions ϕ_ℓ, $\ell = 0, 1$ are determined using Matlab/Simulink (using zero crossing detections to determine the values of τ_{mati} and τ_{mad} accurately according to (23)). The condition (23a) indicates that τ_{mati} is determined by the intersection of ϕ_0 and the constant line with value $\lambda^2\phi_1(0)$ and condition (23b) states that τ_{mad} is determined by the intersection of ϕ_0 and ϕ_1 (as long as $\phi_0(0) \leq \phi_1(0)$). For the specific situation depicted in Fig. 3 this would result in $\tau_{mati} = 0.008794$ and $\tau_{mad} = 0.005062$, meaning that UGES is guaranteed for transmission intervals up to 0.008794 and transmission delays up to 0.005062. Interestingly, the initial conditions of both functions ϕ_0 and ϕ_1 can be used to make design tradeoffs. For instance, by taking $\phi_1(0)$ larger, the allowable delays become larger (as the solid line indicated by 'o' shifts upwards), while the maximum transmission interval becomes smaller as the dashed line indicated by '•' will shift upwards as well causing its intersection with ϕ_0 (dotted line indicated by '+') to occur for a lower value of τ. Hence, once the hypotheses of Theorem IV.2 are satisfied, a continuum of Lyapunov functions is available leading to different combinations of MATI and MAD. This shows that tradeoff curves between τ_{mad} and τ_{mati} can indeed be constructed. Following this procedure for various increasing values of $\phi_1(0)$, while keeping $\phi_0(0)$ equal to $\lambda_{TOD}^{-1} = \sqrt{2}$, provides the graph in Fig. 4, where

$$A_{11} = \begin{pmatrix} 1.3800 & -0.2077 & 6.7150 & -5.6760 & 0 & 0 \\ -0.5814 & -15.6480 & 0 & 0.6750 & -11.3580 & 0 \\ -14.6630 & 2.0010 & -22.3840 & 21.6230 & -2.2720 & -25.1680 \\ 0.0480 & 2.0010 & 1.3430 & -2.1040 & -2.2720 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 1.0000 & 0 & 1.0000 & -1.0000 & 0 & 0 \end{pmatrix}; \; A_{12} = \begin{pmatrix} 0 & 0 \\ 0 & -11.3580 \\ -15.7300 & -2.2720 \\ 0 & -2.2720 \\ 0 & 1.0000 \\ 1.0000 & 0 \end{pmatrix};$$
$$A_{21} = \begin{pmatrix} 13.3310 & 0.2077 & 17.0120 & -18.0510 & 0 & 25.1680 \\ 0.5814 & 15.6480 & 0 & -0.6750 & 11.3580 & 0 \end{pmatrix}; \; A_{22} = \begin{pmatrix} 15.7300 & 0 \\ 0 & 11.3580 \end{pmatrix};$$
$$A_{13} = \begin{pmatrix} 10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 5 & 0 & 5 & 0 & 0 \end{pmatrix}^\top; \; A_{23} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \; C = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \; D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{56}$$

the particular point $\tau_{mati} = 0.008794$ and $\tau_{mad} = 0.005062$ corresponding to Fig. 3 is highlighted. Note that the graph ends where $\tau_{mati} = \tau_{mad}$ as the developed model does not include delays larger than the transmission interval.

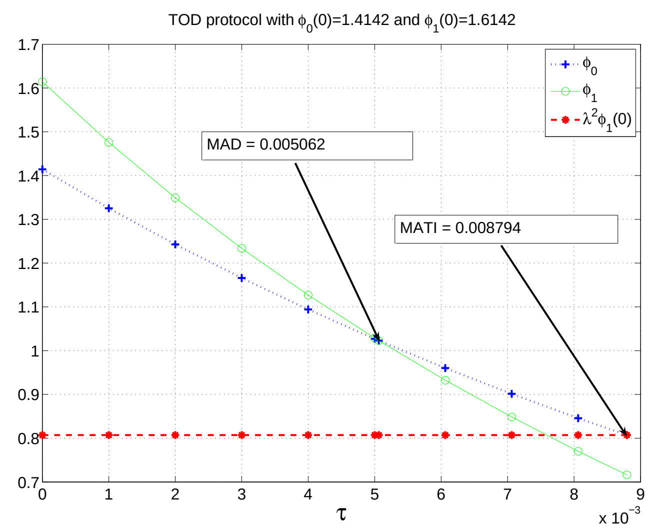

Fig. 3. Batch reactor functions ϕ_ℓ, $\ell = 0, 1$ with $\phi_0(0) = 1.4142$ and $\phi_1(0) = 1.6142$.

In case of the RR protocol we obtain the values $L_0 = 22.2456$, $L_1 = 44.4912$, $\gamma_0 = 22.5093$ and $\gamma_1 = 45.0185$ if we invoke Theorem V.3 directly. However, improved values for L_0 and L_1 can be obtained by exploiting the special structure in the matrix A_{22} as was also done in [38, Ex. 3]. This is achieved by deriving directly the condition (19) instead of following the general approach based on (49)-(52) using (40) and (41) in the proof of Theorem V.3. Indeed, using that $W(i,e) = |D(i)e|$ for a diagonal matrix $D(i)$ (with the values $a_i(j)$ as in the proof of Lemma V.4 on the diagonal), we can derive directly for almost all e and s

$$\begin{aligned} &\left\langle \frac{\partial W(i,e+s)}{\partial e}, g(x,e,0) \right\rangle \leq |D(i)\dot{e}| \\ &= |D(i)A_{22}e| + |D(i)A_{21}x| \\ &\leq |A_{22}D(i)e| + \sqrt{l}|A_{21}x| \leq |A_{22}|W(i,e) + \underbrace{\sqrt{l}|A_{21}x|}_{=H_0(x)=H_1(x)}, \end{aligned}$$

where we used that $A_{22}D(i) = D(i)A_{22}$ as both matrices are diagonal. Using this sharper result in (49)-(52) instead of (40) and (41), we obtain the improved values $L_0 = 15.7300$, $L_1 = 31.4600$, $\gamma_0 = 22.5093$ and $\gamma_1 = 45.0185$, similar to [5] and [38], which leads for the delay-free case to $\tau_{mati} = 0.0090$ (recovering the result in [5], which outperforms the values found in [38], [39], [51]). The tradeoff curve between MATI and MAD is also given in Fig. 4. In this figure also the delay-free case with $\tau_{mad} = 0$ and $\tau_{mati} = 0.0090$ is visualized. These tradeoff curves can be used to impose conditions or select a suitable network with certain communication delay and bandwidth requirements (note that MATI is inversely proportional to the bandwidth).

Also different protocols can be compared with respect to each other. In Fig. 4, it is seen that for the task of stabilization of the unstable batch reactor the TOD protocol outperforms the RR protocol in the sense that it can allow for larger delays and larger transmission intervals. The difference between the tradeoff curves for different protocols is caused by different values for the parameters λ, $\overline{\alpha}_W$, M_1 and λ_W (see Lemma V.4), which in turn induce different values for the parameters L_0, L_1, γ_0 and γ_1 (see Theorem V.3) and thus different solutions to the differential equations (22). This results in different combinations of τ_{mad} and τ_{mati} that guarantee stability due to Theorem IV.2.

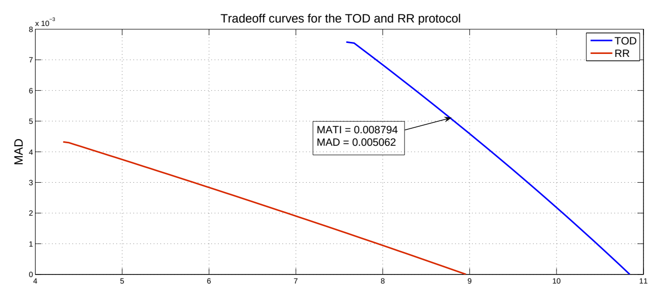

Fig. 4. Tradeoff curves between MATI and MAD.

B. $\mathcal{L}_2$ gain analysis

To apply the developed framework for $\mathcal{L}_2$ gain analysis, the effect of the additional disturbance input w on the controlled variable z is studied.

We take $M_e = |A_{22}| := \sqrt{\lambda_{max}(A_{22}^\top A_{22})}$ and $m(x,w) = |A_{21}x + A_{23}w|$ in (41). To verify (45) we compute simultaneously an upper bound on the $\mathcal{L}_2$ gain from $W(\kappa,e) = W_{TOD}(\kappa,e) = |e|$ to $m(x,w)$ and on the $\mathcal{L}_2$ gain from w to z by finding Pareto minimal values for γ and θ subject to the matrix inequalities in the matrix P and multiplier $\mu > 0$ as given in (58). These matrix inequalities above demonstrate that there will be a tradeoff between performance in terms of the $\mathcal{L}_2$ gain from w to z reflected in θ on the one hand and the size of τ_{mati} and τ_{mad} on the other as reflected in γ. Recall that γ directly influences MATI and MAD through γ_0 and γ_1 in (48). The tradeoff between τ_{mati} and τ_{mad} can still be made by varying $\phi_0(0)$ and $\phi_1(0)$ whilst guaranteeing a certain $\mathcal{L}_2$ gain θ for the NCS.

To make these observations quantitative, we fix θ at various values and search for the smallest value of γ such that there exist P and $\mu > 0$ satisfying (58). Note that (58) is an LMI when θ is fixed and hence, can be solved efficiently. As a lower bound on θ we take the $\mathcal{L}_2$ gain θ^* from w to z of the system without the network ($e = 0$). This value is found by solving the standard $\mathcal{L}_2$-gain / $\mathcal{H}_\infty$ LMI for linear systems (see e.g. [1]) using, again, the SEDUMI solver and the YALMIP interface. This yields $\theta^* = 1.9597$. Using this as a lower bound on the considered values of the desirable performance level in terms of the $\mathcal{L}_2$ gain θ, we search minimal values for γ (corresponding to the selected value of θ) under the feasibility of the LMI (57). These minimal values yield the "Pareto optimal curves" for (γ, θ) as shown in Fig. 5. This figure demonstrates that the minimal value of γ approaches 15.9165 for large values of θ, which was the $\mathcal{L}_2$ gain from $|e| = W_{TOD}(\kappa, e)$ to $m_x(x) = |A_{21}x|$ as computed in Section VI-A for the stability analysis. This is expected as the value 15.9165 corresponds to the smallest value of γ found in Section VI-A if only stability is required (without any additional $\mathcal{L}_2$ performance conditions, so the $\mathcal{L}_2$ gain θ approaches infinity). The other extreme, when γ approaches infinity, recovers the situation where the θ values approach the asymptotic value of $\theta^* = 1.9597$ being the optimal network-free $\mathcal{L}_2$ gain from w to z.

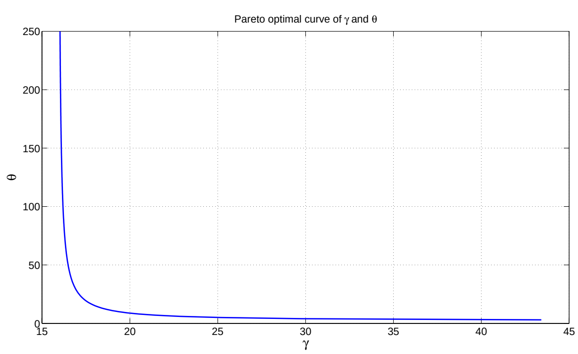

Fig. 5. Pareto optimal curves for (γ, θ) for the TOD protocol.

As we can see, a smaller $\mathcal{L}_2$ gain θ requires a larger value of γ, which will in turn result in smaller values for τ_{mati} and τ_{mad}. We will demonstrate this for the TOD protocol. The results for the RR protocol can be obtained using the same approach. The computed values of (γ, θ) are now used in (45) and the values for the constants λ, $\underline{\alpha}_W$, $\overline{\alpha}_W$ and λ_W can be obtained from Lemma V.4, as in the stability analysis above. Then Theorem V.3 can be applied to construct a continuum of suitable storage functions for the closed-loop NCS system choosing a certain combination of (γ, θ) in a similar manner as for the stability analysis. This leads to combinations of $(\tau_{mati}, \tau_{mad}, \theta)$ such that the NCS has an $\mathcal{L}_2$ gain from w to z smaller than θ for transmission intervals smaller than τ_{mati} and communication delays smaller than τ_{mad}. The tradeoff curves for various levels of the $\mathcal{L}_2$ performance θ are provided in Fig. 6. For the value of $\theta = 200$ we (almost) recover the tradeoff plot as in Fig. 4 for the TOD protocol, because, loosely speaking, the $\mathcal{L}_2$ gain requirement is very mild as $\theta = 200$ is a relatively large value that practically approaches the condition that the NCS should be UGES only. In the other extreme, if very high requirements are given with respect to robustness to disturbances w (in the sense of a very low $\mathcal{L}_2$ gain from w to z approaching the network-free $\mathcal{L}_2$ gain $\theta^* = 1.9597$), the values for MATI and MAD that guarantee this $\mathcal{L}_2$ gain are approaching 0. This is clearly shown in Fig. 6 as the tradeoff curve for the value $\theta = 2 \approx \theta^*$ corresponds to values of τ_{mati} and τ_{mad} close to zero. The limit case $\theta \to \theta^*$ would actually correspond to $\tau_{mati} \to 0$ and $\tau_{mad} \to 0$. Hence, the control and network engineers have to make clear design tradeoffs between MATI, MAD, robustness in terms of $\mathcal{L}_2$ performance *and* the choice of the protocol. The provided framework supports the engineers to make these design choices in a quantified manner.

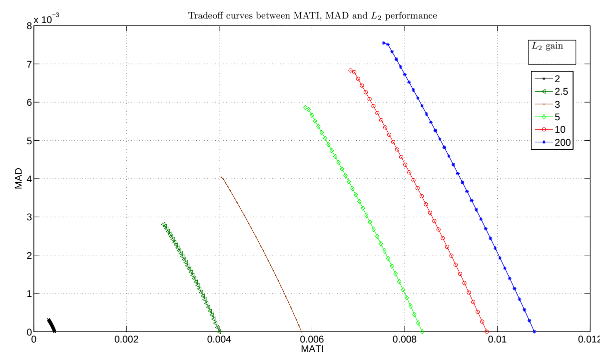

Fig. 6. Tradeoff curves between MATI and MAD for various levels of the $\mathcal{L}_2$ gain of the NCS with the TOD protocol.

VII. Conclusions

In this paper we presented a framework for studying the stability of a NCS, which both involves communication constraints (only one node accessing the network per transmission) and varying transmission intervals and varying transmission delays. Based on a newly developed model, a Lyapunov-based characterization of stability was provided and explicit bounds on the MATI and MAD were obtained. We explicitly showed how a continuum of Lyapunov functions can be constructed from the commonly adopted conditions for the delay-free case. The application of the results on a benchmark example showed how these tradeoff curves between MATI and MAD can be computed providing designers of NCS with proper tools to support their design choices. Interestingly, recently developed improvements in [5] leading to sharper bounds for the MATI (the non-delay case) are included as a

$$\begin{pmatrix} A_{11}^\top P + PA_{11} + A_{21}^\top A_{21} + \mu C^\top C & PA_{12} & A_{21}^\top A_{23} + PA_{13} + \mu C^\top D \\ A_{12}^\top P & -\gamma^2 I & 0 \\ A_{13}^\top P + A_{23}^\top A_{21} + \mu D^\top C & 0 & \mu D^\top D + A_{23}^\top A_{23} - \mu\theta^2 I \end{pmatrix} \preceq 0, \qquad P = P^\top \succ 0. \tag{58}$$

special case in this more general framework. Additionally, we have analyzed the $\mathcal{L}_p$ performance of NCS and provided the theoretical framework that shows how MATI, MAD and $\mathcal{L}_p$ can be traded quantitatively against each other. Under weaker conditions, we provided also semiglobal practical stability results for the NCS.

Future work will involve the consideration of the large delay case (delays larger than the transmission interval) and the development of an analysis framework that also includes quantization effects. Extending the current framework so as to include quantization effects would provide the means to study NCS incorporating all the types of network phenomena mentioned in the introduction (as packet dropouts can be modeled using a prolongation of the MATI as discussed in Remark II.4). We foresee that such an extension would be extremely valuable to network and control system designers, provided that the extended framework leads to quantitative tradeoff curves between the various network parameters (MATI, MAD, quantization error, bandwidth, etc.) and the performance of the overall control loop. Some preliminary results in this direction can be found in [21].

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