Robustness of nonlinear control systems with quantized feedback☆

Tatiana Kameneva and Dragan Nešić

T.Kameneva is with National ICT Australia, t.kameneva@ee.unimelb.edu.au. D.Nesic is with the Department of Electrical and Electronic Engineering, University of Melbourne, Parkville, 3052, Victoria, Australia, d.nesic@ee.unimelb.edu.au.

Abstract

This paper analyzes the stability of nonlinear systems with quantized feedback in the presence of exogenous disturbances. This paper is an extension of [Liberzon, Nešić (2007)] to nonlinear systems. Under appropriate assumptions using nonlinear modification of the scheme proposed in [Liberzon, Nešić (2007)], it is shown here that it is possible to achieve input-to-state and nonlinear gain l_2 stability for nonlinear systems with quantized feedback.

Key words: Quantized control, nonlinear systems, input-to-state stability, l_2 stability

1. Introduction

In the literature on networked control systems, stability properties of systems are addressed in a wide range of problems. However robustness of the systems is still an area that requires much more attention and careful examination. Robust stabilization and estimation was considered, for instance, in [2], [6], [10], [13], [12], [14], [18]. Even though the stability results are important, the controllers that satisfy stability properties may not be implementable in practice if these controllers lack appropriate robustness properties. Indeed, the controller designed to stabilize the system in the absence of disturbances may lead to instability when a disturbance is introduced into the system. Analysis and design of systems with disturbances are, in general, different from systems without disturbances. Since disturbances almost always appear in practice, there is strong motivation to take this particular research direction.

This paper focuses on robust stability of control systems when measurements between a plant and a controller are transmitted via a limited capacity channel, in particular, on achieving robustness with respect to exogenous disturbances. One of the robustness properties that is considered here is characterized by the input-to-state stability property (Definition 1) that was introduced by Sontag in [22]. It involves nonlinear gains from the initial state and the supremum norm of the disturbance to the supremum norm

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of the state and also the supremum limit of the state. It is a particular version of the "bounded input implies bounded state" property, that is fully compatible with the Lyapunov techniques and is becoming a popular form of external stability for analysis and design of nonlinear control systems. This property proved to be natural and useful in a range of control problems [1], [8], [7], [15], [23]. On the other hand, l_p, particulary l_2, stability plays a special role in systems analysis. If $u(t)$ is thought of as a current or voltage, then $u(t)u^T(t)$ is proportional to the instantaneous power of the signal and integral (sum for discrete time systems) over all time is a measure of the energy of the signal. It is natural to work with square-integrable signals which can be viewed as finite energy signals.

Our study has been mainly motivated by the work of Liberzon and Nešić [12] and Martins [13]. Liberzon and Nešić consider the problem of achieving input-to-state stability with respect to external disturbances for control systems with linear dynamics and quantized state measurements. Liberzon and Nešić's trajectory-based proof is novel and utilizes a cascade structure of the closed-loop hybrid system. This paper's results are heavily based on the time-sampling scheme proposed in [12]. We adopt Liberzon and Nešić's strategy and closely follow their sampled-data scheme. This scheme is used as a representative example of other quantized control schemes that have adaptive quantization as their main feature, that is: the quantizer's range and quantization error are changing adaptively depending on the quantized measurements of the plant. We believe, that the qualitative results derived using the scheme in [12] will hold for other adaptive quantization schemes. Our Definition 2 in Section 3 employs the notion of nonlinear gains to explore l_2 stability properties of the system. This is consistent with the result by Martins [13], where it is shown that linear (finite) gains are not achievable when quantized control with finitely many levels is used. Later, in [9] it was shown that it is possible to achieve Nonlinear Gain (NG) l_2 stability for quantized control systems. All results developed in [12] and [9] are limited to linear systems. The present work generalizes the contributions of [12] and [9] to nonlinear systems.

In this paper, we consider nonlinear time-invariant feedback systems with quantized measurements, when the system is perturbed by bounded disturbances. Under appropriate assumptions, the objective of this work is to find the conditions under which the closed-loop system is input-to-state and NG l_2 stable with respect to bounded disturbances. Building on the earlier work from [12] and [9], our main result, Theorem 1 in Section 5 shows that if the parameters of the switching scheme and the parameters of the quantizer are adjusted appropriately, then the nonlinear plant is input-to-state and NG l_2 stable with respect to bounded disturbances.

The remainder of the paper is organized as follows. In Section 2 the definitions that are used in the sequel are given. The problem formulation is given in Section 3. More details on the dynamics of the closed loop system, switching rules and protocol are given in Section 4. The main results are presented in Section 5. Section 6 offers the conclusions. The proofs and technical lemmas are given in the appendix.

2. Notation and preliminaries

In this section some notation is introduced and the definitions that make the discussed concepts precise are given. The two-norm of the vector is denoted as follows: $|z| := \sqrt{\sum_{i=1}^{n}(z^i)^2}$, where $z = (z^1, z^2, \ldots, z^n)$, n is the dimension of the vector z. The sequence of vectors z_k for $k \in [k_1, k_2]$, is denoted as $z_{[k_1,k_2]}$. The two-norm of a sequence of vectors on a time-interval $[k_1, k_2]$ is denoted as $\|z\|_{[k_1,k_2]} := \sqrt{\sum_{k=k_1}^{k_2} |z_k|^2}$. The infinity-norm of a sequence of vectors on a time-interval $[k_1, k_2]$ is denoted $\|z_{[k_1,k_2]}\|_\infty := \sup_{k\in[k_1,k_2]} |z_k|$.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and unbounded. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to be class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ is strictly increasing and $\beta(0, s) = 0$, and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. For $\chi \in \mathcal{K}_\infty$ the composition is defined as $\chi\circ\chi(s)$. We define $\chi\circ\chi(s) \equiv \chi^2(s)$, then for n iterations we have $\chi^n(s)$. A function γ is subadditive if the following holds: $\gamma(x + y) \leq \gamma(x) + \gamma(y)$ $\forall x, y \geq 0$. A function γ is superadditive if the following holds: $\gamma(x) + \gamma(y) \leq \gamma(x + y)$ $\forall x, y \geq 0$. We denote as $\mathcal{K}_+$ the class of functions that are of class $\mathcal{K}_\infty$ and are subadditive. Similarly, we denote by $\mathcal{K}^+$ the class of functions that are of class $\mathcal{K}_\infty$ and are superadditive. The following lemma will be used:

Lemma 1. *[9] For any $\gamma \in \mathcal{K}_\infty$, there exist $\gamma_1 \in \mathcal{K}^+$ and $\gamma_2 \in \mathcal{K}_+$ such that: $\gamma(s) \leq \gamma_1 \circ \gamma_2(s)$ $\forall s \geq 0$.*

3. Problem formulation

In this section the system under consideration and stability definitions are introduced. The system's closed-loop dynamics is a generalization of the sampled-data scheme described in [12] to the nonlinear feedback systems. In order to control the system (2) we use a nonlinear version of the sampled-data scheme employed in [9] and [12]. We use this scheme as a representative example of a broad class of the adaptive-quantization schemes. Consider the continuous-time nonlinear system with a control input:

$$\dot{x}(t) = f(x(t), u(t), w(t)), \quad x(0) \in \mathbb{R}^n \tag{1}$$

where f is well defined for all x, u, w and is locally Lipschitz in x and w for each u; $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is a control input and $w \in \mathbb{R}^l$ is an unknown bounded disturbance. The closed-loop system, induced by the sampled-data plant (1) is given in Figure 1. There are two approaches for designing discrete time control systems for continuous time plants. A general framework for the design of nonlinear digital controllers using the emulation method is reported in [15]. The second approach is to derive a discrete time equivalent of the plant and then directly design a discrete time controller to control the discretized plant. Throughout the paper the second approach is utilized. Define $t_k = kT$ for $k = 0, 1, 2, \ldots$, where $T > 0$ is a given sampling period. Denote $x(t_k) = x_k$ and similarly for all other variables. Let $u(t) = u_k =$ const. and $w(t) = w_k = const.$ for all

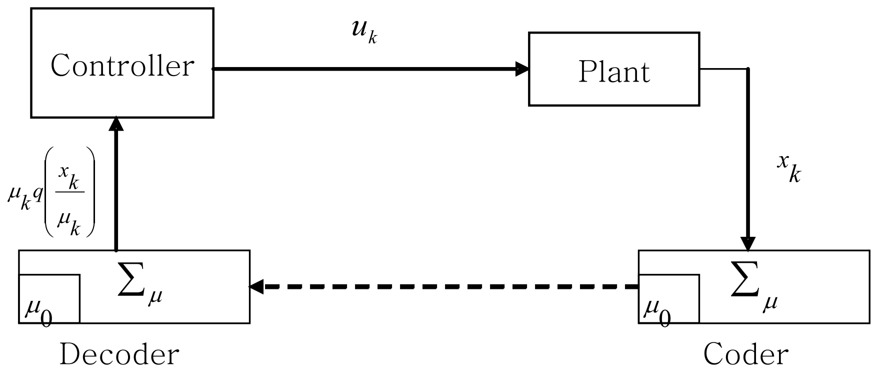

Figure 1: A feedback system with a digital channel.

$t \in [kT, (k+1)T)$[1]. The exact discrete time plant model of the sampled-data plant (1) is the following:

$$x_{k+1} = x_k + \int_{kT}^{(k+1)T} f(x(\tau), u_k, w_k)d\tau = F(x_k, u_k, w_k), \tag{2}$$

where $x_0 \in \mathbb{R}^n$ and $F(x_k, u_k, w_k)$ is the solution of (1) at a time T starting at x_k and with the constant inputs u_k and w_k. This model is well-defined when the continuous model (1) does not exhibit finite-escape time. The closed-loop dynamics consists of the plant (2) and the controller and zooming protocol described below by the following equations:

$$u_k = \begin{cases} 0 & \text{if } \Omega_k = \Omega_{out} \\ \kappa(q_k) & \text{if } \Omega_k = \Omega_{in}. \end{cases} \tag{3}$$

$$\mu_{k+1} = \begin{cases} \chi(\mu_k) + c & \text{if } \Omega_k = \Omega_{out}, \quad c > 0, \quad \mu_0 \in \mathbb{R}_{>0}, \\ \psi(\mu_k) & \text{if } \Omega_k = \Omega_{in}, \end{cases} \tag{4}$$

$$\Omega_k = \begin{cases} \Omega_{out} & \text{if } |q_k| > l_{out}\mu_k \\ \Omega_{in} & \text{if } |q_k| < l_{in}\mu_k \\ \Omega_{k-1} & \text{if } |q_k| \in [l_{in}\mu_k, l_{out}\mu_k], \quad \Omega_{-1} = \Omega_{out}, \end{cases} \tag{5}$$

where Ω_k can take only two strictly positive values Ω_{out} and Ω_{in}. If $\Omega_k = \Omega_{out}$ it is said that the zoom-out condition is triggered at a time k. If $\Omega_k = \Omega_{in}$ is it said that the zoom-in condition is triggered at a time k. The functions $\chi, \psi \in \mathcal{K}_\infty$, more conditions on χ and ψ are imposed later (in particular, Assumption 2 and conditions $(v), (vi)$ of Theorem 1). l_{out} and l_{in} are strictly positive numbers such that $l_{out} > l_{in}$, that will be defined later. Finally, q_k is a family of dynamic quantizers in the form $q_k := \mu_k q(\frac{x_k}{\mu_k}), \mu_k > 0$, that satisfy Assumption 1 below. μ_k is an adjustable parameter, called a "zoom" variable, that is updated at discrete instants of time and depends only on the quantized measurements of the state q_k. For each fixed μ the range of the quantizer is $M\mu$ and the quantization error is $\Delta\mu$, M, Δ come from the assumption below.

[1]The assumption that $w_k = const.$ can be relaxed, for details refer to [15].

Assumption 1. *There exist strictly positive numbers $M > \Delta > 0$, Δ_0 such that the following holds: (i) If $|z| \leq M$ then $|q(z) - z| \leq \Delta$; (ii) If $|z| > M$ then $|q(z)| > M - \Delta$; (iii) For all $|z| \leq \Delta_0$ we have that $q(z) = 0$.*

M is called the range of the quantizer, Δ is called the quantization error, Δ_0 is a dead-zone. The first condition gives a bound on the quantization error when the state is in the range of the quantizer, the second gives the possibility to detect saturation, the third condition is needed to preserve the origin.

The main goal of this work is to show that nonlinear systems can be rendered input-to-state and NG l_2 stable when the parameters of the coder/decoder/quantizer are adjusted appropriately. The precise definitions of ISS and NG l_2 stability are given below.

Definition 1. *The system (2) - (5) is said to be input-to-state stable (ISS) if for all $\mu_0 > 0$ there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for any initial conditions x_0 and every bounded disturbance w we have that μ_k is bounded for all $k \geq 0$ and:*

$$|x_k| \leq \gamma_1(|x_0|) + \gamma_2(\|w\|_\infty) \quad \forall k \geq 0, \tag{6}$$

$$\limsup_{k\to\infty} |x_k| \leq \gamma_3(\limsup_{k\to\infty} |w_k|). \tag{7}$$

Definition 2. *The system (2) - (5) is said to be Nonlinear Gain (NG) l_2 stable if for every $\mu_0 > 0$ there exist $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for every initial conditions x_0 and every disturbance w the following holds:*

$$\|x_{[0,k]}\|_2 \leq \gamma_1(|x_0|) + \gamma_2\left(\sum_{i=0}^{k-1} \gamma_3(|w_i|)\right) \quad \forall k \geq 0. \tag{8}$$

Remark 1. *Note, that the stability bound that is valid at sampling instants t_k can be extended to a bound that is valid for all $t \geq 0$ if the intersampling behavior of the system is bounded. Note, that it is assumed that the exact discrete time plant model is well defined, which is the case when the continuous model does not exhibit finite-escape time. In other words, there exists a bound on the state trajectories between sampling instances. We will analyze only the stability properties of the discrete-time system (2) with (3) - (5) induced by the sampled-data system (1). It was shown in [16] how to use the discrete-time model stability to conclude appropriate stability properties of the sampled-data system.*

Note, that the gain functions $\gamma_1, \gamma_2, \gamma_3$ in Definitions 1 and 2 may depend on the choice of the initial value μ_0 of the zoom variable μ (but not on x_0 and w). These functions in Definitions 1 and 2 are not the same.

It was shown in [20] that for continuous-time systems, the property expressed by inequalities (6), (7) is equivalent to input-to-state stability with respect to w. In the present case, the closed-loop system contains an additional state μ and we talk about a partial stability property (in x) of the closed loop system. With some abuse of terminology, we will refer to the previous property as ISS of the closed-loop system.

The definition of NG l_2 stability above employs a concept of nonlinear gains, similar to [21], to describe robustness properties of systems with respect to external disturbances. This is consistent with the result in [13], where Martins showed that nonlinear gains are necessary when formulating properties of disturbance attenuation for linear discrete-time systems with feedback from a finite set. Martins proved that linear (finite) l_p, $p \in [1, \infty]$, gains are not achievable when quantized control with finitely many levels is used. We remark here that in [24] it was shown how to construct a finite gain stabilizing controller with countably infinite number of control choices. Zhang and Dullerud [24] show that static logarithmic memoryless quantizer is sufficient for finite l_p gain stabilization. Definition 2 shows explicitly what we mean by nonlinear gains and is in a slightly different form than the definition of NG l_2 stability used by Martins in [13], where the property

$$\|x\|_2 \leq \gamma_1(|x_0|) + \gamma_M(\|w\|_2) \tag{9}$$

is considered. Note that for a particular choice of γ_2, one can get a property (9) when $\gamma_3(s) = s^2$, since the disturbance gain in (9) γ_M can be always expressed via disturbance gain γ_2.

4. Modes of the operation

In this section the control policy (3), switching rules (5) and the dynamics of the adjustable parameter μ (4) are described in details. The control policy is composed of two stages: zoom-out and zoom-in. During the zoom-out stage the system is running in open loop, that is we apply $u_k = 0$. During the zoom-in stage a certainty-equivalence feedback $u_k = \kappa(q_k)$ is applied. Hysteresis switching is used to switch between the zoom-in and zoom-out stages. This is not necessary and it is introduced to simplify the analysis.

During the zoom-out stage the value of the adjustable parameter μ is increased until the state of the system can be adequately measured. During the zoom-in stage the value of the adjustable parameter μ is decreased in such a way as to drive the state to the origin. The adjustment policy for μ_k can be thought of as being implemented synchronously on both ends of the communication channel from some known initial value μ_0. In other words, two dynamical systems (e.g. copies of the plant model) have to be run on both ends of the communication channel (at the coder and at the decoder).

Some notation is introduced. For each $k \geq 0$ there are two possible cases: $\Omega_k = \Omega_{out}$ or $\Omega_k = \Omega_{in}$. Given an initial condition and disturbance there is a sequence of the zoom-out and zoom-in intervals. We introduce $k_j \in \mathbb{N}$ such that

$$\begin{array}{lll} \Omega_k = \Omega_{out} & \text{if } k \in [k_{2i}, k_{2i+1} - 1], & i = 0, 1, \ldots, N \\ \Omega_k = \Omega_{in} & \text{if } k \in [k_{2i+1}, k_{2i+2} - 1]. & \end{array}$$

That is: k_{2i+1} is the time instant at which the plant switches from the zoom-out stage to the zoom-in stage; k_{2i+2} is the time instant at which the plant switches from the zoom-in stage to the zoom-out stage. Let $k_0 = 0$. For simplicity it is assumed that the first interval is always the zoom-out: $\Omega_{-1} = \Omega_{out}$. An example of the switching intervals is

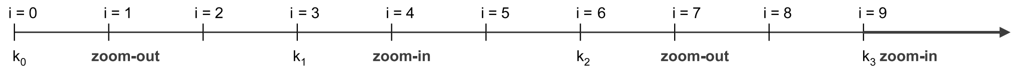

Figure 2: Zooming Intervals.

given in Figure 2: the initial zoom-out stage for $i = 0, 1, 2$ is followed by the zoom-in stage for $i = 3, 4, 5$, then the zoom-out again for $i = 6, 7, 8$ and the last zoom-in stage for $i \geq 9$. On the figure k_1, k_2, k_3 are the switching instances between the zoom-out and zoom-in stages.

In order to understand the operation of the system during the zoom-out and zoom-in stage, the dynamics of the switching variable $\xi_k := \frac{x_k}{\mu_k}$ has to be considered. Indeed, in (5) let $l_{out} := M - \Delta$ and $l_{in} := \Delta_M - \Delta$.[2] Consider the zoom-in switching condition in (5). Note, that whenever $\left|\frac{x_k}{\mu_k}\right| < l_{in} - \Delta$ holds, $\left|\mu_k q\left(\frac{x_k}{\mu_k}\right)\right| < l_{in}\mu_k$ holds. Also, the zoom-out switching condition in (5) $\left|\mu_k q\left(\frac{x_k}{\mu_k}\right)\right| > l_{out}\mu_k$ implies that $\left|\frac{x_k}{\mu_k}\right| > l_{out} + \Delta$. Therefore, it can be concluded that the switching is governed by the variable ξ_k. During the zoom-out stage, the dynamics of ξ_k evolves according to the following for all $k \in [k_{2i}, k_{2i+1} - 1]$:

$$\xi_{k+1} = \frac{x_{k+1}}{\mu_{k+1}} = \frac{F(x_k, 0, w_k)}{\chi(\mu_k) + c} = \frac{F(\mu_k \xi_k, 0, w_k)}{\chi(\mu_k) + c} =: F_{out}(\xi_k, \mu_k, w_k). \tag{10}$$

During the zoom-in stage, the dynamics of ξ_k evolves according to the following for all $k \in [k_{2i+1}, k_{2i+2} - 1]$:

$$\begin{aligned}\xi_{k+1} &= \frac{x_{k+1}}{\mu_{k+1}} = \frac{F(x_k, u_k, w_k)}{\psi(\mu_k)} = \frac{F(x_k, \kappa(\mu_k(q(\xi_k))), w_k)}{\psi(\mu_k)} \\ &= \frac{F(\mu_k\xi_k, \kappa(\mu_k\xi_k + \mu_k(q(\xi_k) - \xi_k)), w_k)}{\psi(\mu_k)} =: F_{in}(\xi_k, \mu_k, \nu_k, w_k),\end{aligned} \tag{11}$$

where $\nu_k := q(\xi_k) - \xi_k$ with $|\nu_k| \leq \Delta$ due to the fact that during the zoom-in stage $|\xi_k| \leq l_{out} + \Delta = M$ and due to the first condition of Assumption 1.

The following results, that are similar to Lemma III.2 and Corollary III.3 [12] for linear systems, are stated. It is assumed that there exists $\psi \in \mathcal{K}_\infty$ such that during the zoom-in stage ξ_k dynamics is ISS with respect to the bounded error, ν, and the bounded disturbance, w, uniformly in μ.

Assumption 2. *Assume that there exists $\psi \in \mathcal{K}_\infty$ such that for all $\mu_0 > 0$ there exist functions $\bar\beta_1 \in \mathcal{KL}$ and $\bar\gamma_1, \bar\gamma_2 \in \mathcal{K}_\infty$ such that for every initial condition ξ_0 and every ν_k the following holds for the trajectories of the system (11) $\forall k \in [k_{2i+1}, k_{2i+2}]$:*

$$|\xi_k| \leq \bar\beta_1(|\xi_{k_{2i+1}}|, k) + \bar\gamma_1(\|\nu_{[k_{2i+1}, k-1]}\|_\infty) + \bar\gamma_2(\|\zeta_{[k_{2i+1}, k-1]}\|_\infty),$$

where $\zeta_k := w_k/\mu_k$.

[2] This choice becomes clearer after Corollary 1 is introduced at the end of this section.

Note that Assumption 2 requires uniformity of ISS property in μ, which can be restrictive. In other words, Assumption 2 is equivalent to saying that there exists a smooth function $\bar{V}(\xi,\mu) : \mathbb{R}^n : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ such that for some class $\mathcal{K}_\infty$ functions $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \tilde{\rho}_1^*, \tilde{\rho}_2^*$, for all $\xi, \nu \in \mathbb{R}^n$ we have:

$$\bar{\alpha}_1(|\xi|) \leq \bar{V}(\xi,\mu) \leq \bar{\alpha}_2(|\xi|) \tag{12}$$

and

$$|\xi| \geq \tilde{\rho}_1^*(|\nu|) + \tilde{\rho}_2^*(|w|) \Longrightarrow \bar{V}(F_{in}(\xi,\mu,\nu,w), \psi(\mu)) - \bar{V}(\xi,\mu) \leq -\bar{\alpha}_3(|\xi|). \tag{13}$$

In analogy to Assumption 2 for nonlinear systems, it is shown in [12] that for linear systems ξ dynamics is ISS with respect to the error, ν, and the disturbance, w. Classes of plants that satisfy Assumption 2 include linear in control plants, refer to Example 1 below.

Example 1. *Consider the following system:*

$$x_{k+1} = F(x_k) + u_k + w_k, \tag{14}$$

where F is Globally Lipshitz with Lipshitz constant L:

$$\|F(x) - F(x+e)\| \leq L\|e\|.$$

Let $u_k = -F(x_k+e_k) - 0.5(x_k+e_k)$, where e_k is a quantization error, $e_k = \mu_k q\left(\frac{x_k}{\mu_k}\right) - x_k$. Then

$$x_{k+1} = F(x_k) - F(x_k+e_k) - 0.5(x_k+e_k) + w_k.$$

Since $V = \|x\|$ is a Lyapunov function for the system (14), then

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= \|F(x_k) - F(x_k+e_k) - 0.5(x_k+e_k) + w_k\| - \|x_k\| \\ &\leq 0.5\|x_k\| + (L+0.5)\|e_k\| + \|w_k\| \end{aligned}$$

possesses ISS-property, since e and w are bounded, therefore (20) holds.

Now we show that Assumption 2 holds for $\xi-$system, uniformly in μ. Due to the Global Lipshitz property of $F(x)$, we can take a linear law of μ update: $\mu_{k+1} = \Omega_{in}\mu_k$. Also, we define $\zeta_k := \frac{w_k}{\mu_k}$ and the quantization error e_k can be written in the following way: $e_k = \mu_k q\left(\frac{x_k}{\mu_k}\right) - \mu_k\frac{x_k}{\mu_k} = \mu_k\nu_k$, where $\nu_k = q(\xi_k) - \xi_k$. We have:

$$\xi_{k+1} := \frac{x_{k+1}}{\mu_{k+1}} = \frac{F(x_k) - F(x_k+e_k) - 0.5(x_k+e_k) + w_k}{\Omega_{in}\mu_k}. \tag{15}$$

Since $V = \|\xi\|$ is a Lyapunov function for the system (15), then

$$\begin{aligned} V(\xi_{k+1}) - V(\xi_k) &\leq \frac{0.5\|x_k\| + (L+0.5)\|e_k\| + \|w_k\|}{\Omega_{in}\mu_k} - \frac{\|x_k\|}{\Omega_{in}\mu_k} \\ &= \frac{1}{\Omega_{in}}0.5\|\xi_k\| + \frac{1}{\Omega_{in}}(L+0.5)\|\nu_k\| + \frac{1}{\Omega_{in}}\|\zeta_k\| \end{aligned}$$

possesses ISS-property, since ν and ζ are bounded, therefore (13) holds.

The following corollary that basically says that if the range of the quantizer M is large enough compared to the quantization error Δ (i.e. the quantizer takes sufficiently many levels), then $|\xi_k|$ and $|q(\xi_k)|$ are bounded. The condition on the number of quantization levels (16) can be interpreted as a condition on the data-rate of the channel.

Corollary 1. *Let $\bar{\beta}, \bar{\gamma}_1, \bar{\gamma}_2$ come from Assumption 2 and let strictly positive M and Δ be such that the following holds:*

$$M > \bar{\beta}(\Delta, 0) + \bar{\gamma}_1(\Delta) + \bar{\gamma}_2(\Delta) + 2\Delta. \tag{16}$$

Then there exist $\Delta_M > 0$ with $\Delta_M - \Delta > 0$ and $\Delta_w > 0$, such that whenever $|\xi_0| \leq \Delta_M$, $|\nu| \leq \Delta$ and $|\zeta| < \Delta_w$, we have:

$$|q(\xi_k)| \leq M - \Delta \quad \textit{and} \quad |\xi_k| \leq M \quad \forall k \geq 0. \tag{17}$$

The proof of Corollary 1 is given in the appendix. Corollary 1 motivates the choice of $l_{out} := M - \Delta$ and $l_{in} := \Delta_M - \Delta$ that are the same as for linear systems. These parameters will be used in Theorem 1 in the next section.

5. Main result

The main result of this paper is presented in this section. It is shown that using a nonlinear generalization of the sampled-data scheme employed in [9] and [12], given by (3) - (5), it is possible using appropriate assumptions to adjust the parameters of the switching scheme and the parameters of the quantizer so that the plant with nonlinear dynamics is ISS and NG l_2 stable. Note, in [11] it was shown that the techniques developed in [12] can be extended to nonlinear systems that are input-to-state stable with respect to the measurement errors. The system considered in [11] does not have input disturbances, in this paper however we consider robustness properties of the system with respect to input disturbances. The main result of this paper is the following theorem, which presents conditions under which the system (2) with (3) - (5) is ISS and NG l_2 stable with respect to input disturbances.

Theorem 1. *Consider the closed-loop system (2) - (5). The system (2) - (5) is ISS from w to x and NG l_2 stable if the following conditions hold:*

(i) *q is a quantizer fulfilling Assumption 1.*

(ii) *There exist φ_1, $\varphi_2, \varphi_3 \in \mathcal{K}_\infty$ and $\bar{c} = const.$ such that for the trajectories of (2) with $u_k = 0$ $\forall k \in [k_{2i}, k_{2i+1}]$*

$$|x_k| \leq \varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w_{[k_{2i},k-1]}\|_\infty) + \varphi_3(k - k_{2i}) + \bar{c}.$$

(iii) *There exist $\chi_1, \chi_2 \in \mathcal{K}_\infty$ such that for any $\bar{T} \in [k_{2i+1}, k_{2i+2})$ for the trajectories of the system (2) - (5) for all $k \in [k_{2i+1}, \bar{T}]$*

$$|x_k| \leq \chi_1(|x_{k_{2i+1}}|) + \chi_2(\|w_{[k_{2i+1},k-1]}\|_\infty).$$

(*iv*) *For all* $\mu_0 > 0$ *there exist* $\beta \in \mathcal{KL}$ *and* $\gamma_0, \bar{\gamma}_0 \in \mathcal{K}_\infty$ *such that for every initial condition* x_0 *and every* e_k, w_k *the corresponding solution of (2) - (5) with* $u_k = \kappa(x_k + e_k)$ *satisfies*

$$|x_k| \leq \beta(|x_{k_{2i+1}}|, k) + \gamma_0(\|e_{[k_{2i+1},k-1]}\|_\infty) + \bar{\gamma}_0(\|w_{[k_{2i+1},k-1]}\|_\infty) \quad \forall k \geq 0.$$

(*v*) $\psi(s)$ *is such that* $\psi(s) < s \ \forall s > 0$ *and Assumption 2 holds.*

(*vi*) $\chi(s)$ *is such that* $\chi(s) > as, a > 1$ *and*

$$\chi(s) > \varphi_1(s) + \varphi_2(s) + \varphi_3(s) + \bar{c} \quad \forall s > 0,$$

where $\varphi_1, \varphi_2, \varphi_3, \bar{c}$ *come from condition* (*ii*) *above and* $c > 0$ *comes from (4).*

(*vii*) M, Δ *are such that (16) holds and* $l_{out} := M - \Delta$, $l_{in} := \Delta_M - \Delta$, *where* Δ_M *comes from Corollary 1.*

The proofs for both, ISS and NG l_2 stability rely on Lemmas 2 - 6 that are given in the appendix. Also, to prove NG l_2 stability an additional Lemma 7 is used. The proof of Lemma 7 is given in [9]. Note that under the same conditions it is possible to show ISS and NG l_2 stability of nonlinear control system with quantized state measurements in the presence of exogenous disturbances. The proof of Theorem 1 to conclude ISS is similar to the proof of Theorem 2 from [12] with the only difference being that $\bar{\gamma}_4$ in (29) is $\mathcal{K}_\infty$ function, not a positive constant as in [12]. The proof of Theorem 1 to conclude ISS is given in the appendix. The proof of Theorem 1 to conclude NG l_2 stability is similar to the proof of Theorem 1 from [9] for linear systems. The proof relies on the properties of $\mathcal{K}_\infty$ functions that in addition are subadditive or superadditive. The proof is given in the appendix for completeness.

Next the conditions of Theorem 1 are discussed. Condition (*i*) of the theorem is the condition on the quantizer. This type of adaptive quantizers are extensively used in the work by Liberzon, i.e. [11].

Condition (*ii*) of the theorem implies that during the zoom-out stage the system (2) with $u_k = 0$ is forward complete [22]. Note, that linear systems always satisfy condition (*ii*), however a nonlinear systems's state $|\cdot|$ can go to infinity in a finite time. Note that when k_{2i+1} is sufficiently large, since the disturbance w is bounded, by condition (*ii*) there exists a time $k^* \in [k_{2i}, k_{2i+1}]$ such that the following holds $\forall k \in [k^*, k_{2i+1}]$:

$$|x_k| \leq \varphi_1(k - k_{2i}) + \varphi_2(k - k_{2i}) + \varphi_3(k - k_{2i}) + \bar{c}. \tag{18}$$

Condition (*iii*) of Theorem 1 is similar to the assumption of forward completeness. In analogy to condition (*ii*) during the zoom-out stage, condition (*iii*) is used during the zoom-in stage to deal with the phenomenon that trajectories of the nonlinear system can escape to infinity in finite time.

Condition (*iv*) of Theorem 1 implies that during the zoom-in stage the closed-loop system (2) - (5) with $u_k = \kappa(x_k + e_k)$ is ISS with respect to the measurement error, e, and the disturbance, w. Condition (*iv*) (x_k is ISS) and Assumption 2 (x_k/μ_k is ISS) together guarantee that during the zoom-in stage the adjustable parameter μ_k decreases slow enough. Note, that for linear systems $x_{k+1} = (\Phi+\Gamma K)x_k + \Gamma K\left(q\left(\frac{x_k}{\mu_k}\right) - \frac{x_k}{\mu_k}\right) + w_k$, the assumption that the matrix K is such that $\Phi+\Gamma K$ is Schur is used, which implies that condition (iv) holds. We remark that condition (iv) can be restrictive (for continuous-time systems see [3]). This condition is equivalent to saying that there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that for some class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3, \bar{\rho}_1^*, \bar{\rho}_2^*$, for all $x, e \in \mathbb{R}^n$ we have:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{19}$$

and

$$|x| \geq \bar{\rho}_1^*(|e|) + \bar{\rho}_2^*(|w|) \Longrightarrow V(F(x, \kappa(x+e), w)) - V(x) \leq -\alpha_3(|x|). \tag{20}$$

Classes of plants that satisfy ISS-property with respect to the measurement errors are explored in [4], [5], [17], [19] and references therein. In [4], Freeman shows that time-varying feedback renders feedback passive systems ISS with respect to measurement noise. In [5], Freeman and Kokotović show that a class of single-input systems in a strict feedback form satisfies the aforementioned assumption. In [17], Nešić and Sontag find a stabilizing controller which is robust with respect to the observation noise for a controllable and observable system for which only magnitudes of outputs are measured. In [19], Sanfelice and Teel describe a hybrid controller that renders a subclass of control affine systems to possess the above property.

Condition (v) states how slow the adjustable parameter μ is decreased during the zoom-in. During the zoom-in stage, while the state is in the range of the quantizer ($|x_k| \leq M\mu_k$), the quantization error is decreased (by decreasing the value of the adjustable parameter μ_k) in such way as to drive the state to the origin.

Condition (vi) states how fast the adjustable parameter μ is increased during the zoom-out. During the zoom-out stage, while the state is in the saturation region ($|x_k| > M\mu_k$), the range of the quantizer is increased by increasing μ_k in a piecewise constant fashion, fast enough to dominate the rate of growth of $|x_k|$. In particular, if k_{2i+1} is sufficiently large, there exists a time instant $\bar{k} \in [k_{2i}, k_{2i+1}]$ such that the following holds:

$$\mu_{\bar{k}+1} = \chi(\mu_{\bar{k}}) + c > \varphi_1(\bar{k} - k_{2i}) + \varphi_2(\bar{k} - k_{2i}) + \varphi_3(\bar{k} - k_{2i}) + \bar{c} > |x_{\bar{k}+1}|.$$

The specification on the switching parameters are given in the condition of Theorem 1. Condition (vii) also guarantees that the range of the quantizer is large enough compared to the quantization error. Since this can be interpreted as a condition on the data-rate of the channel, it can be said that condition (vii) requires that the data-rate of the channel is sufficiently high.

The last condition of the theorem implies that the considered system can be stabilized when the data is transmitted via a channel with finite data-rate. Therefore we can say that the system under consideration is a control system with finite set quantized feedback. Martins [13] showed that linear gains are not achievable with finite quantized feedback. Our Theorem 1, on the other hand, shows what kind of nonlinear l_2 gains are achievable for nonlinear systems with the controller from Section 3. To illustrate the necessity of nonlinear gains for control systems with (finite set) quantized feedback the following example is presented. The example illustrates that even for a linear system the property of NG l_2 stability considered in Theorem 1 does not hold when the disturbance gain is linear.

Example 2. *For clarity we duplicate the NG l_2 stability property (9) considered by Martins:*

$$\|x\|_2 \leq \gamma_1(|x_0|) + \gamma_M(\|w\|_2). \tag{21}$$

Through simulations we show that the property (21) does not hold when γ_M is a linear function. Suppose γ_M is a linear function, then with $x_0 = 0$ $\forall |w|$ there exist $G = const$ such that the following holds:

$$\frac{\|x\|_2}{\|w\|_2} \leq G. \tag{22}$$

We construct a sequence of disturbance inputs and show that the upperbound on the ratio $\frac{\|x\|_2}{\|w\|_2}$ blows up for small and large disturbances. Consider a linear system, that is: $F(x_k, u_k, w_k) = \Phi x_k + \Gamma u_k + w_k$, $\kappa(q_k) = K q_k$, $\chi(\mu_k) = \Omega_{out}\mu_k$, $\psi(\mu_k) = \Omega_{in}\mu_k$, refer to [12]. We simulated the control algorithm with the following parameters:

$$\Phi = \begin{pmatrix} 2 & 1.5 \\ 0 & 1.5 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$\Delta = 0.5$, $K = (-0.87 - 0.72)$, $\Omega_{in} = 0.5$, $\Omega_{out} = 3$, $c = 1$, $w_k \equiv 0$.

We simulated the discrete-time plant and the simulation results show a qualitative picture of $|x_k|$ dynamics. Note, in simulations $k_0 = 1$. The results of the simulations are shown in Figure 3, where we plotted $\frac{\|x\|_2}{\|w\|_2}$ verses different values of $|w|$ to approximate G in (22). Consider two cases. Note in both case the disturbance is from a class of a finite energy signals.

Case 1. $|w|$ is small. As a particular choice we take $w_k = W_s \exp(-k)$ $\forall k \geq 0$. $W_s \geq 1$ is a constant which is set to different values for different simulation's runs. Figure 3a shows that as $|w|$ becomes smaller, the ratio $\frac{\|x\|_2}{\|w\|_2}$ becomes larger. In other words, there does not exist G large enough such that $\frac{\|x\|_2}{\|w\|_2}$ can be overbounded by G $\forall |w|$.

Case 2. $|w|$ is large. As a particular choice we take $w_k = W_L$ for k from 0 to 10 and $w_k = 0$ $\forall k > 10$. $W_L \geq 100$ is a constant which is set to different values for different simulation's runs. Figure 3b shows that as $|w|$ becomes larger, the ratio $\frac{\|x\|_2}{\|w\|_2}$ becomes larger. It is observed in Figure 3b that there does not exist G large enough such that $\frac{\|x\|_2}{\|w\|_2}$ can be overbounded $\forall |w|$.

In other words, we came to the contradiction that there exists a constant G such that (22) holds for $\forall |w|$. Simulations show that γ_M has to be nonlinear function. Simulations confirm the results [13] and support the employment of nonlinear gains in [12] and in this paper.

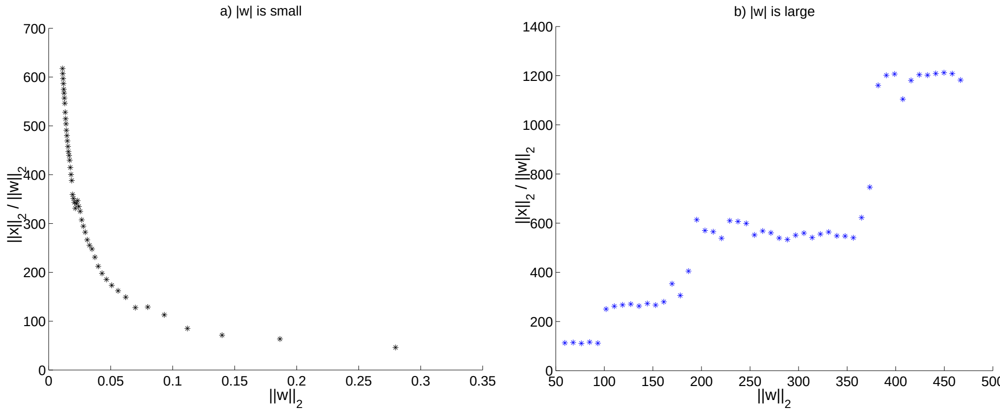

Figure 3: Simulation results for Example 2. Approximation of γ_M in $\|x\|_2 \leq \gamma_M(\|w\|_2)$.

6. Summary

In this paper a stabilization problem for quantized feedback systems with nonlinear dynamics in the presence of exogenous disturbances is addressed. The approach of this work fits into the framework of control with limited information in the sense that the state of the system is not completely known. Assuming that the transmission data rate is above the required minimum, the focus of this paper is on the issue of the controller/coder/decoder design. It is shown that using appropriate assumptions, it is possible to adjust the parameters of the switching scheme and the parameters of the quantizer so that a nonlinear plant is input-to-state and NG l_2 stable.

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Appendix

Proof of Corollary 1. Since (16) is a strict inequality there exist $\Delta_M > 0$ and $\Delta_w > 0$ sufficiently close to Δ such that the following holds:

$$M > \bar{\beta}(\Delta_M, 0) + \bar{\gamma}_1(\Delta) + \bar{\gamma}_2(\Delta_w) + 2\Delta.$$

During the zoom-in stage, due to Assumption 2, whenever $|\xi_0| \leq \Delta_M$, $|\nu| \leq \Delta$, $|\zeta| < \Delta_w$ and (16) holds, we have for all $k \in [k_{2i+1}, k_{2i+2}]$:

$$\begin{aligned}|\xi_k| &\leq \bar{\beta}_1(|\xi_{k_{2i+1}}|, k) + \bar{\gamma}_1(\|\nu_{[k_{2i+1}, k-1]}\|_\infty) + \bar{\gamma}_2(\|\zeta_{[k_{2i+1}, k-1]}\|_\infty) \\ &\leq \bar{\beta}_1(\Delta_M, k) + \bar{\gamma}_1(\Delta) + \bar{\gamma}_2(\Delta_w) < M.\end{aligned}$$

Also whenever $|\xi_0| \le \Delta_M$, $|\nu| \le \Delta$, $|\zeta| < \Delta_w$ and (16) holds, we can write the following for all $k \in [k_{2i+1}, k_{2i+2}]$:

$$|\mu_k q(\xi_k)| = |\mu_k(q(\xi_k) - \xi_k) + \mu_k \xi_k| < \mu_k \Delta + \mu_k(M - 2\Delta) < (M - \Delta)\mu_k.$$

Cancelling μ_k, we can conclude that whenever $|\xi_0| \le \Delta_M$, $|\nu| \le \Delta$, $|\zeta| < \Delta_w$ and (16) holds, we have that (17) holds. ■

Now we present Lemmas 2 - 6. These lemmas are similar to Lemmas IV.5 - IV.9 in [12] for the systems with linear dynamics. The following Lemma 2 implies that the zoom-out condition can be only triggered for finitely many time steps. Hence, if N is finite, then $k_{2N+2} = \infty$. In other words, there exists a $k_{2N+1} \in \mathbf{N}$ such that the zoom-in condition is triggered on the interval $[k_{2n+1}, \infty)$. Moreover, Lemma 2 establishes a bound on the state x during the zoom-out interval.

Lemma 2. *Consider the system (2) with (3) - (5). Let q be a quantizer fulfilling Assumption 1. Let all conditions of Theorem 1 hold. Then there exist $\bar{\varphi}_1, \bar{\varphi}_2, \rho_1, \rho_2 \in \mathcal{K}_\infty$ such that for all $i = 0, 1, \dots, N, x_{k_{2i}} \in \mathbb{R}^n, \mu_{k_{2i}} > 0, w \in \mathbb{R}^l$ the following holds:*

$$k_{2i+1} - k_{2i} \le \bar{\varphi}_1(|x_{k_{2i}}|) + \bar{\varphi}_2(\|w_{[k_{2i}, k_{2i+1}-1]}\|_\infty) \tag{23}$$

$$|x_k| \le \rho_1(|x_{k_{2i}}|) + \rho_2(\|w_{[k_{2i}, k_{2i+1}-1]}\|_\infty), \quad k \in [k_{2i}, k_{2i+1}]. \tag{24}$$

Note, that functions $\bar{\varphi}_i$ and ρ_i, $i = 1, 2$, are independent of μ.

Proof of Lemma 2. The proof will be carried out by contradiction. Suppose the zoom-out interval is unbounded, that is $k_{2i+1} = \infty$. For all $k \in [k_{2i}, k_{2i+1}]$ by Assumption **??** we have the following:

$$|x_k| \le \varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w_{[k_{2i}, k-1]}\|_\infty) + \varphi_3(k - k_{2i}) + \bar{c}.$$

Dividing both sides of the inequality above by μ_k we have for all $k \in [k_{2i}, k_{2i+1}]$:

$$\begin{aligned}|\xi_k| = \left|\frac{x_k}{\mu_k}\right| \le \\ \le \frac{\varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w_{[k_{2i}, k-1]}\|_\infty) + \varphi_3(k - k_{2i}) + \bar{c}}{\chi(\mu_{k-1}) + c} \\ \le \frac{\varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w_{[k_{2i}, k-1]}\|_\infty) + \varphi_3(k - k_{2i}) + \bar{c}}{\chi^{k-k_{2i}-1}(\mu_{k_{2i}})}.\end{aligned}$$

Since the disturbance w is bounded, there exists a time instant $k^* \in [k_{2i}, k_{2i+1}]$ such that the following holds for all $k \in [k^*, k_{2i+1}]$:

$$\begin{aligned}|\xi_k| \le \frac{\varphi_1(k - k_{2i}) + \varphi_2(k - k_{2i}) + \varphi_3(k - k_{2i}) + \bar{c}}{\chi^{k-k_{2i}-1}(\mu_{k_{2i}})} \\ < \frac{\chi(k - k_{2i})}{\chi(a^{k-k_{2i}-1}\mu_{k_{2i}})},\end{aligned}$$

where the last inequality above is due to condition (v) of Theorem 1. Since $a > 1$, for sufficiently large k the following holds: $a^{k-k_{2i}-1}\mu_{k_{2i}} > k - k_{2i}$. We have for all $k \in [k^*, k_{2i+1}]$:

$$\lim_{k\to\infty} \frac{\chi(k-k_{2i})}{\chi(a^{k-k_{2i}-1}\mu_{k_{2i}})} = 0.$$

We can conclude, that the variable ξ_k is decreasing and eventually we must have $|\xi_k| < l_{in} - \Delta$, which implies that $|\mu_k q(\xi_k)| < l_{in}\mu_k$ and the zoom-in stage is triggered in a finite time. Hence, we came to the contradiction of the claim that $k_{2i+1} = \infty$ and we can conclude that $k_{2i+1} - k_{2i} - 1$ is bounded. Moreover, we can write the following for some $\bar{\varphi}$ function that is continuous bounded and nondecreasing as a function of each single argument when the other one is fixed:

$$k_{2i+1} - k_{2i} - 1 \leq \bar{\varphi}(|x_{k_{2i}}|, \|w_{[k_{2i},k_{2i+1}-1]}\|_\infty). \tag{25}$$

Note, that we can let $\bar{\varphi}(0,0) = 0$ since if $x_{k_{2i}} = 0$ then $k_{2i+1} - k_{2i} = 1$. Hence, we can find $\bar{\varphi}_1, \bar{\varphi}_2 \in \mathcal{K}_\infty$ so that (23) holds. Note also that for all $k \in [k_{2i}, k_{2i+1}]$ we have the following:

$$\begin{aligned}
|x_k| &\leq \varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w_{[k_{2i},k_{2i+1}-1]}\|_\infty) + \varphi_3(k - k_{2i}) + c \\
&\leq \varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w_{[k_{2i},k_{2i+1}-1]}\|_\infty) + \bar{\varphi}_3(k_{2i+1} - k_{2i}) \\
&\leq \varphi_1(|x_{k_{2i}}|) + \varphi_2(\|w_{[k_{2i},k_{2i+1}-1]}\|_\infty) \\
&\quad + \bar{\varphi}_3(\bar{\varphi}_1(|x_{k_{2i}}|) + \bar{\varphi}_2(\|w_{[k_{2i},k_{2i+1}-1]}\|_\infty)) \\
&\leq \rho_1(|x_{k_{2i}}|) + \rho_2(\|w_{[k_{2i},k_{2i+1}-1]}\|_\infty),
\end{aligned}$$

where $\rho_1, \rho_2 \in \mathcal{K}_\infty$ are such that $\rho_1(s) := \varphi_1(s) + \bar{\varphi}_3(2\bar{\varphi}_1(s))$ and $\rho_2(s) := \varphi_2(s) + \bar{\varphi}_3(2\bar{\varphi}_2(s))$. ■

The following Lemma 3 establishes a bound on μ at the end of each zoom-out interval in terms of the values of μ and x at the beginning of that interval and the infinity norm of the disturbance during the zoom-in interval.

Lemma 3. *Consider the system (2) with (3) - (5) and let q be a quantizer fulfilling Assumption 1. Let all conditions of Theorem 1 hold. Then there exists a continuous bounded function ρ_μ^{out} such that for any $\mu > 0$ we have $\rho_\mu^{out}(\mu, 0, 0) > 0$ and the following is true for all $i = 0, 1, \ldots, N$ and all $\mu_{k_{2i}} > 0$, $x_{k_{2i}} \in \mathbb{R}^n$, $w \in \mathbb{R}^l$:*

$$|\mu_{k_{2i+1}}| \leq \rho_\mu^{out}(\mu_{k_{2i}}, |x_{k_{2i}}|, \|w_{[k_{2i},k_{2i+1}-1]}\|_\infty).$$

Proof of Lemma 3. Note that we can find a continuous bounded strictly increasing function $\bar{\chi}$ such that $\bar{\chi}(s) > \chi(s) + c$. Then we have that the following holds:

$$|\mu_{k_{2i+1}}| < \underbrace{\bar{\chi} \circ \bar{\chi} \circ \ldots \bar{\chi}}_{k_{2i+1}-k_{2i}\ \text{times}} (\mu_{k_{2i}}) = \bar{\chi}^{k_{2i+1}-k_{2i}}(\mu_{k_{2i}}).$$

Since $k_{2i+1} - k_{2i} - 1$ is bounded by (23), we can find continuous bounded function ρ_μ^{out} such that the following holds:

$$\mu_{k_{2i+1}} \leq \bar{\chi}^{\lceil \bar{\varphi}_1(|x_{k_{2i}}|) + \bar{\varphi}_2(\|w_{[k_{2i},k_{2i+1}-1]}\|_\infty)\rceil + 1}(\mu_{k_{2i}})$$

$$\leq \rho_\mu^{out}(\mu_{k_{2i}}, |x_{k_{2i}}|, \|w_{[k_{2i},k_{2i+1}-1]}\|_\infty), \tag{26}$$

where the function ρ_μ^{out} depends on $|x_{k_{2i}}|$ and $\|w_{[k_{2i},k_{2i+1}-1]}\|_\infty$ since the number of composition of the $\bar{\chi}$ function depends on $|x_{k_{2i}}|$ and $\|w_{[k_{2i},k_{2i+1}-1]}\|_\infty$. ■

The following Lemma 4 establishes an appropriate bound on the state x during the zoom-in intervals. This bound is a direct consequence of the fact that during the zoom-in interval the system behaves as a cascade of $x-$ and $\mu-$subsystems. The $x-$subsystem is ISS when μ is regarded as an input, and the $\mu-$subsystem is globally asymptotically stable.

Lemma 4. *Consider the system (2) with (3) - (5) and let q be a quantizer fulfilling Assumption 1. Let all conditions of Theorem 1 hold. Then there exist functions $\bar{\beta} \in \mathcal{KL}$ and $\bar{\gamma}_3 \in \mathcal{K}_\infty$ such that for all $k \in [k_{2i+1}, k_{2i+2}]$ the following holds:*

$$|x_k| \leq \bar{\beta}(|x_{k_{2i+1}}| + \mu_{k_{2i+1}}, k) + \bar{\gamma}_3(\|w_{[k_{2i+1},k-1]}\|_\infty). \tag{27}$$

Proof of Lemma 4. Consider the closed-loop system

$$x_{k+1} = F(x_k, \kappa(x_k + e_k), w_k),$$

where the measurement error e_k is defined as $e_k = |\mu_k q(x_k/\mu_k) - x_k|$. During the zoom-in interval $[k_{2i+1}, k_{2i+2}]$ due to condition (iv) of Theorem 1 the $x-$subsystem satisfies the following:

$$|x_k| \leq \beta(|x_{k_{2i+1}}|, k) + \gamma_0(\|e_{[k_{2i+1},k-1]}\|_\infty) + \bar{\gamma}_0(\|w_{[k_{2i+1},k-1]}\|_\infty),$$

and the $\mu-$subsystem evolves according to the following:

$$\mu_{k+1} = \psi(\mu_k).$$

The $x-$subsystem is ISS when μ is regarded as an input, and the $\mu-$subsystem is globally asymptotically stable since $\psi(s)$ is a contraction map with $\psi(s) < s \ \forall s$. This is a cascade of an ISS and GAS systems and, hence, the overall system during the zoom-in interval is ISS and (27) follows immediately. ■

The following Lemma 5 establishes an appropriate bound on the state x during the zoom-in intervals.

Lemma 5. *Consider the system (2) with (3) - (5) and let q be a quantizer fulfilling Assumption 1. Let all conditions of Theorem 1 hold. Then there exists a continuous function $\rho_x^{in} : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $\rho_x^{in}(\mu, 0, 0) = 0$ for all $\mu > 0$, and such that for any $s \geq 0$, $\rho_x^{in}(\cdot, \cdot, s)$ is nondecreasing in each of its first two single arguments when the other ones are fixed and for any $i = 0, 1, \dots, N$, the following holds for all $\mu_{k_{2i+1}}, x_{k_{2i+1}}, w \ \forall k \in [k_{2i+1}, k_{2i+2}]$:*

$$|x_k| \leq \rho_x^{in}(\mu_{k_{2i+1}}, |x_{k_{2i+1}}|, \|w_{[k_{2i+1},k_{2i+2}-1]}\|_\infty). \tag{28}$$

Proof of Lemma 5. In order to obtain the desired bound, we consider two cases:

- Case 1: $|x_{k_{2i+1}}| \geq \|w_{[k_{2i+1},k_{2i+2}-1]}\|_\infty$,
- Case 2: $|x_{k_{2i+1}}| \leq \|w_{[k_{2i+1},k_{2i+2}-1]}\|_\infty$.

Case 1: Introduce T_x^* such that

$$\psi^{T_x^*}(\mu_{k_{2i+1}}) \leq \mu_{k_{2i+1}}|x_{k_{2i+1}}|.$$

Hence, for all $k \geq k_{2i+1} + T^*$ we have that the following holds:

$$|x_k| \leq M\mu_k \leq M\psi^{k-k_{2i+1}}(\mu_{k_{2i+1}}) \leq M\psi^{T_x^*}(\mu_{k_{2i+1}})$$
$$\leq M\ \mu_{k_{2i+1}}\ |x_{k_{2i+1}}| =: \chi_1^x(\mu_{k_{2i+1}}, |x_{k_{2i+1}}|),$$

where $\chi_1^x(\mu, \cdot) \in \mathcal{K}_\infty$ for each fixed $\mu > 0$.

Note that for any T_x^* Assumption condition (iii) of Theorem 1 holds and for all $k \in [k_{2i+1}, k_{2i+1} + T_x^*]$ we have:

$$|x_k| \leq \chi_1(|x_{k_{2i+1}}|) + \chi_2(\|w_{[k_{2i+1},k-1]}\|_\infty)$$
$$\leq \chi_1(|x_{k_{2i+1}}|) + \chi_2(|x_{k_{2i+1}}|) := \chi_2^x(|x_{k_{2i+1}}|).$$

Note that $\chi_2^x(\cdot) \in \mathcal{K}_\infty$. Let $\chi^x(\mu, s) := \chi_1^x(\mu, s) + \chi_2^x(s)$, then for all $k \in [k_{2i+1}, k_{2i+2}]$ we have that the following holds:

$$|x_k| \leq \chi^x(\mu_{k_{2i+1}}, |x_{k_{2i+1}}|),$$

where $\chi^x(\mu, \cdot) \in \mathcal{K}_\infty$ for each fixed $\mu > 0$.

Case 2: The proof of this case follows exactly the same steps as the proof of Case 1 with the following modification. Introduce T_w^* such that

$$\psi^{T_w^*}(\mu_{k_{2i+1}}) \leq \mu_{k_{2i+1}}\|w_{[k_{2i+1},k_{2i+2}-1]}\|_\infty.$$

Hence, for all $k \geq k_{2i+1} + T_w^*$ we have the following:

$$|x_k| \leq M\mu_k \leq M\psi^{k-k_{2i+1}}(\mu_{k_{2i+1}}) \leq M\psi^{T_x^*}(\mu_{k_{2i+1}})$$
$$\leq M\ \mu_{k_{2i+1}}\ \|w_{[k_{2i+1},k_{2i+2}-1]}\|_\infty$$
$$=: \chi_1^w(\mu_{k_{2i+1}}, \|w_{[k_{2i+1},k_{2i+2}-1]}\|_\infty),$$

where $\chi_1^w(\mu, \cdot) \in \mathcal{K}_\infty$ for each fixed $\mu > 0$.

Note that for any T_x^* condition (iii) of Theorem 1 holds and for all $k \in [k_{2i+1}, k_{2i+1} + T_w^*]$ we have:

$$|x_k| \leq \chi_1(|x_{k_{2i+1}}|) + \chi_2(\|w_{[k_{2i+1},k-1]}\|_\infty)$$
$$\leq \chi_1(\|w\|_{[k_{2i+1},k_{2i+2}-1]}) + \chi_2(|x_{k_{2i+1}}|)$$
$$:= \chi_2^w(\|w_{[k_{2i+1},k_{2i+2}-1]}\|_\infty).$$

Note that $\chi_2^w(\cdot) \in \mathcal{K}_\infty$. Let $\chi^w(\mu, s) := \chi_1^w(\mu, s) + \chi_2^w(s)$, then for all $k \in [k_{2i+1}, k_{2i+2}]$ we have:

$$|x_k| \leq \chi^w(\mu_{k_{2i+1}}, \|w_{[k_{2i+1}, k_{2i+2}-1]}\|_\infty),$$

where $\chi^w(\mu, \cdot) \in \mathcal{K}_\infty$ for each fixed $\mu > 0$. The conclusion of the lemma follows by defining $\rho_x^{in}(\mu, s, p) := \chi^x(\mu, s) + \chi^w(\mu, p)$ and noting that χ^x and χ^w are nondecreasing in μ. ■

The following Lemma 6 establishes that if the zoom-in interval is bounded (i.e., is followed by the zoom-out interval) then at the end of the zoom-in interval we have that x and μ are bounded by the function of the disturbance only, i. e., the initial conditions are "forgotten".

Lemma 6. *Consider the system (2) with (3) - (5) and let q be a quantizer fulfilling Assumption 1. Let all conditions of Theorem 1 hold. Consider arbitrary $i \in \{0, 1, \ldots, N\}$. If $k_{2i+2} < +\infty$, then $i < N - 1$ and there exists $\bar{\gamma}_4 \in \mathcal{K}_\infty$ such that the following holds:*

$$\max\left\{|x_{k_{2i+2}}|, \mu_{k_{2i+2}}\right\} \leq \bar{\gamma}_4(\|w_{[k_{2i+1}, k_{2i+2}-1]}\|_\infty). \tag{29}$$

Proof of Lemma 6. The inequality $i < N - 1$ follows by the definition of N. Note, that by Corollary 1 a zoom-out can occur after a zoom-in only if there exists $k^* \in [k_{2i+1}, k_{2i+2} - 1]$ such that

$$\Delta_w^{-1}|w_{k^*}| \geq \mu_{k^*}.$$

Indeed, if $\Delta_w^{-1}|w_k| \leq \mu_k$ for all k during the zoom-in interval, then we have from Corollary 1 $|\zeta_k| = |w_k/\mu_k| \leq \Delta_w$ and hence $|x_k| \leq M\mu_k$ for all k. Moreover, during the zoom-in interval we must have $|x_{k^*}| \leq M\mu_{k^*}$ and also

$$\Delta_w^{-1}M|w_{k^*}| \geq |x_{k^*}|.$$

Using (27) with $k = k^*$ we have:

$$|x_{k_{2i+2}}| \leq \beta(|x_{k^*}| + \mu_{k^*}, k^*) + \bar{\gamma}_3(\|w_{[k_{2i+1}, k_{2i+2}-1]}\|_\infty)$$

$$\leq \beta(\Delta_w^{-1}M|w_{k^*}| + \Delta_w^{-1}|w_{k^*}|, k^*) + \bar{\gamma}_3(\|w_{[k_{2i+1}, k_{2i+2}-1]}\|_\infty).$$

From here we can find a function $\bar{\gamma}_4 \in \mathcal{K}_\infty$ such that (29) holds. ■

Lemma 7 combines the results of Lemmas 2 - 6 (when the upper bounds with two-norms instead of supremum-norms are used) and shows a bound from w to x.

Lemma 7. *Consider the system (2) with (3) - (5). Suppose that Lemmas 2 - 6 hold. Then for every $\mu_0 > 0$ there exist γ_1, $\hat{\varphi}_2, \hat{\varphi}_3 \in \mathcal{K}_\infty$ such that for every initial condition x_{k_0}, μ_{k_0} and any bounded disturbance w there exist k_i, $i = 0, 1, \ldots, N$ (N may be infinity) such that the following holds:*

$$\|x_{[k_0, k_N]}\|_2 \leq \gamma_1(|x_{k_0}|) + \sum_{l=0}^{N-1} \hat{\varphi}_2\left(\sum_{j=k_l}^{k_{l+1}-1} \hat{\varphi}_3(|w_j|)\right) \tag{30}$$

The proof of Lemma 7 is given in [9]. Note that the bound (30) depends on the switching times k_l, that in turn depend on x_0, μ_0 and w. Nevertheless, the bound (30) implies NG l_2 stability via Lemma 1 from Section 2.

The proof of Theorem 1 to conclude ISS. The proof is done by induction. *First we prove that (6) holds.*

Step $i = 0$**.** Let $k_0 = 0$. Suppose that the first interval is the zoom-out: $\Omega_0 = \Omega_{out}$. From Lemma 2 we have the following for all $k \in [k_0, k_1]$:

$$|x_k| \leq \rho_1(|x_0|) + \rho_2(\|w_{[k_0,k_1-1]}\|_\infty).$$

Now we use Lemmas 3 and 5, the fact that the zoom-out interval is bounded (refer to (23)) and that ρ_x^{in} is nondecreasing in its first two arguments. We have the following for all $k \in [k_1, k_2]$:

$$\begin{aligned} |x_k| &\leq \rho_x^{in}(\mu_{k_1}, |x_{k_1}|, \|w_{[k_1,k-1]}\|_\infty) \leq \rho_x^{in}(\rho_\mu^{out}, \rho_1 + \rho_2, \|w_{[k_1,k-1]}\|_\infty) \\ &\leq \gamma_1^x(\mu_0, |x_0|) + \gamma_2^x(\mu_0, \|w_{[k_1,k-1]}\|_\infty), \end{aligned}$$

where γ_1^x, γ_2^x are a class $\mathcal{K}_\infty$ functions for each fixed μ and are nondecreasing in μ for a fixed value of the second argument. If $k_2 = \infty$, then the proof is complete. If $k_2 < \infty$, then from Lemma 6 we have the following:

$$\max\{|x_{k_2}|, \mu_{k_2}\} \leq \bar{\gamma}_4(\|w_{[k_1,k_2-1]}\|_\infty). \tag{31}$$

Now we proceed to **Step** $i = 1$**.** From Lemmas 2 and (31) we have the following for all $k \in [k_2, k_3]$:

$$|x_k| \leq \rho_1(\bar{\gamma}_4(\|w_{[k_1,k_2-1]}\|_\infty)) + \rho_2(\|w_{[k_2,k_3-1]}\|_\infty) \leq \gamma^x(\|w_{[k_1,k-1]}\|_\infty),$$

where γ^x is independent of μ since $\rho_1, \rho_2, \bar{\gamma}_4$ are independent of μ. A function γ^x is defined as the follows: $\gamma^x(s) := \rho_1(\bar{\gamma}_4(s)) + \rho_2(s)$. Now we again use Lemmas 2, 5 and (31). We can write the following for all $k \in [k_3, k_4]$:

$$\begin{aligned} |x_k| &\leq \rho_x^{in}(\mu_{k_3}, |x_{k_3}|, \|w_{[k_3,k-1]}\|_\infty) \leq \rho_x^{in}(\rho_\mu^{out}, \rho_1 + \rho_2, \|w_{[k_3,k-1]}\|_\infty) \\ &\leq \gamma_1^x(\mu_2, |x_2|) + \gamma_2^x(\mu_2, \|w_{[k_2,k-1]}\|_\infty) \end{aligned}$$

$$\begin{aligned} &\leq \gamma_1^x(\bar{\gamma}_4(\|w_{[k_1,k_2-1]}\|_\infty), \bar{\gamma}_4(\|w_{[k_1,k_2-1]}\|_\infty)) + \gamma_2^x(\bar{\gamma}_4(\|w_{[k_1,k_2-1]}\|_\infty), \|w_{[k_2,k-1]}\|_\infty) \\ &\leq \hat{\gamma}^x(\|w_{[k_1,k-1]}\|_\infty), \end{aligned}$$

where the function $\hat{\gamma}^x$ is independent of μ since $\bar{\gamma}_4$ does not depend on μ. The function $\hat{\gamma}^x$ is defined as follows: $\hat{\gamma}^x(s) := \gamma_1^x(\bar{\gamma}_4(s), \bar{\gamma}_4(s)) + \gamma_2^x(\bar{\gamma}_4(s), s)$. If $k_4 = \infty$, then the proof is complete. If $k_4 < \infty$, then from Lemma 6 we have:

$$\max\{|x_{k_4}|, \mu_{k_4}\} \leq \bar{\gamma}_4(\|w_{[k_3,k_4-1]}\|_\infty).$$

Now we proceed to **Step** $i \geq 1$**.** Similar to the previous argument, we have that for any $i \in \{1, 2, \ldots, N\}$ the following holds:

$$|x_k| \leq \gamma^x(\|w_{[k_{2i-1},k-1]}\|_\infty), \quad k \in [k_{2i}, k_{2i+1}],$$
$$|x_k| \leq \hat{\gamma}^x(\|w_{[k_{2i-1},k-1]}\|_\infty), \quad k \in [k_{2i+1}, k_{2i+2}].$$

By induction, we can conclude, that (6) holds with $\gamma_1(\mu, s) := \max\{\rho_1(s), \gamma_1^x(\mu, s)\}$ and $\gamma_2(\mu, s) := \max\{\rho_2(s), \gamma_2^x(\mu, s), \gamma^x(s), \hat{\gamma}^x\}$.

The proof of (7) is done similar to the induction argument above. If N is finite, then the last stage is zooming-in and Lemma 4 guarantees that

$$\lim_{k\to\infty} \sup |x_k| \leq \bar{\gamma}_3(\lim_{k\to\infty} \sup |w_k|).$$

If $N = \infty$, then it is already proven above that for all $k \in [k_{2i}, k_{2i+1}]$ the following holds:

$$|x_k| \leq \gamma^x(\|w_{[k_{2i-1},k-1]}\|_\infty)$$

and for all $k \in [k_{2i+1}, k_{2i+2}]$ the following holds:

$$|x_k| \leq \hat{\gamma}^x(\|w_{[k_{2i-1},k-1]}\|_\infty).$$

hence, we can take $\gamma_3(s) := \max\{\bar{\gamma}_3(s), \gamma^x(s), \hat{\gamma}^x(s)\}$. Again, γ_3 is independent of μ_0 since $\bar{\gamma}_3, \gamma^x, \hat{\gamma}^x$ are independent of μ_0. ∎

The proof of Theorem 1 to conclude NG l_2 stability. From Lemma 7 for any $\mu_0 > 0$ there exist γ_1, $\hat{\varphi}_2$ and $\hat{\varphi}_3$ such that (30) holds. Then for any fixed initial conditions x_{k_0}, μ_{k_0} and any bounded disturbance w we can write:

$$\|x_{[k_0,k_N]}\|_2 \leq \gamma_1(|x_{k_0}|) + \sum_{l=0}^{N-1} \hat{\varphi}_2 \left(\sum_{j=k_l}^{k_{l+1}-1} \hat{\varphi}_3(|w_j|) \right)$$

$$\leq \gamma_1(|x_{k_0}|) + \sum_{l=0}^{N-1} \varphi_{11} \left(\varphi_{12} \left(\sum_{j=k_l}^{k_{l+1}-1} \hat{\varphi}_3(|w_j|) \right) \right) \tag{32}$$

$$\leq \gamma_1(|x_{k_0}|) + \varphi_{11} \left(\sum_{l=0}^{N-1} \varphi_{12} \left(\sum_{j=k_l}^{k_{l+1}-1} \hat{\varphi}_3(|w_j|) \right) \right) \tag{33}$$

$$\leq \gamma_1(|x_{k_0}|) + \varphi_{11} \left(\sum_{l=0}^{N-1} \sum_{j=k_l}^{k_{l+1}-1} \varphi_{12} \circ \hat{\varphi}_3(|w_j|) \right) \tag{34}$$

$$= \gamma_1(|x_{k_0}|) + \gamma_2 \left(\sum_{j=k_0}^{k-1} \gamma_3(|w_j|) \right) \tag{35}$$

where the inequality (32) comes from Lemma 1 [9] since $\hat{\varphi}_2 \leq \varphi_{11} \circ \varphi_{12}$, $\varphi_{11} \in \mathcal{K}^+$ and $\varphi_{12} \in \mathcal{K}_+$. The inequality (33) is true since $\varphi_{11} \in \mathcal{K}^+$. The inequality (34) is true since $\varphi_{12} \in \mathcal{K}_+$. The last equality (35) comes from the fact that we denote $\gamma_2(s) := \varphi_{11}(s)$ and $\gamma_3(s) := \varphi_{12} \circ \hat{\varphi}_3(s)$. ∎