# On intuitionistic modal and tense logics and their classical companion logics: topological semantics and bisimulations*

J.M. Davoren

Department of Electrical & Electronic Engineering,
The University of Melbourne, VIC 3010 AUSTRALIA
davoren@unimelb.edu.au

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**Abstract**

We take the well-known intuitionistic modal logic of Fischer Servi with semantics in bi-relational Kripke frames, and give the natural extension to topological Kripke frames. Fischer Servi's two interaction conditions relating the intuitionistic pre-order (or partial-order) with the modal accessibility relation generalise to the requirement that the relation and its inverse be lower semi-continuous with respect to the topology. We then investigate the notion of topological bisimulation relations between topological Kripke frames, as introduced by Aiello and van Benthem, and show that their topology-preserving conditions are equivalent to the properties that the inverse-relation and the relation are lower semi-continuous with respect to the topologies on the two models. Our first main result is that this notion of topological bisimulation yields semantic preservation w.r.t. topological Kripke models for both intuitionistic tense logics, and for their classical companion multi-modal logics in the setting of the Gödel translation. After giving canonical topological Kripke models for the Hilbert-style axiomatizations of the Fischer Servi logic and its classical companion logic, we use the canonical model in a second main result which characterizes a Hennessy-Milner class of topological models between any pair of which there is a maximal topological bisimulation that preserve the intuitionistic semantics. The Hennessy-Milner class we identify includes transition system representations of hybrid automata over a product state space whose factors are a Euclidean space and a finite discrete space equipped with an Alexandrov topology determined by a pre-order.

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# 1 Introduction

Topological semantics for intuitionistic logic and for the classical modal logic S4 have a long history going back to Tarski and co-workers in the 1930s and 40s, predating the relational Kripke semantics for both [29, 36]. A little earlier again is the 1933 Gödel translation GT [23] of intuitionistic logic into classical S4. The translation makes perfect sense within the topological semantics: where the S4 $\Box$ is interpreted by topological interior, the translation $\mathrm{GT}(\neg\varphi) = \Box\neg\,\mathrm{GT}(\varphi)$ says that intuitionistic negation calls for the *interior* of the complement, and not just the complement. In the topological semantics, a basic semantic object is the *denotation set* $[\![\,\varphi\,]\!]^{\mathcal{M}}$ of a formula $\varphi$, consisting of the set of all states/worlds of the model $\mathcal{M}$ at which the formula is true, and the semantic clauses of the logic are given in terms of operations on sets of states. The intuitionistic requirement on the semantics is that all formulas must denote open sets: that is, sets that are equal to their own interior. Any formula $\varphi$ partitions the state space $X$ into three disjoint sets: the two open sets $[\![\,\varphi\,]\!]^{\mathcal{M}}$ and $[\![\,\neg\,\varphi\,]\!]^{\mathcal{M}}$, and the closed set $bd([\![\,\varphi\,]\!]^{\mathcal{M}})$, with the points in the topological boundary set $bd([\![\,\varphi\,]\!]^{\mathcal{M}})$ falsifying the law of excluded middle, since they neither satisfy nor falsify $\varphi$.

For the extension from intuitionistic propositional logics to intuitionistic modal logics, Fischer Servi in the 1970s [18, 19, 20] developed semantics over bi-relational Kripke frames, and this work has generated a good deal of research [11, 17, 22, 25, 34, 37, 41, 42]. In bi-relational frames $(X, \preccurlyeq, R)$ where $\preccurlyeq$ is a pre-order (quasi-order) for the intuitionistic semantics, and $R$ is a binary accessibility relation on $X$ for the modal operators, the two Fischer Servi conditions are equivalent to the following relation inclusions [20, 34, 37]:

$$(R^{-1}\circ \preccurlyeq)\ \subseteq\ (\preccurlyeq \circ R^{-1}) \qquad \text{and} \qquad (R\circ \preccurlyeq)\ \subseteq\ (\preccurlyeq \circ R) \tag{1}$$

where $\circ$ is relational/sequential composition, and $(\cdot)^{-1}$ is relational inverse. Axiomatically, the base Fischer Servi modal logic **IK** has normality axioms for both the modal box $\boxdot$ and the diamond $\diamonddot$, as well as the additional two axiom schemes:

$$\mathbf{FS1}:\ \diamonddot(\varphi \to \psi) \to (\boxdot\varphi \to \diamonddot\psi)\ \text{ and }\ \mathbf{FS2}:\ (\diamonddot\varphi \to \boxdot\psi) \to \boxdot(\varphi \to \psi) \tag{2}$$

A study of various normal extensions of **IK** is given in [37], and the finite model property and decidability of **IK** is established in [25] and further clarified in [22]. Earlier, starting from the 1950s, the intuitionistic S5 logic **MIPC** [35, 10] was given algebraic semantics in the form of *monadic Heyting algebras* [6, 31, 32, 39, 40][1] and later as bi-relational frames with an equivalence relation for the S5 modality [7, 16, 32, 39]. This line of work has focused on $\mathbf{MIPC} = \mathbf{IK} \oplus \mathbf{T}\boxdot\diamonddot \oplus \mathbf{5}\boxdot\diamonddot$ and its normal extensions[2], and translations into intuitionistic and intermediate predicate logics. Within algebraic semantics, topological spaces arise in the context of Stone duality, and in [6, 7, 16], the focus restricts to Stone spaces (compact, Hausdorff and totally disconnected, having as a basis the Boolean algebra of closed-and-open sets).

[1] The additional *monadic* operators are $\forall$ and $\exists$ unary operators behaving as S5 box and diamond modalities, and come from Halmos' work on monadic Boolean algebras.

[2] Here, $\mathbf{T}\boxdot\diamonddot$ is the conjunction of the separate $\boxdot$ and $\diamonddot$ characteristic schemes for reflexivity, and likewise $\mathbf{5}\boxdot\diamonddot$ for Euclideanness, so together they characterize equivalence relations.

In this paper, following [14], we give semantics for intuitionistic modal logic over topological Kripke frames $\mathcal{F} = (X, \mathcal{T}, R)$, where $(X, \mathcal{T})$ is a topological space and $R \subseteq X \times X$ is an accessibility relation for the modalities; the Fisher Servi bi-relational semantics are straight-forwardly extended from pre-orders $\preccurlyeq$ on X and their associated *Alexandrov topology* $\mathcal{T}_{\preccurlyeq}$, to arbitrary topological spaces $(X, \mathcal{T})$[3]. Over topological Kripke frames, the two Fischer Servi bi-relational conditions on the interaction between modal and intuitionistic semantics ((1) above) generalize to *semi-continuity* properties of the relation $R$, and of its inverse $R^{-1}$, with respect to the topology. As for the base logic, Fischer Servi's extension of the Gödel translation reads as a direct transcription of the topological semantics. The translation $\mathrm{GT}(\boxdot\varphi) = \Box\boxdot\,\mathrm{GT}(\varphi)$ says that the intuitionistic box requires the interior of the classical box operator, since the latter is defined by an intersection and may fail to preserve open sets. In contrast, the translation clause $\mathrm{GT}(\diamonddot\varphi) = \diamonddot\,\mathrm{GT}(\varphi)$ says that, semantically, the operator $\diamonddot$ preserves open sets. This condition is exactly the lower semi-continuity (l.s.c.) condition on the accessibility relation, and corresponds to the first Fischer Servi bi-relational inclusion $R^{-1}\circ \preccurlyeq \subseteq \preccurlyeq \circ R^{-1}$ in (1), and it is this condition that is required to verify topological soundness of the axiom scheme **FS1** in (2)[4]. Similarly, Fischer Servi's second bi-relational inclusion $R \circ \preccurlyeq \subseteq \preccurlyeq \circ R$ generalizes to the l.s.c. property of the $R^{-1}$ relation, where the latter is required to verify topological soundness of the axiom scheme **FS2** in (2).

The symmetry of the interaction conditions on the modal relation $R$ and its inverse $R^{-1}$ means that we can – with no additional semantic assumptions – lift the topological semantics to intuitionistic tense logics extending Fischer Servi's modal logic (introduced by Ewald in [17]), with modalities in pairs $\diamonddot$, $\boxdot$, and $\blacklozenge$, $\blacksquare$, for future and past along the accessibility relation. It soon becomes clear that the resulting semantics and meta theoretic results such as completeness come out *cleaner and simpler* for the tense logic than they do for the modal logic. We can often streamline arguments involving the box modality $\boxdot$ by using its adjoint diamond $\blacklozenge$, which like $\diamonddot$, preserves open sets. Furthermore, with regard to applications of interest, the flexibility of having both forwards and backwards modalities is advantageous.

For example, the core of transition system representations of dynamical systems (with discrete, continuous or hybrid evolution) is the *reachability relation*: one state has a second as a reachability successor iff there is a trajectory of the dynamical system leading from the first state to the second, and in general the dynamics are non-deterministic in the sense that there are multiple trajectories leading out of any state [3, 5, 13]. In this setting, a formula $\blacklozenge p$ denotes the set of states *reachable from* the $p$ states, with $p$ considered as a

[3]Other work giving topological semantics for intuitionistic modal logics is [41], further investigated in [27]. This logic is properly weaker than Fischer Servi's as its intuitionistic diamond is not required to distribute over disjunction (hence is sub-normal). Both the bi-relational and topological semantics in [41] and the *relational spaces* in [27] have *no* conditions on the interaction of the intuitionistic and modal semantic structures, and the semantic clauses for both box and diamond require application of the interior operator to guarantee open sets.

[4]In the algebraic setting of Monteiro and Varsavsky's work [31] w.r.t. the logic MIPC, a special case of the l.s.c. property is anticipated: the lattice of open sets of a topological space is a complete Heyting algebra, and the structure yields a monadic Heyting algebra when the space is further equipped with an equivalence relation $R$ with the property that the $R$-expansion of an open set is open.

source or initial state set, while the forward modal diamond formula $\Diamonddot p$ denotes the set of states *from which* $p$ states *can be reached*, here $p$ denoting a target or goal state set. The compound formula $\boxdot(\Diamonddot p \wedge \blacklozenge p)$ denotes the states from which the dynamics *always recurrently visits* $p$ states, in the sense that along every trajectory from such a state, at every point, the trajectory leads to a $p$-state in the future and leads from a $p$-state in the past. The state spaces of dynamical systems (of varying sorts) can be equipped with natural topologies (of varying sorts). For continuous dynamical systems, and the continuous components of hybrid systems, under some standard regularity assumptions on the differential inclusions or equations defining the dynamics [4, 5], the reachability relation $R$ and its inverse will be l.s.c. (as well as reflexive and transitive).

As an example of a topological concept expressible in the logics, consider the notion from [2] of a subset $A \subseteq X$ being *topologically stable* under a relation $R \subseteq X \times X$ in a topological space $(X, \mathcal{T})$ if for all open sets $U \in \mathcal{T}$, if $A \subseteq U$, then there exists an open set $V \in \mathcal{T}$ such that $A \subseteq V$ and for all $x, x' \in X$, if $x \in V$ and $x\,R\,x'$, then $x' \in U$. In words: if you start within the neighborhood $V$ of $A$, then all your $R$-successors lie in the given neighborhood $U$ of $A$. Let $\mathcal{M} = (X, \mathcal{T}, R, v)$ be a topological Kripke model, with valuation $v(p_0) = [\![\, p_0 \,]\!]^{\mathcal{M}} = A$. We can express the topological stability property of the set $A$ under the classical semantics by the inference rule: *from* $p_0 \rightarrow \Box\psi$, *infer* $p_0 \rightarrow \Box\boxdot\Box\psi$, or using the universal modality[5] by the formula scheme:

$$\boxed{\mathrm{u}}(p_0 \rightarrow \Box\psi) \rightarrow \boxed{\mathrm{u}}(p_0 \rightarrow \Box\boxdot\Box\psi)$$

*provided* the model $\mathcal{M}$ is such that the topology $\mathcal{T}$ is suitably 'saturated' in $\mathcal{M}$, in the sense that the family of all denotation sets $[\![\, \Box\psi \,]\!]^{\mathcal{M}}$, for $\psi$ ranging over all formulas, constitutes a *basis* for the topology $\mathcal{T}$. Under the intuitionistic semantics, where all formulas denote open sets, and in particular, $A = [\![\, p_0 \,]\!]^{\mathcal{M}}$ must be open, the topological stability property reduces to invariance for $p_0$, expressed by the validity of $p_0 \rightarrow \boxdot p_0$ in the model $\mathcal{M}$.

We continue on the theme of semi-continuity properties of relations in our second topic of investigation, namely that of *topological bisimulations* between topological Kripke models. A bisimulation notion for topological spaces $(X, \mathcal{T})$ has recently been developed by Aiello and van Benthem (e.g. [1], Def. 2.1). We show below that their forth and back topology-preserving conditions are equivalent to the lower semi-continuity of the inverse relation and of the relation, respectively. The first main result of the paper is that this notion of topological bisimulation yields the semantic preservation property w.r.t. topological Kripke models for both intuitionistic tense logics, and for their classical companion multi-modal logics in the setting of the Gödel translation, where semantic preservation means that bisimilar states satisfy the same set of formulas in their respective models, and thus are indistinguishable in the logic.

In the next part of the paper, we give canonical topological Kripke models for the Hilbert-style axiomatizations of the Fischer Servi logics and their classical companions logics – over the set of prime theories of the intuitionistic logic and the set of ultrafilters of the companion classical logic, respectively, with topologies on the spaces that are neither

[5] In a multi-modal language including $\boxed{\mathrm{u}}$, the classical semantics in a model $\mathcal{M}$ with state space $X$ are that $[\![\, \boxed{\mathrm{u}}\varphi \,]\!]^{\mathcal{M}} = X$ if $[\![\, \varphi \,]\!]^{\mathcal{M}} = X$, and otherwise $[\![\, \boxed{\mathrm{u}}\varphi \,]\!]^{\mathcal{M}} = \varnothing$.

Alexandrov nor Stone. While the canonical models are of interest in their own right, the primary use made of them here is as a means to establish the second main result of the paper, which addresses the question of which classes of models have the *Hennessy-Milner property* that indistinguishability under a topological bisimulation coincides with indistinguishability in the intuitionistic logic, or with indistinguishability in companion classical logics. For both semantics, we identify a class of models $\mathcal{M}$ with the property that the natural (single-valued) map from $\mathcal{M}$ into the canonical model is a topological bisimulation. Then for any two models $\mathcal{M}$ and $\mathcal{M}'$ in the class, the composition of the natural map from $\mathcal{M}$ with the inverse of the map from $\mathcal{M}'$ will be a topological bisimulation which maximally preserves indistinguishability in the semantics. We first give a logical characterization of the Hennessy-Milner class in terms of 'saturation' concepts developed for the classical topological and intuitionistic semantics, and then identify a set of purely topological conditions that together are sufficient for a model to be in the given Hennesy-Milner class.

The paper is organized as follows. Section 2 covers preliminaries from general topology, particularly continuity of relations or set-valued maps. Section 3 sets out the syntax and topological semantics of Fischer Servi intuitionistic modal and tense logics, and their classical companion logics, while Section 4 introduces topological bisimulations and includes the semantic preservation results. In Section 5, we give canonical topological models for axiomatizations of the Ewald's intuitionistic tense logic and its classical companion. The lengthy Section 6 is mostly devoted to the Hennessy-Milner property for the intuitionistic semantics, ending with a brief sketch of an analogous result for the classical topological semantics, and in Section 7, we investigate the given Hennessy-Milner class for the intuitionistic semantics, and characterize in purely topological terms a sub-class of the given class.

# 2 Preliminaries from general topology

We adopt the notation from set-valued analysis [4] in writing $r : X \rightsquigarrow Y$ to mean both that $r : X \to \mathcal{P}(Y)$ is a *set-valued map*, with (possibly empty) set-values $r(x) \subseteq Y$ for each $x \in X$, and equivalently, that $r \subseteq X \times Y$ is a *relation*. The expressions $y \in r(x)$, $(x, y) \in r$ and $x\, r\, y$ are synonymous. For a map $r : X \rightsquigarrow Y$, the *inverse* $r^{-1} : Y \rightsquigarrow X$ given by: $x \in r^{-1}(y)$ iff $y \in r(x)$; the *domain* is $dom(r) := \{x \in X \mid r(x) \neq \varnothing\}$, and the *range* is $ran(r) := dom(r^{-1}) \subseteq Y$. A map $r : X \rightsquigarrow Y$ is *total on* $X$ if $dom(r) = X$, and *surjective on* $Y$ if $ran(r) = Y$. We write (as usual) $r : X \to Y$ to mean $r$ is a *function*, i.e. a single-valued map total on $X$ with values written $r(x) = y$ (rather than $r(x) = \{y\}$). For $r_1 : X \rightsquigarrow Y$ and $r_2 : Y \rightsquigarrow Z$, we write their relational composition as $r_1 \circ r_2 : X \rightsquigarrow Z$ given by $(r_1 \circ r_2)(x) := \{z \in Z \mid (\exists y \in Y)\, [(x, y) \in r_1 \wedge (y, z) \in r_2]\}$. Recall that $(r_1 \circ r_2)^{-1} = r_2^{-1} \circ r_1^{-1}$. A *pre-order* (*quasi-order*) is a reflexive and transitive binary relation, and a *partial-order* is a pre-order that is also anti-symmetric.

A relation $r : X \rightsquigarrow Y$ determines two *pre-image operators* (predicate transformers). The *existential* (or *lower*) pre-image is of type $r^{-\exists} : \mathcal{P}(Y) \to \mathcal{P}(X)$ and the *universal* (or *upper*) pre-image $r^{-\forall} : \mathcal{P}(Y) \to \mathcal{P}(X)$ is its dual w.r.t. set-complement:

$$\begin{array}{rcl} r^{-\exists}(Z) & := & \{x \in X \mid (\exists y \in Y)[\,(x,y) \in r \quad \wedge \quad y \in Z]\} \\ & = & \{x \in X \mid Z \cap r(x) \neq \varnothing\} \\ r^{-\forall}(Z) & := & X - r^{-\exists}(Y - Z) \;=\; \{x \in X \mid r(x) \subseteq Z\} \end{array}$$

for all $Z \subseteq Y$. The operator $r^{-\exists}$ distributes over arbitrary unions, while $r^{-\forall}$ distributes over arbitrary intersections: $r^{-\exists}(\varnothing) = \varnothing$, $r^{-\exists}(Y) = dom(r)$, $r^{-\forall}(\varnothing) = X - dom(r)$, and $r^{-\forall}(Y) = X$. Note that when $r : X \to Y$ is a function, the pre-image operators reduce to the standard *inverse-image* operator; i.e. $r^{-\exists}(Z) = r^{-\forall}(Z) = r^{-1}(Z)$. The pre-image operators respect relational inclusions: if $r_1 \subseteq r_2 \subseteq X \times Y$, then for all $W \subseteq Y$, we have $r_1^{-\exists}(W) \subseteq r_2^{-\exists}(W)$, but reversing to $r_2^{-\forall}(W) \subseteq r_1^{-\forall}(W)$. For the case of binary relations $r \colon X \rightsquigarrow X$ on a space $X$, the pre-images express in operator form the standard relational Kripke semantics for the (future) diamond and box modal operators determined by $r$. The operators on sets derived from the inverse relation $r^{-1}$ are usually called the *post-image operators* $r^{\exists}$, $r^{\forall} : \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by $r^{\exists} := (r^{-1})^{-\exists}$ and $r^{\forall} := (r^{-1})^{-\forall}$; these arise in the relational Kripke semantics for the *past* diamond and box operators in tense and temporal logics. The fundamental relationship between pre- and post-images is the *adjoint property*:

$$\forall W \subseteq X,\ \forall Z \subseteq Y, \quad W \subseteq r^{-\forall}(Z) \quad \text{iff} \quad r^{\exists}(W) \subseteq Z\,. \tag{3}$$

Note that for compositions of relations, with $r_1 : X \rightsquigarrow Y$ and $r_2 : Y \rightsquigarrow Z$, the pre- and post-image operators satisfy $(r_1 \circ r_2)^{-\mathrm{Q}}(Z) = r_1^{-\mathrm{Q}}(r_2^{-\mathrm{Q}}(Z))$ and $(r_1 \circ r_2)^{\mathrm{Q}}(W) = r_2^{\mathrm{Q}}(r_1^{\mathrm{Q}}(W))$ for quantifiers $\mathrm{Q} \in \{\exists, \forall\}$, and sets $Z \subseteq Y$ and $W \subseteq X$.

A *topology* $\mathcal{T} \subseteq \mathcal{P}(X)$ on a set $X$ is a family of subsets of $X$ closed under arbitrary unions and finite intersections. The extreme cases are the *discrete* topology $\mathcal{T}_D = \mathcal{P}(X)$, and the *trivial* (or *indiscrete*) topology $\mathcal{T}_\varnothing = \{\varnothing, X\}$. The *interior operator* $int_{\mathcal{T}} : \mathcal{P}(X) \to \mathcal{P}(X)$ determined by $\mathcal{T}$ is given by $int_{\mathcal{T}}(W) := \bigcup\{U \in \mathcal{T} \mid U \subseteq W\}$. Sets $W \in \mathcal{T}$ are called *open* w.r.t. $\mathcal{T}$, and this is so iff $W = int_{\mathcal{T}}(W)$. Sets $W \subseteq X$ such that $(X - W) \in \mathcal{T}$ are called *closed* w.r.t. $\mathcal{T}$, and this is so iff $W = cl_{\mathcal{T}}(W)$, where the dual *closure operator* $cl_{\mathcal{T}} : \mathcal{P}(X) \to \mathcal{P}(X)$ is given by $cl_{\mathcal{T}}(W) := X - cl_{\mathcal{T}}(X - W)$, and the topological *boundary* is $bd_{\mathcal{T}}(W) := cl_{\mathcal{T}}(W) - int_{\mathcal{T}}(W)$. A family of open sets $\mathcal{B} \subseteq \mathcal{T}$ constitutes a *basis* for a topology $\mathcal{T}$ on $X$ if every open set $W \in \mathcal{T}$ is a union of basic opens in $\mathcal{B}$, and for every $x \in X$ and every pair of basic opens $U_1, U_2 \in \mathcal{B}$ such that $x \in U_1 \cap U_2$, there exists $U_3 \in \mathcal{B}$ such that $x \in U_3 \subseteq (U_1 \cap U_2)$. A family of sets $\{W_i\}_{i \in I}$ in $X$ has the *finite intersection property* if the intersection of every finite sub-family is non-empty; i.e. for every finite subset $F \subseteq I$ of indices, $\bigcap_{i \in F} W_i \neq \varnothing$. An elementary result we use is that a topological space $(X, \mathcal{T})$ is *compact* iff for every family of sets $\{W_i\}_{i \in I}$ with the finite intersection property, the intersection of all the closures is non-empty: $\bigcap_{i \in I} cl_{\mathcal{T}}(W_i) \neq \varnothing$.

The purely topological notion of *continuity* for a function $f \colon X \to Y$ is that the inverse image $f^{-1}(U)$ is open whenever $U$ is open. Analogous notions for relations/set-valued maps were first introduced by Kuratowski and Bouligand in the 1920s. Given two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$, a map $R : X \rightsquigarrow Y$ is called: *lower semi-continuous* (l.s.c.) if for every $\mathcal{S}$-open set $U$ in $Y$, $R^{-\exists}(U)$ is $\mathcal{T}$-open in $X$; *upper semi-continuous* (u.s.c.) if for every $\mathcal{S}$-open set $U$ in $Y$, $R^{-\forall}(U)$ is $\mathcal{T}$-open in $X$; and *Vietoris continuous* if it is both l.s.c. and u.s.c. [4, 9, 28, 38]. The u.s.c. condition is equivalent to $R^{-\exists}(V)$ is $\mathcal{T}$-closed in $X$ whenever $V$ is $\mathcal{S}$-closed in $Y$. Moreover, we have: $R : X \rightsquigarrow Y$ is l.s.c. iff $R^{-\exists}(int_{\mathcal{S}}(W)) \subseteq int_{\mathcal{T}}(R^{-\exists}(W))$ for all $W \subseteq Y$; and $R : X \rightsquigarrow Y$ is u.s.c. iff $R^{-\forall}(int_{\mathcal{S}}(W)) \subseteq int_{\mathcal{T}}(R^{-\forall}(W))$ for all $W \subseteq Y$ ([28], Vol. I, §18.I, p. 173). The two semi-continuity properties reduce to the standard notion of continuity for functions $R : X \to Y$, and both are preserved under relational composition, and also under finite unions of relations. We also make a limited use of yet another notion, that of *outer semi-continuity* (o.s.c.) which holds of a map $R : X \rightsquigarrow Y$ if $R(x)$ is closed for all $x \in dom(R)$ ($R$ is *closed-valued*) and $R^{-\exists}(V)$ is $\mathcal{T}$-closed in $X$ whenever $V$ is $\mathcal{S}$-compact in $Y$. A map $R : X \rightsquigarrow Y$ is called *Fell continuous* if it is l.s.c. and o.s.c. [6] If $Y$ is Hausdorff and $R$ is image-closed, then $R$ being u.s.c. implies $R$ is o.s.c.

We note the subclass of *Alexandrov topologies* because of their correspondence with Kripke relational semantics for classical S4 and intuitionistic logics. e.g. [1, 30]. A topological space $(X, \mathcal{T})$ is called *Alexandrov* if for every $x \in X$, there is a *smallest* open set $U \in \mathcal{T}$ such that $x \in U$. In particular, every *finite* topology (i.e. only finitely many open sets) is Alexandrov. There is a one-to-one correspondence between pre-orders on $X$ and Alexandrov topologies on $X$. Any pre-order $\preccurlyeq$ on $X$ induces an Alexandrov topology $\mathcal{T}_{\preccurlyeq}$ by taking $int_{\mathcal{T}_{\preccurlyeq}}(W) := (\preccurlyeq)^{-\forall}(W)$, which means $U \in \mathcal{T}_{\preccurlyeq}$ iff $U$ is upwards-$\preccurlyeq$-closed. In particular, $\mathcal{T}_{\preccurlyeq}$ is closed under arbitrary intersections as well as arbitrary unions, and $-\mathcal{T}_{\preccurlyeq} = \mathcal{T}_{\succcurlyeq}$. Conversely, for any topology, define a pre-order $\preccurlyeq_{\mathcal{T}}$ on $X$, known as the *specialisation pre-order*: $x \preccurlyeq_{\mathcal{T}} y$ iff $(\forall U \in \mathcal{T})\,[x \in U \Rightarrow y \in U]$. For any pre-order, $\preccurlyeq_{\mathcal{T}_{\preccurlyeq}} = \preccurlyeq$, and for any topology, $\mathcal{T}_{\preccurlyeq_{\mathcal{T}}} = \mathcal{T}$ iff $\mathcal{T}$ is Alexandrov (e.g. see [1]). Alexandrov topologies have weak separation properties: the only Alexandrov topology that is Hausdorff is the discrete topology.

# 3 Syntax and topological semantics

Fix a countable set $AP$ of atomic propositions. The propositional language $\mathcal{L}_0$ is generated from $p \in AP$ using the connectives $\vee$, $\wedge$, $\to$ and the constant $\perp$. As usual, define further connectives: $\neg\,\varphi := \varphi \to \perp$ and $\varphi_1 \leftrightarrow \varphi_2 := (\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$, and $\top := \perp \to \perp$. Let $\mathcal{L}_{0,\Box}$ be the mono-modal language extending $\mathcal{L}_0$ with the addition of the unary modal operator $\Box$. A further modal operator $\Diamond$ can be defined as the classical dual: $\Diamond\varphi := \neg\Box\neg\varphi$.

For the intuitionistic modal and tense languages, let $\mathcal{L}^{\mathbf{m}}$ ($\mathcal{L}^{\mathbf{t}}$) be the modal (tense) language extending $\mathcal{L}_0$ with the addition of two (four) modal operators $\diamonddot$ and $\boxdot$ (and $\blacklozenge$ and $\blacksquare$) , generated by the grammar:

$$\varphi \ ::= \ p \mid \perp \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \to \varphi_2 \mid \diamonddot\varphi \mid \boxdot\varphi \quad ( \mid \blacklozenge\varphi \mid \blacksquare\varphi \ )$$

[6] The Fell and Vietoris topologies are two possible structures on the set of $\mathrm{Cl}_{\mathcal{T}}[X]$ of closed subsets of a space, and continuity of set-valued maps $R : X \rightsquigarrow Y$ can be equivalently formulated as continuity of functions $R : X \to \mathrm{Cl}_{\mathcal{T}}[Y]$ when all images $R(x)$ are closed sets in $Y$. When $Y$ is compact Hausdorff, the Fell and Vietoris topologies on $\mathrm{Cl}_{\mathcal{T}}[Y]$ coincide, and for such $Y$, if $R$ is image-closed, then the u.s.c. and o.s.c. properties are equivalent.

for $p \in AP$. Likewise, for the classical topological modal and tense logics, let $\mathcal{L}^{\mathbf{m}}_{\Box}$ ($\mathcal{L}^{\mathbf{t}}_{\Box}$) be the modal (tense) language extending $\mathcal{L}_{0,\Box}$ with the addition of $\Diamonddot$ and $\boxdot$ (and $\blacklozenge$ and $\blacksquare$).

The original Gödel translation [23], as a function $\mathrm{GT} : \mathcal{L}_0 \to \mathcal{L}_{0,\Box}$, simply prefixes $\Box$ to *every* subformula of a propositional formula. Reading the S4 $\Box$ as topological interior, this means we force every propositional formula to intuitionistically denote an open set. In Fischer Servi's extension of the Gödel translation [20, 18], the clauses for the propositional fragment are from a variant translation used by Fitting [21], who shows it to be equivalent to Gödel's original ([21], Ch. 9, # 20). Define the function $\mathrm{GT} : \mathcal{L}^{\mathbf{t}} \to \mathcal{L}^{\mathbf{t}}_{\Box}$ by induction on formulas as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
\mathrm{GT}(p) & := & \Box p \ \text{ for } p \in AP & & & \\
\mathrm{GT}(\varphi_1 \to \varphi_2) & := & \Box\,(\mathrm{GT}(\varphi_1) \to \mathrm{GT}(\varphi_2)) & \mathrm{GT}(\bot) & := & \bot \\
\mathrm{GT}(\varphi_1 \vee \varphi_2) & := & \mathrm{GT}(\varphi_1) \vee \mathrm{GT}(\varphi_2) & \mathrm{GT}(\varphi_1 \wedge \varphi_2) & := & \mathrm{GT}(\varphi_1) \wedge \mathrm{GT}(\varphi_2) \\
\mathrm{GT}(\Diamonddot\varphi) & := & \Diamonddot\,\mathrm{GT}(\varphi) & \mathrm{GT}(\blacklozenge\varphi) & := & \blacklozenge\,\mathrm{GT}(\varphi) \\
\mathrm{GT}(\boxdot\varphi) & := & \Box\boxdot\,\mathrm{GT}(\varphi) & \mathrm{GT}(\blacksquare\varphi) & := & \Box\blacksquare\,\mathrm{GT}(\varphi)
\end{array}
$$

In topological terms, the only clauses in the translation where it is essential to have an explicit $\Box$ to guarantee openness of denotation sets are for atomic propositions, for implication $\to$, and for the box modalties $\boxdot$ and $\blacksquare$. There is no such need in the clauses for $\vee$ and $\wedge$ because finite unions and finite intersections of open sets are open. For the diamond modalties, the semi-continuity conditions that $R$ and its inverse $R^{-1}$ are both l.s.c. ensure that the semantic operators $R^{-\exists}$ and $R^{\exists}$ interpretting $\Diamonddot$ and $\blacklozenge$ must preserve open sets. We now explain this generalization, which was first presented in [14].

The bi-relational semantics of Fischer Servi [18, 19], and Plotkin and Stirling [34, 37] are over Kripke frames $\mathcal{F} = (X, \preccurlyeq, R)$, where $\preccurlyeq$ is a pre-order on $X$ and $R : X \rightsquigarrow X$ is the modal accessibility relation. Using the induced Alexandrov topology $\mathcal{T}_{\preccurlyeq}$, a bi-relational Kripke frame $\mathcal{F}$ is equivalent to the topological frame $(X, \mathcal{T}_{\preccurlyeq}, R)$. A set is open in $\mathcal{T}_{\preccurlyeq}$ exactly when it is $\preccurlyeq$-*persistent* or upward-$\preccurlyeq$-closed. The four bi-relational conditions identified in [34], and also familiar as the forth ("Zig") and back ("Zag") conditions for *bisimulations* (e.g. [8], Ch. 2), can be cleanly transcribed as *semi-continuity conditions* on the relations $R : X \rightsquigarrow X$ and $R^{-1} : X \rightsquigarrow X$ with respect to the topology $\mathcal{T}_{\preccurlyeq}$.

**Definition 3.1** *Let $\mathcal{F} = (X, \preccurlyeq, R)$ be a bi-relational frame. Four conditions expressing interaction between $\preccurlyeq$ and $R$ are identified as follows:*

$$
\begin{array}{ll}
\mathbf{Zig}(\preccurlyeq, R): & \textit{if}\ \ x \preccurlyeq y\ \ \textit{and}\ \ x\,R\,x'\ \textit{then}\ (\exists y' \in X)\big[\ y\,R\,y'\ \ \textit{and}\ \ x' \preccurlyeq y'\ \big] \\
\mathbf{Zag}(\preccurlyeq, R): & \textit{if}\ \ x \preccurlyeq y\ \ \textit{and}\ \ y\,R\,y'\ \textit{then}\ (\exists x' \in X)\big[\ x\,R\,x'\ \ \textit{and}\ \ x' \preccurlyeq y'\ \big] \\
\mathbf{Zig}(\preccurlyeq, R^{-1}): & \textit{if}\ \ x \preccurlyeq y\ \ \textit{and}\ \ x'R\,x\ \textit{then}\ (\exists y' \in X)\big[\ y'\,R\,y\ \ \textit{and}\ \ x' \preccurlyeq y'\ \big] \\
\mathbf{Zag}(\preccurlyeq, R^{-1}): & \textit{if}\ \ x \preccurlyeq y\ \ \textit{and}\ \ y'\,R\,y\ \textit{then}\ (\exists x' \in X)\big[\ x'R\,x\ \ \textit{and}\ \ x' \preccurlyeq y'\ \big]
\end{array}
$$

$$\begin{tikzcd} x \arrow[r, "\preccurlyeq"] \arrow[d, "R"'] & y \arrow[d, dashed, "R"] \\ x' \arrow[r, dashed, "\preccurlyeq"'] & y' \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\preccurlyeq"] \arrow[d, dashed, "R"'] & y \arrow[d, "R"] \\ x' \arrow[r, dashed, "\preccurlyeq"'] & y' \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\preccurlyeq"] & y \\ x' \arrow[u, "R"] \arrow[r, dashed, "\preccurlyeq"'] & y' \arrow[u, dashed, "R"'] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\preccurlyeq"] & y \\ x' \arrow[u, dashed, "R"] \arrow[r, dashed, "\preccurlyeq"'] & y' \arrow[u, "R"'] \end{tikzcd}$$

**Zig**$(\preccurlyeq, R)$ **Zag**$(\preccurlyeq, R)$ **Zig**$(\preccurlyeq, R^{-1})$ **Zag**$(\preccurlyeq, R^{-1})$

From earlier work [12], we know these bi-relational conditions correspond to semi-continuity properties of $R$ with respect to the Alexandrov topology $\mathcal{T}_{\preccurlyeq}$.

**Proposition 3.2** ([12]) *Let $\mathcal{F} = (X, \preccurlyeq, R)$ be a bi-relational frame, with $\mathcal{T}_{\preccurlyeq}$ it induced topology. The conditions in each row below are equivalent.*

| | | | |
|---|---|---|---|
| *1.* | **Zig**$(\preccurlyeq, R)$ | $(R^{-1}\circ \preccurlyeq) \subseteq (\preccurlyeq \circ R^{-1})$ | $R$ *is* l.s.c. *in* $\mathcal{T}_{\preccurlyeq}$ |
| *2.* | **Zag**$(\preccurlyeq, R)$ | $(\preccurlyeq \circ R) \subseteq (R \circ \preccurlyeq)$ | $R$ *is* u.s.c. *in* $\mathcal{T}_{\preccurlyeq}$ |
| *3.* | **Zig**$(\preccurlyeq, R^{-1})$ | $(R \circ \preccurlyeq) \subseteq (\preccurlyeq \circ R)$ | $R^{-1}$ *is* l.s.c. *in* $\mathcal{T}_{\preccurlyeq}$ |
| *4.* | **Zag**$(\preccurlyeq, R^{-1})$ | $(\preccurlyeq \circ R^{-1}) \subseteq (R^{-1}\circ \preccurlyeq)$ | $R^{-1}$ *is* u.s.c. *in* $\mathcal{T}_{\preccurlyeq}$ |

The Fischer Servi interaction conditions between the intuitionistic and modal relations, introduced in [19] and used in [17, 20, 25, 34, 37], are the first and third bi-relational conditions **Zig**$(\preccurlyeq, R)$ and **Zig**$(\preccurlyeq, R^{-1})$. In Kripke frames meeting these conditions, one can give semantic clauses for the diamond and box that are natural under the intuitionistic reading of the restricted $\exists$ and $\forall$ quantification with respect to $R$-successors. More precisely, the resulting logic is faithfully embedded into intuitionistic first-order logic by the standard modal to first-order translation, and a natural extension of the Gödel translation faithfully embeds it into the classical bi-modal logic combining **S4**$\Box$ with **K** or extensions.

Since the Fischer Servi interaction conditions for the *forward* or *future* modal operators $\diamonddot$ and $\boxdot$ for $R$ require the same l.s.c. property of both $R$ and $R^{-1}$, this means that, *at no extra cost* in semantic assumptions, we can add on the *backward* or *past* modal operators $\blacklozenge$ and $\blacksquare$ for $R^{-1}$, and obtain the desired interaction condition for $R^{-1}$ *for free.*

**Definition 3.3** *A* topological frame *is a structure $\mathcal{F} = (X, \mathcal{T}, R)$ where $(X, \mathcal{T})$ is a topological space and $R : X \rightsquigarrow X$ is a binary relation. $\mathcal{F}$ is an* l.s.c. topological frame *if both $R$ and $R^{-1}$ are l.s.c. in $\mathcal{T}$. A* model *over $\mathcal{F}$ is a structure $\mathcal{M} = (\mathcal{F}, v)$ where $v : AP \rightsquigarrow X$ is an atomic valuation relation. A model $\mathcal{M}$ is an* open model *if for each $p \in AP$, the denotation set $v(p)$ is open in $\mathcal{T}$. For open models $\mathcal{M}$ over l.s.c. frames $\mathcal{F}$, the intuitionistic denotation map $[\![\cdot]\!]^{\mathcal{M}}_{\mathbf{I}} : \mathcal{L}^{\mathbf{t}} \rightsquigarrow X$ (or $[\![\cdot]\!]^{\mathcal{M}}_{\mathbf{I}} : \mathcal{L}^{\mathbf{m}} \rightsquigarrow X$) is defined by:*

$$\begin{array}{rcl@{\qquad}rcl}
[\![p]\!]^{\mathcal{M}}_{\mathbf{I}} & := & v(p) \quad \textit{for } p \in AP & [\![\bot]\!]^{\mathcal{M}}_{\mathbf{I}} & := & \varnothing \\
[\![\varphi_1 \to \varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}} & := & int_{\mathcal{T}}((X - [\![\varphi_1]\!]^{\mathcal{M}}_{\mathbf{I}}) \cup [\![\varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}}) & & & \\
[\![\varphi_1 \vee \varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}} & := & [\![\varphi_1]\!]^{\mathcal{M}}_{\mathbf{I}} \cup [\![\varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}} & [\![\varphi_1 \wedge \varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}} & := & [\![\varphi_1]\!]^{\mathcal{M}}_{\mathbf{I}} \cap [\![\varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}} \\
[\![\diamonddot\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} & := & R^{-\exists}([\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}}) & [\![\boxdot\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} & := & int_{\mathcal{T}}\big(R^{-\forall}([\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}})\big) \\
[\![\blacklozenge\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} & := & R^{\exists}([\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}}) & [\![\blacksquare\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} & := & int_{\mathcal{T}}\big(R^{\forall}([\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}})\big)\ .
\end{array}$$

*A formula* $\varphi$ *is* satisfiable *in* $\mathcal{M}$ *if* $[\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}} \neq \varnothing$*, and* $\varphi$ *is* falsifiable *in* $\mathcal{M}$ *if* $[\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}} \neq X$*. For a formula* $\varphi \in \mathcal{L}^{\mathbf{t}}$ *(or* $\varphi \in \mathcal{L}^{\mathbf{m}}$*), we write* $\mathcal{M} \Vdash \varphi$*, if* $[\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}} = X$*, and for an l.s.c. frame* $\mathcal{F} = (X, \mathcal{T}, R)$*, we write* $\mathcal{F} \Vdash \varphi$*, if* $\mathcal{M} \Vdash \varphi$ *for all open models* $\mathcal{M}$ *over* $\mathcal{F}$*.*

*Let* $\mathbb{LSC}$ *denote the class of all l.s.c. topological frames. For any class of frames* $\mathbb{F} \subseteq \mathbb{LSC}$*, define the* intuitionistic theory *of* $\mathbb{F}$ *to be:*

$$\mathrm{Th}_{\mathbf{I}}(\mathbb{F}) := \{ \varphi \in \mathcal{L}^{\mathbf{t}} \mid (\forall \mathcal{F} \in \mathbb{F})\ \mathcal{F} \Vdash \varphi \}.$$

The property that every denotation set $[\![ \varphi ]\!]_{\mathbf{I}}^{\mathcal{M}}$ is open in $\mathcal{T}$ follows immediately from the openness condition on $v(p)$, the l.s.c. properties of $R^{-\exists}$ and $R^{\exists}$, and the extra interior operation in the semantics for $\rightarrow$, $\boxdot$ and $\blacksquare$.

**Definition 3.4** *For the tense (modal) language* $\mathcal{L}^{\mathbf{t}}_{\Box}$ *( and* $\mathcal{L}^{\mathbf{m}}_{\Box}$ *), we define the classical denotation map* $[\![ \cdot ]\!]^{\mathcal{M}} : \mathcal{L}^{\mathbf{t}}_{\Box} \rightsquigarrow X$ *(* $[\![ \cdot ]\!]^{\mathcal{M}} : \mathcal{L}^{\mathbf{m}}_{\Box} \rightsquigarrow X$ *) with respect to arbitrary topological models* $\mathcal{M} = (X, \mathcal{T}, R, v)$*, where* $v : AP \rightsquigarrow X$ *is unrestricted. The map* $[\![ \cdot ]\!]^{\mathcal{M}}$ *is defined the same way as* $[\![ \cdot ]\!]_{\mathbf{I}}^{\mathcal{M}}$ *for atomic* $p \in AP$*,* $\bot$*,* $\vee$*,* $\wedge$*,* $\diamonddot$ *and* $\blacklozenge$*, but differs on the following clauses:*

$$[\![\varphi_1 \rightarrow \varphi_2]\!]^{\mathcal{M}} := (X - [\![\varphi_1]\!]^{\mathcal{M}}) \cup [\![\varphi_2]\!]^{\mathcal{M}} \qquad [\![ \Box\varphi ]\!]^{\mathcal{M}} := int_{\mathcal{T}}([\![ \varphi ]\!]^{\mathcal{M}})$$
$$[\![ \boxdot\varphi ]\!]^{\mathcal{M}} := R^{-\forall}([\![\varphi]\!]^{\mathcal{M}}) \qquad [\![ \blacksquare\varphi ]\!]^{\mathcal{M}} := R^{\forall}([\![\varphi]\!]^{\mathcal{M}}).$$

*For a formula* $\varphi \in \mathcal{L}^{\mathbf{t}}_{\Box}$ *(or* $\varphi \in \mathcal{L}^{\mathbf{m}}_{\Box}$*), we write* $\mathcal{M} \models \varphi$*, if* $[\![\varphi]\!]^{\mathcal{M}} = X$*, and for a topological frame* $\mathcal{F} = (X, \mathcal{T}, R)$*, we write* $\mathcal{F} \models \varphi$*, if* $\mathcal{M} \models \varphi$ *for all models* $\mathcal{M}$ *over* $\mathcal{F}$*.*

*Let* $\mathbb{T}$ *denote the class of all topological frames. For any class of topological frames* $\mathbb{F} \subseteq \mathbb{T}$*, define the* classical theory *of* $\mathbb{F}$ *to be:*

$$\mathrm{Th}(\mathbb{F}) := \{ \psi \in \mathcal{L}^{\mathbf{t}}_{\Box} \mid (\forall \mathcal{F} \in \mathbb{F})\ \mathcal{F} \models \psi \}.$$

For Fischer Servi's extension of Gödel's translation, Definitions 3.3 and 3.4 imply that for any model $\mathcal{M} = (\mathcal{F}, v)$ over an l.s.c. topological frame $\mathcal{F}$, if $\mathcal{M}' = (\mathcal{F}, v')$ is the variant open model with $v'(p) := int_{\mathcal{T}}(v(p))$, then $\forall \varphi \in \mathcal{L}^{\mathbf{t}}$:

$$[\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}'} = [\![ \mathrm{GT}(\varphi) ]\!]^{\mathcal{M}} = [\![ \Box\mathrm{GT}(\varphi) ]\!]^{\mathcal{M}}. \tag{4}$$

Consequently, we have semantic faithfulness, as well as the openness property: for all $\varphi \in \mathcal{L}^{\mathbf{t}}$, the formula $\mathrm{GT}(\varphi) \leftrightarrow \Box\mathrm{GT}(\varphi)$ is in $\mathrm{Th}_{\mathbf{I}}(\mathbb{LSC})$.

**Proposition 3.5** [Extended Gödel translation: semantic faithfulness]
*For all* $\varphi \in \mathcal{L}^{\mathbf{t}}$*,* $\varphi \in \mathrm{Th}_{\mathbf{I}}(\mathbb{LSC})$ *iff* $\mathrm{GT}(\varphi) \in \mathrm{Th}(\mathbb{LSC})$*.*

The semi-continuity conditions can be cleanly characterized in the companion classical multi-modal logics, as given in [15].

**Proposition 3.6** [[15] Modal characterization of semi-continuity conditions]
*Let* $\mathcal{F} = (X, \mathcal{T}, R)$ *be a topological frame and let* $p \in AP$*. In the following table, the conditions listed across each row are equivalent.*

| | | | |
|---|---|---|---|
| (1.) | $R$ *is l.s.c. in* $\mathcal{T}$ | $\mathcal{F} \models \Diamonddot\Box p \to \Box\Diamonddot p$ | $\mathcal{F} \models \Diamonddot\Box p \leftrightarrow \Box\Diamonddot\Box p$ |
| (2.) | $R$ *is u.s.c. in* $\mathcal{T}$ | $\mathcal{F} \models \boxdot\Box p \to \Box\boxdot p$ | |
| (3.) | $R^{-1}$ *is l.s.c. in* $\mathcal{T}$ | $\mathcal{F} \models \Box\boxdot p \to \boxdot\Box p$ | $\mathcal{F} \models \blacklozenge\Box p \leftrightarrow \Box\blacklozenge\Box p$ |
| (4.) | $R^{-1}$ *is u.s.c. in* $\mathcal{T}$ | $\mathcal{F} \models \Box\Diamonddot p \to \Diamonddot\Box p$ | |

For naturally occurring l.s.c. topological frames, consider frames $\mathcal{F}$ where $X \subseteq \mathbb{R}^n$, with norm $\|\cdot\|$ on $\mathbb{R}^n$ inducing the standard Euclidean topology $\mathcal{T}_{\mathrm{E}}$ on $X$ (as a subspace of $\mathbb{R}^n$). Let $\mathsf{AC}(X)$ be the set of all functions $\gamma : [0, \tau] \to X$ such that $\tau \in \mathbb{R}^+_\circ := [0, \infty)$ and $\gamma$ is *absolutely continuous* on the real interval $[0, \tau]$. A *differential inclusion* is described by a set-valued map $F : X \rightsquigarrow \mathbb{R}^n$, and *solutions* to the inclusion $\dot{x} \in F(x)$ starting at a state $x \in X$ are defined by: $\mathsf{Sol}_F(x) := \big\{\, \gamma \in \mathsf{AC}(X) \mid \gamma(0) = x \wedge \big(\frac{d}{dt}\gamma\big)(s) \in F(\gamma(s))$ for almost all $s \in [0, \tau] \,\big\}$. The set $\mathsf{Sol}_F(x)$ is partially ordered by inclusion (considering solution curves as subsets $\gamma \subset \mathbb{R}^+_\circ \times X$). To ensure the existence of non-trivial solutions from each $x \in cl(dom(F))$, one needs to impose regularity assumptions on $F : X \rightsquigarrow \mathbb{R}^n$, such as the *Marchaud* conditions [5][7]. The *reachability relation* $R_F : X \rightsquigarrow X$ is defined by $(x, x') \in R_F$ iff there exists $\gamma \in \mathsf{Sol}_F(x)$ such that $\gamma(t) = x'$ for some $t \in dom(\gamma)$. Clearly, $R_F$ is reflexive and transitive, so the $\Diamonddot$ and $\boxdot$ modalities will satisfy the axioms of S4. Under the Marchaud conditions (and weaker assumptions) on $F$, both the forwards and backwards relations $R_F$ and $R_F^{-1}$ will be l.s.c., thus $\mathcal{F} = (X, \mathcal{T}_{\mathrm{E}}, R_F)$ will be an l.s.c. topological frame.

A *hybrid automaton* $H$ (see, e.g., [3]) with continuous dynamics in $\mathbb{R}^n$ consists of a finite family of differential inclusion maps $F_q : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and mode domains $D_q \subseteq \mathbb{R}^n$ indexed by $q \in Q$, with $Q$ the space of discrete modes, together with a transition graph $E : Q \rightsquigarrow Q$ and family of *reset* or *switching* relations $S_{q,q'} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ for each $(q, q') \in E$, describing when and how discrete changes of mode and dynamics are permitted. The system $H$ has the state space $X_H := \{(q, x) \in Q \times \mathbb{R}^n \mid x \in D_q\}$, and reachability relation $R_H : X_H \rightsquigarrow X_H$ such that $(q, x)\, R_H\, (q', x')$ iff there is a $H$-trajectory of finite duration leading from $(q, x)$ to $(q', x')$. Equipping $X_H$ with the product topology $\mathcal{T}_H$ arising from an Alexandrov topology $\mathcal{T}_{\preccurlyeq}$ from a pre-prder $\preccurlyeq$ on $Q$ and the Euclidean topology $\mathcal{T}_{\mathrm{E}}$ on $\mathbb{R}^n$, we have a topological frame $\mathcal{F}_H = (X_H, \mathcal{T}_H, R_H)$. The product topology $\mathcal{T}_H$ will then be Hausdorff only in special cases, when the pre-order $\preccurlyeq$ is identity and $\mathcal{T}_{\preccurlyeq}$ is discrete, or when the mode domains $D_q$ are pair-wise disjoint. Assume the reset relations $S_{q,q'}$ and their inverses are l.s.c. with respect to $\mathcal{T}_{\mathrm{E}}$ on $\mathbb{R}^n$ (which implies that the *transition guard* regions $dom(S_{q,q'})$ and the *post-transition* sets $ran(S_{q,q'})$ are open), and assume regularity conditions on the continuous dynamics $F_q$ and the domains $D_q$ sufficient for the l.s.c. property for their reachability relations $R_q : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and their inverses. Further assume that the discrete transition relation $E : Q \rightsquigarrow Q$ is such that $(E^{-1} \circ \preccurlyeq) \subseteq (\preccurlyeq \circ E^{-1})$ and $(E \circ \preccurlyeq) \subseteq (\preccurlyeq \circ E)$, and hence $E$ and $E^{-1}$ are l.s.c. with respect to $\mathcal{T}_{\preccurlyeq}$. Then the hybrid reachability relation $R_H$ and its inverse will be l.s.c., and thus the frame $\mathcal{F}_H$ will be l.s.c..

[7](a) $F$ is total on $X$; (b) $F \subseteq X \times \mathbb{R}^n$ is a closed set; (c) the image set $F(x)$ is convex and compact in $\mathbb{R}^n$ for every $x \in dom(F)$; and (d) there exists a real constant $c > 0$ such that $\sup\{\, \|y\| \mid y \in F(x)\} \leqslant c(\|x\| + 1)$ for all $x \in dom(F)$.

# 4 Topological bisimulations

Aiello and van Benthem's notions of topological simulation and bisimulation between classical S4 topological models are as follows.

**Definition 4.1** [[1], Definition 2.1] *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces, let $v_1 : AP \rightsquigarrow X_1$ and $v_2 : AP \rightsquigarrow X_2$ be valuations of atomic propositions, and let $\mathcal{M}_1 = (X_1, \mathcal{T}_1, v_1)$ and $\mathcal{M}_2 = (X_2, \mathcal{T}_2, v_2)$ be topological models.*
*A relation $B : X_1 \rightsquigarrow X_2$ is a* topo-bisimulation *between $\mathcal{M}_1$ and $\mathcal{M}_2$ if*

(i.a) $\forall x \in X_1,\ \forall y \in X_2,\ \forall p \in AP,$ if $x\,B\,y$ and $x \in v_1(p)$ then $y \in v_2(p)$ ;
(i.b) $\forall x \in X_1,\ \forall y \in X_2,\ \forall p \in AP,$ if $x\,B\,y$ and $y \in v_2(p)$ then $x \in v_1(p)$ ;
(ii.a) $\forall x \in X_1,\ \forall y \in X_2,\ \forall U \in \mathcal{T}_1,$ if $x\,B\,y$ and $x \in U$
then $\exists V \in \mathcal{T}_2$ with $y \in V$ and $\forall y' \in V,\ \exists x' \in U$ such that $x'\,B\,y'$ ;
(ii.b) $\forall x \in X_1,\ \forall y \in X_2,\ \forall V \in \mathcal{T}_2,$ if $x\,B\,y$ and $y \in V$
then $\exists U \in \mathcal{T}_1$ with $x \in U$ and $\forall x' \in U,\ \exists y' \in V$ such that $x'\,B\,y'$ .

*If only conditions* (i.a) *and* (ii.a) *hold of a relation $B : X_1 \rightsquigarrow X_2$, then $B$ is called a* topo-simulation *of $\mathcal{M}_1$ by $\mathcal{M}_2$.*

Our first observation is that the topological conditions (ii.b) and (ii.a) are equivalent to lower semi-continuity properties of the relation $B$ and its inverse.

**Proposition 4.2** *Given a map $B : X_1 \rightsquigarrow X_2$ between $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$,*
(1.) *$B$ satisfies condition* (ii.a) *of Definition 4.1 iff $B^{-1}$ is* l.s.c.*;*
(2.) *$B$ satisfies condition* (ii.b) *of Definition 4.1 iff $B$ is* l.s.c..

**Proof.** By rewriting in terms of the pre- and post-image set-operators, it is easy to show that conditions (ii.a) and (ii.b) are equivalent to the following:

$$\begin{array}{lllll} (\text{ii.a}^\sharp) & \forall U \in \mathcal{T}_1, & B^{\exists}(U) & \subseteq & int_{\mathcal{T}_2}\big(B^{\exists}(U)\big) \\ (\text{ii.b}^\sharp) & \forall V \in \mathcal{T}_2, & B^{-\exists}(V) & \subseteq & int_{\mathcal{T}_1}\big(B^{-\exists}(V)\big) \end{array}$$

Clearly, (ii.a$^\sharp$) says that $B^{\exists}(U)$ is open in $X_2$ whenever $U$ open in $X_1$, while (ii.b$^\sharp$) says that $B^{-\exists}(V)$ is open in $X_1$ whenever $V$ open in $X_2$. $\dashv$

For a suitable notion of topological bisimulation between topological Kripke models for the intuitionistic and classical companion modal and tense logics under study here, we need to put together the topology-preserving conditions (ii.a) and (ii.b) above with the standard clauses respecting the modal/tense semantic structure.

**Definition 4.3** *Let $\mathcal{M}_1 = (X_1, \mathcal{T}_1, R_1, v_1)$ and $\mathcal{M}_2 = (X_2, \mathcal{T}_2, R_2, v_2)$ be two topological models. A map $B : X_1 \rightsquigarrow X_2$ will be called a* tense topo-bisimulation *between $\mathcal{M}_1$ and $\mathcal{M}_2$ if for all atomic $p \in AP$:*

$$\begin{array}{llll} (\text{i.a}) & B^{\exists}(v_1(p)) \subseteq v_2(p) & (\text{i.b}) & B^{-\exists}(v_2(p)) \subseteq v_1(p) \\ (\text{ii.a}) & B^{-1} : X_2 \rightsquigarrow X_1 \textit{ is l.s.c.} & (\text{ii.b}) & B : X_1 \rightsquigarrow X_2 \textit{ is l.s.c.} \\ (\text{iii.a}) & (B^{-1} \circ R_1) \subseteq (R_2 \circ B^{-1}) & (\text{iii.b}) & (B \circ R_2) \subseteq (R_1 \circ B) \\ (\text{iv.a}) & (B^{-1} \circ R_1^{-1}) \subseteq (R_2^{-1} \circ B^{-1}) & (\text{iv.b}) & (B \circ R_2^{-1}) \subseteq (R_1^{-1} \circ B) \end{array}$$

*If only conditions* (i.a), (ii.a) *and* (iii.a) *hold of the map* $B : X_1 \rightsquigarrow X_2$*, then* $B$ *is called a* modal topo-simulation *of* $\mathcal{M}_1$ *by* $\mathcal{M}_2$*; if all but conditions* (iv.a) *and* (iv.b) *hold, then* $B$ *is a* modal topo-bisimulation *between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*.*

What we discover is that this notion of bisimulation between models yields the same semantic preservation property for *both* the intuitionistic and the classical semantics. Otherwise put, the specifically *topological* requirement that the operators $B^{\exists}$ and $B^{-\exists}$ preserve open sets is enough to push through the result for intuitionistic modal and tense logics.

Conditions (i.a) and (i.b) give the base case for atomic propositions, in an induction on formulas $\varphi \in \mathcal{L}^{\mathbf{t}}$ and for open l.s.c. models $\mathcal{M}_1$ and $\mathcal{M}_2$, for the following semantic preservation inclusions in set-operator form:

$$B^{\exists}(\llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}_1}) \subseteq \llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}_2} \quad \text{and} \quad B^{-\exists}(\llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}_2}) \subseteq \llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}_1} \tag{5}$$

and likewise for classical denotation maps $\llbracket \varphi \rrbracket^{\mathcal{M}_i}$, without restriction on the topological models. We will also use the dual versions under the adjoint equivalence (3). These are:

$$\llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}_1} \subseteq B^{-\forall}(\llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}_2}) \quad \text{and} \quad \llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}_2} \subseteq B^{\forall}(\llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}_1}) \tag{6}$$

and likewise for $\llbracket \varphi \rrbracket^{\mathcal{M}_i}$.

Combining all the four Zig-Zag conditions (iii.a), (iii.b), (iv.a) and (iv.b), one obtains two relational equalities: $(R_1 \circ B) = (B \circ R_2)$ and $(R_2 \circ B^{-1}) = (B^{-1} \circ R_1)$. The inductive cases for $\blacklozenge$ and $\diamondsuit$ in the proof of the set-operator semantic preservation inclusions (5), for both the intuitionistic and the classical semantics, are easy consequences of these two equalities.

The topological condition (ii.a) requiring that $B^{-1} : X_2 \rightsquigarrow X_1$ be l.s.c. has a further equivalent set-operator characterization: $int_{\mathcal{T}_1}(B^{-\forall}(Z)) \subseteq B^{-\forall}(int_{\mathcal{T}_2}(Z))$, for all $Z \subseteq X_2$; symmetrically, condition (ii.b) requiring that $B : X_1 \rightsquigarrow X_2$ be l.s.c. is equivalent to $int_{\mathcal{T}_2}(B^{\forall}(W)) \subseteq B^{\forall}(int_{\mathcal{T}_1}(W))$, for all $W \subseteq X_1$. These are generalizations of the characterization for binary relations on a single space $X$ that is formalized in Proposition 3.6, Row (*3.*).

The set-operator semantic preservation inclusions (5), for a formula $\varphi \in \mathcal{L}^{\mathbf{t}}$ and a tense topo-bisimulation $B$ from model $\mathcal{M}_1$ to model $\mathcal{M}_2$, are equivalent to the condition that for all states $x$ in $\mathcal{M}_1$ and $y$ in $\mathcal{M}_2$, if $x\,B\,y$ then the formula $\varphi$ is either satisfied by both $x$ in $\mathcal{M}_1$ and $y$ in $\mathcal{M}_2$, or else it is not satisfied by either of them. Semantic preservation thus means that bisimilar states satisfy the same formulas in their respective models.

**Definition 4.4** *Let* $\mathcal{M} = (X, \mathcal{T}, R, v)$ *be a topological model. Define the* classical theory map $\mathrm{Th}^{\mathcal{M}} \colon X \rightsquigarrow \mathcal{L}^{\mathbf{t}}_{\Box}$ *to be the inverse of the denotation map* $\llbracket \cdot \rrbracket^{\mathcal{M}} \colon \mathcal{L}^{\mathbf{t}}_{\Box} \rightsquigarrow X$*; that is,* $\mathrm{Th}^{\mathcal{M}}(x) := \{\psi \in \mathcal{L}^{\mathbf{t}}_{\Box} \mid x \in \llbracket \psi \rrbracket^{\mathcal{M}}\}$*. When* $\mathcal{M}$ *is open and l.s.c., likewise define the* intuitionistic theory map $\mathrm{Th}_{\mathrm{I}}^{\mathcal{M}} \colon X \rightsquigarrow \mathcal{L}^{\mathbf{t}}$ *to be the inverse of the denotation map* $\llbracket \cdot \rrbracket_{\mathrm{I}}^{\mathcal{M}} \colon \mathcal{L}^{\mathbf{t}} \rightsquigarrow X$*, so* $\mathrm{Th}_{\mathrm{I}}^{\mathcal{M}}(x) := \{\varphi \in \mathcal{L}^{\mathbf{t}} \mid x \in \llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}}\}$*.*

It is immediate that $\mathrm{Th}_{\mathrm{I}}^{\mathcal{M}_1}(x) = \mathrm{Th}_{\mathrm{I}}^{\mathcal{M}_2}(y)$ holds exactly when $x \in \llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}_1}$ iff $y \in \llbracket \varphi \rrbracket_{\mathrm{I}}^{\mathcal{M}_2}$ for all $\varphi \in \mathcal{L}^{\mathbf{t}}$, and likewise for the classical semantics.

**Theorem 4.5** [Semantic preservation for tense topo-bisimulations]
*Let $\mathcal{M}_1 = (X_1, \mathcal{T}_1, R_1, v_1)$ and $\mathcal{M}_2 = (X_2, \mathcal{T}_2, R_2, v_2)$ be any two topological models, and let $B : X_1 \rightsquigarrow X_2$ be a tense topo-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$.*

(1.) *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are open and l.s.c., then for all $x \in X_1$ and $y \in X_2$:*

$$x\,B\,y \qquad \textit{implies} \qquad \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}_1}(x) = \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}_2}(y)\ .$$

(2.) *For all $x \in X_1$ and $y \in X_2$:*

$$x\,B\,y \qquad \textit{implies} \qquad \mathrm{Th}^{\mathcal{M}_1}(x) = \mathrm{Th}^{\mathcal{M}_2}(y)\ .$$

**Proof.** The proof proceeds as usual, by induction on the structure of formulas, to establish the two inclusions displayed in (5), or their analogs for the classical denotation maps. As noted already, the base case for atomic propositions is given by conditions (i.a) and (i.b), and the induction case for $\blacklozenge$ and $\diamonddot$ are immediate from conditions (iii) and (iv). For the classical semantics in Part (2.), the argument is completely standard for the propositional and modal/tense operators, and the case for topological $\Box$ is given in [1]. For the intuitionistic semantics in Part (1.), we give the cases for implication $\to$ and for box $\boxdot$. Assume the result holds for $\varphi_1$ and $\varphi_2$ in $\boldsymbol{\mathcal{L}^{\mathbf{t}}}$. In particular, from inclusions (5) and (6), we have: $(X_1 - [\![\varphi_1]\!]_{\mathbf{I}}^{\mathcal{M}_1}) \subseteq (X_1 - B^{-\exists}([\![\varphi_1]\!]_{\mathbf{I}}^{\mathcal{M}_2}))$, and $[\![\varphi_2]\!]_{\mathbf{I}}^{\mathcal{M}_1} \subseteq B^{-\forall}([\![\,\varphi_2\,]\!]_{\mathbf{I}}^{\mathcal{M}_2})$. Now:

$$\begin{array}{cll}
 & B^{\exists}\,([\![\,\varphi_1 \to \varphi_2\,]\!]_{\mathbf{I}}^{\mathcal{M}_1}) & \\
= & B^{\exists}\,(int_{\mathcal{T}_1}\,(\,(\,X_1 - [\![\varphi_1]\!]_{\mathbf{I}}^{\mathcal{M}_1}\,)\,\cup\,[\![\varphi_2]\!]_{\mathbf{I}}^{\mathcal{M}_1})) & \\
\subseteq & B^{\exists}\left(int_{\mathcal{T}_1}\left(X_1 - B^{-\exists}([\![\varphi_1]\!]_{\mathbf{I}}^{\mathcal{M}_2})\,\cup\,B^{-\forall}([\![\varphi_2]\!]_{\mathbf{I}}^{\mathcal{M}_2})\right)\right) & \text{by induction hypothesis} \\
= & B^{\exists}\left(int_{\mathcal{T}_1}\left(B^{-\forall}(X_2 - [\![\varphi_1]\!]_{\mathbf{I}}^{\mathcal{M}_2})\,\cup\,B^{-\forall}([\![\varphi_2]\!]_{\mathbf{I}}^{\mathcal{M}_2})\right)\right) & \text{by duality } B^{-\forall}\,/\,B^{-\exists} \\
\subseteq & int_{\mathcal{T}_2}\left(B^{\exists}\left(B^{-\forall}(X_2 - [\![\varphi_1]\!]_{\mathbf{I}}^{\mathcal{M}_2})\,\cup\,B^{-\forall}([\![\varphi_2]\!]_{\mathbf{I}}^{\mathcal{M}_2})\right)\right) & \text{by } B^{-1} \text{ being l.s.c.} \\
\subseteq & int_{\mathcal{T}_2}\left(B^{\exists}\left(B^{-\forall}\left(\,(X_2 - [\![\varphi_1]\!]_{\mathbf{I}}^{\mathcal{M}_2})\,\cup\,[\![\varphi_2]\!]_{\mathbf{I}}^{\mathcal{M}_2}\,\right)\right)\right) & \text{by monotonicity of } B^{-\forall} \\
\subseteq & int_{\mathcal{T}_2}\,(\,(\,X_2 - [\![\varphi_1]\!]_{\mathbf{I}}^{\mathcal{M}_2}\,)\,\cup\,[\![\varphi_2]\!]_{\mathbf{I}}^{\mathcal{M}_2}\,) & \text{by adjoint property} \\
= & [\![\,\varphi_1 \to \varphi_2\,]\!]_{\mathbf{I}}^{\mathcal{M}_2} &
\end{array}$$

Verifying that $B^{-\exists}\,([\![\,\varphi_1 \to \varphi_2\,]\!]_{\mathbf{I}}^{\mathcal{M}_2}) \subseteq [\![\,\varphi_1 \to \varphi_2\,]\!]_{\mathbf{I}}^{\mathcal{M}_1}$ proceeds similarly, using from the induction hypothesis: $(X_2 - [\![\varphi_1]\!]_{\mathbf{I}}^{\mathcal{M}_2}) \subseteq (X_2 - B^{\exists}([\![\varphi_1]\!]_{\mathbf{I}}^{\mathcal{M}_1}))$, and $[\![\varphi_2]\!]_{\mathbf{I}}^{\mathcal{M}_2} \subseteq B^{\forall}([\![\,\varphi_2\,]\!]_{\mathbf{I}}^{\mathcal{M}_1})$. For the $\boxdot$ case:

$$\begin{array}{cll}
 & [\![\,\boxdot\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}_1} & \\
= & int_{\mathcal{T}_1}\left(R_1^{-\forall}\,([\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}_1})\right) & \\
\subseteq & int_{\mathcal{T}_1}\left(R_1^{-\forall}\left(B^{-\forall}\,([\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}_2})\right)\right) & \text{by induction hypothesis} \\
\subseteq & int_{\mathcal{T}_1}\left(B^{-\forall}\left(R_2^{-\forall}\,([\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}_2})\right)\right) & \text{since } R_1 \circ B = B \circ R_2 \\
\subseteq & B^{-\forall}\left(int_{\mathcal{T}_2}\left(R_2^{-\forall}\,([\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}_2})\right)\right) & \text{by } B^{-1} \text{ being l.s.c. (dual } B^{-\forall} \text{ form)} \\
= & B^{-\forall}\,([\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}_2}) &
\end{array}$$

The argument for $\blacksquare$ symmetrically appeals to $B$ being l.s.c. (dual $B^{\forall}$ form). $\dashv$

In Section 6 below, we give a partial converse (Hennessy-Milner type result) by proving that a certain class of open l.s.c. models has the property that for any two models $\mathcal{M}$

and $\mathcal{M}'$ in the class, there is a tense topo-bisimulation $B$ between them that maximally preserves the intuitionistic semantics, in the sense that for all $x \in X$ and $y \in X'$:

$$x \, B \, y \qquad \text{iff} \qquad \mathrm{Th}^{\mathcal{M}}_{\mathbf{I}}(x) = \mathrm{Th}^{\mathcal{M}'}_{\mathbf{I}}(y) \; .$$

A key ingredient in the Hennessy-Milner type result is the *canonical topological model* of the base Fisher Servi tense logic, as developed in the next section.

# 5 Axiomatizations and canonical models

Let $\mathbf{IPC} \subseteq \mathcal{L}_0$ be the set of intuitionistic propositional theorems, and abusing notation, let $\mathbf{IPC}$ also denote a standard axiomatisation for that logic. Likewise, let $\mathbf{S4}\Box \subseteq \mathcal{L}_{0,\Box}$ be the set of theorems of classical S4, and let $\mathbf{S4}\Box$ also denote any standard axiomatisation of classical S4. To be concrete, let $\mathbf{S4}\Box$ contain all instances of classical propositional tautologies in the language $\mathcal{L}_{0,\Box}$, and of the axiom schemes:

$$\mathbf{N}\Box \; : \; \Box\top \qquad\qquad \mathbf{T}\Box \; : \; \Box\varphi \to \varphi$$
$$\mathbf{R}\Box \; : \; \Box(\varphi_1 \wedge \varphi_2) \leftrightarrow \Box\varphi_1 \wedge \Box\varphi_2 \qquad\qquad \mathbf{4}\Box \; : \; \Box\varphi \to \Box\Box\varphi$$

and be closed under the inference rules of *modus ponens* (**MP**), *uniform substitution* (**Subst**) (of formulas for atomic propositions), and $\Box$-monotonicity (**Mono**$\Box$): from $\varphi_1 \to \varphi_2$ infer $\Box\varphi_1 \to \Box\varphi_2$.

On notation, for any axiomatically presented logic $\Lambda$ in a language $\mathcal{L}$, set of formulas $\Psi \subseteq \mathcal{L}$ and formula $\varphi \in \mathcal{L}$, we write $\Psi \vdash_\Lambda \varphi$ to mean that there exists a finite set $\{\psi_1, \ldots, \psi_n\} \subseteq \Psi$ of formulas such that $(\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$ is a theorem of $\Lambda$ (allowing $n = 0$ and $\varphi$ is a theorem of $\Lambda$). The relation $\vdash_\Lambda \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ is the consequence relation of $\Lambda$. We will abuse notation (as we have with **IPC** and $\mathbf{S4}\Box$) and identify $\Lambda$ with its set of theorems: i.e. $\Lambda = \{\varphi \in \mathcal{L} \mid \varnothing \vdash_\Lambda \varphi\}$.

Let **IK** be the axiomatic system of Fischer Servi [20, 11, 17, 25], which is equivalent to an alternative axiomatisation given in [34, 37]; **IK** also goes by the name **FS** in [25] and [22, 42, 43]. **IK** has as axioms all instances in the language $\mathcal{L}^{\mathbf{m}}$ of the axiom schemes of **IPC**, and the following further axiom schemes:

$$\mathbf{R}\diamonddot \; : \; \diamonddot(\varphi \vee \psi) \leftrightarrow (\diamonddot\varphi \vee \diamonddot\psi) \qquad\qquad \mathbf{N}\diamonddot \; : \; \neg\diamonddot\bot$$
$$\mathbf{R}\boxdot \; : \; \boxdot(\varphi \wedge \psi) \leftrightarrow (\boxdot\varphi \wedge \boxdot\psi) \qquad\qquad \mathbf{N}\boxdot \; : \; \boxdot\top$$
$$\mathbf{F1}\boxdot\diamonddot \; : \; \diamonddot(\varphi \to \psi) \to (\boxdot\varphi \to \diamonddot\psi) \qquad\qquad \mathbf{F2}\boxdot\diamonddot \; : \; (\diamonddot\varphi \to \boxdot\psi) \to \boxdot(\varphi \to \psi)$$

and is closed under the inference rules (**MP**) and (**Subst**), and the rule (**Mono**$\diamonddot$): from $\varphi_1 \to \varphi_2$ infer $\diamonddot\varphi_1 \to \diamonddot\varphi_2$, and likewise (**Mono**$\boxdot$).

With regard to notation for combinations of modal logics, we follow that of [22]. If $\Lambda_1$ and $\Lambda_2$ are axiomatically presented modal logics in languages $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then the *fusion* $\Lambda_1 \otimes \Lambda_2$ is the smallest multi-modal logic in the language $\mathcal{L}_1 \otimes \mathcal{L}_2$ containing $\Lambda_1$ and $\Lambda_2$, and closed under all the inference rules of $\Lambda_1$ and $\Lambda_2$, where $\mathcal{L}_1 \otimes \mathcal{L}_2$ denotes the least common extension of the languages $\mathcal{L}_1$ and $\mathcal{L}_2$. If $\Lambda$ is a logic in language $\mathcal{L}$, and $\Gamma$ is a finite list of schemes in $\mathcal{L}$, then the *extension* $\Lambda \oplus \Gamma$ is the smallest logic in $\mathcal{L}$ extending $\Lambda$, containing the schemes in $\Gamma$ as additional axioms, and closed under the rules of $\Lambda$. The basic system in [42], under the name **IntK**, is such that: $\mathbf{IK} = \mathbf{IntK} \oplus \mathbf{F1}\boxdot\Diamonddot \oplus \mathbf{F2}\boxdot\Diamonddot$. The latter two schemes were identified by Fischer Servi in [20][8].

For the extension to tense logics with forwards and backwards modalities, let $\mathbf{IK^t}$ be Ewald's [17] deductive system, which is the fusion of $\mathbf{IK}\Diamonddot\boxdot := \mathbf{IK}$ with the "mirror" system $\mathbf{IK}\blacklozenge\blacksquare$ having axiom schemes $\mathbf{R}\blacklozenge$, $\mathbf{N}\blacklozenge$, $\mathbf{R}\blacksquare$, $\mathbf{N}\blacksquare$, $\mathbf{F1}\blacksquare\blacklozenge$ and $\mathbf{F2}\blacksquare\blacklozenge$, and inference rules ($\mathbf{Mono}\blacklozenge$) and ($\mathbf{Mono}\blacksquare$), which is then further extended with four axiom schemes expressing the *adjoint property* (Assertion (3)) of the operators interpreting the tense modalities:

$$\mathbf{Ad1} : \varphi \to \boxdot\blacklozenge\varphi \qquad \mathbf{Ad2} : \varphi \to \blacksquare\Diamonddot\varphi \qquad \mathbf{Ad3} : \Diamonddot\blacksquare\varphi \to \varphi \qquad \mathbf{Ad4} : \blacklozenge\boxdot\varphi \to \varphi\ .$$

Thus $\mathbf{IK^t} := (\mathbf{IK}\Diamonddot\boxdot \otimes \mathbf{IK}\blacklozenge\blacksquare) \oplus \mathbf{Ad1} \oplus \mathbf{Ad2} \oplus \mathbf{Ad3} \oplus \mathbf{Ad4}$.

We now identify the companion classical logics. Let $\mathbf{K}\boxdot$ be the minimal normal modal logic (over a classical propositional base), let $(\mathbf{S4}\Box \otimes \mathbf{K}\boxdot)$ be the bi-modal fusion of $\mathbf{S4}\Box$ and $\mathbf{K}\boxdot$, and let $\mathbf{K^mLSC} := (\mathbf{S4}\Box \otimes \mathbf{K}\boxdot) \oplus (\Diamonddot\Box\varphi \to \Box\Diamonddot\varphi) \oplus (\Box\boxdot\varphi \to \boxdot\Box\varphi)$ be the extension of $(\mathbf{S4}\Box \otimes \mathbf{K}\boxdot)$ with characteristic modal schemes for the $R$-l.s.c. and $R^{-1}$-l.s.c. frame conditions, from Proposition 3.6 (and as identified in [18]). Likewise, $\mathbf{K^t} := (\mathbf{K}\boxdot \otimes \mathbf{K}\blacksquare) \oplus \mathbf{Ad1} \oplus \mathbf{Ad2}$ is the minimal normal tense logic, and $\mathbf{K^tLSC} := (\mathbf{S4}\Box \otimes \mathbf{K^t}) \oplus (\Diamonddot\Box\varphi \to \Box\Diamonddot\varphi) \oplus (\blacklozenge\Box\varphi \to \Box\blacklozenge\varphi)$, here using instead the tense scheme for $R^{-1}$-l.s.c. from Proposition 3.6. (A related logic is $\mathbf{S4LSC} := (\mathbf{S4}\Box \otimes \mathbf{S4}\boxdot) \oplus (\Diamonddot\Box\varphi \to \Box\Diamonddot\varphi)$ of the fusion of $\mathbf{S4}\Box$ and $\mathbf{S4}\boxdot$, studied in [15], where it goes under the working name of **LSC**).

In what follows, we will deal generically with extensions $\mathbf{IK^t} \oplus \Gamma$ for subsets $\Gamma$ of the five axiom schemes below or their $\blacksquare$-$\blacklozenge$ mirror images:

$$\begin{array}{ll}
\mathbf{T}\boxdot\Diamonddot : (\boxdot\varphi \to \varphi) \wedge (\varphi \to \Diamonddot\varphi) & \mathbf{B}\boxdot\Diamonddot : (\varphi \to \boxdot\Diamonddot\varphi) \wedge (\Diamonddot\boxdot\varphi \to \varphi) \\
\mathbf{D}\Diamonddot : \Diamonddot\top & \\
\mathbf{4}\boxdot\Diamonddot : (\boxdot\varphi \to \boxdot\boxdot\varphi) \wedge (\Diamonddot\Diamonddot\varphi \to \Diamonddot\varphi) & \mathbf{5}\boxdot\Diamonddot : (\Diamonddot\boxdot\varphi \to \boxdot\varphi) \wedge (\Diamonddot\varphi \to \boxdot\Diamonddot\varphi)
\end{array} \tag{7}$$

where the schemes characterize, in turn, the properties of relations $R : X \rightsquigarrow X$ of reflexivity, symmetry, totality (seriality), transitivity and Euclideanness, and the mirror image scheme characterize relations $R$ such that $R^{-1}$ has the property[9]. For a set $\Gamma$ of formula schemes, let $\mathcal{L}^{\mathbf{t}}(\Gamma)$ denote the set of all instances of schemes in $\Gamma$ in the language $\mathcal{L}^{\mathbf{t}}$, and let $\mathbb{LSC}_{\mathbf{I}}(\Gamma)$ denote the class of all l.s.c. topological frames $\mathcal{F}$ such that $\mathcal{F} \Vdash \varphi$ for every formula $\varphi \in \mathcal{L}^{\mathbf{t}}(\Gamma)$. Likewise, for the companion classical logics, let $\mathcal{L}^{\mathbf{t}}_{\Box}(\Gamma)$ denote the set of all instances of schemes in $\Gamma$ in the language $\mathcal{L}^{\mathbf{t}}_{\Box}$, and let $\mathbb{T}(\Gamma)$ denote the class of all topological frames $\mathcal{F}$ such that $\mathcal{F} \models \psi$ for every formula $\psi \in \mathcal{L}^{\mathbf{t}}_{\Box}(\Gamma)$.

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[8]The intuitionistic modal logics considered in [41] and [27] are yet weaker sub-systems: they have the normality schemes $\mathbf{R}\boxdot$ and $\mathbf{N}\boxdot$ for $\boxdot$, but $\Diamonddot$ is sub-normal – they include the scheme $\mathbf{N}\Diamonddot$, but $\mathbf{R}\Diamonddot$ is replaced by $(\boxdot\varphi \wedge \Diamonddot\psi) \to \Diamonddot(\varphi \wedge \psi)$.

[9]Note that $R$ has reflexivity, symmetry or transitivity iff $R^{-1}$ has the same property, so the mirrored tense schemes $\mathbf{T}\blacksquare\blacklozenge$, $\mathbf{B}\blacksquare\blacklozenge$ and $\mathbf{4}\blacksquare\blacklozenge$ are semantically equivalent to their un-mirrored modal versions.

The topological soundness of $\mathbf{IK^t}$ and of $\mathbf{K^tLSC}$ are easy verifications. For example, the soundness of the Fischer Servi scheme $\mathbf{F1}\boxdot\diamondsuit$ is equivalent to the assertion that, for all open sets $U, V \in \mathcal{T}$:

$$R^{-\exists}\left(int_{\mathcal{T}}(-U \cup V)\right) \subseteq int_{\mathcal{T}}\big(-int_{\mathcal{T}}(R^{-\forall}(U)) \cup R^{-\exists}(V)\big).$$

The inclusion $R^{-\exists}\left(int_{\mathcal{T}}(-U \cup V)\right) \subseteq int_{\mathcal{T}}\left(R^{-\exists}(-U \cup V)\right)$ follows from $R$ being l.s.c. Applying distribution over unions, duality, and monotonicity, we can get $int_{\mathcal{T}}\left(R^{-\exists}(-U \cup V)\right) \subseteq int_{\mathcal{T}}\left(-int_{\mathcal{T}}(R^{-\forall}(U)) \cup R^{-\exists}(V)\right)$, so we are done. $R$ being l.s.c. is also used for soundness of the adjoint axioms **Ad2** and **Ad3**.

From Proposition 3.5 and topological completeness in Proposition 5.2 below, we can derive deductive faithfulness of the extended Gödel translation.

**Proposition 5.1** [Extended Gödel translation: deductive faithfulness]
*Let $\Gamma$ be any finite set of schemes in $\mathcal{L}^{\mathbf{t}}$ from the list in* (7) *above.*
*For all $\varphi \in \mathcal{L}^{\mathbf{t}}$, $\varphi \in \mathbf{IK^t} \oplus \Gamma$ iff* $\mathrm{GT}(\varphi) \in \mathbf{K^tLSC} \oplus \Gamma$.

This result can also be derived from a general result for (an equivalent) Gödel translation given in [43], Theorem 8, on the faithful embedding of modal logics $L = \mathbf{IntK} \oplus \Gamma_1$ (including $\mathbf{IK} \oplus \Gamma = \mathbf{IntK} \oplus \mathbf{F1}\boxdot\diamondsuit \oplus \mathbf{F2}\boxdot\diamondsuit \oplus \Gamma$) into bi-modal logics in the interval between $(\mathbf{S4}\Box \otimes \mathbf{K}\boxdot) \oplus \mathrm{GT}(\Gamma_1)$ and $(\mathbf{Grz}\Box \otimes \mathbf{K}\boxdot) \oplus \mathrm{GT}(\Gamma_1) \oplus \mathbf{mix}$, where $\mathbf{Grz}\Box = \mathbf{S4}\Box \oplus \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$ and $\mathbf{mix} = (\Box\boxdot\varphi \leftrightarrow \boxdot\varphi) \wedge (\boxdot\Box\varphi \leftrightarrow \Box\varphi)$. Since that level of generality is not sought here, we have restricted the schemes in $\Gamma$ to those from a "safe" list of relational properties that *don't* require translating, since the schemes characterize the same relations in the intuitionistic and classical semantics.

Recall that for a logic $\Lambda$ in a language $\mathcal{L}$ with deductive consequence relation $\vdash_\Lambda$, a set of formulas $x \subseteq \mathcal{L}$ is said to be $\Lambda$-*consistent* if $x \nvdash_\Lambda \bot$; $x$ is $\Lambda$-*deductively closed* if $x \vdash_\Lambda \varphi$ implies $\varphi \in x$ for all formulas $\varphi \in \mathcal{L}$; and $x$ is *maximal* $\Lambda$-*consistent* if $x$ is $\Lambda$-consistent, and no proper superset of $x$ is $\Lambda$-consistent. A set $x \subseteq \mathcal{L}$ is a *prime theory* of $\Lambda$ if $\Lambda \subseteq x$, and $x$ has the disjunction property, and is $\Lambda$-consistent, and $\Lambda$-deductively closed.

It will follow as a consequence of topological completeness (and be used in Section 6 below) that for every open and l.s.c. model $\mathcal{M}$, and for all states $w$ in $\mathcal{M}$, the set of formulas $\mathrm{Th}^{\mathcal{M}}_{\mathrm{I}}(w)$ is a prime theory of $\mathbf{IK^t}$. Likewise, in the classical semantics, for every l.s.c. topological model $\mathfrak{M}$ and every state $w$, the set of formulas $\mathrm{Th}^{\mathcal{M}}(w)$ is maximal $\mathbf{K^tLSC}$-consistent.

Completeness w.r.t. bi-relational frames for $\mathbf{IK}$ and $\mathbf{IK^t}$ is proved in [20, 37] and [17] by building a canonical model over the state space $X_{\mathrm{IP}}$ defined to be the set of all sets of formulas $x \subseteq \mathcal{L}^{\mathbf{t}}$ that are prime theories of $\mathbf{IK^t}$. The space $X_{\mathrm{IP}}$ is partially ordered by inclusion, so we have available an Alexandrov topology $\mathcal{T}_{\subseteq}$. One then defines the modal accessibility relation $R_0$ in an "almost classical" way, the only concession to intuitionistic semantics being clauses in the definition for both $\diamondsuit$ and $\boxdot$. As verified in [20, 37, 11] for the modal logic, and [17] for the tense logic, the relations $R_0$ and $R_0^{-1}$ satisfy the frame conditions $\mathbf{Zig}(\subseteq, R_0)$ and $\mathbf{Zig}(\subseteq, R_0^{-1})$. So we get an l.s.c. topological frame $\mathcal{F}_0 = (X_{\mathrm{IP}}, \mathcal{T}_{\subseteq}, R_0)$, and with the canonical valuation $u : AP \rightsquigarrow X_{\mathrm{IP}}$ given by $u(p) = \{x \in X_{\mathrm{IP}} \mid$

$p \in x\}$; one then proves of the model $\mathcal{M}_0 = (\mathcal{F}_0, u)$ the "Truth Lemma": for all $\varphi \in \mathcal{L}^{\mathbf{t}}$ and $x \in X_{\mathrm{IP}}$, $x \in [\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}_0}$ iff $\varphi \in x$.

Adapting [1], Sec. 3, on classical **S4**, to the classical companion logics here, we can go beyond the pre-orders of the bi-relational Kripke semantics by equipping the space of maximal consistent sets of formulas with a natural topology that is neither Alexandrov nor Stone, but rather is the intersection of those two topologies.

**Proposition 5.2** [Topological soundness and completeness]
*Let $\Gamma$ be any finite set of axiom schemes from $\mathcal{L}^{\mathbf{t}}$ from the list in* (7) *above.*
(1.) *For the classical logic,* $\mathbf{K^tLSC} \oplus \Gamma \;=\; \mathrm{Th}(\mathbb{LSC} \cap \mathbb{T}(\Gamma))$.
(2.) *For the intuitionistic logic,* $\mathbf{IK^t} \oplus \Gamma \;=\; \mathrm{Th}_{\mathbf{I}}(\mathbb{LSC}_{\mathbf{I}}(\Gamma))$.

In what follows, we use **IL** and $\mathbf{L}_\Box$, respectively, as abbreviations for the axiomatically presented logics $\mathbf{IK^t} \oplus \Gamma$ and $\mathbf{K^tLSC} \oplus \Gamma$. Taking soundness as established, we sketch completeness by describing the canonical models.

For the classical companion $\mathbf{L}_\Box$, define a model $\mathcal{M}_\Box = (Y_{\Box\mathrm{M}}, \mathcal{S}_\Box, Q_\Box, v_\Box)$ as follows:

$Y_{\Box\mathrm{M}} := \{y \subseteq \mathcal{L}^{\mathbf{t}}_\Box \mid y \text{ is a maximal } \mathbf{L}_\Box\text{-consistent set of formulas}\}$;
$\mathcal{S}_\Box$ is the topology on $Y_{\Box\mathrm{M}}$ which has as a basis the family
$\{\, V(\Box\psi) \mid \psi \in \mathcal{L}^{\mathbf{t}}_\Box \,\}$ where $V(\Box\psi) := \{y \in Y_{\Box\mathrm{M}} \mid \Box\psi \in y\}$;
$Q_\Box : Y_{\Box\mathrm{M}} \rightsquigarrow Y_{\Box\mathrm{M}}$ defined for all $y \in Y_{\Box\mathrm{M}}$ by
$Q_\Box(y) \;:=\; \{\, y' \in Y_{\Box\mathrm{M}} \mid \{\diamondsuit\psi \mid \psi \in y'\} \subseteq y \text{ and } \{\blacklozenge\psi \mid \psi \in y\} \subseteq y' \,\}$;
$v_\Box : AP \rightsquigarrow Y_{\Box\mathrm{M}}$ defined for all $p \in AP$ by $v_\Box(p) := \{y \in Y_{\Box\mathrm{M}} \mid p \in y\}$.

As noted in [1], the topology $\mathcal{S}_\Box$ on $Y_{\Box\mathrm{M}}$ is the intersection the "default" Alexandrov topology from the canonical relational Kripke model, and the standard Stone topology on $Y_{\Box\mathrm{M}}$ which has as a basis all sets of the form $V(\psi)$ for all formulas $\psi \in \mathcal{L}^{\mathbf{t}}_\Box$, not just the $V(\Box\psi)$ ones. Moreover, the space $(Y_{\Box\mathrm{M}}, \mathcal{S}_\Box)$ is compact and dense-in-itself (has no isolated points). Verification that $Q_\Box$ and $Q_\Box^{-1}$ are l.s.c. reduces to establishing that for all $\psi \in \mathcal{L}^{\mathbf{t}}_\Box$:

$$Q_\Box^{-\exists}(V(\Box\psi)) \;=\; V(\Box\diamondsuit\Box\psi) \quad\text{and}\quad Q_\Box^{\exists}(V(\Box\psi)) \;=\; V(\Box\blacklozenge\Box\psi).$$

The "Truth Lemma" here is $y = \mathrm{Th}^{\mathcal{M}_\Box}(y)$ for all $y \in Y_{\Box\mathrm{M}}$, which means $y \in [\![\varphi]\!]^{\mathcal{M}_\Box}$ iff $\psi \in y$, for all $\psi \in \mathcal{L}^{\mathbf{t}}_\Box$ and all $y \in Y_{\Box\mathrm{M}}$.

For the intuitionistic logic **IL**, define an open model $\mathcal{M}_\star = (X_{\mathrm{IP}}, \mathcal{T}_\star, R_\star, u_\star)$ as follows:

$X_{\mathrm{IP}} := \{x \subseteq \mathcal{L}^{\mathbf{t}} \mid x \text{ is a prime } \mathbf{IL}\text{-theory}\}$;
$\mathcal{T}_\star$ is the topology on $X_{\mathrm{IP}}$ which has as a basis the family
$\{\, U(\varphi) \mid \varphi \in \mathcal{L}^{\mathbf{t}} \,\}$ where $U(\varphi) := \{x \in X_{\mathrm{IP}} \mid \varphi \in x\}$;
$R_\star : X_{\mathrm{IP}} \rightsquigarrow X_{\mathrm{IP}}$ defined for all $x, x' \in X_{\mathrm{IP}}$ by $R_\star := R_0$; i.e. $x\, R_\star\, x'$ iff
$\{\diamondsuit\psi \mid \psi \in x'\} \subseteq x$ and $\{\psi \mid \boxdot\psi \in x\} \subseteq x'$ and
$\{\blacklozenge\psi \mid \psi \in x\} \subseteq x'$ and $\{\psi \mid \blacksquare\psi \in x'\} \subseteq x$;
$u_\star : AP \rightsquigarrow X_{\mathrm{IP}}$ defined for all $p \in AP$ by $u_\star(p) := U(p)$.

Here, the toplogical space $(X_{\text{IP}}, \mathcal{T}_\star)$ has a *spectral topology* (see, for example, [38], Sec.4), which means it is compact and $\mathrm{T}_0$; the family of compact and open sets in $\mathcal{T}_\star$ gives a basis for the topology; and $\mathcal{T}_\star$ is sober, i.e. for every completely prime filter $\mathcal{F}$ in the lattice $\mathcal{T}_\star$, there exists a (unique) point $x \in X_{\text{IP}}$ such that $\mathcal{F} = \mathcal{N}_x := \{U \in \mathcal{T}_\star \mid x \in U\}$, the filter of neighbourhoods of $x$. An equivalent characterization of a topology being sober is that every *irreducible* closed set is the closure of exactly one singleton set, where a closed set is irreducible if it is not the union of two proper closed subsets.

The hardest parts of the verification that $\mathcal{M}_\star$ is a model are the l.s.c. properties for $R_\star$ and $R_\star^{-1}$. The task reduces to establishing that for all $\varphi \in \mathcal{L}^{\mathbf{t}}$:

$$R_\star^{-\exists}(U(\varphi)) = U(\diamondsuit\varphi) \qquad \text{and} \qquad R_\star^{\exists}(U(\varphi)) = U(\blacklozenge\varphi)\ .$$

To prove the right-to-left inclusions, a recursive Henkin-style construction can be used to produce a prime **IL**-theory $x'$ such that $x\, R_\star\, x'$ and $\varphi \in x'$, to derive $x \in R_\star^{-\exists}(U(\varphi))$ given $x \in U(\diamondsuit\varphi)$, and symmetrically for the $R_\star^{\exists}(U(\varphi))$ inclusion. The required "Truth Lemma" is that $x = \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}_\star}(x)$ for all $x \in X_{\text{IP}}$, which means $x \in [\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}_\star}$ iff $\varphi \in x$ for all $\varphi \in \mathcal{L}^{\mathbf{t}}$ and all $x \in X_{\text{IP}}$.

# 6 Hennessy-Milner classes

Given a modal language $\mathcal{L}$ and semantics $[\![\cdot]\!]^{\mathcal{M}} : \mathcal{L} \rightsquigarrow X$ in models $\mathcal{M}$ with state spaces $X$, together with a notion of bisimulation between models, a class $\mathfrak{C}$ of models is said to have the *Hennessy-Milner property* [26, 24] if for every two models $\mathcal{M}, \mathcal{M}' \in \mathfrak{C}$, there is a tense topo-bisimulation $B : X \rightsquigarrow X'$ that maximally preserves the semantics, in the sense that for all $x \in X$ and $y \in X'$:

$$x\, B\, y \qquad \text{iff} \qquad \mathrm{Th}^{\mathcal{M}}(x) = \mathrm{Th}^{\mathcal{M}'}(y)$$

where $\mathrm{Th}^{\mathcal{M}}(x) = \{\varphi \in \mathcal{L} \mid x \in [\![\varphi]\!]^{\mathcal{M}}\}$ is the theory map inverse to the denotation map.

In classical modal (tense) logics, the original Hennessy-Milner class consisted of all models whose accessibility relations $R$ are *image-finite* (*bi-image-finite*); i.e. for all $x \in X$, the set $R(x)$ is finite (and also for $R^{-1}(x)$ for tense logics). A natural generalization of the image-finite condition is that of a relation $R$ being *modally saturated* in a model $\mathcal{M}$: for every state $x \in X$ and every set $\Psi \subseteq \mathcal{L}^{\mathrm{m}}$, if for every finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Psi$, there is an $R$-successor $x' \in R(x)$ that satisfies each $\varphi_k$ for $1 \leq k \leq n$, then there is an $R$-successor $x_\star \in R(x)$ that satisfies every formula in $\Psi$ (and likewise for $\Psi \subseteq \mathcal{L}^{\mathrm{t}}$ and $R^{-1}$ for tense logics); see [8], Ch. 2. For a diamond modality $\diamondsuit$ interpreted by $R$, classical modal saturation is equivalent to the condition that $\diamondsuit$ is *descriptive* of $R$ in $\mathcal{M}$: i.e. $x\, R\, x'$ iff for all $\varphi \in \mathcal{L}^{\mathrm{m}}$, if $x' \in [\![\varphi]\!]^{\mathcal{M}}$ then $x \in [\![\diamondsuit\varphi]\!]^{\mathcal{M}}$. Informally, this condition says that the relation $R$ is "recoverable" from the algebra of denotation sets $\{[\![\varphi]\!]^{\mathcal{M}} \mid \varphi \in \mathcal{L}^{\mathrm{m}}\}$ in the model $\mathcal{M}$, in the same way that the usual relation in the canonical Kripke model for a modal logic is so "recoverable" (and more generally, as it is for descriptive general frames; e.g. [8, 24]).

Addressing the question for the intuitionistic semantics, we start with the canonical model $\mathcal{M}_\star$ for the base logic $\mathbf{IK^t}$. For any open l.s.c. model $\mathcal{M}$ over state space $X$, consider the natural single-valued function $B_\star\colon X \to X_{\text{IP}}$ given by:

$$B_\star(w) \;:=\; \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}}(w)$$

for all $w \in X$. From topological completeness, each such set of formulas $\mathrm{Th}_{\mathbf{I}}^{\mathcal{M}}(w)$ is a prime theory of $\mathbf{IK^t}$, so the function $B_\star$ is total and well-defined (single-valued). In the special case when $\mathcal{M} = \mathcal{M}_\star$, the map $B_\star$ is the identity function, since by the "Truth Lemma", $x = \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}_\star}(x)$ for all $x \in X_{\text{IP}}$.

Our question then becomes: *For which class of open l.s.c. models $\mathfrak{C}$ is it the case that for each model $\mathcal{M}$ in $\mathfrak{C}$, the natural map $B_\star$ is a tense topo-bisimulation between $\mathcal{M}$ and the canonical model $\mathcal{M}_\star$?*

Suppose we had identified such a class $\mathfrak{C}$, and we pick arbitrary models $\mathcal{M}, \mathcal{M}' \in \mathfrak{C}$ with their canonical model bisimulations $B_\star\colon X \to X_{\text{IP}}$ and $B'_\star\colon X' \to X_{\text{IP}}$. We can then define the set-valued map $B\colon X \rightsquigarrow X'$ by $B = B_\star \circ (B'_\star)^{-1}$ to get a tense topo-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ that maximally preserves the intuitionistic semantics, since for all $w \in X$ and $z \in X'$, we will have:

$$w\,B\,z \quad \text{iff} \quad B_\star(w) = B'_\star(z) \quad \text{iff} \quad \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}}(w) = \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}'}(z)\ .$$

Hence such a class $\mathfrak{C}$ has the Hennessy-Milner property. Moreover, such a class $\mathfrak{C}$ will be maximal w.r.t. the Hennessy-Milner property iff it is maximal w.r.t. the property that for every model in the class, the natural map $B_\star$ is a tense topo-bisimulation into the canonical model.

In answering the question, we first need to determine, for an arbitrary open l.s.c. model $\mathcal{M}$, how "far short" the map $B_\star$ is from being a tense topo-bisimulation (how many of the clauses of Definition 4.3 "come for free"), and then to determine what additional "saturation" or "recoverability" properties are required of $\mathcal{M}$ in order to fill the short-fall.

First, consider an arbitrary open l.s.c. model $\mathcal{M}$ and a formula $\varphi \in \mathcal{L}^{\mathbf{t}}$ with (open) denotation sets $[\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}}$ in $\mathcal{M}$ and $[\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}_\star} = U(\varphi) = \{x \in X_{\text{IP}} \mid \varphi \in x\}$. Evaluating $\exists$-pre- and $\exists$-post-images of the map $B_\star$, we get:

$$(B_\star)^{-\exists}(U(\varphi)) \;=\; [\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}} \quad \text{and} \quad (B_\star)^{\exists}\,(\,[\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}}\,) \;=\; U(\varphi)\ . \tag{8}$$

Direct from equations (8), clauses (i.a) and (i.b) of Definition 4.3 for atomic $p \in AP$, are immediately satisfied.

The second semi-continuity clause (ii.b) calling for $B_\star : X \to X_{\text{IP}}$ to be l.s.c. is also an immediate consequence of the first equation in (8): every open set $V$ in the canonical model $\mathcal{M}_\star$ is a union of a family of basic opens $U(\varphi)$ indexed by a set of formulas $\Psi \subseteq \mathcal{L}^{\mathbf{t}}$, hence:

$$(B_\star)^{-\exists}(V) \;=\; \bigcup_{\varphi\in\Psi}(B_\star)^{-\exists}(U(\varphi)) \;=\; \bigcup_{\varphi\in\Psi}[\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}}\ .$$

Thus $(B_\star)^{-\exists}(V)$ is an open set in the given model $\mathcal{M}$.

The first semi-continuity clause (ii.a), calling for $B_\star^{-1} : X_{\text{IP}} \rightsquigarrow X$ to be l.s.c., requires that the topology be "recoverable" from the algebra of denotation sets in the model. Before developing that notion, we first examine more closely the inverse map $B_\star^{-1}$ and the image sets $(B_\star)^\exists(W)$ for $W \subseteq X$.

In an arbitrary open l.s.c. model $\mathcal{M}$, and for any $x \in X_{\text{IP}}$ a prime theory of $\mathbf{IK^t}$,

$$B_\star^{-1}(x) \;=\; \{w \in X \mid x = \mathrm{Th}^{\mathcal{M}}(w)\}\;.$$

In examining the state set $B_\star^{-1}(x)$, we need to address not only the *satisfiability* of formulas in the set $x$, but also the *falsifiability* of formulas not in $x$ (as in [33]; see also [22], Proposition 10.12, in the setting of general frames). The formulas not in $x$ have to be separated into those whose negations are in $x$, and those such that neither they nor their negations are in $x$; for the second, the *boundaries* of their denotation sets are crucial.

**Lemma 6.1** *Given an open l.s.c. topological model $\mathcal{M} = (X, \mathcal{T}, R, v)$ and canonical model map $B_\star \colon X \to X_{\text{IP}}$, the inverse map $B_\star^{-1} : X_{\text{IP}} \rightsquigarrow X$ is such that for all $x \in X_{\text{IP}}$,*

$$B_\star^{-1}(x) \;=\; \left(\bigcap_{\varphi\in x} [\![\,\varphi\,]\!]^{\mathcal{M}}_{\mathbf{I}}\right) \cap \left(\bigcap_{\psi\in\partial x} bd_{\mathcal{T}}([\![\,\psi\,]\!]^{\mathcal{M}}_{\mathbf{I}})\right) \tag{9}$$

*where $\partial x \;:=\; \{\psi \in \mathcal{L}^{\mathbf{t}} \mid \psi \notin x \wedge \neg\psi \notin x\}$.*

**PROOF.** Direct from the definition of $B_\star^{-1}$, it is immediate that:

$$B_\star^{-1}(x) \;=\; \left(\bigcap_{\varphi\in x} [\![\,\varphi\,]\!]^{\mathcal{M}}_{\mathbf{I}}\right) \cap \left(\bigcap_{\psi\notin x} (\,X - [\![\,\psi\,]\!]^{\mathcal{M}}_{\mathbf{I}}\,)\right).$$

Since we have the disjoint union $X - [\![\,\psi\,]\!]^{\mathcal{M}}_{\mathbf{I}} \;=\; [\![\,\neg\psi\,]\!]^{\mathcal{M}}_{\mathbf{I}} \cup bd_{\mathcal{T}}([\![\,\psi\,]\!]^{\mathcal{M}}_{\mathbf{I}})$, and we also have $bd_{\mathcal{T}}([\![\,\psi\,]\!]^{\mathcal{M}}_{\mathbf{I}}) = (X - [\![\,\psi\,]\!]^{\mathcal{M}}_{\mathbf{I}}) \cap (X - [\![\,\neg\psi\,]\!]^{\mathcal{M}}_{\mathbf{I}})$, equation (9) follows directly. $\dashv$

Note that for each prime theory $x \in X_{\text{IP}}$, the set $B_\star^{-1}(x)$ consists of all the states $w$ in the model $\mathcal{M}$ that *realize* $x$ as a theory in $\mathcal{M}$, since $w \in B_\star^{-1}(x)$ iff $x = \mathrm{Th}^{\mathcal{M}}_{\mathbf{I}}(w)$.

In developing a notion of the topology being "recoverable" from the algebra of denotation sets in the model, we draw on ideas in [24], here also giving the corresponding notion for the companion classical logics.

**Definition 6.2** *Given an open and l.s.c. topological model $\mathcal{M} = (X, \mathcal{T}, R, v)$, define $\mathcal{B}^{\mathcal{M}}_{\mathbf{I}}$ to be the family of opens:*

$$\mathcal{B}^{\mathcal{M}}_{\mathbf{I}} \;:=\; \{\; [\![\,\varphi\,]\!]^{\mathcal{M}}_{\mathbf{I}} \;\mid\; \varphi \in \mathcal{L}^{\mathbf{t}} \;\}$$

*and let $\mathcal{T}^{\mathcal{M}}_{\mathbf{I}}$ be the smallest sub-topology of $\mathcal{T}$ containing the family $\mathcal{B}^{\mathcal{M}}_{\mathbf{I}}$. We say the topology $\mathcal{T}$ is* saturated *in the model $\mathcal{M}$ if $\mathcal{T}^{\mathcal{M}}_{\mathbf{I}} = \mathcal{T}$.*

*Given an arbitrary topological model $\mathcal{M} = (X, \mathcal{T}, R, v)$, let $\mathcal{B}^{\mathcal{M}}_{\Box}$ be the family of open sets:*

$$\mathcal{B}^{\mathcal{M}}_{\Box} \;:=\; \{\; [\![\,\Box\psi\,]\!]^{\mathcal{M}} \;\mid\; \psi \in \mathcal{L}^{\mathbf{t}}_{\Box} \;\}$$

*and let $\mathcal{T}^{\mathcal{M}}_{\Box}$ be the smallest sub-topology of $\mathcal{T}$ containing the family $\mathcal{B}^{\mathcal{M}}_{\Box}$. We say the topology $\mathcal{T}$ is* $\Box$-saturated *in $\mathcal{M}$ if $\mathcal{T}^{\mathcal{M}}_{\Box} = \mathcal{T}$.*

It is readily seen that the families $\mathcal{B}_{\mathbf{I}}^{\mathcal{M}}$ and $\mathcal{B}_{\Box}^{\mathcal{M}}$ are closed under finite intersections, just by taking conjunctions of formulas; thus they constitute a basis for the topologies $\mathcal{T}_{\mathbf{I}}^{\mathcal{M}}$ and $\mathcal{T}_{\Box}^{\mathcal{M}}$, respectively. A topology $\mathcal{T}$ is saturated in an open l.s.c. model $\mathcal{M}$ if there are no *other* open sets in $\mathcal{T}$ besides the ones you get by taking unions of intuitionistic denotation sets of formulas. If $\mathcal{T}$ is saturated in an open l.s.c. model $\mathcal{M}$, then $\mathcal{T}$ is also $\Box$-saturated in $\mathcal{M}$, under the extended Gödel translation.

From our example class of frames $\mathcal{F}$ over $X \subseteq \mathbb{R}^n$, the Euclidean topology $\mathcal{T}_{\mathrm{E}}$ has as a basis the countable family of all metric $\delta$-balls $B_\delta(x)$ where $\delta \in \mathbb{Q}^+$ is positive rational and the centers $x \in (X \cap \mathbb{Q}^n)$; here, $B_\delta(x) := \{y \in \mathbb{R}^n \mid d(x,y) < \delta\}$. Thus we can make the topology $\mathcal{T}_{\mathrm{E}}$ saturated in a model $\mathcal{M}$ over $\mathcal{F}$ if the atomic valuation of $\mathcal{M}$ maps surjectively onto this family. More generally, if the topological space $(X, \mathcal{T})$ has a countable basis, and the maps $R$ and $R^{-1}$ are l.s.c., then indexing the basic opens via the countable set $AP$ of atomic propositions, we can form an open and l.s.c. model $\mathcal{M}$ in which $\mathcal{T}$ is saturated.

We now verify that this notion of topological saturation is sufficient to push through the first semi-continuity clause (ii.a).

**Lemma 6.3** *Given an open l.s.c. topological model $\mathcal{M} = (X, \mathcal{T}, R, v)$ and canonical model map $B_\star \colon X \to X_{\mathrm{IP}}$, if the topology $\mathcal{T}$ is saturated in $\mathcal{M}$, then $B_\star^{-1} : X_{\mathrm{IP}} \rightsquigarrow X$ is l.s.c.*

**PROOF.** Suppose $\mathcal{T}$ is saturated in $\mathcal{M}$. Then each open set $W$ in $\mathcal{T}$ is the union of a family of basic opens $[\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}}$ indexed by a set of formulas $\Psi \subseteq \mathcal{L}^{\mathbf{t}}$, hence:

$$(B_\star)^{\exists}(W) \;=\; (B_\star)^{\exists}\left(\bigcup_{\varphi\in\Psi} [\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}}\right) \;=\; \bigcup_{\varphi\in\Psi}(B_\star)^{\exists}([\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}}) \;=\; \bigcup_{\varphi\in\Psi} U(\varphi)\,.$$

Thus $(B_\star)^{\exists}(W)$ is an open set in the canonical model $\mathcal{M}_\star$. $\dashv$

The remaining clauses in Definition 4.3 of a tense topo-bisimulation are the Zig-Zag clauses (iii.a), (iii.b), (iv.a) and (iv.b), which together give the equalities $R \circ B_\star = B_\star \circ R_\star$ and $B_\star^{-1} \circ R_\star = R \circ B_\star^{-1}$. Using equations (8) together with $R_\star^{-\exists}(U(\varphi)) = U(\diamondsuit\varphi)$ and $R_\star^{\exists}(U(\varphi)) = U(\blacklozenge\varphi)$ from the proof of topological completeness, it is readily verified that for all formulas $\varphi \in \mathcal{L}^{\mathbf{t}}$:

$$\begin{array}{rcl@{\qquad}rcl}
[\![\,\diamondsuit\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}} &=& (R \circ B_\star)^{-\exists}\,(U(\varphi)) & [\![\,\blacklozenge\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}} &=& (R \circ B_\star^{-1})^{\exists}\,(U(\varphi)) \\
&=& (B_\star \circ R_\star)^{-\exists}\,(U(\varphi)) & &=& (B_\star^{-1} \circ R)^{\exists}\,(U(\varphi)) \\
U(\blacklozenge\varphi) &=& (R \circ B_\star)^{\exists}\,([\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}}) & U(\diamondsuit\varphi) &=& (R \circ B_\star^{-1})^{-\exists}\,([\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}}) \\
&=& (B_\star \circ R_\star)^{\exists}\,([\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}}) & &=& (B_\star^{-1} \circ R)^{-\exists}\,([\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}})\,.
\end{array} \tag{10}$$

Then define two maps $J_1 : X \rightsquigarrow X_{\mathrm{IP}}$ and $J_2 : X_{\mathrm{IP}} \rightsquigarrow X$ as follows: for all $w \in X$ and $x \in X_{\mathrm{IP}}$,

$$w\,J_1\,x \;\Leftrightarrow\; (\forall\varphi \in \mathcal{L}^{\mathbf{t}})\,[\,[\,\varphi \in x \;\Rightarrow\; w \in [\![\,\diamondsuit\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}}\,] \;\wedge\; [\,w \in [\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}} \;\Rightarrow\; \blacklozenge\varphi \in x\,]\,]$$

$$x\,J_2\,w \;\Leftrightarrow\; (\forall\varphi \in \mathcal{L}^{\mathbf{t}})\,[\,[\,\varphi \in x \;\Rightarrow\; w \in [\![\,\blacklozenge\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}}\,] \;\wedge\; [\,w \in [\![\,\varphi\,]\!]_{\mathbf{I}}^{\mathcal{M}} \;\Rightarrow\; \diamondsuit\varphi \in x\,]\,]$$

The two equalities $B_\star \circ R_\star = J_1$ and $R_\star \circ B_\star^{-1} = J_2$, and the inclusions $R \circ B_\star \subseteq J_1$ and $B_\star^{-1} \circ R \subseteq J_2$, are all easy consequences of equations (10) together with properties of the canonical relation $R_\star$ in $\mathcal{M}_\star$. Thus we get clauses (iii.a) and (iv.a) "for free". The remaining clauses (iii.b) and (iv.b) of Definition 4.3 (ensuring that $B_\star^{-1}$ is a tense topo-simulation of $\mathcal{M}_\star$ by the given model $\mathcal{M}$) require further "saturation"-type conditions. Before investigating such conditions, we summarize our reasoning over the proceeding few pages.

**Proposition 6.4** *Given an open l.s.c. topological model $\mathcal{M} = (X, \mathcal{T}, R, v)$, the canonical model map $B_\star \colon X \to X_{\mathrm{IP}}$ is a tense topo-bisimulation between $\mathcal{M}$ and $\mathcal{M}_\star$ iff the following conditions are satisfied:*

$$\begin{array}{ll} (1) & B_\star^{-1} : X_{\mathrm{IP}} \rightsquigarrow X \text{ is l.s.c.,} \\ (2) & J_1 \subseteq (R \circ B_\star), \text{ and} \\ (3) & J_2 \subseteq (B_\star^{-1} \circ R) \,. \end{array}$$

Now consider more closely condition (2): to show that $J_1 \subseteq R \circ B_\star$, suppose $w\, J_1\, x$ ; we need to find an $R$-successor $w_0 \in R(w)$ such that $x = \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}}(w_0)$; we would then have $w_0 \in R(w) \cap B_\star^{-1}(x)$ which witnesses that $w\,(R \circ B_\star)\, x$, as required. We use the characterization of $B_\star^{-1}(x)$ from Lemma 6.1 in formulating an intuitionistic notion of saturation.

**Definition 6.5** *Let $\mathcal{M} = (X, \mathcal{T}, R, v)$ be any open l.s.c. topological model, and let the relation $S : X \rightsquigarrow X$ be either $S = R$ or $S = R^{-1}$.*

- $S$ *is* image-closed w.r.t. $\mathcal{T}$ *if for each $w \in X$, the set $S(w)$ is closed in $\mathcal{T}$;*
- $S$ *has* negative saturation *in $\mathcal{M}$ if for every set of formulas $\Psi \subseteq \mathcal{L}^{\mathrm{t}}$ and for every $w \in X$, the following holds:*
  if, *for every finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \Psi$, $w \in S^{-\exists}\left(\bigcap_{1 \le j \le m}(X - [\![\, \psi_j \,]\!]_{\mathbf{I}}^{\mathcal{M}})\right)$,*
  then $w \in S^{-\exists}\left(\bigcap_{\psi \in \Psi}(X - [\![\, \psi \,]\!]_{\mathbf{I}}^{\mathcal{M}})\right)$*;*
- $S$ *has* realization saturation *in $\mathcal{M}$ if for every set of formulas $\Psi \subseteq \mathcal{L}^{\mathrm{t}}$ that contains* $\mathbf{IK^t}$ *and is negation-consistent (not both $\varphi \in \Psi$ and $\neg\varphi \in \Psi$), and for every state $w \in X$, the following condition holds:*
  if, *for every finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Psi$, $w \in S^{-\exists}\left(\bigcap_{1 \le k \le n} [\![\, \varphi_k \,]\!]_{\mathbf{I}}^{\mathcal{M}}\right)$,*
  and *for every finite $\{\psi_1, \ldots, \psi_m\} \subseteq \partial\Psi$, $w \in S^{-\exists}\left(\bigcap_{1 \le j \le m} bd_{\mathcal{T}}([\![\, \psi_j \,]\!]_{\mathbf{I}}^{\mathcal{M}})\right)$,*
  then *there exists an $S$-successor $w_0 \in S(w)$ such that $\Psi = \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}}(w_0)$;*
  *i.e. $w_0$ realizes $\Psi$ as a theory in $\mathcal{M}$.*
- $S$ *is* boundary-closed *in $\mathcal{M}$ if for every set of formulas $\Psi \subseteq \mathcal{L}^{\mathrm{t}}$ that contains* $\mathbf{IK^t}$ *and is negation-consistent, and for every finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \partial\Psi$,*
  *if $D := \bigcap_{1 \le j \le m} bd_{\mathcal{T}}([\![\, \psi_j \,]\!]_{\mathbf{I}}^{\mathcal{M}})$ then $S^{-\exists}(D)$ is closed in $\mathcal{T}$.*

Regarding the two distinct notions of intuitionistic relational saturation, the second, realization saturation, will be used immediately to establish the Hennessy-Milner property. The first notion, negative saturation, returns in Section 7, where we establish that it implies realization saturation under further topological hypotheses: compactness of the space, and outer semi-continuity (and hence Fell continuity and image-closedness) of the

the relation. The technical *boundary-closed* property is used in establishing the hypotheses of realization saturation. In the classical companion logics, the standard notion of modal saturation works as usual.

**Definition 6.6** *Let $\mathcal{M} = (X, \mathcal{T}, R, v)$ be any topological model, and let $S : X \rightsquigarrow X$ be either $S = R$ or $S = R^{-1}$. The relation $S$ has* (*classical*) modal saturation *in $\mathcal{M}$ if for every set of formulas $\Psi \subseteq \mathcal{L}^{\mathbf{t}}_{\Box}$ and for every $x \in X$, the following holds:*

if, *for every finite subset $\{\psi_1, \dots, \psi_m\} \subseteq \Psi$, $x \in S^{-\exists}\left(\bigcap_{1 \le j \le m} [\![\, \psi_j \,]\!]^{\mathcal{M}}\right)$,*

then $x \in S^{-\exists}\left(\bigcap_{\psi \in \Psi} [\![\, \psi \,]\!]^{\mathcal{M}}\right)$.

**Definition 6.7** *Let $\mathfrak{C}_0$ denote the class of all open l.s.c. models $\mathcal{M} = (X, \mathcal{T}, R, v)$ such that either $\mathcal{M} = \mathcal{M}_\star$, or else $\mathcal{M}$ is such that:*
*– the topology $\mathcal{T}$ is saturated in $\mathcal{M}$; and*
*– the relations $R$ and $R^{-1}$ both have realization saturation in $\mathcal{M}$; and*
*– both $R$ and $R^{-1}$ are boundary-closed in $\mathcal{M}$.*

*Let $\mathfrak{C}_{1\Box}$ denote the class of all topological models $\mathcal{M} = (X, \mathcal{T}, R, v)$ such that:*
*– the topology $\mathcal{T}$ is $\Box$-saturated in $\mathcal{M}$; and*
*– the relations $R$ and $R^{-1}$ both have classical modal saturation in $\mathcal{M}$.*

Unfortunately, the class $\mathfrak{C}_0$ has a somewhat awkward disjunctive characterization, including the base canonical model as a separate case, because we have been unable to directly verify that $\mathcal{M}_\star$ satisfies all three of the main conditions for the class. It is clear that the topology $\mathcal{T}_\star$ is saturated in $\mathcal{M}_\star$, and with a straight-forward argument by cases, it can be shown that both $R_\star$ and $R_\star^{-1}$ have realization saturation in $\mathcal{M}_\star$. However, we have not been able to settle the question as to whether or not $R_\star$ and $R_\star^{-1}$ are boundary-closed in $\mathcal{M}_\star$. In the classical case, it is clear that canonical models $\mathcal{M}_{\Box\mathrm{M}}$ do satisfy the two conditions for the class $\mathfrak{C}_{1\Box}$.

**Theorem 6.8** [Hennessy-Milner property for class $\mathfrak{C}_0$]
*Let $\mathcal{M} = (X, \mathcal{T}, R, v)$ and $\mathcal{M}' = (X', \mathcal{T}', R', v')$ be two open and l.s.c. topological models in the class $\mathfrak{C}_0$.*
*Then there exists a tense topo-bisimulation $B : X \rightsquigarrow X'$ between $\mathcal{M}$ and $\mathcal{M}'$ such that for all pairs of states $w \in X$ and $z \in X'$:*

$$w\, B\, z \quad \textit{iff} \quad \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}}(w) = \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}'}(z)\ .$$

*The map $B$ will be total when $\mathrm{Th}_{\mathbf{I}}(\mathcal{M}) \subseteq \mathrm{Th}_{\mathbf{I}}(\mathcal{M}')$, and surjective when $\mathrm{Th}_{\mathbf{I}}(\mathcal{M}') \subseteq \mathrm{Th}_{\mathbf{I}}(\mathcal{M})$, where $\mathrm{Th}_{\mathbf{I}}(\mathcal{M}) := \{\mathrm{Th}_{\mathbf{I}}^{\mathcal{M}}(w) \mid w \in X\}$.*

**PROOF.** It suffices to show, for arbitrary $\mathcal{M} = (X, \mathcal{T}, R, v)$ in $\mathfrak{C}_0$, that the canonical model map $B_\star \colon X \to X_{\mathrm{IP}}$ is a tense topo-bisimulation between $\mathcal{M}$ and $\mathcal{M}_\star$. If $\mathcal{M} = \mathcal{M}_\star$, then $B_\star(x) = \mathrm{Th}_{\mathbf{I}}^{\mathcal{M}_\star}(x) = x$ for all $x \in X_{\mathrm{IP}}$ so $B_\star$ is the identity map, and thus trivially a tense topo-bisimulation. So suppose that $\mathcal{M} \neq \mathcal{M}_\star$. Then by Proposition 6.4 and Lemma 6.3, it only remains to show that $J_1 \subseteq (R \circ B_\star)$ and $J_2 \subseteq (B_\star^{-1} \circ R)$.

For the first inclusion, suppose $w\,J_1\,x$. We need to find an $R$-successor $w_0 \in R(w)$ realizing $x$ in $\mathcal{M}$. Since $\mathcal{M} \in \mathfrak{C}_0$, we know $R$ has realization saturation in $\mathcal{M}$. So we want to establish the two hypotheses of that property hold of the state $w \in X$ and the set of formulas $x \in X_{\mathrm{IP}}$, a prime theory of $\mathbf{IK^t}$. So we need to show that for every finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq x$, there is a $w_1 \in R(w)$ such that $w_1 \in \llbracket \varphi_k \rrbracket^{\mathcal{M}}_{\mathrm{I}}$ for each $k \in \{1, \ldots, n\}$, and for every finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \partial x$, there is a $w_2 \in R(w)$ such that $w_2 \in bd_{\mathcal{T}}(\llbracket \psi_j \rrbracket^{\mathcal{M}}_{\mathrm{I}})$ for each $j \in \{1, \ldots, m\}$.

Now if $\{\varphi_1, \ldots, \varphi_n\} \subseteq x$, then $\varphi_0 \in x$, where $\varphi_0 := \bigwedge_{1 \le k \le n} \varphi_k$, since $x$ is $\mathbf{IK^t}$-deductively closed. Then $w\,J_1\,x$ and $\varphi_0 \in x$ together imply that $w \in \llbracket \Diamonddot \varphi_0 \rrbracket^{\mathcal{M}}_{\mathrm{I}}$, and hence there exists a $w_1 \in R(w)$ such that $w_1 \in \llbracket \varphi_k \rrbracket^{\mathcal{M}}_{\mathrm{I}}$ for each $k \in \{1, \ldots, n\}$.

Suppose $\{\psi_1, \ldots, \psi_m\} \subseteq \partial x$. Then $\psi_0 \notin x$, where $\psi_0 := \bigvee_{1 \le j \le m} (\psi_j \vee \neg\psi_j)$, since $x$ has the disjunction property. Then $\blacklozenge\boxdot\psi_0 \notin x$, since $x$ is $\mathbf{IK^t}$-deductively closed (applying axiom **Ad4**). Then $w\,J_1\,x$ together with $\blacklozenge\boxdot\psi_0 \notin x$ imply that we have $w \notin \llbracket \boxdot\psi_0 \rrbracket^{\mathcal{M}}_{\mathrm{I}}$, and hence $w \in cl_{\mathcal{T}}(R^{-\exists}(X - \llbracket \psi_0 \rrbracket^{\mathcal{M}}_{\mathrm{I}}))$. Now set:

$$D \;:=\; X - \llbracket \psi_0 \rrbracket^{\mathcal{M}}_{\mathrm{I}} \;=\; \bigcap_{1 \le j \le m} bd_{\mathcal{T}}(\llbracket \psi_j \rrbracket^{\mathcal{M}}_{\mathrm{I}})$$

Since $\mathcal{M} \in \mathfrak{C}_0$, the relation $R$ is boundary-closed in $\mathcal{M}$, so we can conclude that the set $R^{-\exists}(D)$ is closed in $\mathcal{T}$, so $cl_{\mathcal{T}}(R^{-\exists}(D)) = R^{-\exists}(D)$, and thus $w \in R^{-\exists}(D)$. Hence there exists a $w_2 \in R(w)$ such that $w_2 \in bd_{\mathcal{T}}(\llbracket \psi_j \rrbracket^{\mathcal{M}}_{\mathrm{I}})$ for each $j \in \{1, \ldots, m\}$, as required. Hence $J_1 \subseteq R \circ B_\star$.

The final inclusion $J_2 \subseteq B_\star^{-1} \circ R$ is equivalent to $(J_2)^{-1} \subseteq R^{-1} \circ B_\star$, and this is proved by the symmetric "mirror" argument, obtained by uniformly replacing $J_1$ with $(J_2)^{-1}$, replacing $\Diamonddot$ and $\boxdot$ with $\blacklozenge$ and $\blacksquare$, and replacing $R$ and $R$ with relations $R^{-1}$ and $R^{-1}$. $\dashv$

For the base classical tense logic $\mathbf{K^t} \otimes \mathbf{S4}\Box$, consider the canonical model $\mathcal{M}_\Box$ over the space $Y_{\Box\mathrm{M}}$ of maximal $\mathbf{K^t} \otimes \mathbf{S4}\Box$-consistent sets of formulas. Then for any topological model $\mathcal{M}$ over state space $X$, we can use the natural map $B_\Box \colon X \to Y_{\Box\mathrm{M}}$ given by:

$$B_\Box(w) \;:=\; \mathrm{Th}^{\mathcal{M}}(w)$$

for all $w \in X$. By topological completeness of the classical semantics, the formula set $B_\Box(w)$ will be maximal $(\mathbf{K^t} \otimes \mathbf{S4}\Box)$-consistent, and for $y \in Y_{\Box\mathrm{M}}$, we have $B_\Box^{-1}(y) = \bigcap_{\psi \in y} \llbracket \psi \rrbracket^{\mathcal{M}}$, the classical realization set, with $\partial y = \varnothing$ due to maximal consistency. In the special case when $\mathcal{M} = \mathcal{M}_\Box$, the map $B_\Box$ is the identity function, since by the "Truth Lemma", $y = \mathrm{Th}^{\mathcal{M}_\Box}(y)$ for all $y \in Y_{\Box\mathrm{M}}$. However, no special case argument is required for the classical semantics, since $\mathcal{M}_{\Box\mathrm{M}}$ satisfies the two conditions for the class $\mathfrak{C}_{1\Box}$.

**Proposition 6.9** [Hennessy-Milner property for class $\mathfrak{C}_{1\Box}$]
*Let $\mathcal{M} = (X, \mathcal{T}, R, v)$ and $\mathcal{M}' = (X', \mathcal{T}', R', v')$ be two topological models in the class $\mathfrak{C}_{1\Box}$. Then there exists a tense topo-bisimulation $B : X \rightsquigarrow X'$ between $\mathcal{M}$ and $\mathcal{M}'$ such that for all pairs of states $w \in X$ and $z \in X'$:*

$$w\,B\,z \quad \textit{iff} \quad \mathrm{Th}^{\mathcal{M}}(w) = \mathrm{Th}^{\mathcal{M}'}(z)\ .$$

For models $\mathcal{M}$ in $\mathfrak{C}_{1\Box}$, the verification that the natural map $B_\Box$ is a tense topo-bisimulation proceeds along the same lines as for the intuitionistic semantics, map $B_\star$ and class $\mathfrak{C}_0$, but the arguments are simpler.

It remains an open the question as to what is the *maximal* class of open l.s.c. models with the Hennessy-Milner property for the intuitionistic semantics. The maximality of $\mathfrak{C}_0$ is not at all clear, although what is (painfully) clear is that the current characterization of $\mathfrak{C}_0$ is sub-optimal: the boundary-closed property is a lingering thorny issue. A clean characterization of the maximal Hennessy-Milner class should clearly include the canonical model $\mathcal{M}_\star$. In category theory terms, one would expect $\mathcal{M}_\star$ to be the terminal object in a category of models with morphisms being functional (single-valued) tense topo-bisimulations.

# 7 Investigating the Hennessy-Milner class $\mathfrak{C}_0$

In this final section, we identify some topological sufficient conditions for being in the class $\mathfrak{C}_0$, as a means to identify some naturally occurring models in the class. The main result is the following.

**Theorem 7.1** *Let $\mathcal{M} = (X, \mathcal{T}, R, v)$ be any open l.s.c. model such that:*

(i) $(X, \mathcal{T})$ *is compact;*

(ii) *the topology $\mathcal{T}$ has a countable basis $\mathcal{B} = \{U_n \mid n \in \mathbb{N}\}$ with the atomic valuation $v(p_n) = U_n$, for some enumeration $AP = \{p_n \mid n \in N\}$ with $N \subseteq \mathbb{N}$; and*

(iii) *both $R$ and $R^{-1}$ are o.s.c.* (*and hence Fell continuous and image-closed*) *with respect to $\mathcal{T}$.*

*Then $\mathcal{M} \in \mathfrak{C}_0$.*

It should first be noted that for $\mathcal{M} \in \mathfrak{C}_0$ where the topology is Hausdorff, there is no value in looking at the maximal topo-bisimulation $B$ between $\mathcal{M}$ and itself, because in that case, $B$ will be the identity map: the Hausdorff property together with the topology saturated in the model means that for any states $x$ and $y$, there exists disjoint basic open sets $[\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}}$ and $[\![\psi]\!]^{\mathcal{M}}_{\mathbf{I}}$ such that $x \in [\![\varphi \wedge \neg\psi]\!]^{\mathcal{M}}_{\mathbf{I}}$ and $y \in [\![\psi \wedge \neg\varphi]\!]^{\mathcal{M}}_{\mathbf{I}}$. So in this setting, the Hausdorff property is *too* strong, and for models from continuous dynamical systems over Euclidean spaces, tense topo-bisimulation gives too fine a notion of model equivalence.

For models $\mathcal{M}_H$ arising from the dynamics of a hybrid system $H$, the state space is of the form $X_H := \{(q, x) \in Q \times \mathbb{R}^n \mid x \in D_q\}$ and is equipped with the product topology coming from an Alexandrov (and finite, so compact) topology $\mathcal{T}_{\preccurlyeq}$ on the finite set $Q$ and and the Euclidean topology on $\mathbb{R}^n$. In looking at model equivalence via tense topo-bisimulation, we now have good motivation for examining non-trivial pre-orders $\preccurlyeq$ structuring the discrete component of the state space, and for casting to one side the special cases of $\mathcal{T}_H$ being Hausdorff, arising when the pre-order $\preccurlyeq$ is identity and $\mathcal{T}_{\preccurlyeq}$ is discrete, or when the mode domains $D_q$ are pair-wise disjoint. For a hybrid system model $\mathcal{M}_H$ to meet the hypotheses of Theorem 7.1, one can ask of the discrete transition relation $E : Q \rightsquigarrow Q$ that $(E^{-1}\circ \preccurlyeq) = (\preccurlyeq \circ E^{-1})$ and $(E\circ \preccurlyeq) = (\preccurlyeq \circ E)$, so $E$ will be Vietoris continuous w.r.t. $\mathcal{T}_{\preccurlyeq}$.

To prove Theorem 7.1, we utilize a series of lemmas, the proofs of which are given in an Appendix section. In what follows, let $\mathcal{M}$ be any open l.s.c. model over state space $X$ with canonical model map $B_\star \colon X \to X_{\text{IP}}$.

**Lemma 7.2** *If $\mathcal{T}$ is saturated in $\mathcal{M}$, then for all prime theories $x \in X_{\text{IP}}$:*

$$cl_{\mathcal{T}}(B_\star^{-1}(x)) = \left(\bigcap_{\varphi \in x} cl_{\mathcal{T}}(\llbracket \varphi \rrbracket^{\mathcal{M}}_{\mathbf{I}})\right) \cap \left(\bigcap_{\psi \in \partial x} bd_{\mathcal{T}}(\llbracket \psi \rrbracket^{\mathcal{M}}_{\mathbf{I}})\right)$$

**Lemma 7.3** *If $\mathcal{T}$ is saturated in $\mathcal{M}$, and $R$ has negative saturation in $\mathcal{M}$, then for all $w \in X$ and all prime theories $x \in X_{\text{IP}}$,*
if*, for every finite $\{\psi_1, \ldots, \psi_m\} \subseteq (\mathcal{L}^{\text{t}} - x)$, $w \in R^{-\exists}\left(\bigcap_{1\le j\le m}(X - \llbracket \psi_j \rrbracket^{\mathcal{M}}_{\mathbf{I}})\right)$,*
then *$w \in R^{-\exists}(cl_{\mathcal{T}}(B_\star^{-1}(x)))$.*

**Lemma 7.4** *If $\mathcal{T}$ is compact and saturated in $\mathcal{M}$, then for all prime theories $x \in X_{\text{IP}}$, the set $B_\star^{-1}(x)$ is a compact set w.r.t. $\mathcal{T}$.*

**Lemma 7.5** *If $\mathcal{T}$ is saturated in $\mathcal{M}$, $R$ has negative saturation in $\mathcal{M}$, and $R$ is o.s.c., then $R$ has realization saturation in $\mathcal{M}$.*

**PROOF of Theorem 7.1**. By assumptions (ii) and (iii), the topology $\mathcal{T}$ is clearly saturated in the model $\mathcal{M}$, and we have an open atomic valuation, as well as the l.s.c. and the o.s.c. properties for $R$ and $R^{-1}$; in particular, both maps are image-closed. To see that $R$ (and symmetrically, $R^{-1}$) is boundary-closed in $\mathcal{M}$, fix an prime theory $x \in X_{\text{IP}}$ and a finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \partial x$ and set $D := \bigcap_{1\le j\le m} bd_{\mathcal{T}}(\llbracket \psi_j \rrbracket^{\mathcal{M}}_{\mathbf{I}})$. Being the intersection of boundary sets, $D$ is closed, and since $(X, \mathcal{T})$ is compact, by assumption (i), the set $D$ must be compact. Then since $R$ is o.s.c., by assumption (iii), $R^{-\exists}(D)$ must be a closed set. Thus $R$ is boundary-closed.

We claim that from compactness (assumption (i)) plus the image-closed property (from assumption (iii)), we can prove negative saturation. Then by Lemma 7.5, we will have realization saturation, and so $\mathcal{M} \in \mathfrak{C}_0$.

To see that $R$ (and symmetrically, $R^{-1}$) has negative saturation, fix a set of formulas $\Psi \subseteq \mathcal{L}^{\text{t}}$ and a state $w \in X$, and suppose that for every finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \Psi$, there is an $w' \in R(x)$ such that $w' \notin \llbracket \psi_j \rrbracket^{\mathcal{M}}_{\mathbf{I}}$ for each $j \in \{1, \ldots, m\}$. Then since $R(w)$ is closed, the family of closed sets $\{R(w) \cap (X - \llbracket \psi \rrbracket^{\mathcal{M}}_{\mathbf{I}}) \mid \psi \in \Psi\}$ has the finite intersection property. Then by compactness of the topology $\mathcal{T}$ on $X$, the intersection of the whole family is non-empty, so there exists $w_0 \in R(w)$ such that $w_0 \in \bigcap_{\psi\in\Psi}(X - \llbracket \psi \rrbracket^{\mathcal{M}}_{\mathbf{I}})$. Thus $R$ has negative saturation, as claimed. $\dashv$

# 8 Conclusion

This paper illustrates the way topological structure on the state spaces of Krikpe models provides for a clean and intuitive intuitionistic semantics for modal and tense logics, as well as making perpicuous the semantics content of the Gödel translation into the classical companion modal logics. We then investigate the logics by studying the notion of topological bisimulations between models as relations that preserve logical indistinguishability, and identify classes of models with the Hennessy-Milner property that for any two models in the class, there is a topological bisimulation that maximally preserves logical indistinguishability, for both the intuitionistic modal and tense logics, and for the classical logics into which they are translatable. We leave open the question as to whether the identified Hennessy-Milner class is maximal with respect to the property of preserving intuitionistic logical indistinguishability.

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# 9 Appendix

**PROOF of Lemma 7.2**. To begin with, set:

$$A := \bigcap_{\varphi \in x} [\![\varphi]\!]_{\mathrm{I}}^{\mathcal{M}}, \qquad B := \bigcap_{\psi \in \partial x} bd_{\mathcal{T}}([\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}}), \qquad C := \bigcap_{\varphi \in x} cl_{\mathcal{T}}([\![\varphi]\!]_{\mathrm{I}}^{\mathcal{M}}).$$

So we have $B_\star^{-1}(x) = A \cap B$, and hence the inclusions $B_\star^{-1}(x) \subseteq cl_{\mathcal{T}}(B_\star^{-1}(x)) \subseteq (B \cap C)$. We need to show that $(B \cap C) = cl_{\mathcal{T}}(B_\star^{-1}(x))$. (Note that in the extremal case when $\partial x = \varnothing$ – which is the case iff $x$ is maximal $\mathbf{IK^t}$-consistent – the intersection over an empty family gives $B = X$ and hence $B_\star^{-1}(x) = A$, so we don't need to address this case separately.) For prime theories $x \in X_{\mathrm{IP}}$, let:

$$\sim x := \{\, \psi \in \mathcal{L}^{\mathrm{t}} \mid \psi \notin x \ \wedge\ \neg\psi \in x \,\}$$

so that $x$, $\sim x$ and $\partial x$ form a partition of the language $\mathcal{L}^{\mathrm{t}}$, and the set $x$ itself further divides into $x^- := \{\, \varphi \in x \mid (\exists \psi \notin x)\, \varphi = \neg\psi \}$ and $x^+ := x - x^-$.

Since $\mathcal{T}$ is saturated in the model $\mathcal{M}$, we have as a basis the family of all denotation sets $\mathcal{B}_{\mathrm{I}}^{\mathcal{M}}$. Then:

$$\begin{array}{rcl}
cl_{\mathcal{T}}(B_\star^{-1}(x)) & = & \bigcap\{(X - [\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}}) \mid \psi \in \mathcal{L}^{\mathrm{t}} \ \wedge\ B_\star^{-1}(x) \subseteq (X - [\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}}) \,\} \\
& = & \bigcap\{(X - [\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}}) \mid \psi \notin x \,\} \\
& = & \bigcap\{(X - [\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}}) \mid \psi \in \sim x \ \vee\ \psi \in \partial x \,\} \\
& = & (\bigcap\{(X - [\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}}) \mid \psi \in \sim x \,\}) \ \cap\ (\bigcap\{bd_{\mathcal{T}}([\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}}) \mid \psi \in \partial x \,\}) \\
& = & B \ \cap\ (\bigcap\{(X - [\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}}) \mid \psi \in \sim x \,\}) \\
& = & B \ \cap\ (\bigcap\{cl_{\mathcal{T}}([\![\neg\psi]\!]_{\mathrm{I}}^{\mathcal{M}}) \mid \neg\psi \in x \,\}) \\
& \supseteq & B \cap C
\end{array}$$

Hence $cl_{\mathcal{T}}(B_\star^{-1}(x)) = (B \cap C)$, as required. $\dashv$

**PROOF of Lemma 7.3**. Fix $w \in X$, and suppose that for every finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq (\mathcal{L}^{\mathrm{t}} - x)$, there is a $w' \in R(w)$ such that $w' \notin [\![\psi_j]\!]_{\mathrm{I}}^{\mathcal{M}}$ for each $j \in \{1, \ldots, m\}$. Since $R$ has negative saturation, there exists an $w_0 \in R(w)$ such that $w_0 \in \bigcap_{\psi \notin x}(X - [\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}})$.

Now for any formula $\varphi \in x$, we have $\neg\varphi \notin x$, since $x$ is negation-consistent, hence $w_0 \in (X - [\![\neg\varphi]\!]_{\mathrm{I}}^{\mathcal{M}}) = cl_{\mathcal{T}}([\![\varphi]\!]_{\mathrm{I}}^{\mathcal{M}})$. For any boundary formula $\psi \in \partial x$, we have both $\psi \notin x$ and $\neg\psi \notin x$, and hence $w_0 \in (X - [\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}}) \cap (X - [\![\neg\psi]\!]_{\mathrm{I}}^{\mathcal{M}}) = bd_{\mathcal{T}}([\![\psi]\!]_{\mathrm{I}}^{\mathcal{M}})$. Applying Lemma 7.2, we can conclude that: $w_0 \in R(w) \cap cl_{\mathcal{T}}(B_\star^{-1}(x))$, and thus $w \in R^{-\exists}(cl_{\mathcal{T}}(B_\star^{-1}(x)))$, as required. $\dashv$

**PROOF of Lemma 7.4**. Assume $\mathcal{T}$ is compact and saturated in $\mathcal{M}$, and fix a prime theory $x \in X_{\text{IP}}$; we need to show that the set $B_\star^{-1}(x)$ is a compact set w.r.t. $\mathcal{T}$. First observe that $cl_\mathcal{T}(B_\star^{-1}(x))$ is compact, since it is a closed subset of a compact space. Now suppose, for a contradiction, that $B_\star^{-1}(x)$ is not compact. Then since $\mathcal{T}$ is saturated in $\mathcal{M}$, there exists an infinite set of formulas $\Theta$ such that $B_\star^{-1}(x) \subset \bigcup_{\theta\in\Theta} [\![ \theta ]\!]_{\mathbf{I}}^{\mathcal{M}}$, but for all finite subsets $\{\theta_1, \ldots, \theta_n\} \subset \Theta$, there exists $w \in B_\star^{-1}(x) \cap \bigcap_{1\leq i\leq n} (X - [\![ \theta_i ]\!]_{\mathbf{I}}^{\mathcal{M}})$. Now for any two states $w, w' \in B_\star^{-1}(x)$, we have $\text{Th}_{\mathbf{I}}^{\mathcal{M}}(w) = x = \text{Th}_{\mathbf{I}}^{\mathcal{M}}(w')$. Hence we can conclude that $\theta \notin x$ for all $\theta \in \Theta$, so the set $\Theta$ must be disjoint from $x$. We can also assume that there are no formulas $\theta \in \Theta$ such that $\neg\theta \in x$, for if that were such a formula, we would have $B_\star^{-1}(x) \subseteq (X - [\![ \theta ]\!]_{\mathbf{I}}^{\mathcal{M}})$, and so this formula $\theta$ would contribute nothing to the cover of $B_\star^{-1}(x)$, and could be dropped from $\Theta$ without any loss. Hence we can conclude that for all formulas $\theta \in \Theta$, we have both $\theta \notin x$ and $\neg\theta \notin x$, and thus $\theta \in \partial x$. Hence we have $\Theta \subseteq \partial x$. But this means we have $B_\star^{-1}(x) \subset \bigcap_{\theta\in\Theta} bd_\mathcal{T}([\![ \theta ]\!]_{\mathbf{I}}^{\mathcal{M}})$, which contradicts the supposition that $B_\star^{-1}(x) \subset \bigcup_{\theta\in\Theta} [\![ \theta ]\!]_{\mathbf{I}}^{\mathcal{M}}$. Having obtained a contradiction, we can thus conclude that $B_\star^{-1}(x)$ is compact. $\dashv$

**PROOF of Lemma 7.5**. To prove $R$ has realization saturation in $\mathcal{M}$, fix a prime theory $x \subseteq \mathcal{L}^{\text{t}}$, and fix a state $w \in X$. Suppose that for every finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq x$, we have $w \in R^{-\exists} \left(\bigcap_{1\leq k\leq n} [\![ \varphi_k ]\!]_{\mathbf{I}}^{\mathcal{M}}\right)$, and for every finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \partial x$, we have $w \in R^{-\exists} \left(\bigcap_{1\leq j\leq m} bd_\mathcal{T}([\![ \psi_j ]\!]_{\mathbf{I}}^{\mathcal{M}})\right)$. We need to show that $w \in R^{-\exists}(B_\star^{-1}(x))$.

Since $R$ has negative saturation in $\mathcal{M}$, consider the set of formulas $\Psi = \mathcal{L}^{\text{t}} - x$, and fix an arbitrary finite subset $\{\psi_1, \ldots, \psi_m\} \subseteq \Psi$. For each $\psi_j$, we have either $\psi_j \in {\sim}x$ and $\neg\,\psi_j \in x$, or else $\psi_j \in \partial x$ and thus both $\psi_j \notin x$ and $\neg\,\psi_j \notin x$. We claim that $w \in R^{-\exists} \left(\bigcap_{1\leq j\leq m} X - [\![ \psi_j ]\!]_{\mathbf{I}}^{\mathcal{M}}\right)$, so that by negative saturation and Lemma 7.3, we can conclude that $w \in R^{-\exists}(cl_\mathcal{T}(B_\star^{-1}(x)))$. To prove the claim, we proceed by cases.

*Case I*: $\partial x = \varnothing$. Then we have $\neg\,\psi_j \in x^-$ for every $j \in \{1, \ldots, m\}$, and hence there exists a $w_1 \in R(w) \cap \left(\bigcap_{1\leq j\leq m} [\![ \neg\,\psi_j ]\!]_{\mathbf{I}}^{\mathcal{M}}\right) \subseteq R(w) \cap \left(\bigcap_{1\leq j\leq m} X - [\![ \psi_j ]\!]_{\mathbf{I}}^{\mathcal{M}}\right)$. Hence $w \in R^{-\exists} \left(\bigcap_{1\leq j\leq m} X - [\![ \psi_j ]\!]_{\mathbf{I}}^{\mathcal{M}}\right)$, and so we are done.

*Case II*: $\partial x \neq \varnothing$ but $\neg\,\psi_j \in x^-$ for every $j \in \{1, \ldots, m\}$. Then proceed as for *Case I*.

*Case III*: $\partial x \neq \varnothing$ and $\psi_i \in \partial x$ for at least one $i \in \{1, \ldots, m\}$. Then re-number the formulas so that for some $q \in \{1, \ldots, m\}$, we have $\psi_i \in \partial x$ for $1 \leq i \leq q$, and $\neg\,\psi_j \in x^-$ for $q < j \leq m$. We now claim that $\{\psi'_1, \ldots, \psi'_m\} \subseteq \partial x$, where this finite set of formulas is defined by $\psi'_i := \psi_i$ for $1 \leq i \leq q$, and $\psi'_j := (\psi_1 \vee \psi_j)$ for $q < j \leq m$; then we can conclude that $w \in R^{-\exists} \left(\bigcap_{1\leq i\leq m} bd_\mathcal{T}([\![ \psi'_j ]\!]_{\mathbf{I}}^{\mathcal{M}})\right)$. To prove the claim, there is nothing to do for $\psi'_i = \psi_i \in \partial x$ for $1 \leq i \leq q$. For $\psi'_j$ for $q < j \leq m$, first note that $\psi'_j \notin x$ since $x$ has the disjunction property. Moreover, $(\neg\,\psi_1 \wedge \neg\,\psi_j) \leftrightarrow \neg\,\psi'_j$ is a theorem of $\mathbf{IK^t}$. Since $x$ is $\mathbf{IK^t}$-deductively-closed and negation-consistent, and $\neg\,\psi_1 \notin x$, we can conclude that $\neg\,\psi'_j \notin x$, and hence $\psi'_j \in \partial x$ for every $j \in \{1, \ldots, m\}$, as claimed. Now for $q < j \leq m$, we have:

$$
\begin{aligned}
& bd_{\mathcal{T}}(\llbracket\, \psi'_j \,\rrbracket^{\mathcal{M}}_{\mathbf{I}}) \\
= \;& cl_{\mathcal{T}}(\llbracket\, \psi_1 \vee \psi_j \,\rrbracket^{\mathcal{M}}_{\mathbf{I}}) \cap cl_{\mathcal{T}}(\llbracket\, \neg\, \psi_1 \,\wedge\, \neg\, \psi_j \,\rrbracket^{\mathcal{M}}_{\mathbf{I}}) \\
= \;& (cl_{\mathcal{T}}(\llbracket\, \psi_j \,\rrbracket^{\mathcal{M}}_{\mathbf{I}}) \cap cl_{\mathcal{T}}(\llbracket\, \neg\, \psi_1 \,\wedge\, \neg\, \psi_j \,\rrbracket^{\mathcal{M}}_{\mathbf{I}})) \;\cup\; (cl_{\mathcal{T}}(\llbracket\, \psi_1 \,\rrbracket^{\mathcal{M}}_{\mathbf{I}}) \cap cl_{\mathcal{T}}(\llbracket\, \neg\, \psi_1 \,\wedge\, \neg\, \psi_j \,\rrbracket^{\mathcal{M}}_{\mathbf{I}})) \\
\subseteq \;& bd_{\mathcal{T}}(\llbracket\, \psi_j \,\rrbracket^{\mathcal{M}}_{\mathbf{I}}) \;\cup\; cl_{\mathcal{T}}(\llbracket\, \neg\, \psi_j \,\rrbracket^{\mathcal{M}}_{\mathbf{I}}) \\
\subseteq \;& X - \llbracket\, \psi_j \,\rrbracket^{\mathcal{M}}_{\mathbf{I}}
\end{aligned}
$$

Hence $w \in R^{-\exists}\left(\bigcap_{1 \leq j \leq m} X - \llbracket\, \psi_j \,\rrbracket^{\mathcal{M}}_{\mathbf{I}}\right)$, as required to conclude *Case III*.

So now we have that $w \in R^{-\exists}(cl_{\mathcal{T}}(B_{\star}^{-1}(x)))$. Since $R^{-1}$ is l.s.c. we have the inclusion $R^{-\exists}(cl_{\mathcal{T}}(W)) \subseteq cl_{\mathcal{T}}(R^{-\exists}(W))$, for all subsets $W \subseteq X$. Thus we can conclude that $w \in cl_{\mathcal{T}}(R^{-\exists}(B_{\star}^{-1}(x)))$, where $R^{-\exists}(B_{\star}^{-1}(x)) \neq \varnothing$, and hence also $B_{\star}^{-1}(x) \neq \varnothing$. Now by Lemma 7.4, we know that $B_{\star}^{-1}(x))$ is compact as well as non-empty. Since we have assumed $R$ is o.s.c., we can then conclude that $R^{-\exists}(B_{\star}^{-1}(x))$ is closed, and hence $w \in cl_{\mathcal{T}}(R^{-\exists}(B_{\star}^{-1}(x))) = R^{-\exists}(B_{\star}^{-1}(x))$, and we are done. $\dashv$