# Topological semantics for Intuitionistic modal logics, and spatial discretisation by A/D maps

J.M. Davoren[1⋆], V. Coulthard[1], T. Moor[1], R.P. Goré[1], and A. Nerode[2]

[1] Research School of Information Sciences and Engineering
Australian National University, Canberra ACT 0200 AUSTRALIA
[2] Department of Mathematics, Cornell University
Ithaca New York 14853 USA

**Abstract.** The contribution of this paper is threefold. First, we take the well-known Intuitionistic modal logic of Fischer Servi with semantics in birelational Kripke frames, and give the natural extension to topological Kripke frames where the frame conditions relating the Intuitionistic partial order with the modal relation generalise to semi-continuity properties of the relation with respect to the topology. Second, we develop the theory of an interesting class of topologies arising from spatial discretisation by finitary covers; the motivating case is covers of Euclidean space. We use the name "A/D map" to designate covers of a space whose cover cells do not generate any infinite descending chains; for analog-to-digital conversion, were one seeks a discretised view of a continuous world via the cells of a cover, the limits of discernment should be finite. Third, we give a novel application of Intuitionistic semantics to the problem of approximate model-checking of classical modal formulas in models where the exact evaluation of denotation sets is not possible; such models are the norm in applications of modal logics to the formal analysis and design of hybrid (mixed continuous and discrete) dynamical systems. The main result of the paper is that for the positive fragment of a modal language generated from a finite set of atomic propositions, we can give general lower and upper bounds on the classical denotation set of a formula in a given model. Moreover, these bounds are the Intuitionistic denotation sets of the same formula in two different models, where the lower and upper Intuitionistic models are built from an A/D map and have finitary quotients.

## 1 Introduction and motivation

Topological semantics for Intuitionistic logic and for the classical modal logic S4 have a long history going back to Tarski and co-workers in the 1930s and 40s, predating the relational Kripke semantics for both [15], [18]. A little earlier again is the 1933 Gödel translation [13] of Intuitionistic logic into classical S4. The translation makes perfect sense within the topological semantics: $\Box$ is interpreted by topological interior, and the translation $\mathrm{GT}(\neg\varphi) = \Box\neg\,\mathrm{GT}(\varphi)$ says that Intuitionistic negation calls for the interior of the complement, and not just the complement. For the extension to Intuitionistic modal logics, Fischer Servi developed semantics over birelational Kripke frames in the late 1970s, which has generated a good deal of reasearch [9–11, 17, 19, 21, 14]. What is suprising is that there seems to be nothing in the literature on combining the two: keeping the topology to interpret the Intuitionistic base logic, and adding a binary relation to interpret the modal operators. In this paper, we develop a semantics for Intuitionistic modal logic over topological frames $\mathcal{F} = (X, \mathcal{T}, R)$ where $(X, \mathcal{T})$ is a topological space and $R \subseteq X \times X$ is a relation. In the Fischer Servi bi-relational semantics, there are frame conditions relating the Intuitionistic partial order with the modal relation. We show that over topological frames, these conditions generalise to semi-continuity properties of the relation with respect to the topology. Moreover, Fischer Servi's extension

⋆ Corresponding author; email: `j.m.davoren@anu.edu.au`. Version 3.0; date 22 April 2002.

of the Gödel translation is laden with topological meaning: where $\boxdot$ is the box modality for the relation, and $\Box$ is topological interior, the translation $\mathrm{GT}(\boxdot\varphi) = \Box\boxdot\,\mathrm{GT}(\varphi)$ says that the Intuitionistic box requires the interior of the box operator under the classical semantics. This accords with Intuitionistic semantics for first-order logic, which take the interior of an intersection for $\forall$ quantification.

The second contribution of this paper is a study of an interesting class of topologies that arise from spatial discretisation by finitary covers; the motivating case is covers of Euclidean space. To designate covers of a space whose cover cells do not generate any infinite descending chains, we use so-called *A/D maps*, a notion that is motivated from an engineering perspective. For analog-to-digital conversion, were one seeks a discretised view of a continuous world via the cells of a cover, the limits of discernment should be finite. Our target domain of application is the formal analysis and design of hybrid dynamical systems; these are systems including both continuous and discrete states, and continuous and discrete dynamics, and which involve A/D maps as essential components [7]. Nerode and Kohn in 1993 [16] make an initial study of A/D maps and their topologies; the present paper offers a further development of that theory. A key idea, advanced in [16], is that the open sets in the topology on an A/D map have a particularly nice representation as a union of suitably "small" elements in the lattice of opens.

The third contribution of this paper is a novel application of Intuitionistic semantics to the problem of *approximate model-checking* of classical modal formulas in models where the exact evaluation of denotation sets is not possible; such models are the norm in applications of modal logics to the formal analysis and design of hybrid systems [8]. Exact or symbolic model-checking of modal and temporal formulas in models over $X \subseteq \mathbb{R}^n$ is typically restricted to classes of models $\mathcal{M} = (X, R, \xi)$ in which the atomic sets $[\![p]\!]^{\mathcal{M}} = \xi^{-1}(p) \subseteq X$ and the relation(s) $R \subseteq X \times X$ are definable by *linear* or *polynomial predicates* in $n$ and $2n$ variables respectively [2]. For relations $R$ that are the *orbit relation* of solutions of a differential equation, these restrictions mean that only systems with very simple continuous dynamics can be model-checked exactly. The main result of the paper is that for the positive fragment of a modal language generated from a finite set of atomic propositions, we can give general lower and upper bounds on the classical denotation set of a formula in a given model. Moreover, these bounds are the Intuitionistic denotation sets of the same formula in two different models, where the lower and upper Intuitionistic models are built from an A/D map and have finitary quotients.

The rest of the paper is organised as follows. Section 2 tersely reviews the necessary material from general topology. In Section 3, we set out the topological semantics for Intuitionistic propositional logic and classical S4, and the Gödel translation from the former to the latter. Section 4 develops topological semantics for Intuitionistic modal logic, generalising from known results on bi-relational Kripke semantics. In Section 5, we develop the general topology of covers and A/D maps, and in Section 6, we apply results on A/D maps to give a general recipe for approximately evaluating the classical modal denotation set of a positive formula.

## 2 Preliminaries: some general topology

We adopt the notation from set-valued analysis [3] in writing $r : X \rightsquigarrow Y$ to mean both that $r : X \to 2^Y$ is a *set-valued map*, with (possibly empty) set-values $r(x) \subseteq Y$ for each $x \in X$, and equivalently, that $r \subseteq X \times Y$ is a *relation*. The expressions $y \in r(x)$, $(x, y) \in r$ and $x\,r\,y$ are synonymous. Every set-valued map $r : X \rightsquigarrow Y$ has an *inverse* or *converse* $r^{-1} : Y \rightsquigarrow X$ given by: $x \in r^{-1}(y)$ iff $y \in r(x)$. The *domain* of a set-valued map is $\mathrm{dom}(r) := \{x \in X \mid r(x) \neq \varnothing\}$, and the *range* is $\mathrm{ran}(r) := \mathrm{dom}(r^{-1}) \subseteq Y$. A set-valued map $r : X \rightsquigarrow Y$ is *total* if $\mathrm{dom}(r) = X$, and it is *surjective* if $\mathrm{ran}(r) = Y$. We will write (as usual) $r : X \to Y$ to mean $r$ is a total and single-valued function with values $r(x) = y$ (rather than $r(x) = \{y\}$).

For $r_1 : X \rightsquigarrow Y$ and $r_2 : Y \rightsquigarrow Z$, we write their relational/sequential composition as $r_1 \circ r_2 : X \rightsquigarrow Z$ given by $(r_1 \circ r_2)(x) := \{z \in Z \mid (\exists y \in Y)\,[(x,y) \in r_1 \wedge (y,z) \in r_2]\}$, in left-to-right word order.

On notation, for partial orders or preorders $\subseteq$, $\preccurlyeq$, $\sqsubseteq$, we write $\subset$, $\prec$, $\sqsubset$ for the corresponding strict partial orders or preorders; i.e. $x \prec x'$ iff $x \preccurlyeq x'$ and not $x' \preccurlyeq x$. Likewise, we write $\supseteq$, $\succcurlyeq$, $\sqsupseteq$ for the corresponding converse partial orders or preorders. Recall that a *preorder* is a reflexive and transitive binary relation, and a *partial order* is a preorder that is also antisymmetric. A *lattice* is a partial order in which every pair of elements has an l.u.b. and a g.l.b. With regard to families $\mathcal{A} \subseteq 2^X$ of subsets of $X$, a family $\mathcal{A}$ is called a *ring of sets* if it contains the extremal elements $\varnothing$ and $X$, and is closed under finite unions and finite intersections, and $\mathcal{A}$ is called a *field of sets* if it is a ring of sets that is also closed under set-complement in $X$. Under the natural order of set-inclusion, a ring of sets is a distributive lattice with least and greatest elements, and a field of sets is a Boolean algebra.

A relation $r : X \rightsquigarrow Y$ determines two *pre-image operators* (predicate transformers). The *lower* or *existential* pre-image $pre^{\exists}(r) : 2^Y \to 2^X$ given by

$$\begin{aligned} pre^{\exists}(r)(W) &:= \{x \in X \mid (\exists y \in Y)[\,(x,y) \in r \quad \wedge \quad y \in W]\} \\ &= \{x \in X \mid W \cap r(x) \neq \varnothing\} \end{aligned}$$

for $W \subseteq Y$. The *upper* or *universal* pre-image operator $pre^{\forall}(r) : 2^Y \to 2^X$ is the dual under set-theoretic complement:

$$pre^{\forall}(r)(W) \ := \ X - pre^{\exists}(r)(Y - W) \ = \ \{x \in X \mid r(x) \subseteq W\}$$

In words, $x \in pre^{\exists}(r)(W)$ iff *some* $r$-successor of $x$ lies in $W$, while $x \in pre^{\forall}(r)(W)$ iff *all* $r$-successors of $x$ lie in $W$, including $x \notin \mathrm{dom}(r)$. The operator $pre^{\exists}(r)$ distributes over arbitrary unions, while $pre^{\forall}(r)$ distributes over arbitrary intersections; $pre^{\exists}(r)(\varnothing) = \varnothing$, $pre^{\exists}(r)(Y) = \mathrm{dom}(r)$, $pre^{\forall}(r)(\varnothing) = X - \mathrm{dom}(r)$, and $pre^{\forall}(r)(Y) = X$.

For relations $r\colon X \rightsquigarrow X$ on a space $X$, the pre-images express in operator form the standard relational Kripke semantics for the (future) diamond and box modal operators determined by $r$. Note that when $r : X \to Y$ is single-valued, the pre-image operators reduce to the standard *inverse-image* operator; i.e. $pre^{\exists}(r)(W) = pre^{\forall}(r)(W) = r^{-1}(W)$. The relationally converse operators are the *post-image operators* $post^{\exists}(r)$, $post^{\forall}(r) : 2^X \to 2^Y$ given by $post^{\star}(r) := pre^{\star}(r^{-1})$ for $\star \in \{\exists, \forall\}$. These operators arise in the relational Kripke semantics for the *past* diamond and box modal operators in tense and temporal logics. The fundamental relationship between pre- and post-images is the adjoint property:

$$W \subseteq pre^{\forall}(r)(V) \ \text{ iff } \ post^{\exists}(r)(W) \subseteq V\,.$$

Recall that a *topology* $\mathcal{T} \subseteq 2^X$ on a set $X$ is a family of subsets of $X$ that contains $\varnothing$ and $X$, and is closed under arbitrary unions and finite intersections. The extreme cases are the *discrete* topology $\mathcal{T}_D = 2^X$, and the *trivial* topology $\mathcal{T}_{\varnothing} = \{\varnothing, X\}$. The *interior operator* $int_{\mathcal{T}} : 2^X \to 2^X$ determined by $\mathcal{T}$ is given by $int_{\mathcal{T}}(W) := \bigcup\{U \in \mathcal{T} \mid U \subseteq W\}$. A set $W \subseteq X$ is *open* w.r.t. $\mathcal{T}$ if $W \in \mathcal{T}$, and this is so iff $W = int_{\mathcal{T}}(W)$. So $\mathcal{T}$ is a ring of sets that is complete w.r.t. unions. Let $-\mathcal{T}$ denote the dual lattice under complement; i.e. $-\mathcal{T} := \{V \subseteq X \mid (X - V) \in \mathcal{T}\}$. Then $-\mathcal{T}$ is also a ring of sets, and is complete w.r.t. intersections. The dual *closure operator* $cl_{\mathcal{T}} : 2^X \to 2^X$ is given by $cl_{\mathcal{T}}(W) := \bigcap\{V \in -\mathcal{T} \mid W \subseteq V\}$. A set $W \subseteq X$ is *closed* w.r.t. $\mathcal{T}$ if $W \in -\mathcal{T}$, and this is so iff $W = cl_{\mathcal{T}}(W)$.

The purely topological notion of *continuity* for a single-valued function $f\colon X \to Y$ is that the inverse image $f^{-1}(U)$ is open whenever $U$ is open. Analogously, the pre-image operators can be used to characterise purely topological notions of continuity for relations/set-valued maps, as introduced by Kuratowski and Bouligand in the 1920s. A relation $r : (X, \mathcal{T}) \rightsquigarrow (Y, \mathcal{S})$ is called: *lower semi-continuous* (l.s.c.) if for every $\mathcal{S}$-open set $U$ in $Y$, $pre^{\exists}(r)(U)$ is $\mathcal{T}$-open in $X$; *upper semi-continuous* (u.s.c.) if for every $\mathcal{S}$-open set $U$ in $Y$, $pre^{\forall}(r)(U)$ is $\mathcal{T}$-open in $X$; and simply *continuous* if it is both l.s.c. and u.s.c. The u.s.c. condition is equivalent to $pre^{\exists}(r)(V)$ is $\mathcal{T}$-closed in $X$ whenever $V$ is $\mathcal{S}$-closed in $Y$. Each of the semi-continuity conditions reduce to the standard functional continuity when $r : X \to Y$ is single-valued. The semi-continuity properties are preserved under relational composition, and also under finite unions of relations.

In what follows, we make particular use of *Alexandroff topologies*, which are also called *Kripke* or *cone topologies*. A topology $\mathcal{T}$ on $X$ is called Alexandroff if for every $x \in X$, there is a *smallest* open set $U \in \mathcal{T}$ such that $x \in U$. In particular, every *finite* topology $\mathcal{T}$ is Alexandroff. There is a one-to-one correspondance between preorders on $X$ and Alexandroff topologies on $X$. Any preorder $\preccurlyeq$ on $X$ induces an Alexandroff topology $\mathcal{T}_{\preccurlyeq}$ by taking $int_{\mathcal{T}_{\preccurlyeq}}(W) := pre^{\forall}(\preccurlyeq)(W)$, which means $U \in \mathcal{T}_{\preccurlyeq}$ iff $U$ is up-$\preccurlyeq$-closed, and $V \in -\mathcal{T}_{\preccurlyeq}$ iff $V$ is down-$\preccurlyeq$-closed, and $cl_{\mathcal{T}_{\preccurlyeq}}(W) := \mathrm{Pre}^{\exists}(\preccurlyeq)(W)$. In particular, $\mathcal{T}_{\preccurlyeq}$ is closed under arbitrary intersections as well as arbitrary unions. The sets $B_{\preccurlyeq}(x) := \{y \in X \mid x \preccurlyeq y\}$ are sometimes called *$\preccurlyeq$-cones*, and they form a basis for the topology $\mathcal{T}_{\preccurlyeq}$. Conversely, for any topology, define a preorder $\preccurlyeq_{\mathcal{T}}$ on $X$, known as the *specialisation preorder*: $x \preccurlyeq_{\mathcal{T}} y$ iff $(\forall U \in \mathcal{T})\,[x \in U \Rightarrow y \in U]$. For any preorder, $\preccurlyeq_{\mathcal{T}_{\preccurlyeq}} = \preccurlyeq$, and $\mathcal{T}_{\preccurlyeq_{\mathcal{T}}} = \mathcal{T}$ iff $\mathcal{T}$ is Alexandroff. The relation $\approx_{\mathcal{T}}$ on $X$ of *topological equivalence* under $\mathcal{T}$ is given by $\approx_{\mathcal{T}} := (\preccurlyeq_{\mathcal{T}} \cap \succcurlyeq_{\mathcal{T}})$, so $x \approx_{\mathcal{T}} y$ means $x$ and $y$ belong to all the same $\mathcal{T}$-open sets. A topology $\mathcal{T}$ has $\mathrm{T}_0$ separation iff the preorder $\preccurlyeq_{\mathcal{T}}$ is a partial order iff $\approx_{\mathcal{T}}$ is identity. Alexandroff topologies lack separation properties: the only Alexandroff topology that is $\mathrm{T}_1$ is the discrete topology, recalling that $\mathrm{T}_0$ is weaker than $\mathrm{T}_1$ which is weaker than $\mathrm{T}_2$ = Hausdorff.

## 3 Topological semantics for classical S4 and Intuitionistic propositional logic

Fix a countably infinite set $AP$ of atomic propositions. The propositional language $\mathcal{L}_0$ is generated from $p \in AP$ by the grammar:

$$\varphi ::= p \mid \bot \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \to \varphi_2$$

As usual, further connectives are defined by $\neg\varphi := \varphi \to \bot$ and $\varphi_1 \leftrightarrow \varphi_2 := (\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$, and the constant $\top := \bot \to \bot$. Let $\mathcal{L}_\Box$ be the monomodal language extending $\mathcal{L}_0$ with the addition of the unary modal operator $\Box$. A further modal operator $\Diamond$ can be defined as the classical dual: $\Diamond\varphi := \neg\Box\neg\varphi$.

Let $\mathbf{IPC} \subseteq \mathcal{L}_0$ be the set of Intuitionistic propositional theorems, and abusing notation, let **IPC** also denote one's favourite axiomatisation for Intuitionistic propositional logic. Likewise, let $\mathbf{S4}_\Box \subseteq \mathcal{L}_\Box$ be the set of theorems of classical S4, and let $\mathbf{S4}_\Box$ also denote any standard axiomatisation of classical S4. To be concrete, let $\mathbf{S4}_\Box$ contain all instances of classical propositional tautologies in the language $\mathcal{L}_\Box$, and the axiom schemes

$$\begin{array}{llll} \mathbf{N}_\Box : & \Box\top & \qquad \mathbf{T}_\Box : & \Box\varphi \to \varphi \\ \mathbf{R}_\Box : & \Box(\varphi_1 \wedge \varphi_2) \leftrightarrow \Box\varphi_1 \wedge \Box\varphi_2 & \qquad \mathbf{4}_\Box : & \Box\varphi \to \Box\Box\varphi \end{array}$$

and be closed under the inference rules of *modus ponens* (**MP**) and $\Box$-monotonicity ($\mathbf{Mono}_\Box$): from $\varphi_1 \to \varphi_2$, infer $\Box\varphi_1 \to \Box\varphi_2$.

The Gödel translation $\mathrm{GT} : \mathcal{L}_0 \to \mathcal{L}_\Box$ is defined by:

$$\begin{array}{rcll} \mathrm{GT}(p) & := & \Box p & \qquad \text{for atomic } p \in AP \\ \mathrm{GT}(\bot) & := & \bot & \\ \mathrm{GT}(\varphi_1 \vee \varphi_2) & := & \mathrm{GT}(\varphi_1) \vee \mathrm{GT}(\varphi_2) & \\ \mathrm{GT}(\varphi_1 \wedge \varphi_2) & := & \mathrm{GT}(\varphi_1) \wedge \mathrm{GT}(\varphi_2) & \\ \mathrm{GT}(\varphi_1 \to \varphi_2) & := & \Box\,(\mathrm{GT}(\varphi_1) \to \mathrm{GT}(\varphi_2)) & \end{array}$$

In particular, Intuitionistic negation comes out as $\mathrm{GT}(\neg\varphi) = \mathrm{GT}(\varphi \to \bot) = \Box\,(\mathrm{GT}(\varphi) \to \bot) = \Box\neg\,\mathrm{GT}(\varphi)$ and for double negation, $\mathrm{GT}(\neg\neg\varphi) = \Box\neg\Box\neg\,\mathrm{GT}(\varphi) = \Box\Diamond\,\mathrm{GT}(\varphi)$. Reading the

S4 $\Box$ as topological interior, we can read off the Intuitionistic topological semantics directly from the clauses of the Gödel translation.

**Definition 3.1.** *Given a topological space $\mathcal{F} = (X, \mathcal{T})$, a* model *over $\mathcal{F}$ is a structure $\mathcal{M} = (X, \mathcal{T}, \xi)$ where $\xi : X \rightsquigarrow AP$ is a set-valued map, the* atomic valuation relation. *For each $p \in AP$, the set $\xi^{-1}(p) = \{x \in X \mid p \in \xi(x)\}$ is the* denotation *of $p$ in $\mathcal{M}$. A model $\mathcal{M} = (X, \mathcal{T}, \xi)$ is an* open model *if for each $p \in AP$, the denotation set $\xi^{-1}(p)$ is open in $\mathcal{T}$. For open models $\mathcal{M}$, the Intuitionistic denotation map $[\![\cdot]\!]^{\mathcal{M}}_{\text{INT}} : \mathcal{L}_0 \to 2^X$ is defined by:*

$$
\begin{aligned}
[\![p]\!]^{\mathcal{M}}_{\text{INT}} &:= \xi^{-1}(p) \\
[\![\bot]\!]^{\mathcal{M}}_{\text{INT}} &:= \varnothing \\
[\![\varphi_1 \vee \varphi_2]\!]^{\mathcal{M}}_{\text{INT}} &:= [\![\varphi_1]\!]^{\mathcal{M}}_{\text{INT}} \cup [\![\varphi_2]\!]^{\mathcal{M}}_{\text{INT}} \\
[\![\varphi_1 \wedge \varphi_2]\!]^{\mathcal{M}}_{\text{INT}} &:= [\![\varphi_1]\!]^{\mathcal{M}}_{\text{INT}} \cap [\![\varphi_2]\!]^{\mathcal{M}}_{\text{INT}} \\
[\![\varphi_1 \to \varphi_2]\!]^{\mathcal{M}}_{\text{INT}} &:= int_{\mathcal{T}}\left((X - [\![\varphi_1]\!]^{\mathcal{M}}_{\text{INT}}) \cup [\![\varphi_2]\!]^{\mathcal{M}}_{\text{INT}}\right)
\end{aligned}
$$

*A formula $\varphi \in \mathcal{L}_0$ is* Intuitionistically topologically valid *in an open model $\mathcal{M}$, abbreviated* Int-top valid *in $\mathcal{M}$, if $[\![\varphi]\!]^{\mathcal{M}}_{\text{INT}} = X$, and it is* Int-top valid *in $\mathcal{F} = (X, \mathcal{T})$, written $\mathcal{F} \vDash_{Int} \varphi$, if it is Int-top valid in $\mathcal{M}$ for all open models $\mathcal{M}$ over $\mathcal{F}$. Let $\mathbb{IT}$ be the set of all $\varphi \in \mathcal{L}_0$ such that $\mathcal{F} \vDash_{Int} \varphi$ for every topological space $\mathcal{F}$.*

It is immediate that in an open model $\mathcal{M}$, the denotation set $[\![\varphi]\!]^{\mathcal{M}}_{\text{INT}}$ is open in $\mathcal{T}$ for all $\varphi \in \mathcal{L}_0$, and this corresponds to the $\preccurlyeq$-*persistence* or up-$\preccurlyeq$-closed property in the relational Kripke semantics. Note also that if $\mathcal{M} = (X, \mathcal{T}, \xi)$ is open then the map $\xi : (X, \mathcal{T}) \rightsquigarrow (AP, \mathcal{T}_A)$ is l.s.c., where $\mathcal{T}_A \subseteq 2^{AP}$ is *any* topology on $AP$, because $pre^{\exists}(\xi)(Y) = \bigcup \left\{\xi^{-1}(p) \mid p \in Y\right\}$ is open in $\mathcal{T}_A$ for *any* set $Y \subseteq AP$. Conversely, if $\xi : (X, \mathcal{T}) \rightsquigarrow (AP, \mathcal{T}_D)$ is l.s.c., where $\mathcal{T}_D \subseteq 2^{AP}$ is the *discrete* topology on $AP$, then $\mathcal{M} = (X, \mathcal{T}, \xi)$ is an open model.

**Definition 3.2.** *For the modal language $\mathcal{L}_\Box$, we define the (classical) denotation map $[\![\cdot]\!]^{\mathcal{M}} : \mathcal{L}_\Box \to 2^X$ with respect to arbitrary topological models $\mathcal{M} = (X, \mathcal{T}, \xi)$, where $\xi : X \rightsquigarrow AP$ is unrestricted; the map is defined the same way as $[\![\cdot]\!]^{\mathcal{M}}_{\text{INT}}$ for atomic $p \in AP$, $\bot$, $\vee$ and $\wedge$, but differs on the clauses:*

$$
\begin{aligned}
[\![\varphi_1 \to \varphi_2]\!]^{\mathcal{M}} &:= (X - [\![\varphi_1]\!]^{\mathcal{M}}) \cup [\![\varphi_2]\!]^{\mathcal{M}} \\
[\![\Box\varphi]\!]^{\mathcal{M}} &:= int_{\mathcal{T}}\left([\![\varphi]\!]^{\mathcal{M}}\right)
\end{aligned}
$$

*A formula $\varphi \in \mathcal{L}_\Box$ is* modal-top valid *in $\mathcal{F} = (X, \mathcal{T})$, written $\mathcal{F} \vDash \varphi$, if $[\![\varphi]\!]^{\mathcal{M}} = X$ for all models $\mathcal{M} = (X, \mathcal{T}, \xi)$ over $\mathcal{F}$. Let $\mathbb{T}$ be the set of all $\varphi \in \mathcal{L}_\Box$ such that $\mathcal{F} \vDash \varphi$ for every topological space $\mathcal{F}$.*

Topological soundness is a simple verification, and topological completeness can be obtained cheaply from Kripke completeness with respect to frames $\mathcal{F} = (X, \preccurlyeq)$ where $\preccurlyeq$ is a preorder on $X$, using the correspondance between preorders and Alexandroff topologies.

**Proposition 3.3.** [Topological completeness]

– *For all $\varphi \in \mathcal{L}_0$, $\varphi$ is a theorem of* **IPC** *iff* $\varphi \in \mathbb{IT}$.
– *For all $\psi \in \mathcal{L}_\Box$, $\psi$ is a theorem of* **S4**$_\Box$ *iff* $\psi \in \mathbb{T}$.

**Proposition 3.4.** [Gödel translation] *For all $\varphi \in \mathcal{L}_0$,*

– *$\varphi$ is a theorem of* **IPC** *iff* $\mathrm{GT}(\varphi)$ *is a theorem of* **S4**$_\Box$.
– $\mathrm{GT}(\varphi) \leftrightarrow \Box\mathrm{GT}(\varphi)$ *is a theorem of* **S4**$_\Box$.

While we can get topological completeness cheaply, it can also be obtained at greater expense with respect to particular classes of topological spaces. The classic McKinsey and Tarski result [15] is topological completeness of **S4** in separable dense-in-itself metric spaces;

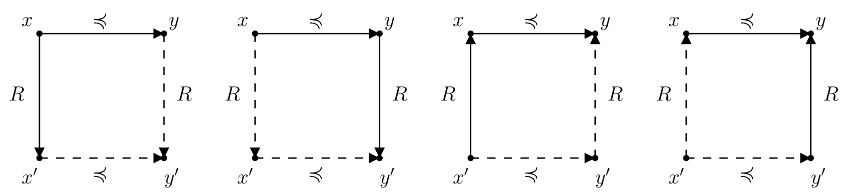
in particular, $\mathbb{R}^n$ equipped with the standard Euclidean topology $\mathcal{T}_E$. The reader is referred to the recent thesis of Aiello [1] which gives new proofs of the McKinsey and Tarski theorem, and of completeness in Cantor space.

## 4 Topological semantics for Intuitionistic modal logics

We give a quite straightforward topological extension of the birelational semantics of Fischer Servi [9, 10], and Plotkin and Stirling [17] over Kripke frames $\mathcal{F} = (X, \preccurlyeq, R)$, where $\preccurlyeq$ is a preorder on $X$, and $R : X \rightsquigarrow X$ is a relation. In the birelational semantics, the central concern is the connection between the two relations: the preorder $\preccurlyeq$ as the Intuitionistic information ordering, and the relation $R$ as the modal accessibility relation. Using the induced Alexandroff topology $\mathcal{T}_{\preccurlyeq}$, a birelational Kripke frame $\mathcal{F}$ is equivalent to the topological frame $(X, \mathcal{T}_{\preccurlyeq}, R)$. The four birelational conditions identified in [17] can be cleanly transcribed as *semi-continuity conditions* on the relation $R : X \rightsquigarrow X$ with respect to the topology $\mathcal{T}_{\preccurlyeq}$, which then generalise to abitrary topologies.

**Definition 4.1.** *Let $\mathcal{F} = (X, \preccurlyeq, R)$ be a birelational frame. Four conditions on the connection between $\preccurlyeq$ and $R$ are identified as follows:*

***Zig***-$\preccurlyeq, R$ : *if* $x \preccurlyeq y$ *and* $x\,R\,x'$ *then* $(\exists y' \in X)\big[\; y\,R\,y'$ *and* $x' \preccurlyeq y' \;\big]$
***Zag***-$\preccurlyeq, R$ : *if* $x \preccurlyeq y$ *and* $y\,R\,y'$ *then* $(\exists x' \in X)\big[\; x\,R\,x'$ *and* $x' \preccurlyeq y' \;\big]$
***Zig***-$\preccurlyeq, R^{-1}$ : *if* $x \preccurlyeq y$ *and* $x'\,R\,x$ *then* $(\exists y' \in X)\big[\; y'\,R\,y$ *and* $x' \preccurlyeq y' \;\big]$
***Zag***-$\preccurlyeq, R^{-1}$ : *if* $x \preccurlyeq y$ *and* $y'\,R\,y$ *then* $(\exists x' \in X)\big[\; x'\,R\,x$ *and* $x' \preccurlyeq y' \;\big]$




**Zig-**$\preccurlyeq$, $R$ **Zag-**$\preccurlyeq$, $R$ **Zig-**$\preccurlyeq$, $R^{-1}$ **Zag-**$\preccurlyeq$, $R^{-1}$

The conditions correspond to the four ways of "completing the square" where two directed edges labelled with $\preccurlyeq$ and $R$ are given, and the square must be completed with two more directed edges labelled with $\preccurlyeq$ and $R$, and edges with matching labels must be parallel and in the same direction. We use the "Zig" and "Zag" names because **Zig-**$\preccurlyeq, R$ and **Zag-**$\preccurlyeq, R$ are exactly the well-known forth and back conditions on $\preccurlyeq$ being a *bisimulation* on the frame $(X, R)$, which are known by those names. From earlier work [5], we know the bisimulation conditions correspond to semi-continuity properties.

**Proposition 4.2.** ([5]) *Let $\mathcal{F} = (X, \preccurlyeq, R)$ be a birelational frame, with $\mathcal{T}_{\preccurlyeq}$ the induced Alexandroff topology. In the following table, the conditions listed along each row are equivalent.*

| | | | | |
|---|---|---|---|---|
| *1.* | **Zig**-$\preccurlyeq, R$ | $(\succcurlyeq \circ R) \subseteq (R \circ \succcurlyeq)$ | | $R$ *is l.s.c. in* $\mathcal{T}_{\preccurlyeq}$ |
| *2.* | **Zag**-$\preccurlyeq, R$ | $(\preccurlyeq \circ R) \subseteq (R \circ \preccurlyeq)$ | | $R$ *is u.s.c. in* $\mathcal{T}_{\preccurlyeq}$ |
| *3.* | **Zig**-$\preccurlyeq, R^{-1}$ | $(\succcurlyeq \circ R^{-1}) \subseteq (R^{-1} \circ \succcurlyeq)$ | $(R \circ \preccurlyeq) \subseteq (\preccurlyeq \circ R)$ | $R^{-1}$ *is l.s.c. in* $\mathcal{T}_{\preccurlyeq}$ |
| *4.* | **Zag**-$\preccurlyeq, R^{-1}$ | $(\preccurlyeq \circ R^{-1}) \subseteq (R^{-1} \circ \preccurlyeq)$ | $(R \circ \succcurlyeq) \subseteq (\succcurlyeq \circ R)$ | $R^{-1}$ *is u.s.c. in* $\mathcal{T}_{\preccurlyeq}$ |

In the birelational semantics introduced by Fischer Servi in [10] and used in [11, 17, 19, 14], it is the first and third frame conditions **Zig-**$\preccurlyeq, R$ and **Zig-**$\preccurlyeq, R^{-1}$ that are identified as those needed to give an Intuitionistic semantics for modalities based on $R$. In frames meeting these conditions, one can give semantic clauses for the diamond and box that are natural under the Intuitionistic reading of the restricted $\exists$ and $\forall$ quantification with respect to $R$-successors. More precisely, the resulting logic is faithfully embedded into Intuitionistic first-order logic by the standard modal to first-order translation, and a natural extension of the Gödel translation faithfully embeds it into the classical bimodal logic combining $\mathbf{S4}_\Box$ with $\mathbf{K}$ or extensions. Various further motivation for these two frame conditions are offered in the literature, to which the reader is referred. From the proposition above, we see that these two conditions correspond to the lower semi-continuity of $R$ and $R^{-1}$ in $\mathcal{T}_{\preccurlyeq}$, and we use this observation to generalise the semantics to arbitary topologies.

Let $\mathcal{L}_{\Diamond\!\!\!\cdot\,\boxdot}$ be the modal language extending $\mathcal{L}_0$ with the addition of two modal operators $\Diamond\!\!\!\cdot\,$ and $\boxdot$, generated by the grammar:

$$\varphi ::= p \mid \bot \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \to \varphi_2 \mid \Diamond\!\!\!\cdot\,\varphi \mid \boxdot\varphi$$

Likewise, let $\mathcal{L}_{\Box\Diamond\!\!\!\cdot\,\boxdot}$ be the modal language extending $\mathcal{L}_\Box$ with the addition of $\Diamond\!\!\!\cdot\,$ and $\boxdot$.

**Definition 4.3.** *A* topological frame *is a structure $\mathcal{F} = (X, \mathcal{T}, R)$ where $(X, \mathcal{T})$ is a topological space and $R : X \rightsquigarrow X$ is a relation. We say $\mathcal{F}$ is an* l.s.c. topological frame *if both $R : (X, \mathcal{T}) \rightsquigarrow (X, \mathcal{T})$ is l.s.c. and $R^{-1} : (X, \mathcal{T}) \rightsquigarrow (X, \mathcal{T})$ is l.s.c. A* model *over $\mathcal{F}$ is a structure $\mathcal{M} = (X, \mathcal{T}, R, \xi)$ where $\xi : X \rightsquigarrow AP$ is an atomic valuation relation. As before, a model $\mathcal{M}$ will be called an* open model *if for each $p \in AP$, the denotation set $\xi^{-1}(p)$ is open in $\mathcal{T}$. For open models $\mathcal{M}$ over l.s.c. frames $\mathcal{F}$, the Intuitionistic denotation map $[\![\cdot]\!]^{\mathcal{M}}_{\mathrm{INT}} : \mathcal{L}_{\Diamond\!\!\!\cdot\,\boxdot} \to 2^X$ is defined the same way as for $\mathcal{L}_0$, with the additional clauses:*

$$[\![\Diamond\!\!\!\cdot\,\varphi]\!]^{\mathcal{M}}_{\mathrm{INT}} := pre^{\exists}(R)\left([\![\varphi]\!]^{\mathcal{M}}_{\mathrm{INT}}\right)$$
$$[\![\boxdot\varphi]\!]^{\mathcal{M}}_{\mathrm{INT}} := int_{\mathcal{T}}\left(pre^{\forall}(R)\left([\![\varphi]\!]^{\mathcal{M}}_{\mathrm{INT}}\right)\right)$$

*A formula $\varphi \in \mathcal{L}_{\Diamond\!\!\!\cdot\,\boxdot}$ will be called* Int-modal-top valid *in an l.s.c. frame $\mathcal{F} = (X, \mathcal{T}, R)$ if $[\![\varphi]\!]^{\mathcal{M}}_{\mathrm{INT}} = X$ for all open models $\mathcal{M}$ over $\mathcal{F}$. Let $\mathbb{IKT}$ be the set of all $\varphi \in \mathcal{L}_{\Diamond\!\!\!\cdot\,\boxdot}$ such that $\varphi$ is Int-modal-top valid in every l.s.c. topological frame $\mathcal{F}$.*

The property that every denotation set $[\![\varphi]\!]^{\mathcal{M}}_{\mathrm{INT}}$ is open in $\mathcal{T}$ follows immediately from the openness condition on $\xi^{-1}(p)$, the l.s.c. condition on $pre^{\exists}(R)$, and the extra interior operation in the semantics for $\boxdot\varphi$. The semantic clauses are exactly as one would expect, given the standard modal to first-order translation, and the topological semantics for Intuitionistic first-order logic, where $\forall$ is evaluated by the interior of an intersection, and $\exists$ is evaluated by a union.

**Definition 4.4.** *For the modal language $\mathcal{L}_{\Box\Diamond\!\!\!\cdot\,\boxdot}$, we define the (classical) denotation map $[\![\cdot]\!]^{\mathcal{M}} : \mathcal{L}_{\Box\Diamond\!\!\!\cdot\,\boxdot} \to 2^X$ with respect to arbitrary topological models $\mathcal{M} = (X, \mathcal{T}, R, \xi)$, where $\xi : X \rightsquigarrow AP$ is unrestricted; the map is defined the same way as for $\mathcal{L}_\Box$, with the additional clauses:*

$$[\![\Diamond\!\!\!\cdot\,\varphi]\!]^{\mathcal{M}} := pre^{\exists}(R)\left([\![\varphi]\!]^{\mathcal{M}}\right)$$
$$[\![\boxdot\varphi]\!]^{\mathcal{M}} := pre^{\forall}(R)\left([\![\varphi]\!]^{\mathcal{M}}\right)$$

*A formula $\varphi \in \mathcal{L}_{\Box\Diamond\!\!\!\cdot\,\boxdot}$ will be called* modal-top valid *in $\mathcal{F} = (X, \mathcal{T}, R)$, written $\mathcal{F} \vDash \varphi$, if $[\![\varphi]\!]^{\mathcal{M}} = X$ for all models $\mathcal{M}$ over $\mathcal{F}$. Let $\mathbb{KT}$ be the set of all $\varphi \in \mathcal{L}_{\Box\Diamond\!\!\!\cdot\,\boxdot}$ such that $\varphi$ is modal-top valid in $\mathcal{F}$ for every topological frame $\mathcal{F}$, and let $\mathbb{LST}$ be the set of all $\varphi \in \mathcal{L}_{\Box\Diamond\!\!\!\cdot\,\boxdot}$ such that $\varphi$ is modal-top valid in every l.s.c. topological frame.*

The Fischer Servi extension of the Gödel translation [9, 11], extends to $\mathrm{GT} : \mathcal{L}_{\Diamond\!\!\!\cdot\,\boxdot} \to \mathcal{L}_{\Box\Diamond\!\!\!\cdot\,\boxdot}$ with the additional clauses:

$$\mathrm{GT}(\Diamond\!\!\!\cdot\,\varphi) := \Diamond\!\!\!\cdot\,\mathrm{GT}(\varphi)$$
$$\mathrm{GT}(\boxdot\varphi) := \Box\boxdot\,\mathrm{GT}(\varphi)$$

The extension continues with the clause-by-clause syntactic replication of the topological semantics. The semi-continuity frame conditions also have clean modal characterisations, as we have observed in [6].

**Proposition 4.5.** ([6]) *Let $\mathcal{F} = (X, \mathcal{T}, R)$ be a topological frame and let $p \in AP$. In the following tables, the conditions listed across each row are equivalent.*

| | | | |
|---|---|---|---|
| *1.* | *$R$ is l.s.c. in $\mathcal{T}$* | $\mathcal{F} \models \diamonddot\Box p \to \Box\diamonddot p$ | $\mathcal{F} \models \diamonddot\Box p \leftrightarrow \Box\diamonddot\Box p$ |
| *2.* | *$R$ is u.s.c. in $\mathcal{T}$* | $\mathcal{F} \models \boxdot\Box p \to \Box\boxdot p$ | $\mathcal{F} \models \Diamond\diamonddot p \to \diamonddot\Diamond p$ |
| *3.* | *$R^{-1}$ is l.s.c. in $\mathcal{T}$* | $\mathcal{F} \models \Box\boxdot p \to \boxdot\Box p$ | $\mathcal{F} \models \diamonddot\Diamond p \to \Diamond\diamonddot p$ |
| *4.* | *$R^{-1}$ is u.s.c. in $\mathcal{T}$* | $\mathcal{F} \models \Box\diamonddot p \to \diamonddot\Box p$ | |

This modal syntactic characterisation can shed some further light on the Intuitionistic frame conditions. The $R$-l.s.c. scheme $\diamonddot\Box p \leftrightarrow \Box\diamonddot\Box p$ expresses directly that $\diamonddot$ applied to an open set $\Box p$ must be open, being $\Box\diamonddot\Box p$. The $R^{-1}$-l.s.c. scheme is not quite so clear: it says that $\Box$ can be "pushed through" $\boxdot$ in one direction. We return to these modal characterisations of the semi-continuity frame conditions later in this section when deriving theorems on semi-duality between Intuitionistic $\diamonddot$ and $\boxdot$ via the Gödel translation.

Let $\mathbf{IK}_{\diamonddot\boxdot}$ be the axiomatic system of Fischer Servi [11, 14], which is equivalent to an alternative axiomatisation given in [17, 19]; $\mathbf{IK}_{\diamonddot\boxdot}$ also goes by the name **FS** in [14] and [21]. $\mathbf{IK}_{\diamonddot\boxdot}$ has as axioms all instances in the language $\mathcal{L}_{\diamonddot\boxdot}$ of Intuitionistic propositional theorems, and further axiom schemes:

$$\mathbf{R}_{\diamonddot} : \diamonddot(\varphi \vee \psi) \leftrightarrow (\diamonddot\varphi \vee \diamonddot\psi)$$
$$\mathbf{R}_{\boxdot} : \boxdot(\varphi \wedge \psi) \leftrightarrow (\boxdot\varphi \wedge \boxdot\psi)$$
$$\mathbf{N}_{\neg\diamonddot} : \neg\diamonddot\bot$$
$$\mathbf{N}_{\boxdot} : \boxdot\top$$
$$\mathbf{F1}_{\boxdot\diamonddot} : \diamonddot(\varphi \to \psi) \to (\boxdot\varphi \to \diamonddot\psi)$$
$$\mathbf{F2}_{\boxdot\diamonddot} : (\diamonddot\varphi \to \boxdot\psi) \to \boxdot(\varphi \to \psi)$$

$\mathbf{IK}_{\diamonddot\boxdot}$ is closed under the inference rules of *modus ponens* (**MP**) and the rule ($\mathbf{Mono}_{\diamonddot}$): from $\varphi_1 \to \varphi_2$, infer $\diamonddot\varphi_1 \to \diamonddot\varphi_2$, and likewise ($\mathbf{Mono}_{\boxdot}$). The theorems of $\mathbf{IK}_{\diamonddot\boxdot}$ include the following schema expressing semi-duality laws for $\diamonddot$ and $\boxdot$.

$$\mathbf{SD}1 : \quad \diamonddot\varphi \to \neg\boxdot\neg\varphi$$
$$\mathbf{SD}2 : \quad \diamonddot\neg\varphi \to \neg\boxdot\varphi$$
$$\mathbf{SD}3 : \quad \neg\diamonddot\varphi \leftrightarrow \boxdot\neg\varphi$$

$\mathbf{IK}_{\diamonddot\boxdot}$ has characteristically Intuitionistic features.

**Proposition 4.6.** ([17])

*1.* Disjunction property*: for all $\varphi, \psi \in \mathcal{L}_{\diamonddot\boxdot}$, if $\varphi \vee \psi$ is a theorem of $\mathbf{IK}_{\diamonddot\boxdot}$, then either $\varphi$ is a theorem of $\mathbf{IK}_{\diamonddot\boxdot}$ or $\psi$ is a theorem of $\mathbf{IK}_{\diamonddot\boxdot}$.*
*2.* Classical collapse*: the axiom system consisting of $\mathbf{IK}_{\diamonddot\boxdot}$ together with* excluded middle $\varphi \vee \neg\varphi$ *and* modal duality $\diamonddot\varphi \leftrightarrow \neg\boxdot\neg\varphi$*, captures exactly the minimal classical normal modal logic $\mathbf{K}_{\boxdot}$.*

Let $(\mathbf{S4}_{\Box} \otimes \mathbf{K}_{\boxdot})$ be the bimodal *fusion* of $\mathbf{S4}_{\Box}$ and $\mathbf{K}_{\boxdot}$, i.e. the smallest normal bimodal logic in the language $\mathcal{L}_{\Box\diamonddot\boxdot}$ containing both $\mathbf{S4}_{\Box}$ and $\mathbf{K}_{\boxdot}$, which includes closure under the modal inference rules ($\mathbf{Mono}_{\Box}$), ($\mathbf{Mono}_{\diamonddot}$) and ($\mathbf{Mono}_{\boxdot}$). With regard to notation for combinations of modal logics, we follow that used in the work of Wolter and Zakharyaschev [21, 22]. If $L_1$ and $L_2$ are modal logics in languages $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then the fusion $L_1 \otimes L_2$ is the smallest (poly-)modal logic in the language $\mathcal{L}_1 \otimes \mathcal{L}_1$ containing $L_1$ and $L_2$, and closed under all the inference rules of $L_1$ and $L_2$; we use $\mathcal{L}_1 \otimes \mathcal{L}_1$ to denote the least common extension of the languages $\mathcal{L}_1$ and $\mathcal{L}_2$. If $L$ is a logic in language $\mathcal{L}$, and $\Gamma$ is a finite list of schema in $\mathcal{L}$, then the *extension* $L \oplus \Gamma$ is the smallest logic in $\mathcal{L}$ extending $L$, containing the schema in $\Gamma$ as additional axioms, and closed under the rules of $L$.

Let $\mathbf{KLSC} := (\mathbf{S4}_\Box \otimes \mathbf{K}_{\boxdot}) \oplus (\Diamond\!\!\!\cdot\,\Box\varphi \to \Box\Diamond\!\!\!\cdot\,\varphi) \oplus (\Box\boxdot\varphi \to \boxdot\Box\varphi)$ be the extension of $(\mathbf{S4}_\Box \otimes \mathbf{K}_{\boxdot})$ with the characteristic schema for the $R$-l.s.c. and $R^{-1}$-l.s.c. frame conditions. The closely related logic $(\mathbf{S4}_\Box \otimes \mathbf{S4}_{\boxdot}) \oplus (\Diamond\!\!\!\cdot\,\Box\varphi \to \Box\Diamond\!\!\!\cdot\,\varphi)$ is studied by Davoren and Goré [6], under the working name of **LSC**, and given semantics in topological frames where $R$ is l.s.c. and a preorder. That logic was motivated by a modal investigation of continuous dynamics, as the class of frames characterised by **LSC** includes frames over Euclidean space where $R$ is the positive orbit relation of a differential equation with unique solutions. With a stronger requirement of invertibility on the flow of the differential equation, both $R$ and $R^{-1}$ will be l.s.c., so those topological frames can be models of **KLSC** as well.

As a point of comparison, the basic system of Intuitionistic modal logic in [21], under the name $\mathbf{IntK}_{\Diamond\!\!\!\cdot\,\boxdot}$, is weaker than $\mathbf{IK}_{\Diamond\!\!\!\cdot\,\boxdot}$ in that it has only the axiom schema $\mathbf{R}_{\Diamond\!\!\!\cdot}$, $\mathbf{R}_{\boxdot}$, $\mathbf{N}_{\neg\Diamond\!\!\!\cdot}$, and $\mathbf{N}_{\boxdot}$, plus the rules $(\mathbf{Mono}_{\Diamond\!\!\!\cdot})$ and $(\mathbf{Mono}_{\boxdot})$. So our $\mathbf{IK}_{\Diamond\!\!\!\cdot\,\boxdot} = \mathbf{IntK}_{\Diamond\!\!\!\cdot\,\boxdot} \oplus \{\mathbf{F1}_{\boxdot\Diamond\!\!\!\cdot}, \mathbf{F2}_{\boxdot\Diamond\!\!\!\cdot}\}$. The additional schemes $\mathbf{F1}_{\boxdot\Diamond\!\!\!\cdot}$ and $\mathbf{F2}_{\boxdot\Diamond\!\!\!\cdot}$ are identified in [11].

The topological soundness of $\mathbf{IK}_{\Diamond\!\!\!\cdot\,\boxdot}$ and **KLSC** are easy verifications. To prove topological completeness, we can get a free ride from the corresponding proofs for birelational frames. Completeness for $\mathbf{IK}_{\Diamond\!\!\!\cdot\,\boxdot}$ [11], [19], is proved using a canonical model over the state space $X_{\mathbf{IK}_{\Diamond\!\!\!\cdot\,\boxdot}}$ as the set of all *prime theories* $T \subseteq \mathcal{L}_{\Diamond\!\!\!\cdot\,\boxdot}$ such that $\mathbf{IK}_{\Diamond\!\!\!\cdot\,\boxdot} \subseteq T$, where $T$ is prime if it is deductively closed, consistent, and has the disjunction property. The space is partially ordered by inclusion, inducing an Alexandroff topology $\mathcal{T}_{\subseteq}$, and the modal relation is: $T_1\,R\,T_2$ iff $\{\Diamond\!\!\!\cdot\,\varphi \mid \varphi \in T_2\} \subseteq T_1$ and $\{\psi \mid \boxdot\psi \in T_1\} \subseteq T_2$, which is the conjunction of the usual (classically equal) relations determined by $\Diamond\!\!\!\cdot$ and $\boxdot$. Topological completeness for $\mathbf{S4}_\Box \otimes \mathbf{K}_{\boxdot}$ and **KLSC** comes straightforwardly via the classical canonical model over the space of maximal consistent theories, with the preorder determined by $\Box$, and $R$ determined by $\boxdot$ [6].

**Proposition 4.7.** [Topological completeness] *For all* $\varphi \in \mathcal{L}_{\Diamond\!\!\!\cdot\,\boxdot}$,

- $\varphi$ *is a theorem of* $\mathbf{IK}_{\Diamond\!\!\!\cdot\,\boxdot}$ *iff* $\varphi \in \mathbb{IKT}$.
- *For all* $\psi \in \mathcal{L}_{\Box\Diamond\!\!\!\cdot\,\boxdot}$, $\psi$ *is a theorem of* $\mathbf{S4}_\Box \otimes \mathbf{K}_{\boxdot}$ *iff* $\psi \in \mathbb{KT}$.
- *For all* $\psi \in \mathcal{L}_{\Box\Diamond\!\!\!\cdot\,\boxdot}$, $\psi$ *is a theorem of* **KLSC** *iff* $\psi \in \mathbb{LST}$.

**Proposition 4.8.** [Extended Gödel translation] (Fischer Servi [10]) *For all* $\varphi \in \mathcal{L}_{\Diamond\!\!\!\cdot\,\boxdot}$,

- $\varphi$ *is a theorem of* $\mathbf{IK}_{\Diamond\!\!\!\cdot\,\boxdot}$ *iff* $\mathrm{GT}(\varphi)$ *is a theorem of* **KLSC**
- $\mathrm{GT}(\varphi) \leftrightarrow \Box\mathrm{GT}(\varphi)$ *is a theorem of* **KLSC**.

To prove $(\varphi \to \psi)$ is a theorem of $\mathbf{IK}_{\Diamond\!\!\!\cdot\,\boxdot}$, it suffices to prove that $\mathrm{GT}(\varphi) \to \mathrm{GT}(\psi)$ is a theorem of **KLSC**. This is so because $\mathrm{GT}(\varphi \to \psi) = \Box(\mathrm{GT}(\varphi) \to \mathrm{GT}(\psi))$, which is derivable from $\mathrm{GT}(\varphi) \to \mathrm{GT}(\psi)$ by the derived rule of $\Box$-necessitation. We prove instances of the semi-duality properties by the Gödel translation, with explicit appeal to the modal semi-continuity schema.

$$
\begin{array}{lll}
\mathbf{SD}1: \quad \mathrm{GT}(\Diamond\!\!\!\cdot\,p) & \leftrightarrow \Diamond\!\!\!\cdot\,\Box p & \\
& \leftrightarrow \Box\Diamond\!\!\!\cdot\,\Box p & R\text{-l.s.c. } (\Diamond\!\!\!\cdot\,\Box\varphi \leftrightarrow \Box\Diamond\!\!\!\cdot\,\Box\varphi) \\
& \to \Box\Diamond\Diamond\!\!\!\cdot\,\Diamond\Box p & \mathbf{T}_\Diamond \text{ scheme \& } \mathbf{Mono} \text{ rules} \\
& \leftrightarrow \Box\neg\Box\neg\Diamond\!\!\!\cdot\,\neg\Box\neg\Box p & \text{duality } \Box/\Diamond \\
& \leftrightarrow \Box\neg\Box\boxdot\Box\neg\Box p & \text{duality } \boxdot/\Diamond\!\!\!\cdot \\
& \leftrightarrow \mathrm{GT}(\neg\boxdot\neg p) &
\end{array}
$$

$$
\begin{aligned}
\mathbf{SD2}: \quad \mathrm{GT}(\Diamonddot\neg p) &\leftrightarrow \Diamonddot\Box\neg\Box p \\
&\leftrightarrow \Box\Diamonddot\Box\neg\Box p && R\text{-l.s.c.}(\Diamonddot\Box\varphi \leftrightarrow \Box\Diamonddot\Box\varphi) \\
&\rightarrow \Box\Diamonddot\neg\Box p && \mathbf{T}_\Box \text{ \& } \mathbf{Mono} \text{ rules} \\
&\rightarrow \Box\Diamond\Diamonddot\neg\Box p && \mathbf{T}_\Diamond \text{ \& } \mathbf{Mono} \text{ rules} \\
&\leftrightarrow \Box\neg\Box\boxdot\Box p && \text{duality } \Box/\Diamond \text{ \& } \boxdot/\Diamonddot \\
&\leftrightarrow \mathrm{GT}(\neg\boxdot p)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{SD3}: \quad \mathrm{GT}(\neg\Diamonddot p) &\leftrightarrow \Box\neg\Diamonddot\Box p \\
&\leftrightarrow \Box\boxdot\neg\Box p && \text{duality } \boxdot/\Diamonddot \\
&\leftrightarrow \Box\Box\boxdot\neg\Box p && \mathbf{4}_\Box \text{ and } \mathbf{T}_\Box \\
&\rightarrow \Box\boxdot\Box\neg\Box p && R^{-1}\text{-l.s.c. } (\Box\boxdot\varphi \rightarrow \boxdot\Box\varphi) \text{ \& } \mathbf{Mono}_\Box \\
&\leftrightarrow \mathrm{GT}(\boxdot\neg\varphi)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{GT}(\boxdot\neg\varphi) &\leftrightarrow \Box\boxdot\Box\neg\Box p \\
&\rightarrow \Box\boxdot\neg\Box p && \mathbf{T}_\Box \text{ \& } \mathbf{Mono} \text{ rules} \\
&\leftrightarrow \Box\neg\Diamonddot\Box p && \text{duality } \boxdot/\Diamonddot \\
&\leftrightarrow \mathrm{GT}(\neg\Diamonddot p)
\end{aligned}
$$

On the subject of decidability, there are some positive results on extensions $\mathbf{IK}_{\Diamonddot\boxdot} \oplus \Gamma$ for some subsets $\Gamma$ of the five axiom schemes below:

$$
\begin{aligned}
\mathbf{T}_{\boxdot\Diamonddot} &: (\boxdot\varphi \rightarrow \varphi) \wedge (\varphi \rightarrow \Diamonddot\varphi) \\
\mathbf{B}_{\boxdot\Diamonddot} &: (\varphi \rightarrow \boxdot\Diamonddot\varphi) \wedge (\Diamonddot\boxdot\varphi \rightarrow \varphi) \\
\mathbf{D}_{\Diamonddot} &: \Diamonddot\top \\
\mathbf{4}_{\boxdot\Diamonddot} &: (\boxdot\varphi \rightarrow \boxdot\boxdot\varphi) \wedge (\Diamonddot\Diamonddot\varphi \rightarrow \Diamonddot\varphi) \\
\mathbf{5}_{\boxdot\Diamonddot} &: (\Diamonddot\boxdot\varphi \rightarrow \boxdot\varphi) \wedge (\Diamonddot\varphi \rightarrow \boxdot\Diamonddot\varphi)
\end{aligned}
$$

Simpson [19] proves the finite model property over birelational frames for extensions where $\Gamma \subseteq \{\mathbf{T}_{\boxdot\Diamonddot}, \mathbf{B}_{\boxdot\Diamonddot}, \mathbf{D}_{\Diamonddot}\}$, and Grefe [14] proves the result independently for $\mathbf{IK}_{\Diamonddot\boxdot}$; earlier work via algebraic models proves the finite model property over birelational frames for $\mathbf{IS5}_{\Diamonddot\boxdot} := \mathbf{IK}_{\Diamonddot\boxdot} \oplus \mathbf{T}_{\boxdot\Diamonddot} \oplus \mathbf{5}_{\boxdot\Diamonddot}$, also known as **MIPC**. Decidability and the finite model property for the other combinations, including $\mathbf{IS4}_{\Diamonddot\boxdot} := \mathbf{IK}_{\Diamonddot\boxdot} \oplus \mathbf{T}_{\boxdot\Diamonddot} \oplus \mathbf{4}_{\boxdot\Diamonddot}$, remain open questions. For the bimodal logics, decidability for **KLSC**, and for the related S4-based **LSC** logic of [6] are also open questions. Wolter and Zakharyaschev also have finite model property results for quite a number of extensions of the weaker logic $\mathbf{IntK}_{\Diamonddot\boxdot}$, summarised in [21].

## 5 General topology of covers and A/D maps

We now turn our attention to a particularly interesting class of topologies identified in [16] that arise in the course of spatial discretisation.

**Definition 5.1.** *On an arbitrary set $X \neq \varnothing$, a* cover *is a total and surjective map $\alpha\colon X \rightsquigarrow Y$. The* cells *of the cover $\alpha$ are the sets $A_y := \alpha^{-1}(y) = \{x \in X \mid y \in \alpha(x)\}$, for each $y \in Y$, and we call $Y$ the* index set *or* range *of $\alpha$.*

Note that the totality condition on $\alpha$ ensures that $X = \bigcup_{y\in Y} A_y$, and the surjectivity condition ensures that $\alpha^{-1}$ is total, which means $A_y \neq \varnothing$ for each $y \in Y$. Observe that an arbitrary map $\beta\colon X \rightsquigarrow Z$ can be readily transformed into a total and surjective $\alpha\colon X \rightsquigarrow Y$ by taking $Y = \mathrm{ran}(\beta) \cup \{\$\}$ with $\alpha(x) = \beta(x) \in Z$ for $x \in \mathrm{dom}(\beta)$, and $\alpha(x) = \{\$\}$ for $x \notin \mathrm{dom}(\beta)$, and $\$ \notin Z$ a new element. If $\alpha\colon X \rightarrow Y$ is single-valued, then $\alpha$ is a quotient map and the $\alpha$-cells are partition blocks or equivalence classes. Let $Cov(X)$ be the family of all covers of a set $X$. Extremal elements of $Cov(X)$ are the constant function $c : X \rightarrow \{0\}$ onto a one point space, and at the other end of the spectrum, bijective functions $f : X \rightarrow Y$. The family $Cov(X)$ is partially ordered by the following notion of *refinement.*

**Definition 5.2.** *For two covers $\alpha\colon X \rightsquigarrow Y$ and $\beta\colon X \rightsquigarrow Z$ of $X$, we say $\alpha$ is* refined by *$\beta$, written $\alpha \Subset \beta$, if the following two conditions hold:*
*(1) for all $y \in Y$, there exists $z_1, \ldots, z_n \in Z$ such that $A_y = B_{z_1} \cup \cdots \cup B_{z_n}$;*
*(2) for all $z \in Z$, there exists $y \in Y$ such that $B_z \subseteq A_y$.*

The refinement relation $\Subset$ can be thought of as an information ordering: the constant function is the minimal element with *least* information, while the maximal elements with *perfect* information are the bijective functions on $X$.

With the notion of a cover, we have placed no constraint on the cardinality of the range $Y$. For a cover map $\alpha\colon X \rightsquigarrow Y$ to be appropriate for *analog-to-digital conversion* from an uncountable state space, say $X \subseteq \mathbb{R}^n$, onto a space $Y$, we place a constraint that the *limits of discernment* available through $\alpha$ are *finite*. This constraint can be simply satisfied by taking $Y$ to be finite (as we did in [7] and in [16]). More generally, we can require that the $\alpha$-cells $A_y$ do not generate any infinite descending chains of subsets of $X$.

**Definition 5.3.** *Given a cover $\alpha\colon X \rightsquigarrow Y$ of $X$, let $\mathcal{T}_\alpha \subseteq 2^X$ be the* topology *generated by the $\alpha$-cells; i.e. the collection of all sets $U \subseteq X$ formed from $\alpha$-cells by closing under arbitrary unions and finite intersections.*
*We call a cover $\alpha\colon X \rightsquigarrow Y$ an* A/D map *if as a lattice ordered by set-inclusion, $\mathcal{T}_\alpha$ has only finite-length chains.*

Simple examples of A/D maps include covers $\alpha\colon X \rightsquigarrow Y$ where $Y$ is finite, in which case the topology $\mathcal{T}_\alpha$ is finite, and in cardinality, $|\mathcal{T}_\alpha| \leq 2^{|Y|}$. Further examples include covers $\alpha\colon X \rightsquigarrow Y$ of $X \subseteq \mathbb{R}^n$ where the cells $A_y$ are of a fixed geometric shape and size, and are laid out in a regular grid; in this case, if $X$ is unbounded then the range set $Y$ will be countably infinite, and if $X$ is bounded then $Y$ will be finite. On the other hand, if $X \subseteq \mathbb{R}^n$, and the $\alpha$-cells constitute a *sub-basis* for the standard Euclidean topology $\mathcal{T}_E$ on $X$, so $\mathcal{T}_\alpha = \mathcal{T}_E$, then $\alpha$ clearly fails to have the finite discernment property.

Observe also that our notion of cover refinement respects the subtopology relation. It is immediate that if $\alpha \Subset \beta$ then $\mathcal{T}_\alpha \subseteq \mathcal{T}_\beta$. Conversely, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ are topologies on $X$, then there exist covers $\alpha\colon X \rightsquigarrow Y$ and $\beta\colon X \rightsquigarrow Z$ of $X$ such that $\alpha \Subset \beta$ and $\mathcal{T}_\alpha = \mathcal{T}_1$ and $\mathcal{T}_\beta = \mathcal{T}_2$; to see this, take the $\mathcal{T}_1$-open sets to be $\alpha$-cells and the $\mathcal{T}_2$-open sets to be $\beta$-cells.

Drawing on old results in lattice theory, we can use the finite discernment property of an A/D map $\alpha$ to give an elegant representation of the lattice of $\mathcal{T}_\alpha$-open sets. The key step, following [16], is to distinguish suitable "minimal" elements in the lattice $\mathcal{T}_\alpha$.

**Definition 5.4.** (Birkhoff [4], III §3) *Let $\mathcal{A}$ be a ring of sets. A set $B \in \mathcal{A}$ is called* join irreducible *in $\mathcal{A}$ if $B \neq \varnothing$ and for all $A_1, A_2 \in \mathcal{A}$,*

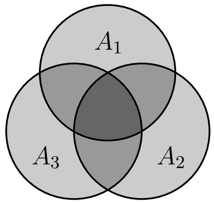
$$\textit{if} \quad B = A_1 \cup A_2 \quad \textit{then} \quad B = A_1 \quad \text{or} \quad B = A_2.$$

As established in [4], III §3, the join-irreducible property is equivalent to: $B \neq \varnothing$ and for all $A_1, A_2 \in \mathcal{A}$,

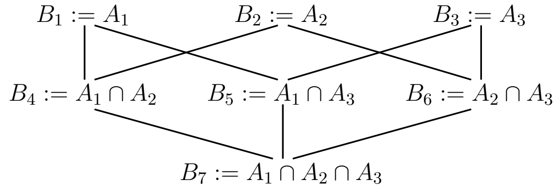
$$\text{if} \quad B \subseteq A_1 \cup A_2 \quad \text{then} \quad B \subseteq A_1 \quad \textit{or} \quad B \subseteq A_2.$$

**Proposition 5.5.** (Birkhoff [4], III §3) *If $\mathcal{A}$ is a ring of sets with only finite-length chains, then each set $A \in \mathcal{A}$ has a unique representation as a union of join-irreducibles, where uniqueness comes from the union being irredundant; i.e. any smaller union is properly contained in $A$.*

**Definition 5.6.** *Given an A/D map $\alpha\colon X \rightsquigarrow Y$, let $\mathcal{J}_\alpha \subseteq \mathcal{T}_\alpha$ be the set of all join-irreducibles of $\mathcal{T}_\alpha$. We suppose the elements of $\mathcal{J}_\alpha$ are indexed by a set $Z_\alpha$, so $B_z$ is the element of $\mathcal{J}_\alpha$ indexed by $z \in Z_\alpha$; thus $\mathcal{J}_\alpha = \{B_z \mid z \in Z_\alpha\}$. Define a map $\bar{\alpha} : X \rightsquigarrow Z_\alpha$ by $\bar{\alpha}(x) := \{z \in Z_\alpha \mid x \in B_z\}$. Also define a partial ordering $\sqsubseteq$ on $Z_\alpha$ by: $z \sqsubseteq z'$ iff $B_z \supseteq B_{z'}$.*




Cover $X = A_1 \cup A_2 \cup A_3$




Partial order of join-irreducible sets $B_z$

A simple illustration of an A/D map $\alpha$ and the partial order $\mathcal{J}_\alpha$ is given above. Note also that every $B_z \in \mathcal{J}_\alpha$ is a finite intersection of $\alpha$-cells; i.e. $B_z = A_{y_1} \cap \cdots \cap A_{y_k}$ for some $y_1, \ldots, y_k \in Y = \mathrm{ran}(\alpha)$. The induced partial order $(Z_\alpha, \sqsubseteq)$ is just a replica of the partial order $(\mathcal{J}_\alpha, \supseteq)$ of join-irreducible sets; as we shall see, they form the fundamental core of the topology $\mathcal{T}_\alpha$.

**Proposition 5.7.** *Let $\alpha\colon X \rightsquigarrow Y$ be any A/D map, with its induced join-irreducible map $\bar{\alpha} : X \rightsquigarrow Z_\alpha$.*

1. *For every state $x \in X$, there is a unique $\subseteq$-smallest join-irreducible $B_z \in \mathcal{J}_\alpha$ such that $x \in B_z$.*
2. *For every $B_z \in \mathcal{J}_\alpha$, there is some $x \in X$ such that $B_z$ is the smallest $\mathcal{T}_\alpha$-open set containing $x$; i.e. $B_z = \bigcap\{U \in \mathcal{T}_\alpha \mid x \in U\}$.*
3. *$\mathcal{T}_\alpha$ is an Alexandroff topology, and the collection $\mathcal{J}_\alpha$ forms a minimal basis for $\mathcal{T}_\alpha$.*
4. *The map $\bar{\alpha} : X \rightsquigarrow Z_\alpha$ is an A/D map on $X$ that refines $\alpha$, and if $\alpha \Subset \beta \Subset \bar{\alpha}$ then $\mathcal{T}_\beta = \mathcal{T}_\alpha$.*

**PROOF.** For 1., fix $x \in X$, and let $\mathcal{U}(x) \subseteq \mathcal{J}_\alpha$ be the family of all join irreducibles $B \in \mathcal{J}_\alpha$ such that $x \in B$. Under inclusion, $\mathcal{U}(x)$ is a sub-partial-order of $\mathcal{T}_\alpha$, hence all chains in $\mathcal{U}(x)$ are of finite length, and thus have *minimum* elements. Let $\mathcal{V}(x) \subseteq \mathcal{U}(x)$ be the collection of all the minimum elements of the chains in $\mathcal{U}(x)$. We claim $\mathcal{V}(x)$ is a singleton, so $\mathcal{V}(x) = \{B_z\}$ for some $z$ and thus $B_z$ is the $\subseteq$-smallest join-irreducible containing $x$. To prove the claim, suppose otherwise, so there exists $B_1, B_2 \in \mathcal{V}(x)$ with $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$. But this means $x \in (B_1 \cap B_2) \in \mathcal{U}(x)$ and $(B_1 \cap B_2) \subset B_1$ and $(B_1 \cap B_2) \subset B_2$, which contradicts the assumption that $B_1$ and $B_2$ are minimum elements of chains in $\mathcal{U}(x)$.

For 2., suppose for a contradiction that there exists a $B_z \in \mathcal{J}_\alpha$ such that for all $x \in B_z$, there exists a $z' \sqsupset z$ such that $x \in B_{z'}$. Hence $B_z = \bigcup\{\ B_{z'} \in \mathcal{J}_\alpha \mid z \sqsubset z'\ \}$, which means $B_z$ could not be a join irreducible.

For 3., the characteristic of Alexandroff topologies is that for every $x \in X$, there is a smallest open set containing $x$, so $\mathcal{T}_\alpha$ being Alexandroff is immediate from 1. The proof that $\mathcal{J}_\alpha$ is a basis for $\mathcal{T}_\alpha$, and that no smaller collection forms a basis, is straight-forward, with an appeal to the representation in Proposition 5.5.

For 4., it is immediate that $\alpha \Subset \bar{\alpha}$ and that $\mathcal{T}_{\bar{\alpha}} = \mathcal{T}_\alpha$ because the $\bar{\alpha}$-cells are just $\mathcal{J}_\alpha$, the join-irreducibles of $\mathcal{T}_\alpha$. If $\alpha \Subset \beta \Subset \bar{\alpha}$, then since $\Subset$ respects the subtopology relation, we have $\mathcal{T}_\alpha \subseteq \mathcal{T}_\beta$ and $\mathcal{T}_\beta \subseteq \mathcal{T}_{\bar{\alpha}} = \mathcal{T}_\alpha$, and hence $\mathcal{T}_\beta = \mathcal{T}_\alpha$. ■

Our next move is to further refine the non-deterministic A/D map $\bar{\alpha} : X \rightsquigarrow Z_\alpha$ to produce a deterministic quotient function. Following [16], the key objects of study are the *essential parts* of the join-irreducibles $B_z \in \mathcal{J}_\alpha$.

**Definition 5.8.** *Let $\alpha\colon X \rightsquigarrow Y$ be any A/D map, with its refined join-irreducible A/D map $\bar{\alpha} : X \rightsquigarrow Z_\alpha$. For each $z \in Z_\alpha$, define the* essential part *$E_z$ of $B_z$ as follows:*

$$E_z \ := \ B_z - \bigcup\{\ B_{z'} \in \mathcal{J}_\alpha \mid z \sqsubset z'\}$$

*Then define a function $q_\alpha : X \to Z_\alpha$ by: $q_\alpha(x) = z$ iff $x \in E_z$.*

In words, the set $E_z = q_\alpha^{-1}(z)$ is that part of the join-irreducible $B_z$ which is not contained in any strictly smaller join-irreducible $B_{z'} \in \mathcal{J}_\alpha$. It must be non-empty, for otherwise $B_z = \bigcup\{\, B_{z'} \in \mathcal{J}_\alpha \mid z \sqsubset z'\}$ which would mean $B_z$ is not join irreducible. The following proposition summarises elementary properties of essential parts and the $q_\alpha$ map; the results are all simple consequences of Proposition 5.7 and the definitions.

**Proposition 5.9.** *For each $z \in Z_\alpha$*

*1. $E_z \neq \varnothing$ is the intersection of a $\mathcal{T}_\alpha$-open set and a $\mathcal{T}_\alpha$-closed set;*
*2. for all $z' \neq z$, $E_z \cap E_{z'} = \varnothing$;*
*3. $E_z = \{x \in B_z \mid B_z$ is the smallest $\mathcal{T}_\alpha$-open set containing $x\}$.*

*Hence:*

*(a) the collection $\{E_z\}_{z \in Z_\alpha}$ forms a total partition of $X$, thus the single-valued function $q_\alpha : X \to Z_\alpha$ is well-defined;*
*(b) for all open sets $U \in \mathcal{T}_\alpha$, and for all $z \in Z_\alpha$, the following are equivalent:*

(i) $E_z \cap U \neq \varnothing$ (ii) $E_z \subseteq U$ (iii) $B_z \subseteq U$ (iv) $B_z \cap U \neq \varnothing$

*(c) for all $x, x' \in X$, if $x \preccurlyeq_{\mathcal{T}_\alpha} x$ then $q_\alpha(x) \sqsubseteq q_\alpha(x')$; thus $x \approx_{\mathcal{T}_\alpha} x$ iff $q_\alpha(x) = q_\alpha(x')$;*
*(d) $\alpha \Subset \bar{\alpha} \Subset q_\alpha$ and as a relation, $q_\alpha \subseteq \bar{\alpha}$*
*(e) $\bar{\alpha} = q_\alpha \circ \sqsupseteq$*
*(f) $q_\alpha(x) = z$ iff $z$ is the $\sqsubseteq$-maximal element of the set $\bar{\alpha}(x)$.*

Observe that if the original A/D map $\alpha$ is actually single-valued, so we start with a partition of $X$, then $\mathcal{T}_\alpha$ is actually a Boolean algebra (closed under complement), and the join-irreducibles are identical to the essential parts, which in turn are the atoms of the Boolean algebra. In this sense, the special case of partitions can be viewed as "collapsing back" into classical reasoning.

**Theorem 5.10.** [Decomposition of open and closed sets of $\mathcal{T}_\alpha$ into essential parts]
*For all join-irreducibles $B_z \in \mathcal{J}_\alpha$,*

$$B_z = \bigcup\{E_{z'} \mid z \sqsubseteq z'\} \tag{1}$$

*Hence for all open sets $U \in \mathcal{T}_\alpha$,*

$$U = \bigcup\{\, E_z \mid z \in Z_\alpha \text{ and } B_z \subseteq U \,\} \tag{2}$$

*and for all closed sets $V \in (-\mathcal{T}_\alpha)$,*

$$V = \bigcup\{\, E_z \mid z \in Z_\alpha \text{ and } V \cap B_z \neq \varnothing \,\} \tag{3}$$

**PROOF.** Fix $z \in Z_\alpha$, and consider the RHS set $R_z := \bigcup\{E_{z'} \mid z \sqsubseteq z'\} = \bigcup\{E_{z'} \mid B_{z'} \subseteq B_z\}$. Since $E_{z'} \subseteq B_{z'} \subseteq B_z$ for each essential part $E_{z'}$ in the disjoint union, we know that $R_z \subseteq B_z$. So it suffices to show that $B_z \subseteq R_z$. Suppose $x \notin R_z$. So for some $z' \in Z_\alpha$, we do *not* have $z \sqsubseteq z'$ and we do have $x \in E_{z'}$. So $B_{z'} \nsubseteq B_z$, and by Proposition 5.9, part (2.), we must have $B_{z'} = \bigcap\{\, U \in \mathcal{T}_\alpha \mid x \in U \,\}$. This implies $x \notin B_z$, as required.

Next, fix any open set $U \in \mathcal{T}_\alpha$. Then

$$\begin{aligned}
U &= \bigcup\{\, B_z \in \mathcal{J}_\alpha \mid B_z \subseteq U \,\} && \text{by Proposition 5.7, part (3.)}\\
&= \bigcup \{\bigcup\{\, E_{z'} \mid z \sqsubseteq z'\} \mid z \in Z_\alpha \text{ and } B_z \subseteq U\} && \text{by equation (1)}\\
&= \bigcup\{\, E_{z'} \mid z' \in Z_\alpha \text{ and } B_{z'} \subseteq U \,\} && \text{by simplification}
\end{aligned}$$

Finally, fix any closed set $V \in (-\mathcal{T}_\alpha)$. So $V = X - U$ for some $U \in \mathcal{T}_\alpha$. Then

$$\begin{aligned}
V &= X - U\\
&= X - \bigcup\{\, E_{z'} \mid z' \in Z_\alpha \text{ and } E_{z'} \subseteq U \,\} && \text{by equation (2)}\\
&= \bigcup\{\, E_z \mid z \in Z_\alpha \text{ and } B_z \nsubseteq U \,\} && \text{since } \{E_z\}_{z \in Z_\alpha} \text{ partitions } X\\
&= \bigcup\{\, E_z \mid z \in Z_\alpha \text{ and } B_z \nsubseteq (X - V) \,\}\\
&= \bigcup\{\, E_z \mid z \in Z_\alpha \text{ and } V \cap B_z \neq \varnothing \,\} && \blacksquare
\end{aligned}$$

We can now derive a simple and clean characterisation of both the interior and the closure operators of $\mathcal{T}_\alpha$ in terms of the essential part partition blocks. The proof is omitted due to space constraints.

**Corollary 5.11.** *The topological operators of $\mathcal{T}_\alpha$ satisfy the following formulas:*

$$int_{\mathcal{T}_\alpha}(W) = \bigcup\{\ E_z \mid z \in Z_\alpha \text{ and } B_z \subseteq W\ \} \tag{4}$$

$$cl_{\mathcal{T}_\alpha}(W) = \bigcup\{\ E_z \mid z \in Z_\alpha \text{ and } W \cap B_z \neq \varnothing\ \} \tag{5}$$

Let $\mathcal{T}_q$ denote the *quotient topology* on $Z_\alpha$ induced by $q_\alpha : X \to Z_\alpha$; that is, the minimal topology on $Z_\alpha$ such that $q_\alpha : (X, \mathcal{T}_\alpha) \to (Z_\alpha, \mathcal{T}_q)$ is a continuous function. Thus

$$\mathcal{T}_q := \{V \subseteq Z_\alpha \mid q_\alpha^{-1}(V) \in \mathcal{T}_\alpha\}$$

where $q_\alpha^{-1}(V) = \bigcup\{E_z \mid z \in V\ \}$. The partial order $\sqsubseteq$ on $Z_\alpha$ gives us a way to "read off" how and when blocks $E_z$ "clump together" to form the open and closed sets in $\mathcal{T}_\alpha$, as captured in the next proposition.

**Proposition 5.12.** *For all subsets $V \subseteq Z_\alpha$, the conditions in each row are equivalent.*

1. *$V$ is open in $\mathcal{T}_q$; $V$ is up-$\sqsubseteq$-closed; $q_\alpha^{-1}(V)$ is open in $\mathcal{T}_\alpha$.*
2. *$V$ is closed in $\mathcal{T}_q$; $V$ is down-$\sqsubseteq$-closed; $q_\alpha^{-1}(V)$ is closed in $\mathcal{T}_\alpha$.*

   *In particular, $\mathcal{T}_q$ and $\mathcal{T}_\sqsubseteq$ are identical topologies on $Z_\alpha$.*

We conclude this section with an analysis of the semi-continuity properties of the refined A/D map $\bar{\alpha}$ and the quotient A/D function $q_\alpha$.

**Theorem 5.13.** *Let $\alpha\colon X \rightsquigarrow Y$ be any A/D map. Then the refined A/D map $\bar{\alpha}\colon X \rightsquigarrow Z_\alpha$ and the quotient function $q_\alpha\colon X \to Z_\alpha$ are such that:*

1. *$\bar{\alpha} : (X, \mathcal{T}_\alpha) \rightsquigarrow (Z_\alpha, \mathcal{T}_q)$ is l.s.c.*
2. *$\bar{\alpha}^{-1} : (Z_\alpha, \mathcal{T}_q) \rightsquigarrow (X, \mathcal{T}_\alpha)$ is u.s.c.*
3. *$q_\alpha : (X, \mathcal{T}_\alpha) \rightsquigarrow (Z_\alpha, \mathcal{T}_q)$ is l.s.c.*
4. *$q_\alpha : (X, \mathcal{T}_\alpha) \rightsquigarrow (Z_\alpha, \mathcal{T}_q)$ is u.s.c.*
5. *$q_\alpha^{-1} : (Z_\alpha, \mathcal{T}_q) \rightsquigarrow (X, \mathcal{T}_\alpha)$ is l.s.c.*
6. *$q_\alpha^{-1} : (Z_\alpha, \mathcal{T}_q) \rightsquigarrow (X, \mathcal{T}_\alpha)$ is u.s.c.*

**PROOF.** (1.) To establish that $\bar{\alpha}$ is l.s.c, first observe that $pre^\exists(\bar{\alpha})(V) = \bigcup\{B_z \in \mathcal{J}_\alpha \mid z \in V\}$ for $V \subseteq Z_\alpha$. We claim that if $V$ is up-$\sqsubseteq$-closed, then $pre^\exists(\bar{\alpha})(V) = q_\alpha^{-1}(V)$. The l.s.c. property for $\bar{\alpha}$ would then follow from the claim together with $\mathcal{T}_q = \mathcal{T}_\sqsubseteq$ and the functional continuity of $q_\alpha : (X, \mathcal{T}_\alpha) \to (Z_\alpha, \mathcal{T}_q)$. To establish the claim, observe that the inclusion $q_\alpha^{-1}(V) \subseteq pre^\exists(\bar{\alpha})(V)$ holds in general, since $E_z \subseteq B_z$. For the converse inclusion, we know that $B_z = \bigcup\{E_{z'} \mid z \sqsubseteq z'\}$, hence $pre^\exists(\bar{\alpha})(V) = \bigcup\{E_{z'} \mid (\exists z)[z \in V \text{ and } z \sqsubseteq z']\}$. Since $V$ is up-$\sqsubseteq$-closed, we know that $z \in V$ and $z \sqsubseteq z'$ implies $z' \in V$. It follows that $pre^\exists(\bar{\alpha})(V) \subseteq \bigcup\{E_z \mid z \in V\ \} = q_\alpha^{-1}(V)$.

(2.) To prove that $\bar{\alpha}^{-1}$ is u.s.c., start by observing that $pre^\forall(\bar{\alpha}^{-1})(U) = \{z \in Z_\alpha \mid B_z \subseteq U\}$ for subsets $U \subseteq X$. Now if $U$ is $\mathcal{T}_\alpha$-open and $B_z \subseteq U$ and $z \sqsubseteq z'$, then $B_{z'} \subseteq B_z \subseteq U$; this shows that $pre^\forall(\bar{\alpha}^{-1})(U)$ is up-$\sqsubseteq$-closed.

(3.) and (4.) are trivial, since $q_\alpha : (X, \mathcal{T}_\alpha) \to (Z_\alpha, \mathcal{T}_q)$ is continuous as a single-valued function, by the very definition of the quotient topology $\mathcal{T}_q$.

(5.) To show that $q_\alpha^{-1}$ is l.s.c., observe that for subsets $U \subseteq X$, we have $pre^\exists(q_\alpha^{-1})(U) = \{z \in Z_\alpha \mid U \cap E_z \neq \varnothing\}$. Now if $U$ is $\mathcal{T}_\alpha$-open and $U \cap E_z \neq \varnothing$, then by Proposition 5.9, part (b), we must have $B_z \subseteq U$. Then if $z \sqsubseteq z'$, we have $B_{z'} \subseteq B_z$ and hence $B_{z'} \subseteq U$ and thus $E_{z'} \subseteq U$ from which we can conclude $U \cap E_{z'} \neq \varnothing$. This reasoning shows that if $U$ is $\mathcal{T}_\alpha$-open then $pre^\exists(q_\alpha^{-1})(U)$ is up-$\sqsubseteq$-closed, hence $q_\alpha^{-1}$ is l.s.c.

(6.) Finally, for the u.s.c. property for $q_\alpha^{-1}$, we claim that if $U$ is $\mathcal{T}_\alpha$-open then $pre^\forall(q_\alpha^{-1})(U) = pre^\exists(q_\alpha^{-1})(U)$; the result would then follow from part (5.). To prove the claim, observe that

$pre^{\forall}(q_\alpha^{-1})(U) = \{z \in Z_\alpha \mid E_z \subseteq U\}$, and that by Proposition 5.9, part (b), $E_z \subseteq U$ iff $U \cap E_z \neq \varnothing$. ■

In general, it seems that $\bar{\alpha} : (X, \mathcal{T}_\alpha) \rightsquigarrow (Z_\alpha, \mathcal{T}_q)$ need not be u.s.c., and $\bar{\alpha}^{-1} : (Z_\alpha, \mathcal{T}_q) \rightsquigarrow (X, \mathcal{T}_\alpha)$ need not be l.s.c. Counter-examples will be given in the full paper.

## 6 Applications of Intuitionistic semantics to approximate model-checking

When it comes real-world application of modal logics, more often than not one finds oneself in a situation where exact model-checking is impossible. This, for example, is the typical case in the formal analysis and design of hybrid dynamical systems. A common approach to overcome this limitation is to discretise the state space by an A/D map, and to investigate relevant reachability relationships through the lenses provided by the cells of the cover; e.g. [12, 20]. In this section, we shall use Intuitionistic semantics in order to relate this sort of discretised view of a model with the original model itself.

Suppose we have a Kripke model $\mathcal{M} = (X, R, \xi)$, and we want to evaluate the (classical) denotation set $[\![\varphi]\!]^{\mathcal{M}}$ for formulas $\varphi \in \mathcal{L}(P_0)^+_{\diamond\boxdot}$, where $P_0 \subset AP$ is a finite set of atomic propositions, and $\mathcal{L}(P_0)^+_{\diamond\boxdot}$ is the *positive* sublanguage of $\mathcal{L}_{\diamond\boxdot}$ generated from $P_0$ under $\vee$, $\wedge$, $\diamond$ and $\boxdot$; i.e. no implications or negations.

---

**Recipe:** model approximation via spatial discretisation

---

**Input:** a Kripke model $\mathcal{M} = (X, R, \xi)$, and a finite set $P_0 \subset AP$ of atomic propositions.

**Step 1:** Design an A/D map $\alpha \colon X \rightsquigarrow Y$ such that for each $p \in P_0$, the denotation set $\xi^{-1}(p) \subseteq X$ is a union of $\alpha$-cells. Then construct the topology $\mathcal{T}_\alpha$, join-irreducible refinement $\bar{\alpha} \colon X \rightsquigarrow Z_\alpha$, the quotient map $q_\alpha \colon X \to Z_\alpha$, and the topology $\mathcal{T}_q = \mathcal{T}_{\sqsubseteq}$ on $Z_\alpha$.

**Step 2:** Design two relations $Q_{un} \colon Z_\alpha \rightsquigarrow Z_\alpha$ and $Q_{ov} \colon Z_\alpha \rightsquigarrow Z_\alpha$ satisfying the following:
**(1)-*un*:** $Q_{un} \colon (Z_\alpha, \mathcal{T}_q) \rightsquigarrow (Z_\alpha, \mathcal{T}_q)$ and $Q_{un}^{-1} \colon (Z_\alpha, \mathcal{T}_q) \rightsquigarrow (Z_\alpha, \mathcal{T}_q)$ are l.s.c.
**(2)-*un*:** $(q_\alpha \circ Q_{un} \circ q_\alpha^{-1}) \subseteq R$
**(3)-*un*:** $(q_\alpha^{-1} \circ R) \subseteq (Q_{un} \circ \sqsubseteq \circ q_\alpha^{-1})$
**(1)-*ov*:** $Q_{ov} \colon (Z_\alpha, \mathcal{T}_q) \rightsquigarrow (Z_\alpha, \mathcal{T}_q)$ and $Q_{ov}^{-1} \colon (Z_\alpha, \mathcal{T}_q) \rightsquigarrow (Z_\alpha, \mathcal{T}_q)$ are l.s.c.
**(2)-*ov*:** $R \subseteq (q_\alpha \circ Q_{ov} \circ q_\alpha^{-1})$
**(3)-*ov*:** $(\sqsubseteq \circ Q_{ov}) \subseteq (\sqsubseteq \circ q_\alpha^{-1} \circ R \circ q_\alpha)$.

**Output:** the topological models $\mathcal{M}_{un} := (X, \mathcal{T}_\alpha, R_{un}, \xi_0)$ and $\mathcal{M}_{ov} := (X, \mathcal{T}_\alpha, R_{ov}, \xi_0)$, where: $R_{un} := q_\alpha \circ Q_{un} \circ q_\alpha^{-1}$, and $R_{ov} := q_\alpha \circ Q_{ov} \circ q_\alpha^{-1}$, and atomic valuation $\xi_0 : X \rightsquigarrow AP$ defined by $p \in \xi_0(x)$ iff ($x \in \xi^{-1}(p)$ and $p \in P_0$), or ($x \in int_{\mathcal{T}_\alpha}(\xi^{-1}(p))$ and $p \notin P_0$).

---

By conditions (1)-*un* and (1)-*ov*, together with Theorem 5.13, parts (3.) and (5.), we know that $R_{un}, R_{un}^{-1}, R_{ov}, R_{ov}^{-1} \colon (X, \mathcal{T}_\alpha) \rightsquigarrow (X, \mathcal{T}_\alpha)$ are all l.s.c. By the design of $\alpha$, the original atomic valuation has $\xi^{-1}(p)$ open in $\mathcal{T}_\alpha$ for $p \in P_0$, and the new valuation $\xi_0$ enforces openness in $\mathcal{T}_\alpha$ for those $p \in AP - P_0$, about which we do not care. Hence $\mathcal{M}_{un}$ and $\mathcal{M}_{ov}$ are l.s.c. and open (Intuitionistic) topological models.

Conditions (2)-*un* and (2)-*ov* are minimal for the approximation task, since they give $R_{un} \subseteq R \subseteq R_{ov}$. Intuitively, conditions (3)-*un* and (3)-*ov* say that $R_{un}$ and $R_{ov}$ additionally have to be "quite close" to the original $R$, where the measure of the closeness correlates with the size of the overlaps of the $\alpha$-cells. In the case that there are no overlaps at all, when $\alpha$ is chosen to be a *partition*, then $\sqsubseteq$ is the identity and (3)-*un* forces $R_{un} = R$, and (3)-*ov* forces $R_{ov} = R$. Hence there is no allowance or "wiggle room" for approximating $R$. This accords with the intuition that if the topology $\mathcal{T}_\alpha$ is actually a Boolean algebra, then one has collapsed back into the classical realm and so one shouldn't expect to draw much benefit from a detour into Intuitionistic logic.

**Theorem 6.1.** *Given $\mathcal{M} = (X, R, \xi)$ and a finite set $P_0 \subset AP$, suppose we have followed the recipe above. Then for all positive formulas $\varphi \in \mathcal{L}(P_0)^+_{\diamondsuit\boxdot}$,*

$$[\![\varphi]\!]^{\mathcal{M}_{un}}_{\text{INT}} \;\subseteq\; [\![\varphi]\!]^{\mathcal{M}} \;\subseteq\; [\![\varphi]\!]^{\mathcal{M}_{ov}}_{\text{INT}} \tag{6}$$

*Moreover, the Intuitionistic denotation sets in $\mathcal{M}_{un}$ and $\mathcal{M}_{ov}$ can be computed from denotation sets in quotient models under $q_\alpha$ on the set $Z_\alpha$. Define models $\mathcal{N}_{un} := (Z_\alpha, \mathcal{T}_q, Q_{un}, \bar{\xi})$ and $\mathcal{N}_{ov} := (Z_\alpha, \mathcal{T}_q, Q_{ov}, \bar{\xi})$ where $\bar{\xi} := q_\alpha^{-1} \circ \xi$. Then for all formulas $\psi \in \mathcal{L}(P_0)_{\diamondsuit\boxdot}$, and for $\circledast \in \{un, ov\}$:*

$$[\![\psi]\!]^{\mathcal{M}_\circledast}_{\text{INT}} = \bigcup \left\{ B_z \mid z \in [\![\psi]\!]^{\mathcal{N}_\circledast}_{\text{INT}} \right\} \tag{7}$$

**PROOF.** We prove the double inclusion (6) by induction on formulas. The base case for $p \in P_0$, we have $[\![p]\!]^{\mathcal{M}_{un}}_{\text{INT}} = \xi_0^{-1}(p) = \xi^{-1}(p) = [\![p]\!]^{\mathcal{M}}$ and likewise $[\![p]\!]^{\mathcal{M}_{ov}}_{\text{INT}} = [\![p]\!]^{\mathcal{M}}$. The induction for $\vee$ and $\wedge$ proceeds straight-forwardly, since $\subseteq$ respects $\cup$ and $\cap$. (Notice why the induction step can't go through for $\to$; its semantics involve set-complement in the antecedent, which reverses inclusions, so the best we can do is a positive fragment.)

For the two modal cases, let $U_1 = [\![\varphi]\!]^{\mathcal{M}_{un}}_{\text{INT}}$, $W = [\![\varphi]\!]^{\mathcal{M}}$ and $U_2 = [\![\varphi]\!]^{\mathcal{M}_{ov}}_{\text{INT}}$, so the induction hypothesis is that $U_1 \subseteq W \subseteq U_2$. The $\diamondsuit$ case is quite simple. By conditions (2)-*un* and (2)-*ov*, we have the inclusions $R_{un} \subseteq R \subseteq R_{ov}$. Hence $U_1 \subseteq W \subseteq U_2$ implies that $pre^\exists(R_{un})(U_1) \subseteq pre^\exists(R)(W) \subseteq pre^\exists(R_{ov})(U_2)$, and thus $[\![\diamondsuit\varphi]\!]^{\mathcal{M}_{un}}_{\text{INT}} \subseteq [\![\diamondsuit\varphi]\!]^{\mathcal{M}} \subseteq [\![\diamondsuit\varphi]\!]^{\mathcal{M}_{ov}}_{\text{INT}}$. The harder part is the $\boxdot$ case. We do the two inclusions separately.

To prove $[\![\boxdot\varphi]\!]^{\mathcal{M}_{un}}_{\text{INT}} \subseteq [\![\boxdot\varphi]\!]^{\mathcal{M}}$, it suffices to prove $int_{\mathcal{T}_\alpha}(pre^\forall(R_{un})(U_1)) \subseteq pre^\forall(R)(U_1)$, since $int_{\mathcal{T}_\alpha}(pre^\forall(R_{un})(U_1)) = [\![\boxdot\varphi]\!]^{\mathcal{M}_{un}}_{\text{INT}}$ and $pre^\forall(R)(U_1) \subseteq pre^\forall(R)(W) = [\![\boxdot\varphi]\!]^{\mathcal{M}}$. Now

$$\begin{array}{lll}
& int_{\mathcal{T}_\alpha}(pre^\forall(R_{un})(U_1)) \subseteq pre^\forall(R)(U_1) & \\
\Leftrightarrow & pre^\exists(R^{-1})\left( int_{\mathcal{T}_\alpha}(pre^\forall(R_{un})(U_1))\right) \subseteq U_1 & \text{adjoint property} \\
\Leftrightarrow & \bigcup\left\{pre^\exists(R^{-1})(E_z) \mid B_z \subseteq pre^\forall(R_{un})(U_1)\right\} \;\subseteq\; U_1 & \text{Thm 5.10 \& distribution} \\
\Leftrightarrow & \bigcup\left\{pre^\exists(R^{-1})(E_z) \mid pre^\exists(R_{un}^{-1})(B_z) \subseteq U_1\right\} \;\subseteq\; U_1 & \text{adjoint property} \\
\Leftrightarrow & (\forall z \in Z_\alpha)[\text{ if } pre^\exists(R_{un}^{-1})(B_z) \subseteq U_1 \text{ then } pre^\exists(R^{-1})(E_z) \subseteq U_1\ ] &
\end{array}$$

So it suffices to show that $pre^\exists(R^{-1})(E_z) \subseteq pre^\exists(R_{un}^{-1})(B_z)$ for all $z \in Z_\alpha$. Now starting from condition (3)-*un*, we can take inverses to conclude that $(R^{-1} \circ q_\alpha) \;\subseteq\; (q_\alpha \circ \sqsupseteq \circ Q_{un}^{-1})$. The map $Q_{un}^{-1}$ is l.s.c. by condition (1)-*un*, hence by Proposition 4.2, part (1.), $(\sqsupseteq \circ Q_{un}^{-1}) \subseteq (Q_{un}^{-1} \circ \sqsupseteq)$. Thus $(R^{-1} \circ q_\alpha) \;\subseteq\; (q_\alpha \circ Q_{un}^{-1} \circ \sqsupseteq)$, and $(q_\alpha \circ Q_{un}^{-1} \circ \sqsupseteq) = (q_\alpha \circ Q_{un}^{-1} \circ (q_\alpha^{-1} \circ q_\alpha) \circ \sqsupseteq) = (R_{un}^{-1} \circ q_\alpha \circ \sqsupseteq) = (R_{un}^{-1} \circ \bar{\alpha})$, by Proposition 5.9, part (e). Hence $(R^{-1} \circ q_\alpha) \;\subseteq (R_{un}^{-1} \circ \bar{\alpha})$. Now observe that $(x, z) \in (R^{-1} \circ q_\alpha)$ iff $x \in pre^\exists(R^{-1})(E_z)$, and $(x, z) \in (R_{un}^{-1} \circ \bar{\alpha})$ iff $x \in pre^\exists(R_{un}^{-1})(B_z)$. Hence for all $z \in Z_\alpha$, we have $pre^\exists(R^{-1})(E_z) \subseteq pre^\exists(R_{un}^{-1})(B_z)$, as required.

To prove $[\![\boxdot\varphi]\!]^{\mathcal{M}} \subseteq [\![\boxdot\varphi]\!]^{\mathcal{M}_{ov}}_{\text{INT}}$, we need $pre^\forall(R)(W) \subseteq int_{\mathcal{T}_\alpha}(pre^\forall(R_{ov})(U_2))$. Since $U_2$ is $\mathcal{T}_\alpha$-open, $(-U_2)$ is $\mathcal{T}_\alpha$-closed; hence by Theorem 5.10, $(-U_2) = \bigcup\{\ E_z \mid B_z \nsubseteq U_2\ \}$. Then

$$\begin{array}{ll}
& pre^\forall(R)(W) \subseteq int_{\mathcal{T}_\alpha}(pre^\forall(R_{ov})(U_2)) \\
\Leftrightarrow & cl_{\mathcal{T}_\alpha}(pre^\exists(R_{ov})(-U_2)) \subseteq pre^\exists(R)(-W) \;\; \text{taking complements}
\end{array}$$

Now

$$\begin{array}{ll}
& cl_{\mathcal{T}_\alpha}(pre^\exists(R_{ov})(-U_2)) \\
= & \bigcup\left\{E_z \mid pre^\exists(R_{ov})(-U_2) \cap B_z \;\neq \varnothing\right\} \qquad \text{Thm 5.11} \\
= & \bigcup\left\{E_z \mid (\exists z' \in Z_\alpha)[\ B_{z'} \nsubseteq U_2 \text{ and } \; pre^\exists(R_{ov})(E_{z'}) \cap B_z \;\neq \varnothing\ ]\right\}
\end{array}$$

On the other side,

$$\begin{array}{ll}
& pre^\exists(R)(-W) \\
\supseteq & pre^\exists(R)(int_{\mathcal{T}_\alpha}(-W)) \\
= & \bigcup\left\{E_z \mid pre^\exists(R)(-W) \cap B_z \neq \varnothing\right\} \qquad \text{Thm 5.11} \\
= & \bigcup\left\{E_z \mid (\exists z' \in Z_\alpha)[\ B_{z'} \nsubseteq W \text{ and } pre^\exists(R)(E_{z'}) \cap B_z \neq \varnothing\ ]\right\}
\end{array}$$

Since $W \subseteq U_2$, if $B_{z'} \nsubseteq U_2$ then $B_{z'} \nsubseteq W$. Hence it suffices to show that for all $z, z' \in Z_\alpha$,

$$(\star) \quad \text{if} \quad pre^\exists(R_{ov})(E_{z'}) \cap B_z \;\neq \varnothing \quad \text{then} \quad pre^\exists(R)(E_{z'}) \cap B_z \neq \varnothing$$

(Note that the converse is already known, since $R \subseteq R_{ov}$ by condition (2)-*ov*.) As to be expected, to prove ($\star$), we appeal to condition (3)-*ov*. It is readily verified that $pre^{\exists}(R_{ov})(E_{z'}) \cap B_z \neq \varnothing$ iff $(z, z') \in (\sqsubseteq \circ q_{\alpha}^{-1} \circ R_{ov} \circ q_{\alpha}) = (\sqsubseteq \circ Q_{ov})$, and $pre^{\exists}(R)(E_{z'}) \cap B_z \neq \varnothing$ iff $(z, z') \in (\sqsubseteq \circ q_{\alpha}^{-1} \circ R \circ q_{\alpha})$. Since condition (3)-*ov* says $(\sqsubseteq \circ Q_{ov}) \subseteq (\sqsubseteq \circ q_{\alpha}^{-1} \circ R \circ q_{\alpha})$, we are done.

The last part of the theorem expresses how the Intuitionistic denotation sets in $\mathcal{M}_{un}$ and $\mathcal{M}_{ov}$ can be computed from denotation sets in the quotient models $\mathcal{N}_{un}$ and $\mathcal{N}_{ov}$. For all formulas $\psi \in \mathcal{L}(P_0)_{\diamondsuit\boxdot}$ generated from $P_0$, here including implications, we have:

$$[\![\psi]\!]^{\mathcal{M}_{\circledast}}_{\text{INT}} = \bigcup \left\{ E_z \mid z \in [\![\psi]\!]^{\mathcal{N}_{\circledast}}_{\text{INT}} \right\}$$

for $\circledast \in \{un, ov\}$. Since $[\![\psi]\!]^{\mathcal{M}_{\circledast}}_{\text{INT}}$ is $\mathcal{T}_{\alpha}$-open, we get equation (7). ■

The stronger conditions (3)-*un* and (3)-*ov* on $Q_{un}$ and $Q_{ov}$ are there to push through the induction for $\boxdot$ formulas. One pragmatic response is to take consolation that, in the absence of conditions (3)-*un* and (3)-*ov*, one can still get the approximation inclusion (6) for the smaller fragment of positive formulas generated from atomic propositions in $P_0$ under $\vee$, $\wedge$ and $\diamondsuit$. Another response is to consider a *strategic refinement* of an A/D map $\alpha$ if it is "too coarse" to allow for the design of approximation relations $Q_{un}$ and $Q_{ov}$ on $Z_{\alpha}$ that will satisfy (3)-*un* and (3)-*ov*. One should use data witnessing the failure of the conditions in seeking a refinement $\beta \Supset \alpha$ that might be better for the job. There is also plenty of scope for identifying particular classes of models $\mathcal{M} = (X, R, \xi)$ for which an approximation triple $(\alpha, Q_{un}, Q_{ov})$ can be explicitly determined, and algorithms for doing so.

## 7 Conclusions

In the course of this paper, we have made three original contributions. We have given a complete topological semantics for the Intuitionistic modal logic of Fischer Servi, including generalising the known bi-relational frame conditions to semi-continuity properties of the relation with respect to the topology. We have developed the general topology of an interesting class of topologies arising from spatial discretisation via A/D maps, which are known from engineering practice in the area of hybrid dynamical systems. And finally, we have produced a novel application of Intuitionistic semantics and our theory of A/D maps to the problem of approximate model-checking of classical modal formulas in models where the exact evaluation of denotation sets is not possible.

## References


1. M. Aiello. *Spatial Reasoning - theory and practice.* PhD thesis, Institute for Logic, Language and Computation - University of Amsterdam, February 2002.
2. R. Alur, T. A. Henzinger, G. Lafferriere, and G. Pappas. Discrete abstractions of hybrid systems. *Proceedings of the IEEE*, 88, July 2000.
3. J-P. Aubin and H. Frankowska. *Set-Valued Analysis.* Birkhäuser, Boston, 1990.
4. G. Birkhoff. *Lattice Theory.* American Math. Society, third edition, 1995. First edition 1940.
5. J. M. Davoren. Topologies, continuity and bisimulations. *Theoretical Informatics and Applications*, 33:357–381, 1999.
6. J. M. Davoren and R. P. Goré. Bimodal logics for reasoning about continuous dynamics. In F. Wolter, H. Wansing, and M. Zakharyaschev, editors, *Advances in Modal Logic*, volume 3. CSLI, 2002. In press.
7. J. M. Davoren, T. Moor, and A. Nerode. Hybrid control loops, A/D maps, and dynamic specifications. In C. J. Tomlin and M. R. Greenstreet, editors, *Hybrid Systems: Computation and Control*, LNCS 2289. Springer-Verlag, 2002.

8. J. M. Davoren and A. Nerode. Logics for hybrid systems. *Proceedings of the IEEE*, 88:985–1010, July 2000.
9. G. Fischer Servi. On modal logic with an Intuitionistic base. *Studia Logica*, (36):141–149, 1977.
10. G. Fischer Servi. Semantics for a class of Intuitionistic modal calculi. In M. L. dalla Chiara, editor, *Italian Studies in the Philosophy of Science*, pages 59–72. D. Reidel, 1981.
11. G. Fischer Servi. Axiomatizations for some Intuitionistic modal logics. *Rend. Sem. Mat. Univers. Politecn. Torino*, (42):179–194, 1984.
12. D. Franke, T. Moor, and J. Raisch. Discrete supervisory control of switched linear systems. *at-Automatisierungstechnik*, 48:9:461–467, 2000.
13. K. Gödel. An interpretation of the Intuitionistic propositional calculus. In S. Feferman, editor, *Collected Works*, volume 1, pages 301–303. Oxford UP, 1933. Publications 1929-1936.
14. C. Grefe. Fischer Servi's Intuitionistic modal logic has the finite model property. In M. Kracht, M. De Rijke, H. Wansing, and M. Zakharyaschev, editors, *Advances in Modal Logic*, volume 1. CSLI, 1998.
15. J. C. C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, pages (45):141–191, 1944.
16. A. Nerode and W. Kohn. Models for hybrid systems: Automata, topologies, controllability, observability. In R. L. Grossman, editor, *Hybrid Systems*, LNCS 736. Springer-Verlag, 1993.
17. G. Plotkin and C. Stirling. A framework for Intuitionistic modal logics. In J. Y. Halpern, editor, *Proceedings of 1986 Conference: Theoretical Aspects of Reasoning About Knowledge*, 1986.
18. H. Rasiowa and R. Sikorski. *Mathematics of Metamathematics*. PWN Warsaw, 1963.
19. A. K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, Department of Computer Science, University of Edinburgh, 1994.
20. M.A. Smith, T. Moor, and B.D.O. Anderson. Restricted reachability operators: safe approximations in the presence of disturbances. Technical report, RSISE, Australian National University, 2002.
21. F. Wolter and M. Zakharyaschev. Intuitionistic modal logic. In *Logic and Foundations of Mathematics*. Kluwer Academic Publishers, 1995.
22. F. Wolter and M. Zakharyaschev. Intuitionistic modal logics as fragments of classical bimodal logics. In E. Orlowska, editor, *Logic at Work*. Kluwer Academic Publishers, 1997.