Advances in Modal Logic, Volume 3

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Bimodal Logics for Reasoning About Continuous Dynamics

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ABSTRACT. We study a propositional bimodal logic consisting of two **S4** modalities $\Box$ and $[a]$, together with the interaction axiom scheme $\langle a\rangle\Box\varphi \to \Box\langle a\rangle\varphi$. In the intended semantics, the plain $\Box$ is given the McKinsey-Tarski interpretation as the interior operator of a topology, while the labelled $[a]$ is given the standard Kripke semantics using a reflexive and transitive binary relation R_a. The interaction axiom expresses the property that the R_a relation is *lower semi-continuous* with respect to the topology. The class of topological Kripke frames characterised by the logic includes all frames over Euclidean space where R_a is the positive flow relation of a differential equation. We establish the completeness of the axiomatisation with respect to the intended class of topological Kripke frames, and investigate tableau calculi for the logic, although tableau completeness and decidability are still open questions.

1 Introduction

We study a propositional bimodal logic consisting of two **S4** modalities $\Box$ and $[a]$, together with a cross or mix axiom scheme:

$$\langle a\rangle\Box\mathbf{lsc}: \qquad \langle a\rangle\Box\varphi \to \Box\langle a\rangle\varphi$$

where $\langle a\rangle\varphi \overset{\text{def}}{=} \neg[a]\neg\varphi$. This particular bimodal logic is specifically motivated by *hybrid dynamical systems*: systems characterised by interacting discrete and continuous dynamics, which are the subject of a rapidly growing research field at the interface of computer science and control engineering. In the basic case, a hybrid system consists of finitely many differential equations together with reset relations prescribing when the

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system can discretely switch from one differential equation to another. Their engineering applications include air-traffic control, robotics and automated manufacturing (Antsaklis 2000).

Formal reasoning about discrete dynamics is well-studied in the analysis and verification of computer hardware and software using temporal and modal logics; the challenge in the hybrid setting is to also incorporate continuous dynamics into a common logical framework. A crucial move is to consider relations $R_a \subseteq S \times S$ as a generic way of representing continuous as well as discrete dynamics (Davoren and Nerode 2000). A basic relation of interest is the positive *flow relation* or *orbit relation* of a differential equation on a space $S \subseteq \mathbb{R}^n$, where x' is a flow successor of x iff there is a solution curve of the differential equation leading from x to x'. This relation is both reflexive and transitive, hence a *preorder*, and motivates our choice of the **S4** axioms for the labelled modal operator $[a]$, given the standard Kripke semantics. In application to hybrid systems, one works with a **PDL**-like polymodal extension, with modal operators $[a]$ labelled by letters $a \in \Sigma$ in an alphabet of relations Σ, as in (Davoren and Moor 2000).

The purpose of the other **S4** modal operator $\Box$ is to give modal representation to *topological* structure and properties, and thence to notions of *continuity for relations*. The plain box $\Box$ is given the McKinsey-Tarski interpretation as the *interior operator* in an arbitrary *topological space* $(S, \mathcal{T})$, as in (McKinsey 1941), (McKinsey and Tarski 1944) and (Rasiowa and Sikorski 1963). So the denotation set of a formula $\Box\varphi$ is the largest open set in $\mathcal{T}$ contained in the denotation set of φ; recall that a set is open if it is equal to its own interior. The dual $\Diamond$ is interpreted by the topological closure operator. The semantics for our bimodal logic are thus over topological Kripke frames $\mathcal{F} = (S, \mathcal{T}, R_a)$.

The interaction axiom scheme $\langle a\rangle\Box$**lsc** is equivalent to the assertion that the semantic operator $\langle a\rangle$ applied to an open set is always an open set. From the general topology of relations/set-valued maps (Kuratowski 1966), there are two quite distinct topological notions of continuity. As we show in Section 2, an arbitrary relation $R_a \subseteq S \times S$ is *lower semi-continuous* (l.s.c.) with respect to $\mathcal{T}$ exactly when the frame $\mathcal{F} = (S, \mathcal{T}, R_a)$ validates the $\langle a\rangle\Box$**lsc** axiom scheme, and is *upper semi-continuous* (u.s.c.) with respect to $\mathcal{T}$ exactly when $\mathcal{F}$ validates

$$[a]\Box\mathbf{usc}: \qquad [a]\Box\varphi \to \Box[a]\varphi$$

which asserts that the operator $[a]$ applied to an open set is an open set. When R_a is a total single-valued map, both notions of semi-continuity reduce to the standard topological notion of continuity for functions. An arbitrary relation $R_a \subseteq S \times S$ is thus called *continuous* if it is both l.s.c.

and u.s.c. ; see Kuratowski (1966), Section 18. As we show in Section 3, the flow relation of a differential equation is always l.s.c. with respect to the standard Euclidean topology on $S \subseteq \mathbb{R}^n$, but it may fail to be u.s.c.

We give the name **LSC** to the logic axiomatised by $\Box\mathbf{S4} + [a]\mathbf{S4} + \langle a\rangle\Box\mathbf{lsc}$. The Hilbert-style axiomatisation is sound and complete with respect to the class $\mathbb{LSC}$ of topological Kripke frames $(S, \mathcal{T}, R_a)$ for R_a a preorder that is l.s.c. with respect to $\mathcal{T}$, an arbitrary topology. For completeness, the standard canonical Kripke model construction extends to this bimodal configuration, resulting in a canonical frame $\mathcal{F}_{\mathbf{LSC}}$ whose topology $\mathcal{T}_{\preccurlyeq}$ is determined by a preorder $\preccurlyeq \subseteq S \times S$. Topologies in this subclass are variously known as *Alexandroff*, *Kripke* or *cone* topologies, and are in one-one correspondence with preorders. Applying earlier results in (Davoren 1999), we show that the two semi-continuity properties translate in the bi-relational setting to quite familiar "diamond" graph properties; they correspond to the Zig-Zag conditions for $\preccurlyeq$ to be a *bisimulation* with respect to the relation R_a, with l.s.c. mapping to the "Zig" condition, and u.s.c. mapping to the "Zag" condition. A corollary of the completeness proof is that logics such as **LSC** and its fraternal twin **USC** cannot distinguish between frames with Alexandroff topologies and frames with arbitrary topologies.

We attempt to develop a tableau calculus for the logic **LSC**, producing a tableau rule encoding the l.s.c. axiom which uses analytic cuts on a restricted class of analytic super-formulae. This work builds on previous studies by Goré (1991, 1999): monomodal $\Box\mathbf{S4.2}$ is the special case where $[a]$ is identical to $\Box$, so the $\langle a\rangle\Box\mathbf{lsc}$ axiom scheme reduces to the *weak directedness* scheme $\Box\mathbf{2}: \ \Diamond\Box\varphi \to \Box\Diamond\varphi$. Our tableau calculus is sound with respect to the class of frames but completeness is currently an open question. We nevertheless give details of our current attempt to prove completeness since it may help others to invent additional rules to plug the gap.

Bimodal and polymodal logics *without* any interaction axioms are well-studied as *fusions* of monomodal logics (Wolter 1999) where our base logic $\Box\mathbf{S4} + [a]\mathbf{S4}$ would be written $\mathbf{S4} \otimes \mathbf{S4}$. In particular, by Corollary 2.4 of (Wolter 1999), the base logic is decidable. The source of the difficulty in **LSC** is, of course, the extra interaction axiom.

The remainder of the paper is organised as follows. In Section 2, we give the syntax and topological semantics for the logics, and establish the modal characterisation of the semi-continuity properties. In Section 3, we discuss the motivating applications from the analysis of continuous and hybrid dynamical systems. Section 4 sets out the relationship between topological semantics and the bi-relational Kripke semantics. In Section 5, we give the Hilbert-style axiomatisations for **LSC**, and establish its soundness and completeness. In Section 6 we give our tableau calculus, with a discussion of how the l.s.c. axiom is coded up as a rule, and prove its soundness. Section 7 gives a counter-example to the completeness of our tableau calculus. Section 8 gives an outline of an alternative calculus for **LSC** which is complete and terminating, but whose soundness is open. Section 9 discusses related work on bimodal and polymodal logics, and the concluding Section 10 discusses further work.

2 Syntax and Topological Semantics

Definition 2.1 Let $\mathrm{AP} = \{p_0, p_1, ..., q_0, q_1, ...\}$ be a fixed countable set of *atomic propositions*. With $p \in \mathrm{AP}$, the set $\mathcal{L}(\Box, [a])$ of bimodal formulae is recursively generated by the grammar:

$$\varphi \ ::= \ \ p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box\varphi \mid [a]\varphi$$

The other Boolean connectives, logical constants $\top$ and $\bot$, and dual modalities are defined in the standard way. For $\varphi \in \mathcal{L}(\Box, [a])$, the (finite) set $Sf\ \varphi$ of subformulae of φ is defined as usual.

Definition 2.2 A *topological Kripke frame*, or *frame*, for $\mathcal{L}(\Box, [a])$ is a structure $\mathcal{F} = (S, \mathcal{T}, R_a)$ where $S \neq \emptyset$ is the state space, $\mathcal{T} \subseteq \mathcal{P}(S)$ is a topology on S (so $S, \emptyset \in \mathcal{T}$ and $\mathcal{T}$ is closed under finite intersections and arbitrary unions), and $R_a \subseteq S \times S$ is a binary relation on S.

A *valuation* in $\mathcal{F}$ is any function $\xi : \mathrm{AP} \to \mathcal{P}(S)$ assigning a set of states $\xi(p) \subseteq S$ to each atomic $p \in \mathrm{AP}$, and a model over $\mathcal{F}$ is a pair $\mathcal{M} = (\mathcal{F}, \xi)$. For each model $\mathcal{M}$ over $\mathcal{F}$, the denotation set $[\![\varphi]\!]^{\mathcal{M}}$ of a formula $\varphi \in \mathcal{L}(\Box, [a])$ is defined by:

$$\begin{array}{rclrcl}
[\![p]\!]^{\mathcal{M}} & \stackrel{\text{def}}{=} & \xi(p) & & & \\
[\![\neg\varphi]\!]^{\mathcal{M}} & \stackrel{\text{def}}{=} & S - [\![\varphi]\!]^{\mathcal{M}} & [\![\varphi_1 \wedge \varphi_2]\!]^{\mathcal{M}} & \stackrel{\text{def}}{=} & [\![\varphi_1]\!]^{\mathcal{M}} \cap [\![\varphi_2]\!]^{\mathcal{M}} \\
[\![\Box\,\varphi]\!]^{\mathcal{M}} & \stackrel{\text{def}}{=} & int_{\mathcal{T}}\left([\![\varphi]\!]^{\mathcal{M}}\right) & [\![\,[a]\varphi]\!]^{\mathcal{M}} & \stackrel{\text{def}}{=} & \mathrm{Pre}^{\forall}[R_a]\left([\![\varphi]\!]^{\mathcal{M}}\right)
\end{array}$$

where the $\forall$-*pre-image operator* $\mathrm{Pre}^{\forall}[R] : \mathcal{P}(S) \to \mathcal{P}(S)$ of R is:

$$\mathrm{Pre}^{\forall}[R](A) \stackrel{\text{def}}{=} \{x \in S \mid (\forall y \in S)[\text{ if } xRy \ \text{ then } \ y \in A\,]\}$$

For formulae $\varphi \in \mathcal{L}(\Box, [a])$, and models over a class of frames $\mathbb{F}$:

- φ is *satisfied* at state s in the model $\mathcal{M}$, written $s \Vdash_{\mathcal{M}} \varphi$, if $s \in [\![\varphi]\!]^{\mathcal{M}}$
- φ is *true* in the model $\mathcal{M}$, written $\mathcal{M} \Vdash \varphi$, if $[\![\varphi]\!]^{\mathcal{M}} = S$
- φ is *valid* in the frame $\mathcal{F}$, written $\mathcal{F} \Vdash \varphi$, if $\mathcal{M} \Vdash \varphi$ for *all* models $\mathcal{M} = (\mathcal{F}, \xi)$ over $\mathcal{F}$ and
- φ is $\mathbb{F}$-*valid*, written $\vDash_{\mathbb{F}} \varphi$, if $\mathcal{F} \Vdash \varphi$ for all frames $\mathcal{F} \in \mathbb{F}$

The universal pre-image operator is the translation of the standard Kripke semantics into operator form, and is a special case of Jónsson and Tarski's operators on a Boolean algebra. It is also Dijkstra's *weakest liberal precondition* operator, and in set-valued analysis, it is known as the *core* operator (Aubin and Frankowska 1990). The semantic operator for $\langle a \rangle$ is the dual $\exists$-pre-image operator:

$$\mathrm{Pre}^{\exists}[R](A) = \{x \in S \mid (\exists y \in S)[\ xRy \quad \text{and} \quad y \in A\ ]\}$$

On notation, we use letters s, x, y, u, v, w for states/worlds in a frame, and we write $s \Vdash \varphi$ and $[\![\varphi]\!]$ without the $\mathcal{M}$ when the model is clear.

Definition 2.3 A topological Kripke frame $\mathcal{F} = (S,\ \mathcal{T},\ R_a)$ is:

- *relationally preordered* if the relation $R_a \subseteq S \times S$ is reflexive and transitive; i.e. a preorder;
- *lower semi-continuous* if R_a is l.s.c. in $\mathcal{T}$; i.e. for all $A \subseteq S$, if $A \in \mathcal{T}$ then $\mathrm{Pre}^{\exists}[R_a](A) \in \mathcal{T}$;
- *upper semi-continuous* if R_a is u.s.c. in $\mathcal{T}$; i.e., for all $A \subseteq S$, if $A \in \mathcal{T}$ then $\mathrm{Pre}^{\forall}[R_a](A) \in \mathcal{T}$.

Let $\mathbb{TK}$ denote the universal class of all topological Kripke frames, let $\mathbb{TS}4$ denote the class of all relationally preordered frames, let $\mathbb{LSC}$ denote the class of all relationally preordered, lower semi-continuous frames, and let $\mathbb{USC}$ denote the class of all relationally preordered, upper semi-continuous frames. So $(\mathbb{LSC} \cup \mathbb{USC}) \subseteq \mathbb{TS}4 \subseteq \mathbb{TK}$. We now give the modal characterisation of the semi-continuity properties.

Proposition 2.4 *For all frames $\mathcal{F} \in \mathbb{TK}$, the following are equivalent:*

(i) *for all formulae* $\varphi \in \mathcal{L}(\Box, [a])$, $\quad \mathcal{F} \Vdash \langle a \rangle \Box \varphi \to \Box \langle a \rangle \varphi$

(ii) *for all formulae* $\varphi \in \mathcal{L}(\Box, [a])$, $\quad \mathcal{F} \Vdash \langle a \rangle \Box \varphi \leftrightarrow \Box \langle a \rangle \Box \varphi$

(iii) *the relation R_a is l.s.c. with respect to $\mathcal{T}$.*

Proof. Conditions **(i)**, **(ii)** and **(iii)** are respectively equivalent to:

(i)* $\quad \mathrm{Pre}^{\exists}[R_a]\,(int_{\mathcal{T}}\,(A)) \subseteq int_{\mathcal{T}}\left(\mathrm{Pre}^{\exists}[R_a]\,(A)\right) \ \ \forall A \in \mathcal{P}(S)$

(ii)* $\quad \mathrm{Pre}^{\exists}[R_a]\,(int_{\mathcal{T}}\,(A)) = int_{\mathcal{T}}\left(\mathrm{Pre}^{\exists}[R_a]\,(int_{\mathcal{T}}\,(A))\right) \ \forall A \in \mathcal{P}(S)$

(iii)* $\quad \mathrm{Pre}^{\exists}[R_a]\,(U) = int_{\mathcal{T}}\left(\mathrm{Pre}^{\exists}[R_a]\,(U)\right) \ \ \forall U \in \mathcal{T}$

Since $U \in \mathcal{T}$ iff $U = int_{\mathcal{T}}\,(U)$ iff $U = int_{\mathcal{T}}\,(A)$ for some $A \in \mathcal{P}(S)$, the equivalence **(ii)**$^* \Leftrightarrow$ **(iii)*** is immediate. So it suffices to show **(i)**$^* \Rightarrow$ **(iii)*** and **(ii)**$^* \Rightarrow$ **(i)***. Since $U = int_{\mathcal{T}}\,(U)$ for all open sets $U \in \mathcal{T}$, we have $\mathrm{Pre}^{\exists}[R_a](U) = \mathrm{Pre}^{\exists}[R_a]\,(int_{\mathcal{T}}\,(U))$. Assuming **(i)***, we get

$$\begin{array}{rcl} int_{\mathcal{T}}\left(\mathrm{Pre}^{\exists}[R_a](U)\right) & \subseteq & \mathrm{Pre}^{\exists}[R_a](U) \\ & = & \mathrm{Pre}^{\exists}[R_a]\,(int_{\mathcal{T}}\,(U)) \\ & \subseteq & int_{\mathcal{T}}\left(\mathrm{Pre}^{\exists}[R_a]\,(U)\right) \end{array}$$

so **(iii)*** holds. From the inclusion-monotonicity of both operators together with $int_{\mathcal{T}}(A) \subseteq A$, we get:

$$int_{\mathcal{T}}\left(\mathrm{Pre}^{\exists}[R_a]\,(int_{\mathcal{T}}\,(A))\right) \subseteq int_{\mathcal{T}}\left(\mathrm{Pre}^{\exists}[R_a]\,(A)\right)$$

So **(ii)*** implies **(i)***. $\dashv$

The "3-to-2" box-diamond equivalences for the l.s.c. property have four bi-dual versions:

$$\begin{array}{rclcrcl} \langle a\rangle\Box\,\varphi & \leftrightarrow & \Box\langle a\rangle\Box\,\varphi & \qquad & \Diamond[a]\Diamond\,\varphi & \leftrightarrow & [a]\Diamond\,\varphi \\ \Diamond[a]\varphi & \leftrightarrow & [a]\Diamond[a]\varphi & \qquad & \langle a\rangle\Box\langle a\rangle\varphi & \leftrightarrow & \Box\langle a\rangle\varphi \end{array}$$

Note that, while the upper two equivalences only require $[a]$ be **K**, the lower pair of equivalences require $[a]$ be **S4**.

The proof of Proposition 2.4 only uses the inclusion-monotonicity of $\mathrm{Pre}^{\exists}[R_a]$, so a uniform substitution of $\mathrm{Pre}^{\forall}$ for $\mathrm{Pre}^{\exists}$ in Proposition 2.4 gives the u.s.c. characterisation, with the characteristic formulae all of the form box-box or diamond-diamond.

Proposition 2.5 *For frames $\mathcal{F} \in \mathbb{TK}$, the following are equivalent:*

(i) *for all formulae* $\varphi \in \mathcal{L}(\Box, [a])$, $\mathcal{F} \Vdash [a]\Box\varphi \to \Box[a]\varphi$

(ii) *for all formulae* $\varphi \in \mathcal{L}(\Box, [a])$, $\mathcal{F} \Vdash [a]\Box\varphi \leftrightarrow \Box[a]\Box\varphi$

(iii) *the relation R_a is u.s.c. with respect to $\mathcal{T}$.*

3 Applications to Continuous and Hybrid Systems

Consider a state space $S \subseteq \mathbb{R}^n$ equipped with the standard Euclidean metric topology $\mathcal{T}_{\mathbb{R}}$, and consider a differential equation $\frac{d}{dt}x(t) = F(x(t))$ on S. Assuming standard conditions on the vector field $F : S \to \mathbb{R}^n$ such as Lipschitz continuity, there is a unique solution $\gamma_x : T \to S$ from each initial condition $x = \gamma_x(0) \in S$, for some time interval $T \subseteq \mathbb{R}$ including 0. Assume further that solutions exist for the whole non-negative time axis $\mathbb{R}^+ = [0, \infty)$, so they can be aggregated as the *semi-flow* $\Phi : S \times \mathbb{R}^+ \to S$ of F, which is a continuous function on $S \times \mathbb{R}^+$ satisfying the *flow laws*: $\Phi(x, 0) = x$ and $\Phi(x, t+s) = \Phi(\Phi(x,t), s)$ and $\frac{\partial}{\partial t}\Phi(x,t) = F(\Phi(x,t))$.

Definition 3.1 Given a semiflow $\Phi : S \times \mathbb{R}^+ \to S$, the *positive orbit relation* or *flow relation* $f \subseteq S \times S$ of Φ is given by:

$$x \xrightarrow{f} x' \qquad \text{iff} \qquad (\exists t \in \mathbb{R}^+)\;\; x' = \Phi(x,t)$$

In words, x' is an f-successor of x iff there is a solution curve of the differential equation leading from x to x'. We show that the resulting frame $\mathcal{F} = (S, \mathcal{T}_{\mathbb{R}}, f)$ lies in $\mathbb{LSC}$. The reflexivity and transitivity of f is immediate from the flow laws. Since time in $\mathbb{R}^+$ is linearly ordered, f is also *weakly connected*, giving the modal logic **S4.3**, but we ignore linearity here. Flow relations of reasonable classes of *differential inclusions* from set-valued vector fields also obey the basic **S4** conditions.

Proposition 3.2 *The positive orbit relation of a semi-flow is l.s.c.*

Proof. We appeal to an alternative formulation of the l.s.c. property for relations on metric spaces from (Aubin and Frankowska 1990): $R \subseteq S \times S$ is l.s.c. iff for all $x \in \mathrm{dom}(R)$, if $(x_n)_{n\in\omega}$ is any sequence converging to x and $x\,R\,y$, then there exists a sequence $(y_n)_{n\in\omega}$ converging to y with $x_n\,R\,y_n$ for all $n \in \omega$. Let $(x_n)_{n\in\omega}$ be a sequence converging to x and suppose $x \xrightarrow{f} y$. So $y = \Phi(x, t_0)$ for some $t_0 \in \mathbb{R}^+$. Since Φ is continuous on its product domain, it is continuous in both arguments separately. Thus the map $\Phi_{t_0} := \Phi(-, t_0) : S \to S$ is continuous, and thus preserves converging sequences. Hence the sequence $(y_n)_{n\in\omega}$ defined by $y_n = \Phi_{t_0}(x_n)$ must converge to y, and $x_n \xrightarrow{f} y_n$ for all $n \in \omega$. $\dashv$

In contrast, the u.s.c. property is possessed by some but not all flow relations. As a concrete example, consider the piecewise-linear differential equation over $S = \mathbb{R}$ given by $\frac{dx}{dt} = -x$ if $x \leq 1$ and $\frac{dx}{dt} = x - 2$ if $x > 1$. The semi-flow is explicitly defined by:

$$\Phi(x,t) = \begin{cases} xe^{-t} & \text{if } x \leq 1 \text{ and } t \in \mathbb{R}^+ \\ e^{\ln(\frac{1}{2-x})-t} & \text{if } 1 < x < 2 \text{ and } t \geq \ln\left(\frac{1}{2-x}\right) \\ (x-2)e^t + 2 & \text{if } (\ 1 < x < 2 \text{ and } 0 \leq t < \ln\left(\frac{1}{2-x}\right)\ ) \\ & \text{or } (\ x \geq 2 \text{ and } t \in \mathbb{R}^+\ ) \end{cases}$$

Then consider the open interval $A = (-\infty, 3)$. For the relation $R_a = f$, the box or universal pre-image $[a]A$ is the *flow-invariant subset* of A, namely the points in A *all* of whose flow successors are in A; this is the set $(-\infty, 2]$, which is closed and not open[1], hence f is not u.s.c. The diamond or existential pre-image $\langle a \rangle A$ is the set of points which have *some* flow successor in A. For $A = (-\infty, 3)$ in the example, this is just A itself.

Kripke models of hybrid systems typically have a state space of the form $S = Q \times X$ where Q is a finite set and $X \subseteq \mathbb{R}^n$. Such systems may either flow according to the differential equation associated with a

[1] This example is due to Thomas Moor, Dept. Systems Engineering, RSISE, ANU.

discrete state $q \in Q$, or they may switch discrete states with an accompanying reset of the real-valued variables (possibly the identity map). Hybrid trajectories consist of a sequence of segments of solution curves of differential equations, with endpoints of successive segments related by discrete reset relations. Each discrete state $q \in Q$ also has associated with it a set $Inv_q \subseteq X$, known as the *mode invariant*, and to be a segment of a hybrid trajectory, a solution curve of the state q differential equation is required to lie entirely inside the set Inv_q. In order to reason about hybrid trajectories in relational terms, using alternating compositions of continuous flow relations and discrete reset relations, we actually need to work with *restricted* flow relations.

Given a semi-flow $\Phi : S \times \mathbb{R}^+ \to S$ and any set $A \subseteq S \subseteq \mathbb{R}^n$, define the relation $e(A, \Phi) \subseteq S \times S$ of *restricted evolution* by:

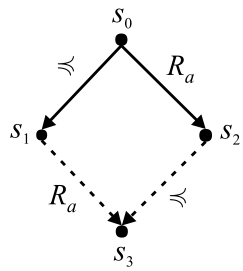
$$x \xrightarrow{e(A,\Phi)} x' \quad \stackrel{\text{def}}{\Leftrightarrow} \quad (\exists t \in \mathbb{R}^+)[\, x' = \Phi(x,t) \;\wedge\; (\forall s \in [0,t])\, \Phi(x,s) \in A\,]$$

The (unrestricted) flow or orbit relation $f = f(\Phi)$ is then the special case given by $f = e(S, \Phi)$.

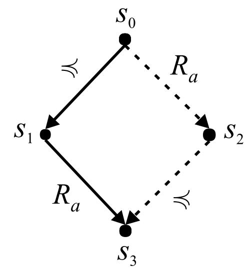
Restricted evolution relations $e(A, \Phi)$ are transitive, weakly connected, and *quasi-reflexive*, in the sense that they are reflexive on their domain A. The axiom schemata $[a]\mathbf{T_Q} : ([a]\varphi \wedge \langle a \rangle \top) \to \varphi$ characterises quasi-reflexivity; note that the domain of the relation R_a is the denotation set of the formula $\langle a \rangle \top$.

Various topological and dynamic conditions on the domain A suffice to ensure that a relation $e(A, \Phi)$ is l.s.c. in the standard Euclidean topology, with quite distinct conditions sufficing for the u.s.c. property. For example, suppose the domain A is *f-convex*, in the sense that if $x_1, x_2 \in A$ and $x_1 \xrightarrow{f} x \xrightarrow{f} x_2$ then $x \in A$. Letting p_A be an atomic proposition denoting the set A, and $\langle f \rangle$ and $[f]$ be the modalities for f, this property is captured modally by the formula $p_A \to [f](\langle f \rangle\, p_A \to p_A)$, and is equivalent to the relational equality $e = f \cap (A \times A)$, where $e = e(A, \Phi)$. In such a case, the modal operators for e are definable in terms of those for f. Under the hypothesis that A is f-convex, e is l.s.c. iff A is an open set. If additionally, f is known to be u.s.c. , then e is u.s.c. iff A is a closed set. As developed in (Davoren and Nerode 2000), a rich array of properties of hybrid trajectories can be encoded as formulae of **PDL**, the modal μ-calculus $\mathbf{L}\mu$, and extensions obtained by adding $\square\mathbf{S4}$ topological modalities.

In addition to the standard Euclidean topology on $S \subseteq \mathbb{R}^n$, work in hybrid systems also leads one to consider *finite* topologies generated from finite covers of S, which are necessarily Alexandroff. Operationally, finite topologies arise from (set-valued) *analog-to-digital* conversion maps, and



(a) "Lower" diamond property



(b) "Upper" diamond property

FIGURE 1 Bi-relational forms of semi-continuity properties

from finite discretisations of S used for computational approximation.

Modal logic has been used quite successfully in (Davoren and Moor 2000) to give a framework and methodology for the design and synthesis of switching controllers for hybrid systems. In that work, the base logic is a polymodal fusion of **K** modalities, which is further extended with axioms for restricted evolution relations, and for *metric tolerance relations* $B_\varepsilon \subseteq S \times S$ defined by: $x\, B_\varepsilon\, y$ iff $d(x, y) < \varepsilon$, where d is a metric on S. The latter relations are captured by the modal logic **KTB** (for reflexivity and symmetry). Note, however, that (Davoren and Moor 2000) does not yet make use of any semi-continuity interactions axioms – although continuity arguments are used in proofs in natural language mathematics. Thus the question of what happens when we *do* add the lower semi-continuity axiom to the basic bi-modal $\square\mathbf{S4} + [a]\mathbf{S4}$ logic is of considerable interest.

4 Bi-relational Kripke Semantics

We now discuss how the relational Kripke semantics for **S4** are a special case of the McKinsey-Tarski topological semantics; see (Davoren 1998, Davoren 1999). We then characterise the semi-continuity properties in bi-relational frames as "diamond" properties, as illustrated in Figure 1.

Given a preorder $\preccurlyeq$ on S, the *Alexandroff topology* $\mathcal{T}_\preccurlyeq$ on S determined by $\preccurlyeq$ is defined by taking $U \in \mathcal{T}_\preccurlyeq$ iff U is *up-closed* with respect to $\preccurlyeq$, *i.e.* if $x \in U$ and $x \preccurlyeq y$ then $y \in U$. Dually, the closed sets under the topology $\mathcal{T}_\preccurlyeq$ are exactly the down-closed sets with respect to $\preccurlyeq$, *i.e.* if $y \in C$ and $x \preccurlyeq y$ then $x \in C$. The topology $\mathcal{T}_\preccurlyeq$ is closed under arbitrary *intersections* as well as unions, and for all sets $A \subseteq S$,

$$int_{\mathcal{T}_\preccurlyeq}(A) = \mathrm{Pre}^{\forall}[\preccurlyeq](A) \qquad \text{and} \qquad cl_{\mathcal{T}_\preccurlyeq}(A) = \mathrm{Pre}^{\exists}[\preccurlyeq](A)$$

More generally, a topology $\mathcal{T}$ on S is called *Alexandroff* if for every point $x \in S$ there is a *smallest open set* containing x. In particular, every *finite* topology on a (arbitrary) set S is Alexandroff. For a preorder $\preccurlyeq$

on S, the topology $\mathcal{T}_{\preccurlyeq}$ is of course Alexandroff. Going the other way, any topology $\mathcal{T}$ on S determines a relation $\preccurlyeq_{\mathcal{T}}$ on S, called the *specialisation preorder* of $\mathcal{T}$, given by:

$$x \preccurlyeq_{\mathcal{T}} y \quad \text{iff} \quad (\forall U \in \mathcal{T})[\ x \in U \ \Rightarrow\ y \in U\ ]$$

Alexandroff topologies are those that can be completely recovered from their specialisation preorder: for any preorder $\preccurlyeq$ on S, $\preccurlyeq_{\mathcal{T}_{\preccurlyeq}} = \preccurlyeq$, and if $\mathcal{T}$ is Alexandroff, then $\mathcal{T}_{\preccurlyeq_{\mathcal{T}}} = \mathcal{T}$. Note that the closure under intersections means that Alexandroff topologies occupy a far corner in the space of all topologies, and look pathological from the viewpoint of general topology: the only Alexandroff topology that is Hausdorff is the discrete topology.

For a class $\mathbb{F}$ of topological Kripke frames, we let $\mathbb{AF}$ denote the subclass of frames $(S, \mathcal{T}, R_a)$ in $\mathbb{F}$ where $\mathcal{T}$ is Alexandroff. We also refer to these as *bi-relational* frames, and write them as $(S, \preccurlyeq, R_a)$.

Proposition 4.1 *Let $\mathcal{F} = (S, \mathcal{T}_{\preccurlyeq}, R_a)$ be any frame with an Alexandroff topology, with $\preccurlyeq$ the corresponding preorder. Then:*

- *The relation R_a is l.s.c. with respect to $\mathcal{T}_{\preccurlyeq}$ iff R_a satisfies the "lower" diamond property w.r.t. $\preccurlyeq$: for all $s_0, s_1, s_2 \in S$*

$$s_0 \preccurlyeq s_1 \ \textit{and}\ \ s_0\, R_a\, s_2 \quad \Rightarrow \quad (\exists s_3 \in S)[\ \ s_1\, R_a\, s_3 \ \textit{and}\ \ s_2 \preccurlyeq s_3\ \ ]$$

- *The relation R_a is u.s.c. with respect to $\mathcal{T}_{\preccurlyeq}$ iff R_a satisfies the "upper" (or "sideways") diamond property w.r.t. $\preccurlyeq$: for all $s_0, s_1, s_3 \in S$*

$$s_0 \preccurlyeq s_1 \ \textit{and}\ \ s_1\, R_a\, s_3 \quad \Rightarrow \quad (\exists s_2 \in S)[\ \ s_0\, R_a\, s_2 \ \textit{and}\ \ s_2 \preccurlyeq s_3\ \ ]$$

Proof. Apply Proposition 2.2 of (Davoren 1999). $\dashv$

The "lower" and "upper" diamond properties are illustrated in Figure 1, parts (a) and (b) respectively, and correspond to the Zig-Zag conditions for $\preccurlyeq$ to be a *bisimulation* with respect to R_a (Davoren 1999).

5 Hilbert-style Axiomatisation

Our Hilbert-style proof system $\mathcal{H}LSC$ for the logic **LSC** contains the tautologies of classical propositional logic, the rule of *modus ponens*, and the axiom schemata and inference rules below:

$\Box\mathbf{K}\wedge$:	$\Box(\varphi_1 \wedge \varphi_2) \leftrightarrow (\Box\varphi_1 \wedge \Box\varphi_2)$	$\Box\mathbf{K}\top$:	$\Box\top$
$\Box\mathbf{T}$:	$\Box\varphi \to \varphi$	$\Box\mathbf{4}$:	$\Box\varphi \to \Box\Box\varphi$
$[a]\mathbf{K}\wedge$:	$[a](\varphi_1 \wedge \varphi_2) \leftrightarrow ([a]\varphi_1 \wedge [a]\varphi_2)$	$[a]\mathbf{K}\top$:	$[a]\top$
$[a]\mathbf{T}$:	$[a]\varphi \to \varphi$	$[a]\mathbf{4}$:	$[a]\varphi \to [a][a]\varphi$

$$\langle a\rangle\Box\mathbf{lsc} : \quad \langle a\rangle\Box\varphi \to \Box\langle a\rangle\varphi$$

$$\Box\text{-}\textit{monotonicity:}\ \ \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \qquad\qquad [a]\text{-}\textit{monotonicity:}\ \ \frac{\varphi \to \psi}{[a]\varphi \to [a]\psi}$$

We write $\vdash_{\mathcal{H}LSC} \varphi$ if the formula $\varphi \in \mathcal{L}(\Box, [a])$ is derivable in $\mathcal{H}LSC$. A set $\Gamma \subseteq \mathcal{L}(\Box, [a]$ of formulae is *$\mathcal{H}LSC$-consistent* if it is not the case that $\vdash_{\mathcal{H}LSC} (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \bot$ for any finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$.

The $\Box\mathbf{S4}$ axiom schemes and the $\Box$-monotonicity rule as presented correspond precisely to Kuratowski's (1966) axiomatisation of the topological interior operator (McKinsey 1941, McKinsey and Tarski 1944), (Rasiowa and Sikorski 1963). Note that the usual box-necessitation rules can be derived from the monotonicity rules using the box-$\mathbf{K}\top$ axioms. Likewise, the $[a]\mathbf{S4}$ axiom schemata capture the properties of the operator $\mathrm{Pre}^{\forall}[R_a]$ for preordered relations R_a. The verification that $\mathcal{F} \Vdash \langle a\rangle\Box\varphi \to \Box\langle a\rangle\varphi$ for all frames $\mathcal{F} \in \mathbb{LSC}$ is given in Proposition 2.4. This establishes soundness. For completeness of the axiomatisation, the standard canonical model construction using maximal consistent sets of formulae suffices, as independently discovered by Fischer-Servi (1981).

Proposition 5.1 (Soundness and completeness of $\mathcal{H}LSC$)
For all $\varphi \in \mathcal{L}(\Box, [a])$, $\models_{\mathbb{LSC}} \varphi$ iff $\vdash_{\mathcal{H}LSC} \varphi$.

Proof. For completeness, define an Alexandroff frame $\mathcal{F}_{\mathbf{LSC}}$ where:

$$\begin{array}{lll} S & := & \{\ s \subseteq \mathcal{L}(\Box, [a]) \mid s \text{ is maximal } \mathcal{H}LSC\text{-consistent}\ \} \\ s_1 \preccurlyeq s_2 & \text{iff} & (\forall\varphi \in \mathcal{L}(\Box, [a]))\ [\ \Box\varphi \in s_1 \ \Rightarrow\ \varphi \in s_2\ ] \\ & \text{iff} & (\forall\varphi \in \mathcal{L}(\Box, [a]))\ [\ \varphi \in s_2 \ \Rightarrow\ \Diamond\varphi \in s_1\ ] \\ s_1\, R_a\, s_2 & \text{iff} & (\forall\varphi \in \mathcal{L}(\Box, [a]))\ [\ [a]\varphi \in s_1 \ \Rightarrow\ \varphi \in s_2\ ] \\ & \text{iff} & (\forall\varphi \in \mathcal{L}(\Box, [a]))\ [\ \varphi \in s_2 \ \Rightarrow\ \langle a\rangle\varphi \in s_1\ ] \end{array}$$

and define the canonical valuation $\xi_0 : \mathrm{AP} \to \mathcal{P}(S)$ as usual by $\xi_0(p) := \{s \in S \mid p \in s\}$. Let $\mathcal{M}_{\mathbf{LSC}} = (\mathcal{F}_{\mathbf{LSC}}, \xi_0)$ denote the canonical model. By the standard arguments, the axioms for $\Box\mathbf{S4}$ and $[a]\mathbf{S4}$ entail that the relations $\preccurlyeq$ and R_a are both preorders, hence $\mathcal{F}_{\mathbf{LSC}} \in \mathbb{ATS}4$.

To establish that $\mathcal{F}_{\mathbf{LSC}}$ is l.s.c. with respect to $\mathcal{T}_{\preccurlyeq}$, and so $\mathcal{F}_{\mathbf{LSC}} \in \mathbb{LSC}$, we need to verify the l.s.c. diamond property: if $s_0 \preccurlyeq s_1$ and

$s_0\, R_a\, s_2$, then there exists a state s_3 such that $s_1\, R_a\, s_3$ and $s_2 \preccurlyeq s_3$. So suppose $s_0 \preccurlyeq s_1$ and $s_0\, R_a\, s_2$. Then consider the set of formulae

$$c = \{\varphi \in \mathcal{L}(\Box, [a]) \mid \quad [a]\varphi \in s_1 \;\text{ or }\; \Box\varphi \in s_2\;\}$$

If c is non-empty and $\mathcal{H}LSC$-consistent, then there exists a maximal $\mathcal{H}LSC$-consistent set $s_3 \supseteq c$, and by construction, $s_1\, R_a\, s_3$ and $s_2 \preccurlyeq s_3$.

The non-emptiness of c is immediate, since $[a]\top \in s$ and $\Box\top \in s$ for all s. To prove that c is $\mathcal{H}LSC$-consistent, suppose, for a contradiction, that it is not. Then there is a finite subset $c_0 = \{\varphi_1, ..., \varphi_n, \psi_1, ..., \psi_m\}$ of c, such that $\vdash_{\mathcal{H}LSC} (\varphi_1 \wedge \cdots \wedge \varphi_n \wedge \psi_1 \wedge \cdots \wedge \psi_m) \rightarrow \bot$, where each $[a]\varphi_i \in s_1$ and each $\Box\psi_j \in s_2$. Let $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_n$ and likewise $\psi = \psi_1 \wedge \cdots \wedge \psi_m$, hence $\vdash_{\mathcal{H}LSC} \psi \rightarrow \neg\varphi$. Since maximal consistent sets s are closed under conjunction, and boxes distribute over conjunction, we have $[a]\varphi \in s_1$ and $\Box\psi \in s_2$. Now $\Box\psi \in s_2$ and $s_0\, R_a\, s_2$ imply $\langle a\rangle\Box\psi \in s_0$, so by the $\langle a\rangle\Box$**lsc** axiom $\langle a\rangle\Box\psi \rightarrow \Box\langle a\rangle\psi$, we then have $\Box\langle a\rangle\psi \in s_0$. By the monotonicity of both diamonds and boxes, $\vdash_{\mathcal{H}LSC} \Box\langle a\rangle\psi \rightarrow \Box\langle a\rangle\neg\varphi$, hence $\Box\langle a\rangle\neg\varphi \in s_0$. Then since $s_0 \preccurlyeq s_1$, we have $\langle a\rangle\neg\varphi \in s_1$, and hence $\neg[a]\varphi \in s_1$. But this contradicts $[a]\varphi \in s_1$, so c must be $\mathcal{H}LSC$-consistent.

The usual induction on formulae establishes the "Truth Lemma": $s \in [\![\varphi]\!]^{\mathcal{M}_{\mathbf{LSC}}}$ iff $\varphi \in s$ for all maximal $\mathcal{H}LSC$-consistent sets $s \subseteq \mathcal{L}(\Box, [a])$ and all formulae $\varphi \in \mathcal{L}(\Box, [a])$. Then for any φ, if it is not the case that $\vdash_{\mathcal{H}LSC} \varphi$, then there is some maximal $\mathcal{H}LSC$-consistent set s such that $\varphi \notin s$, hence $\mathcal{M}_{\mathbf{LSC}} \nVdash \varphi$. $\dashv$

Corollary 5.2 *For all $\varphi \in \mathcal{L}(\Box, [a])$, $\models_{\mathbb{LSC}} \varphi$ iff $\models_{\mathbb{ALSC}} \varphi$.*

Proof. In the non-trivial direction, suppose $\models_{\mathbb{ALSC}} \varphi$. Then in particular, $\mathcal{M}_{\mathbf{LSC}} \Vdash \varphi$ in the canonical model, hence $\vdash_{\mathcal{H}LSC} \varphi$. Then since $\mathcal{H}LSC$ is sound with respect to the whole class $\mathbb{LSC}$, we have $\models_{\mathbb{LSC}} \varphi$. $\dashv$

The canonical model construction can be applied uniformly to the family of bimodal logics obtained by axiomatising the relational modality $[a]$ by any consistent normal canonical monomodal logic, and adding one or both of the two semi-continuity axioms; in particular to the natural twin logic **USC** given by $\Box$**S4**$+[a]$**S4**$+[a]\Box$**usc** (Coulthard 2000). Consequently, the (finitary) bimodal language cannot distinguish between validity in the class of all appropriate frames, with arbitrary topologies, and validity in the class of appropriate frames restricted to Alexandroff topologies. Hence we can restrict attention to the corresponding class of bi-relational frames without any loss.

It remains on open question as to whether standard filtration quotient techniques (see, for example, Chapter 4 of (Goldblatt 1992)) can be successfully extended to the canonical model $\mathcal{M}_{\mathbf{LSC}}$ to prove the finite model property for the logic **LSC**, or analogously for **USC**, or whether filtration breaks down for these logics. What is not clear is how to choose a finite filtration set Γ adequate to push through the semi-continuity properties. As an alternative method of attack we have investigated a tableau calculus for the logic **LSC** as reported next.

6 The Tableau Calculus $\mathcal{T}LSC$

Our tableau calculus $\mathcal{T}LSC$ for $\mathbb{LSC}$ is based upon the tableau calculus for the logic **S4.2** from (Goré 1991). The logic **S4.2** is the monomodal logic obtained from $\mathbb{LSC}$ if $R_a = \preccurlyeq = R$. All the rules for $\mathcal{T}LSC$ are given in Figure 2 where $X, Y \subseteq \mathcal{L}(\Box, [a])$ are finite sets of formulae and $X; \varphi$ means $X \cup \{\varphi\}$: see (Goré 1999) for an introduction to modal tableau calculi. The subset of these rules consisting of the top three rows gives another calculus called $\mathcal{T}(\Box S4 + [a]S4)$, discussed later.

A $\mathcal{T}LSC$-tableau for a finite set of formulae X is an upside down tree with root X, such that each node in the tree is obtained from its parent node by an application of a rule of $\mathcal{T}LSC$. A tableau branch is *closed* if the final rule application on this branch is $(\bot)$, otherwise the branch is *open*. A tableau is open if some branch is open, otherwise it is closed. If there is a closed $\mathcal{T}LSC$-tableau for the finite set $X \cup \{\neg\varphi\}$, we write $X \vdash_{\mathcal{T}LSC} \varphi$. In particular, when X is empty we write $\vdash_{\mathcal{T}LSC} \varphi$.

Following (Goré 1999), all rules are *static* except $(K4\neg\Box)$ and $(K4\neg[a])$ which are *transitional* since they correspond to the creation of successor worlds. The (LSC) rules create bigger formulae from smaller formulae. To keep this process from repeating itself *ad infinitum*, the new formula is marked with a $\star$ and the (LSC) rules are not applicable to such starred formulae. All other rules must treat starred formulae as if they were unstarred. Also, if the formula to be added by an (LSC) rule is already present without a star, then that rule is not applied. This stops a formula from appearing both starred and unstarred in a node.

As usual, each tableau rule can be read downwards as: if the set of formulae above the horizontal line is $\mathbb{LSC}$-satisfiable, then so is at least one of the sets below the horizontal line.

Proposition 6.1 (Soundness of $\mathcal{T}LSC$)
For all formulae $\varphi \in \mathcal{L}(\Box, [a])$, *if* $\vdash_{\mathcal{T}LSC} \varphi$ *then* $\vDash_{\mathbb{LSC}} \varphi$.

Proof. All rules except the (LSC) rules are standard **S4** rules, so consider the (LSC) rules. The premiss of $(LSC\neg\Box)$ is that $X; \neg\Box\varphi$ is **LSC**-satisfiable. Thus there is an $\mathbb{LSC}$-model with a world w_0 such that $w_0 \Vdash X$ and $w_0 \Vdash \neg\Box\varphi$. Now if $w_0 \Vdash [a]\neg\Box\varphi$ then we are done, since

$$(\bot)\ \frac{X;\varphi;\neg\varphi}{\bot} \quad (\wedge)\ \frac{X;\varphi\wedge\psi}{X;\varphi;\psi} \quad (\neg\wedge)\ \frac{X;\neg(\varphi\wedge\psi)}{X;\neg\varphi \mid X;\neg\psi} \quad (\neg\neg)\ \frac{X;\neg\neg\varphi}{X;\varphi}$$

$$(KT\Box)\ \frac{X;\Box\varphi}{X;\Box\varphi;\varphi} \qquad (KT[a])\ \frac{X;[a]\varphi}{X;[a]\varphi;\varphi}$$

$$(K4\neg\Box)\ \frac{X;\Box Y;\neg\Box\varphi}{\Box Y;Y;\neg\varphi} \qquad (K4\neg[a])\ \frac{X;[a]Y;\neg[a]\varphi}{[a]Y;Y;\neg\varphi}$$

$$(cut\neg\Box)\ \frac{X;\neg\Box\varphi}{X;\neg\Box\varphi;\varphi \mid X;\neg\Box\varphi;\neg\varphi} \quad (cut\neg[a])\ \frac{X;\neg[a]\varphi}{X;\neg[a]\varphi;\varphi \mid X;\neg[a]\varphi;\neg\varphi}$$

$$(LSC\neg\Box)\ \frac{X;\neg\Box\varphi}{X;\neg\Box\varphi;\Box(\neg[a]\neg\Box\varphi)^\star \mid X;\neg\Box\varphi;[a]\neg\Box\varphi}\ \neg\Box\varphi \text{ unstarred}$$

$$(LSC\neg[a])\ \frac{X;\neg[a]\varphi}{X;\neg[a]\varphi;[a](\neg\Box\neg[a]\varphi)^\star \mid X;\neg[a]\varphi;\Box\neg[a]\varphi}\ \neg[a]\varphi \text{ unstarred}$$

FIGURE 2 Tableau Calculi $\mathcal{T}(\Box S4 + [a]S4)$ and $\mathcal{T}LSC$

then w_0 satisfies the right child of $(LSC\neg\Box)$. So suppose $w_0 \not\Vdash [a]\neg\Box\varphi$, that is, $w_0 \Vdash \langle a\rangle\Box\varphi$. Then by Proposition 2.4, $w_0 \Vdash \Box\langle a\rangle\Box\varphi$, or equivalently, $w_0 \Vdash \Box\neg[a]\neg\Box\varphi$. Hence w_0 satisfies the left child of $(LSC\neg\Box)$. The soundness of $(LSC\neg[a])$ is proved analogously. $\dashv$

The two branches of $(LSC\neg\Box)$ are mutually exclusive since the right child contains $[a]\neg\Box\varphi$, and an application of $(KT\Box)$ to the left child brings $(\neg[a]\neg\Box\varphi)^\star$ into the left child. If this formula were unstarred then the $(LSC\neg[a])$ rule would be applicable to it, sending the system into an infinite loop. Thus $(LSC\neg\Box)$ encodes a cut on the *larger* formula $[a]\neg\Box\varphi$, for some subformula $\neg\Box\varphi$. The fact that the left child contains $\Box(\neg[a]\neg\Box\varphi)^\star$ rather than just $(\neg[a]\neg\Box\varphi)^\star$ encodes the axiom $\langle a\rangle\Box\varphi \leftrightarrow \Box\langle a\rangle\Box\varphi$, but does not lead to an infinite loop since no rule creates bigger formulae from $\Box$-formulae. The $(LSC\neg[a])$ rule encodes an analogous cut rule and axiom. This "analytic super-formula property" is the key to termination of proof search (Goré 1999).

7 A counter-example to completeness of $\mathcal{T}LSC$

In previous versions of this paper, we conjectured that $\mathcal{T}LSC$ was complete, but the following counter-example to completeness was recently found by Nicolette Bonnette.

Example 7.1 The set $X = \{p_1, p_2, p_3, \neg p_1 \vee \neg p_2 \vee \neg p_3\}$ is classically unsatisfiable and every non-empty proper subset of X is classically satisfiable. Consider the formula

$$\varphi_0 \quad := \quad \Box[a]p_1 \wedge \Diamond[a]p_2 \wedge [a]\Box p_3 \wedge \langle a \rangle \Box(\neg p_1 \vee \neg p_2 \vee \neg p_3)$$

Proposition 7.2 *The formulae φ_0 is not $\mathbb{LSC}$-satisfiable.*

Proof. For a contradiction suppose that there is an $\mathbb{LSC}$-model with a world s_0 such that $s_0 \Vdash \varphi_0$. Then there must exist two worlds s_1 and s_2 such that $s_0 \preccurlyeq s_1$ and $s_0 R_a s_2$, $s_1 \Vdash [a]p_1, [a]p_2$ and $s_2 \Vdash \Box p_3, \Box(\neg p_1 \vee \neg p_2 \vee \neg p_3)$. By the l.s.c. condition there must exist a world s_3 such that $s_1 R_a s_3$ and $s_2 \preccurlyeq s_3$. Then we must have $s_3 \Vdash p_1, p_2, p_3, (\neg p_1 \vee \neg p_2 \vee \neg p_3)$, which we know to be impossible. $\dashv$

However, as a detailed attempt will show, there is no closed $\mathcal{T}LSC$-tableau for the set $\{\varphi_0\}$.

We are currently trying to pinpoint exactly why completeness fails using the model-graph techniques outlined in (Goré 1999) so we can invent new rules to plug the gap. The l.s.c. condition is unusual because of its non-local effects: we require a global view of the counter-model in order to detect and repair incomplete half-diamonds. A similar situation occurs in tense logics where events in the future can effect worlds in the past, and vice-versa. We believe that this is the cause of the incompleteness of our purely local tableau calculus $\mathcal{T}LSC$.

An alternative approach is to use a complete and terminating explicit algorithm, and to prove its soundness, as sketched next.

8 An Alternative Tableau Calculus $\mathcal{T}lscSAT$

For a class of bi-relational frames $\mathbb{B}$, let $\mathbb{B}^{\mathrm{fr}}$ denote the subclass of frames $(S, \preccurlyeq, R_a)$ in $\mathbb{B}$ such that S is finite and the frame consists of a finite rooted tree of nodes: the nodes are finite R_a-clusters and $\preccurlyeq$-clusters when the underlying relations are transitive (Goré 1999).

Let $\mathcal{T}(\Box S4 + [a]S4)$ consist of the rules $(\bot)$, $(\wedge)$, $(\neg\wedge)$, $(\neg\neg)$, $(KT\Box)$, $(KT[a])$, $(K4\neg\Box)$, and $(K4\neg[a])$ from Figure 2. Thus, $\mathcal{T}(\Box S4 + [a]S4)$ is the union of two standard calculi for **S4** (Goré 1999), one for each relation R_a and $\preccurlyeq$.

Proposition 8.1 *For all $\varphi \in \mathcal{L}(\Box, [a])$, the following are equivalent:*
(i) $\models_{\mathbb{TS4}} \varphi$ **(ii)** $\models_{\mathbb{ATS4}} \varphi$ **(iii)** $\models_{\mathbb{ATS4}^{\mathrm{fr}}} \varphi$ **(iv)** $\vdash_{\mathcal{H}(\Box S4+[a]S4)} \varphi$
(v) $\vdash_{\mathcal{T}\Box S4+[a]S4} \varphi$

Given a finite set of formulae X, the systematic tableau procedure (Goré 1999) attempts to construct a closed $\mathcal{T}(\Box S4 + [a]S4)$-tableau for X. If it succeeds then by soundness, the set X is not $\mathbb{TS}4$-satisfiable. If it fails, then no $\mathcal{T}(\Box S4 + [a]S4)$-tableau for X closes so the systematic procedure constructs a (finite) $\mathbb{ATS}4^{\text{fr}}$-model $\mathcal{M}$ whose root satisfies X. In such a representation, the number of actual edges is minimised by including only the essential edges, and implicitly making the edges "transitive". Thus $\mathcal{M}$ is really a data structure (graph), but there is no explicit "database" of tuples representing the relations.

The tableau calculus $\mathcal{T}lscSAT$ is a "hybrid" calculus that uses the tableau calculus $\mathcal{T}(\Box S4 + [a]S4)$ to handle the $\Box\mathbf{S4} + [a]\mathbf{S4}$ aspects of our logic, and uses an additional global data-structure to explicitly track and repair the failures of the l.s.c. condition. A similar technique is used in (Fariñas del Cerro and Gasquet 1999)[2].

In our hybrid method, we start by applying the systematic $\mathcal{T}(\Box S4 + [a]S4)$-tableau construction to $\{\neg\varphi\}$. If this procedure finds a closed $\mathcal{T}(\Box S4 + [a]S4)$-tableau, then we know that φ is $\mathbb{TS}4$-valid, and hence φ is also $\mathbb{ALSC}$-valid. Otherwise, we have at least one finite $\mathbb{ATS}4^{\text{fr}}$-model $\mathcal{M}_1$ whose root satisfies $\neg\varphi$. In constructing $\mathcal{M}_1$, the systematic method chooses either the left or right disjunct when applying the $(\neg\wedge)$ rule. But the other disjunct may also lead to a different finite $\mathbb{ATS}4^{\text{fr}}$-model $\mathcal{M}_2$ for $\{\neg\varphi\}$. In general, there will be some finite number k of such $\mathbb{ATS}4^{\text{fr}}$-models $\mathcal{M}_1, \cdots, \mathcal{M}_k$ for $\{\neg\varphi\}$. We want to extend one of them to an $\mathbb{ALSC}^{\text{fr}}$-model $\mathcal{M}_o$, the root node of which still satisfies $\neg\varphi$. To do this we have to ensure that it satisfies the l.s.c. condition.

Let $(s_0 \preccurlyeq s_1, s_0 R_a s_2)$ stand for three worlds s_0, s_1 and s_2 in $\mathcal{M}_1$ such that $s_0 \preccurlyeq s_1$ and $s_0 R_a s_2$ both hold, but such that there is no x for which both $s_1 R_a x$ and $s_2 \preccurlyeq x$ hold. The l.s.c. condition demands that every such "incomplete lsc-half-diamond" be completed. To create such a world we start a new systematic $\mathcal{T}(\Box S4 + [a]S4)$-tableau construction for the set $x := \{[a]\psi \mid [a]\psi \in s_1\} \cup \{\Box\psi \mid \Box\psi \in s_2\}$. If this procedure finds a closed $\mathcal{T}(\Box S4 + [a]S4)$-tableau then the current model $\mathcal{M}_1$ cannot be extended into an $\mathbb{LSC}$-model so we try $\mathcal{M}_2$, and so on. If the systematic procedure finds no closed $\mathcal{T}(\Box S4 + [a]S4)$-tableau, it will return an $\mathbb{ATS}4^{\text{fr}}$-model $\mathcal{M}_x$ with a root node r containing x. We can safely add the "transitive" edges $s_1 R_a r$ and $s_2 \preccurlyeq r$ as desired, thereby appending $\mathcal{M}_x$ to $\mathcal{M}_1$. If the new model contains no further incomplete lsc-half-diamonds we have produced an $\mathbb{ALSC}^{\text{fr}}$-model which falsifies φ at its

[2]The claim on page 317 of (Fariñas del Cerro and Gasquet 1999) that "a decidability result for confluence with transitivity was an open problem" is erroneous as Segerberg proved the finite model property for **K4.2** in his doctoral dissertation and (Goré 1991) contains a decision procedure for **S4.2**.

root node. Otherwise we must complete all remaining lsc-half-diamonds in the new model. Eventually, we will either report that each original $\mathbb{ATS}4^{\mathrm{fr}}$-model for $\{\neg\varphi\}$ contained at least one lsc-half-diamond which could not be completed, or until we find some $\mathbb{ATS}4^{\mathrm{fr}}$-model for $\{\neg\varphi\}$ in which we can complete all incomplete lsc-half-diamonds.

The procedure $\mathcal{T}lscSAT$ is complete and terminates, but we have been unable to prove its soundness. The main problem is that a node x inserted when completing a half-diamond may appear already higher up in the current counter-model under construction, thereby creating an infinite path. A standard trick is to insert the two required edges so they loop back to the previous occurrence of x, thereby maintaining finiteness at the cost of two loops in the current counter-model. Although this trick can be made to work in the mono-modal setting (Fariñas del Cerro and Gasquet 1999), it is by no means clear how to treat such loops within an inductive argument in the bimodal setting.

9 Related Work on Bimodal and Polymodal Logics

While topological semantics for **S4**-like modalities have been used in a number of recent studies of bimodal logics, our work is the first to directly address continuity properties of *relations* in a bimodal framework. Earlier work by Davoren (1998), and independent work by Kremer and Mints (1997), uses the topological semantics for $\Box$**S4** together with the scheme $[a]\mathbf{F}:\ [a]\varphi \leftrightarrow \langle a\rangle\varphi$, which characterises $R_a = g$ as a *total function* $g : S \to S$. The resulting bimodal logic is adequate to formalise the continuity of functions, via the equivalent schemata $\langle a\rangle\Box$**lsc** or $[a]\Box$**usc**, plus other topological properties such as being an open map or being a homeomorphism. The thesis (Davoren 1998) proves the decidability of the bimodal logic $\Box\mathbf{S4} + [a]\mathbf{KF} + \langle a\rangle\Box\mathbf{usc}$ by giving a sound and complete tableau proof system, and taking a finite quotient of the infinite systematic tableau constructed for completeness. That thesis also begins an investigation of polymodal logics built on $\Box\mathbf{S4} + \mathbf{DPDL}$ (i.e. *deterministic* **PDL**) with the atomic actions $a \in \Sigma$ interpreted by continuous total functions. While the l.s.c. property is inherited under each of the regular expression operations of relational composition, finite unions and Kleene-star, inheritance of the u.s.c. property fails for the Kleene-star (Davoren and Nerode 2000). As noted in an unpublished manuscript by Kremer, following (Kremer and Mints 1997), there are continuous functions whose Kleene-star is not u.s.c.

Dabrowski, Moss and Parikh (1996), and Heinemann (1998, 2000), examine bimodal logics for expressing elementary reasoning about points and sets in general topology, motivated by applications to reasoning about knowledge in multi-agent systems. Their semantics are given over *subset frames* $(S, \mathcal{O})$ where $\mathcal{O}$ is an arbitrary family of non-empty subsets of S, so for a topology $\mathcal{T}$ on S, the family $\mathcal{O} = \mathcal{T} - \{\emptyset\}$ gives a special case. In (Dabrowski et al. 1996, Heinemann 2000) the logics are built on the bimodal base $\Box\mathbf{S4}+K\mathbf{S5}$, where the two box modalities $\Box$ and K are interpreted by different aspects of the relationship between points $s \in S$ and sets $U \in \mathcal{O}$, and are further interconnected by the $K\Box\mathbf{usc}$ axiom scheme; these logics also have *persistence* axioms for literals: $p \to \Box p$ and $\neg p \to \Box\neg p$ for atomic propositions p. When $\mathcal{O}$ comes from a topology, the topological interior operator is recovered by their compound modality $\Diamond K$. The papers (Dabrowski et al. 1996, Heinemann 2000) also extend the canonical Kripke model construction to their bimodal setting, with the two modalities realised by a preorder and an equivalence relation respectively; (Dabrowski et al. 1996) further establishes the decidability of their bimodal logic by taking a finite filtration of the bi-relational canonical Kripke model. In (Heinemann 1998), the **S4** box is weakened to that from a partial function, to give a "next-time" operator.

Fischer-Servi (1981, 1989) studies bimodal logics where two relationally interpreted modalities are subject to interaction axioms of the same form as those we study. These logics have an axiomatisation equivalent to $\Box\mathbf{L} + [a]\mathbf{S4} + \langle a\rangle\Box\mathbf{lsc} + [a]\Box\mathbf{usc}$, where $\mathbf{L}$ is any one of **KT**, **S4**, **S4.1** and **S4.2**. The equivalence can be seen by mapping the box operators L_1 and L_2 of (Fischer-Servi 1981) to our $[a]$ and $\Box$ respectively, and using the equivalence between $\Diamond[a]\varphi \to [a]\Diamond\varphi$ and $\langle a\rangle\Box\mathbf{lsc}$. Fischer-Servi interprets both modalities using relations, varies the axioms for $\Box$ and fixes $[a]$ as an **S4**-modality, whereas we fix $\Box$ as a topologically interpreted **S4**-modality and vary the axioms for $[a]$. The notion of "finite axiomatizability" used in (Fischer-Servi 1989) is *without* the necessitation rules, hence Fischer-Servi's negative results do not directly apply to questions of decidability of our logics.

10 Further Work and Conclusions

We have developed a simple and elegant logical framework in which to reason about continuous dynamics, using the very familiar ingredients of two **S4** modalities together with an interaction axiom. As demonstrated, the interaction is non-trivial, and its origins lie in a natural mathematical phenomenon, giving solid motivation for the resulting logic **LSC**.

We have given a tableau calculus $\mathcal{T}LSC$ which is sound but not complete for **LSC**. We have outlined a procedure $\mathcal{T}lscSAT$ which is complete and terminating for **LSC**, but for which soundness is still open. We have also considered various filtration methods, all to no avail. It is clear that the problems of completeness of $\mathcal{T}LSC$, the soundness of $\mathcal{T}lscSAT$ and the solution to the filtration method are all three vertices of the same problem: if we can solve one, then we can solve them all.

A wide spectrum of interesting bimodal and polymodal logics can be obtained by keeping the topological $\Box$**S4** fixed and varying the axioms for the relational modalities $[a]$ and $\langle a \rangle$. For example, one may strengthen to $[a]$**S4.3** to cover the weak connectedness of flow relations, or consider polymodal variants to cover discrete reset relations. Further investigation is required of logics such as $\Box\mathbf{S4} + \mathbf{PDL}$ and $\Box\mathbf{S4} + \mathbf{L}\mu$, combining the topological modalities with infinitary constructs such as the Kleene-star or more general fixed-point quantification, and the effect on them of adding one or more semi-continuity axioms.

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References

Antsaklis, P.J. (ed.). 2000. *Proceedings of the IEEE*. July. Special Issue on Hybrid Systems: Theory and Applications.

Aubin, J-P., and H. Frankowska. 1990. *Set-Valued Analysis*. Boston: Birkhäuser.

Coulthard, V. 2000. A Study of Topological Polymodal Logics. Technical report. CSL, RSISE, ANU, November. BSc Honours thesis.

Dabrowski, A., L.S. Moss, and R. Parikh. 1996. Topological Reasoning and the Logic of Knowledge. *Annals of Pure and Applied Logic* 78:73–110.

Davoren, J.M. 1998. *Modal Logics for Continuous Dynamics*. Doctoral dissertation, Dept. of Mathematics, Cornell University.

Davoren, J.M. 1999. Topologies, Continuity and Bisimulations. *Theoretical Informatics and Applications* 33:357–381.

Davoren, J.M., and T. Moor. 2000. Logic-based design and synthesis of controllers for hybrid systems. Technical report. Dept. Sys. Eng. , RSISE, ANU, July 2000. `http://csl.anu.edu.au/~davoren/`.

Davoren, J.M., and A. Nerode. 2000. Logics for Hybrid Systems. *Proc. of the IEEE* 88:985–1010.

Fariñas del Cerro, L., and O. Gasquet. 1999. Tableaux Based Decision Procedures for Modal Logics of Confluence and Density. *Fundamenta Informaticae* 40:317–333.

Fischer-Servi, G. 1981. Teoremi di Completezza per Calcoli Bimodali. *Riv. Mat. Univ. Parma* 4(7):347–350.

Fischer-Servi, G. 1989. Finite Axiomatizability in Bimodal Calculi. *Riv. Mat. Univ. Parma* 4(15):271–287.

Goldblatt, R. 1992. *Logics of Time and Computation*. Stanford: CSLI.

Goré, R. 1991. Semi-analytic tableaux for propositional normal modal logics with applications to nonmonotonicity. *Logique et Analyse* 133-134:73–103. (Printed in 1994).

Goré, R. 1999. Tableaux Methods for Modal and Temporal Logics. In *Handbook of Tableaux Methods*, ed. M. D'Agostino *et al.* 297–396. Kluwer.

Heinemann, B. 1998. Topological Nexttime Logic. In *Advances in Modal Logic, Volume 1*, ed. M. Kracht *et al.* 99–113. CSLI.

Heinemann, B. 2000. Generalizing the Modal and Temporal Logic of Linear Time. In *8th International Conference on Algebraic Methodology and Software Technology (AMAST'2000)*, ed. T. Rus *et al.* Berlin. Springer-Verlag.

Kremer, P., and G. Mints. 1997. Dynamic Topological Logic. *Bulletin of Symbolic Logic* 3:371–372.

Kuratowski, K. 1966. *Topology*. New York: Academic Press.

McKinsey, J.C.C. 1941. A solution to the decision problem for the Lewis systems S2 and S4, with an application to topology. *Journal of Symbolic Logic* 6:117–141.

McKinsey, J.C.C., and A. Tarski. 1944. The Algebra of Topology. *Annals of Mathematics* 45:141–191.

Rasiowa, H., and R. Sikorski. 1963. *The Mathematics of Metamathematics*. Warsaw: PWN.

Wolter, Frank. 1999. Decision Problems for Combined (Modal) Logics. Technical report. `http://www.informatik.uni-leipzig.de/~wolter/`: Institüt für Informatik, Univesität Leipzig, September.

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