Topologies, convergence and uniformities in general hybrid path spaces

Jen Davoren
Department of Electrical & Electronic Engineering
The University of Melbourne, VIC 3010 AUSTRALIA
davoren@unimelb.edu.au
and
Inessa Epstein
Department of Mathematics
The University of California at Los Angeles, CA 90095 USA
iepstein@math.ucla.edu

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Abstract:

Topological and/or metric structure on sets of hybrid trajectories is central to stability and robustness theory for hybrid systems, and their generalizations to systems over higher-dimensional heterogeneous signal domains. Recent work by Goebel, Teel and co-workers (2004,2006,2007), and also by Collins (2005,2006), utilizes a 2-dimensional hybrid time structure, and indirectly develops topological structure on hybrid path spaces by studying the convergence of a sequence of paths in terms of the set-convergence of the graphs of those paths, considered as subsets of the product of the time structure and the state space. In this paper, we explicitly develop a topology on spaces of hybrid paths (of both finite and infinite length), where the topology derives from a 3-parameter uniformity giving quantitative measures of closeness. Based on a quite general notion of a time structure as a partially-ordered abelian group equipped with a norm function (which includes 2-dimensional hybrid time), we prove that the path operations of prefix, suffix and fusion all respect the uniformity, and that this uniformity gives topological structure equivalent to that of graph-convergence, which is in turn equivalent to the modified compact-open topology considered by Collins (2005,2006). The uniform topology is metrizable for many spaces of hybrid paths, but it is coarser than the Skorokhod metric topologies considered by Broucke (1998,2002) and Kossentini and Caspi (2004) for a restricted class of hybrid paths.

1 Introduction

One of the challenges in the foundations of hybrid systems is to develop a metric or topology on the space of hybrid paths or trajectories that is rich enough to support concepts of stability and robustness for hybrid systems. Cast in quite general terms, within the framework of differential and difference inclusions [1, 2, 3, 4], the solution paths of a hybrid system consist of a sequence of segments of real-time trajectories that satisfy a differential inclusion $\frac{d}{dt}x \in F(x)$ and are absolutely continuous in real time, where the end-state x'_n of one segment and the start-state x_{n+1} of the successor segment satisfy a difference inclusion $x'_n \in G(x_{n+1})$, the latter modelling the response of the state to a discrete transition event. For example, from [5], consider a system of three particles of varying point-masses moving in one dimension whose state in $\mathbb{R}^6$ consists of the positions and the velocities of each of the particles. While all the particles remain apart, they move with constant velocity, but when any two adjacent particles collide, there is a discrete change in velocity, given by Newton's law of restitution and the conservation of momentum. In the case of simultaneous collision of all three particles, there will be two possible values for the post-collision velocities, so the discrete dynamics are multi-valued, with different values depending upon which order the two 2-way collisions are taken to occur.

Trajectories of hybrid systems fail to be amenable within the classical theories of dynamical and control systems for many reasons: discrete transitions typically give rise to both discontinuities in the state and multi-valued-ness of the state w.r.t. real time; there may be multiple discrete transitions at the same real time; hybrid paths may exhibit Zeno behaviour, which means infinitely many discrete transitions occur in finite real time; variations in the timing of discrete transitions means that two hybrid paths that are "close" may differ in their time domains; and in the passage from finite length paths to infinite length paths, there are multiple distinct ways in which time can "go to infinity", and the notion of a maximal length path becomes more delicate.

One approach addressing several of these issues (proposed independently by Goebel, Teel and co-workers in [2] and by Collins in [6], and employed in [3, 4, 5, 7, 8]) is to model the time domain of a hybrid path as a linearly-ordered subset of the partially-ordered structure $\mathbb{R} \times \mathbb{Z}$, where the real-valued coordinate gives the real time and the discrete coordinate is incremented with each discrete transition; the real-time continuity of each of the segments of a hybrid path is then sufficient to give continuity of the path as a partial function from $\mathbb{R} \times \mathbb{Z}$ into the state-space. This approach is equivalent to the so-called "hybrid time trajectories" used in [1, 9]. Additionally, 2-dimensional time structures linearly-ordered by the lexicographic order have been also used in earlier work on hybrid trajectories within the context of logics and formal methods for hybrid systems in [10, 11, 12, 13, 14], and in behavioural systems approaches to hybrid systems, in [15] and [16].

In this paper, we first develop a quite general notion of a time structure as a partially-ordered abelian group equipped with an order-preserving norm (the latter giving a magnitude measure for the duration of a time position), and then investigate the norm and order topologies on time structures. Time structures so formulated include 2-dimensional hybrid time $\mathbb{R} \times \mathbb{Z}$ as well as higher-dimensional structures for modelling the time domains of hierarchically nested hybrid systems, and also doubly-dense time structures such as $\mathbb{R} \times \mathbb{Q}_{\mathbb{B}}$, where $\mathbb{Q}_{\mathbb{B}} := \{\frac{k}{2^m} \mid k \in \mathbb{Z} \wedge m \in \mathbb{N}\}$ is the binary-codable rationals, which can be used (as suggested in [5]) to model continuations of Zeno trajectories beyond their Zeno time.

We then develop a topology on spaces of hybrid paths (of both finite and infinite length), where the topology derives from a 3-parameter uniformity giving distinct quantitative measures of closeness. We show the topology is metrizable for spaces of hybrid paths all of whose time domains are closed in the norm topology, and that the topology is coarser than the Skorokhod topologies considered by Broucke [17, 18] and Kossentini and Caspi [19] for a more restricted class of hybrid trajectories of infinite real-time duration with a positive lower bound between discrete transitions. We also prove that the path operations of prefix, suffix and fusion all respect the uniform topology. Recent work by Goebel, Teel and co-workers [2, 3, 4, 8], and also by Collins [5, 7], takes an indirect route in developing topological structure on hybrid path spaces by studying the convergence of a sequence of paths in terms of the set-convergence of the graphs of those paths as subsets of $\mathbb{R}^{n+2}$, where the notion of set-convergence is as in texts such as [20]. We prove that for abstract time structures, and under modest assumptions on the state space and time structure, the 3-parameter uniformity gives topological structure equivalent to that of graph-convergence, and also to the modified compact-open topology considered in [5, 7]

The paper is organized as follows. Section 2 develops the basics of time structures and their topologies, while Section 3 concerns compact paths over time structures, their maximal extensions, and general flows. In Section 4, we introduce set-valued retimings and the 2- and 3-parameter uniform topologies, and prove that they are respected by the path operations of prefix, suffix and fusion. In Section 5, we establish the equivalence between the set-convergence of a sequence of graphs of trajectories, and the convergence of the sequence of trajectories in the uniform topology. We also show how the results of this paper can clarify the notion of an *abstract hybrid system* as used in [8].

Some preliminaries: we write $R\colon X \rightsquigarrow Y$ to mean R is a set-valued map, with values $R(x) \subseteq Y$, and inverse $R^{-1}\colon Y \rightsquigarrow X$ given by $R^{-1}(y) := \{x \in X \mid y \in R(x)\}$. We also sometimes write $(x, y) \in R$ as synonymous with $y \in R(x)$, and we do not distinguish notationally between a map R and its graph as a subset of $X \times Y$. The *domain* of a map R is $\mathrm{dom}(R) := \{x \in X \mid R(x) \neq \emptyset\}$, and the *range* is $\mathrm{ran}(R) := \mathrm{dom}(R^{-1})$. A map R is *total* on X if $\mathrm{dom}(R) = X$, and *surjective* onto Y if $\mathrm{ran}(R) = Y$. We write $R\colon X \to Y$ (as is usual) to mean R is a single-valued *function* that is *total* on X, with range contained in Y, and values written $R(x) = y$ (rather than $\{y\}$). Let $[\,X \to Y\,]$ denote the set of all total functions from X to Y. We also distinguish *partial functions*, and write $R\colon X \dashrightarrow Y$ to mean that R is a single-valued function on its domain $\mathrm{dom}(R) \subseteq X$; let $[\,X \dashrightarrow Y\,]$ denote the set of all such maps. For partial functions, we also write $R(x) = y$ if $x \in \mathrm{dom}(R)$, and $R(x) = \textsc{undef}$ if $x \notin \mathrm{dom}(R)$. The empty map ϵ is characterized by $\mathrm{dom}(\epsilon) = \emptyset$, and it is the minimal element of $[\,X \dashrightarrow Y\,]$ and $[\,X \rightsquigarrow Y\,]$ partially-ordered by inclusion $\subseteq$. In general, we have $[\,X \to Y\,] \subseteq [\,X \dashrightarrow Y\,] \subseteq [\,X \rightsquigarrow Y\,]$. For functions as well as relations, we write the *sequential composition* $(R_1 \circ R_2)\colon X \rightsquigarrow Z$ in left-to-right sequence order, for $R_1 : X \rightsquigarrow Y$ and $R_2 : Y \rightsquigarrow Z$; where $(R_1 \circ R_2) := \{(x, z) \in X \times Z \mid \exists y \in Y : (x, y) \in R_1 \wedge (y, z) \in R_2\}$. As usual, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the integers, rationals and reals, respectively. We write $\mathbb{N}$ and $\mathbb{N}^{>0}$ for the natural numbers and strictly positive integers, and $\mathbb{R}^{+}$ and $\mathbb{R}^{>0}$ for the non-negative and strictly positive reals, respectively.

2 Time Structures and Their Topologies

A structure $(S, \leqslant, 0, +, -)$ is an *partially-ordered abelian group* [21] if $(S, \leqslant)$ is a partial order, $(S, 0, +, -)$ is an abelian group, and the strict ordering $<$ is *translation-invariant*:

$$\forall s, t, r \in S, \quad \text{if} \quad s < t \quad \text{then} \quad s + r < t + r\,, \tag{1}$$

where, as is usual, $s < s'$ iff $s \leqslant s'$ and $s \neq s'$. Integer multiplication is defined as iterated addition: for all $s \in S$ and $m \in \mathbb{N}$, $0\,s := 0$, $(m+1)\,s := (m\,s) + s$ and $-m\,s := -(m\,s)$. A strictly positive element $u > 0$ is called an *order-unit* for the partially-ordered group S if for every $s \in S$, there exists an $m \in \mathbb{N}^{>0}$ (depending on s) such that $s \leqslant m\,u$. An order unit u uniquely determines an extended *pseudo-norm* $\|\cdot\|\colon S \to \mathbb{R}^{+\infty}$, as follows:

$$\forall s \in S, \quad \|s\| \;:=\; \inf\left\{ \frac{m}{n} \in \mathbb{Q}^+ \;\mid\; m,n \in \mathbb{N}^{>0} \;\wedge\; -m\,u \;\leqslant\; n\,s \;\leqslant\; m\,u \right\}. \tag{2}$$

Thus $\|u\| = 1$, and for all $s,t \in S$, if $-t \leqslant s \leqslant t$ then $\|s\| \leq \|t\|$. The pseudo-norm satisfies the *difference inequality*: $|\,\|s\| - \|t\|\,| \leq \|\,s - t\,\|$, as well as the *triangle inequality*: $\|s+t\| \leq \|s\| + \|t\|$, and *homogeneity*: $\|m\,s\| = |m|\,\|s\|$ for $m \in \mathbb{Z}$. The pseudo-norm $\|\cdot\|$ is in fact a finite-valued norm if S is *archimedean*, in the sense that if $ks \leqslant t$ for all $k \in \mathbb{N}$, then $s \leqslant 0$ (which is violated if there is a point $t = +\infty$, for any $s > 0$)[1]. For each $s \in S$ and each real $\delta > 0$, the *δ-norm-ball* around s is $B(s,\delta) := \{t \in S \mid \|\,s - t\,\| < \delta\}$. The *positive cone* S^+ of a partially-ordered abelian group is given by $S^+ := \{s \in S \mid 0 \leqslant s\}$, and the set of *strictly positive* elements $S^{>0}$ is given by $S^{>0} := \{s \in S \mid 0 < s\}$.

Given two partially-ordered abelian groups $(S_1, \leqslant_1, 0_1, +_1, -_1)$ and $(S_2, \leqslant_2, 0_2, +_2, -_2)$, a function $g\colon S_1 \to S_2$ is an *order-group-homomorphism* if g is order-preserving, $g(0_1) = 0_2$, $g(s +_1 s') = g(s) +_2 g(s')$, and $g(-_1 s) = -_2 g(s)$ for all $s, s' \in S_1$. Note that if $g\colon S_1 \to S_2$ and $g'\colon S_1 \to S_2$ are both order-group-homomorphisms, then so is their sum, $(g + g')\colon S_1 \to S_2$ given by $(g+g')(s) := g(s) +_2 g'(s)$. If g is an order-group-homomorphism, then for all $s \in S_1$ and $k \in \mathbb{Z}$, we have $g(k\,s) = k\,g(s)$. A *positive order-group-homomorphism* $g\colon S_1 \to S_2$ is one such that if $s \in S_1^+$ then $g(s) \in S_2^+$, for all $s \in S_1$. Given two non-zero ordered groups with distinguished order-units, (S_1, u_1) and (S_2, u_2), a function $g\colon S_1 \to S_2$ is an *normalized positive order-group-homomorphism* (npog-homomorphism) if g is a positive order-group-homomorphism such that $g(u_1) = u_2$. In virtue of preserving both algebraic and order structure, npog-homomorphisms also preserve the norms: given two time structures S_1 and S_2 with norms $\|\cdot\|_i\colon S_i \to \mathbb{R}^+$ for $i = 1,2$, if $g\colon S_1 \to S_2$ is a npog-homomorphism, then $\|g(s)\|_2 \leq \|s\|_1$ for all $s \in S_1$, and if $g\colon S_1 \to S_2$ is a npog-homomorphism, then $\|s\|_1 = \|g(s)\|_2$ for all $s \in S_1$.

Given two partially-ordered abelian groups $(S_1, \leqslant_1, 0_1, +_1, -_1)$ and $(S_2, \leqslant_2, 0_2, +_2, -_2)$, their *direct product* is the partially-ordered abelian group $(S, \leqslant, 0, +, -)$ such that $S := S_1 \times S_2$ and the ordering and group operations are co-ordinate-wise: for all $s_1, t_1 \in S_1$ and $s_2, t_2 \in S_2$, $(s_1, s_2) \leqslant (t_1, t_2)$ iff $s_1 \leqslant_1 t_1$ and $s_2 \leqslant_2 t_2$; the group identity $0 := (0_1, 0_2)$; $(s_1, s_2) + (t_1, t_2) := (s_1 +_1 t_1, s_2 +_2 t_2)$; and $-(s_1, s_2) := (-_1 s_1, -_2 s_2)$. If (S_1, u_1) and (S_2, u_2) are ordered groups with distinguished order-units $u_1 \in S_1^{>0}$ and $u_2 \in S_2^{>0}$, and order-unit norms $\|\cdot\|_1$ and $\|\cdot\|_2$, then $(u_1, u_2) \in S^{>0}$ is an order-unit for $S = S_1 \times S_2$, and $\|(s_1, s_2)\| \geq \max\{\|s_1\|_1, \|s_2\|_2\}$ for all $(s_1, s_2) \in S_1^+ \times S_2^+$.

Definition 2.1 [Time structures]
A time structure $(S, \leqslant, 0, +, -, u, \|\cdot\|)$ *is a non-zero archimedian partially-ordered abelian group with*

[1]Adding points at infinity to "compactify" a partially-ordered or linearly-ordered group will typically result in only the weak or non-strict version of translation-invariance being satisfied, so the structure so formed will fail to be an ordered group.

a distinguished order-unit $u > 0$ and order-unit norm $\|\cdot\|$ determined by u, and with $S \neq \{0\}$. A future time structure *T is the positive cone of a time structure, so $T = S^+$ for some S.*
Given two time structures S_1 and S_2, a time-homomorphism *from S_1 to S_2 is a function $g\colon S_1 \to S_2$ that is both an npog-homomorphism and a continuous function w.r.t. the norm topologies on S_1 and S_2, and $g\colon S_1 \to S_2$ is a* time-isomorphism *if it is a bijective time-homomorphism (and thus a homeomorphism w.r.t. the norm topologies). A time structure S will be called* finite dimensional *if for some integer $n \geq 1$, S is time-isomorphic with an ordered sub-group of* $(\mathbb{R}^n, \mathbf{1}^n)$ *with order-unit the n-vector $\mathbf{1}^n = (1, 1, \ldots, 1)$ (hence S is lattice-ordered); the* dimension *of S,* $\dim(S)$*, is the least such integer n.*

The continuous time structure $\mathbb{R}$ and the discrete time structure $\mathbb{Z}$ are both linearly-ordered abelian groups, and both are archimedean and Dedekind-complete. While any strictly positive element can function as an order-unit, we take 1 in this role as this gives the usual absolute-value norm $\|s\| = |s| = \max(s, -s)$.

The basic 2-dimensional hybrid time structure $\mathbb{Z} \times \mathbb{R}$ is an abelian group with pair-wise addition, $(i, t) + (i', t') = (i + i', t + t')$, with group identity $\mathbf{0} := (0, 0)$, partially-ordered by the pair-wise product order, $(i, t) \leqslant (i', t')$ iff $i \leq i'$ and $t \leq t'$; it is also Dedekind-complete and archimedean. The basic hybrid future time structure $\mathbb{H} := \mathbb{N} \times \mathbb{R}^+$ is the positive cone (and positive quadrant) of $\mathbb{Z} \times \mathbb{R}$. For the order-unit, we can take $u = (1, 1)$, since for every $(i, t) \in \mathbb{H}$, we have $\mathbf{0} \leqslant (i, t) \leqslant (i, r) \leqslant k\,(1, 1)$ when we take $r = \max\{i, t\}$ and $k = \lceil r \rceil$, the least integer greater than or equal to r, and we also have $-k\,(1, 1) \leqslant (-i, -t) \leqslant \mathbf{0}$, and $-k\,(1, 1) \leqslant (-i, t) \leqslant k\,(1, 1)$, and $-k\,(1, 1) \leqslant (i, -t) \leqslant k\,(1, 1)$. The order-unit norm on $\mathbb{Z} \times \mathbb{R}$ then evaluates as $\|\,(i, t)\,\| = \max\{\,|i|, |t|\,\}$. An equivalent norm (giving the same topology) is $|||\,(i, t)\,||| := \frac{1}{2}(\,|i| + |t|\,)$, with $|||\,(1, 1)\,||| = 1$, and which satisfies $\frac{1}{2}\|\,(i, t)\,\| \leq |||\,(i, t)\,||| \leq \|\,(i, t)\,\|$; the second norm is implicitly used in [3, 4, 8] where the quantity $|i| + |t|$ functions as the real magnitude of a hybrid time position $(i, t) \in \mathbb{Z} \times \mathbb{R}$. Being a product of linear-orderings, the hybrid time structure $\mathbb{Z} \times \mathbb{R}$ is also *lattice-ordered*: for all (i, t), $(j, r) \in \mathbb{Z} \times \mathbb{R}$, $(i, t) \vee (j, r) = (\max\{i, j\}, \max\{t, r\})$ and $(i, t) \wedge (j, r) = (\min\{i, j\}, \min\{t, r\})$.

For the modeling and analysis of *discrete-time* hybrid systems, where the dynamics in each discrete mode are typically given by a discrete-time LTI or affine dynamical system, and with event-driven (or time-driven) switching between modes [22], the appropriate future time set is $\mathbb{N} \times \mathbb{N}$, the non-negative quarter of the linearly ordered abelian group $\mathbb{Z} \times \mathbb{Z}$, with the norm $\|(i, n)\| = \max\{i, n\}$ and order-unit $(1, 1)$.

Within real-time hybrid systems, for the modelling of *Zeno trajectories* and their continuation past the Zeno time, one can adapt Collins' proposal in [5] to use the linearly-ordered abelian group of binary-codable rationals:

$$\mathbb{Q}_{\mathbb{B}} := \{\, \frac{k}{2^m} \in \mathbb{Q} \mid k \in \mathbb{Z} \wedge n \in \mathbb{N} \,\}$$

which has $0 = \frac{0}{2^0}$ as the additive identity, $1 = \frac{1}{2^0}$ as the order unit with norm the real magnitude, and when $m = \max(m_1, m_2)$, we have:

$$\frac{k_1}{2^{m_1}} + \frac{k_2}{2^{m_2}} = \frac{2^{m-m_2}k_1 + 2^{m-m_1}k_2}{2^m} \quad \text{and} \quad \frac{k_1}{2^{m_1}} < \frac{k_2}{2^{m_2}} \Leftrightarrow 2^{m-m_2}k_1 < 2^{m-m_1}k_2 \,.$$

We can then take the two dimensional partially-ordered time structure $\mathbb{Q}_{\mathbb{B}} \times \mathbb{R}$ with future time $T = \mathbb{Q}_2^+ \times \mathbb{R}^+$, which has the same identity, order unit and norm as the hybrid time structures $\mathbb{Z} \times \mathbb{R}$ and $\mathbb{H} = \mathbb{N} \times \mathbb{R}^+$. This time structure is lattice-ordered and archimedean, but it is not Dedekind-complete[2], so extra care needs to be taken with arguments depending on the existence of supremums and infimums of bounded subsets. A Zeno trajectory with values in a state space X will then be a partial function $\eta\colon T \dashrightarrow X$ such that:

$$\mathrm{dom}(\eta) = \bigcup_{n\in\mathbb{N}} \{ \frac{2^n - 1}{2^n} \} \times [s_n, s_{n+1}]$$

where the real-valued sequence $\{s_n\}_{n\in\mathbb{N}}$ of switching or event times is such that $\lim_{n\to\infty} s_n = s_*$ with s_* finite. Any trajectory η' which is a continuation of η will continue from the Zeno time $(1, s_*)$.

Our rather general formulation of time structures allows not just for basic hybrid time, but also for time structures suitable for modeling trajectories of more complex systems such as "meta-hybrid automata", represented as a finite state machine with a (standard) hybrid automata at each discrete state/location; the future time structure needed for such a system would be $\mathbb{N} \times \mathbb{N} \times \mathbb{R}^+$, where the first discrete coordinate represents the number of "meta-steps" taken along a trajectory of the system [23]. More generally, this formulation of time structures allows for multi-dimensional time which can be used to explicitly model multiple time scales and dynamics along them, as is commonly found in hierarchical or aggregative systems.

In a time structure S, translation-invariance ensures that for each $r \in S$, the *r-translation function* $\sigma^r\colon S \to S$ is strictly order-preserving, where $\sigma^r(s) := s + r$ for all $s \in S$. Within partial orders, as within linear orders, the basic sets are the *intervals* between points, as well as *up-sets* above or after a given point, *down-sets* below or before a given point, and the *incomparability set* for a given point (which is empty if the ordering is linear); for elements $a, b \in S$, define:

non-strict/strict up-sets:	$[a\uparrow)$	$:=$	$\{ s \in S \mid a \leqslant s \}$	$(a\uparrow)$	$:=$	$\{ s \in S \mid a < s \}$
non-strict/strict down-sets:	$(\downarrow a]$	$:=$	$\{ s \in S \mid s \leqslant a \}$	$(\downarrow a)$	$:=$	$\{ s \in S \mid s < a \}$
non-strict interval:	$[a, b]$	$:=$	$\{ s \in S \mid a \leqslant s \leqslant b \}$		$=$	$[a\uparrow) \cap (\downarrow b]$
strict interval:	(a, b)	$:=$	$\{ s \in S \mid a < s < b \}$		$=$	$(a\uparrow) \cap (\downarrow b)$
order-incomparability set:	$(a\perp)$	$:=$	$\{ s \in S \mid s \nleqslant a \wedge a \nleqslant s \}$		$=$	$S - ([a\uparrow) \cup (a\downarrow])$.

For the semi-strict intervals, $[a, b) := [a\uparrow) \cap (\downarrow b)$ and $(a, b] := (a\uparrow) \cap (\downarrow b]$. Note that, in general, intervals, up-sets and down-sets are only partially-ordered, and not linearly ordered.

In a time structure S with order-unit u, the *unit interval* is $[0, u]$, and the *granularity* of the norm

[2] Take the irrational number π, and consider the set $A = \{t \in \mathbb{Q}_{\mathbb{B}} \mid t < \pi\}$ which is upper-bounded in $\mathbb{Q}_2$ by $3\frac{3}{16} = \frac{51}{2^4}$, but there is no supremum within $\mathbb{Q}_2$.

$\|\cdot\|$ is defined by $\mathrm{gr}(S,u) := \inf\{\, \|s\| \in \mathbb{R}^+ \mid s \in (0,u] \,\}$. Clearly, the densely-ordered time structures $\mathbb{R}$, $\mathbb{Z}\times\mathbb{R}$, and $\mathbb{Q}_{\mathbb{B}}\times\mathbb{R}$ all have granularity 0, while discretely-ordered time $\mathbb{Z}$ has granularity 1. For a fixed positive integer $N \geq 1$, the discretely ordered group $\mathbb{Z}\cdot(\frac{1}{N}) := \{\frac{k}{N} \mid k \in \mathbb{Z}\}$, with order-unit 1, has unit interval $(0,1] = \{\frac{k}{N} \mid 1 \leq k \leq N\}$ and has granularity $\frac{1}{N}$.

Given a time structure S, let $\mathcal{T}_{\leqslant}$ be the order topology on S which has as a basis the collection $\mathcal{B}_{\leqslant}$ of all strict up-sets and down-sets, and their intersections, the strict open intervals:

$$\mathcal{B}_{\leqslant} := \{\, (s\!\uparrow) \mid s \in S \,\} \cup \{\, (s\!\downarrow) \mid s \in S \,\} \cup \{\, (s,t) \mid s \in S \;\wedge\; t \in S \,\}.$$

Let $\mathcal{T}_{\mathrm{norm}}$ be the norm topology on S determined by $\|\cdot\|$ which has as a basis the collection $\mathcal{B}_{\mathrm{norm}}$ of all norm-balls $B(s,\delta) := \{t \in S \mid \|t-s\| < \delta\}$, for $s \in S$ and real $\delta > 0$; $\mathcal{T}_{\mathrm{norm}}$ is also the coarsest topology on S w.r.t. which the norm $\|\cdot\| : S \to \mathbb{R}^+$ is continuous.

Theorem 2.2 [Topologies on time structures]
Let S be a time structure with future time T, and let $\|\cdot\| : S \to \mathbb{R}^+$ be the norm on S.

1. *For all $s \in S$ and for all real $\delta > 0$, there exists a strictly positive $v \in T^+$ such that for all time points $t \in (s-v, s+v)$, we have $\big|\, \|s\| - \|t\| \,\big| < \delta$; hence the norm $\|\cdot\| : S \to \mathbb{R}^+$ is continuous w.r.t. to the order topology $\mathcal{T}_{\leqslant}$ on S*

2. *The norm topology is refined by the order topology; that is: $\mathcal{T}_{\mathrm{norm}} \subseteq \mathcal{T}_{\leqslant}$.*

3. *For each $r \in S$, the translation-map $\sigma^r : S \to S$ is continuous w.r.t. both $\mathcal{T}_{\mathrm{norm}}$ and $\mathcal{T}_{\leqslant}$; being invertible, the translation maps σ^r are thus homeomorphisms in both topologies.*

4. *For all $s,t \in S$, the non-strict interval $[s,t]$, up-set $[s\!\uparrow)$, and down-set $(s\!\downarrow]$, are all closed in $\mathcal{T}_{\mathrm{norm}}$, and the order-incomparability set $(s\!\perp)$ as well as the unions $(s\!\downarrow) \cup (s\!\perp)$ and $(s\!\uparrow) \cup (s\!\perp)$ are all open in $\mathcal{T}_{\mathrm{norm}}$; if S is finite-dimensional and $s \leqslant t$, then the closed interval $[s,t]$ is also compact in $\mathcal{T}_{\mathrm{norm}}$.*

5. *The addition map $(\cdot+\cdot) : S\times S \to S$ and the additive inverse $(-\,\cdot) : S \to S$ are continuous w.r.t. both $\mathcal{T}_{\mathrm{norm}}$ and $\mathcal{T}_{\leqslant}$; in particular, $(S,+,-,0,\mathcal{T}_{\mathrm{norm}})$ is a topological abelian group.*

6. *If S is linearly-ordered, then $\mathcal{T}_{\mathrm{norm}} = \mathcal{T}_{\leqslant}$.*

7. *If S is finite dimensional, then for all $s \in S$ and for all real $\delta > 0$,
if $\mathrm{gr}(S,u) \geq \delta$, then $B(s,\delta) = \{s\}$, and if $\mathrm{gr}(S,u) < \delta$, then:*

$$B(s,\delta) = \bigcup \{\, [s-v, s+v] \mid v \in S^{>0} \;\wedge\; \|v\| < \delta \,\}.$$

8. *If S is finite-dimensional, then for any subset $A \subseteq S$, A is norm-bounded iff A is order-bounded.*

9. *If S is finite-dimensional and Dedekind-complete, then for any subset $A \subseteq S$, A is compact in $\mathcal{T}_{\mathrm{norm}}$ iff A is closed and bounded in $\mathcal{T}_{\mathrm{norm}}$.*

Proof of Theorem 2.2: Observe that *Part 2* is an immediate consequence of *Part 1*, because $\mathcal{T}_{\mathrm{norm}}$ is the coarsest topology on S w.r.t. which the norm $\|\cdot\| : S \to \mathbb{R}^+$ is continuous. For *Part 1*, we consider two exhaustive cases, depending on the granularity $\mathrm{gr}(S,u)$ of the norm $\|\cdot\|$ from u.
Case I: $\mathrm{gr}(S,u) = 0$. So fix $s \in S$ and a real $\delta > 0$. Then choose some $\delta' > 0$ such that $\delta' < \delta$. Since $\mathrm{gr}(S,u) = 0$, we can conclude that there exists a strictly positive element $v \in T^+$ such that $0 < \|v\| \leq \delta'$. Now consider any time point $t \in (s-v, s+v)$. Thus we have $-v < s-t < v$, and hence $\|s-t\| \leq \|v\| \leq \delta < \varepsilon$. Then by the difference inequality for norms, we have $|\,\|s\| - \|t\|\,| \leq \|s-t\| < \varepsilon$, as required.
Case II: $\mathrm{gr}(S,u) > 0$. Now fix $s \in S$ and a real $\delta > 0$. Then we break into further sub-cases, depending on the order relationship between $\mathrm{gr}(S,u)$ and δ.
Sub-case IIa: $0 < \delta \leq \mathrm{gr}(S,u)$. Since $\mathrm{gr}(S,u)$ is the infimum of the values $\|t\|$ for $t \in (0,u]$, and $\mathrm{gr}(S,u) > 0$, this means there exists at least one strictly positive group element $v > 0$ such that $\|v\| \geq c$ and $(s-v, s+v) = \{s\}$. In this case, the only $t \in (s-v, s+v)$ is $t = s$, so we trivially have $|\,\|s\| - \|t\|\,| < \delta$, as required.
Sub-case IIb: $\mathrm{gr}(S,u) < \delta$. In this case, choose a real δ' such that $\mathrm{gr}(S,u) \leq \delta' < \delta$ and there exists a strictly positive group element $v > 0$ such that $\|v\| = \delta'$; since c is the infimum of the values $\|t\|$ for $t \in (0,u]$, there exists at least one such pair δ' and v. Then consider any point $t \in (s-v, s+v)$. So we have $-v < (s-t) < v$, and hence $\|s-t\| \leq \|v\| = \delta' < \delta$. Then by the difference inequality for norms, we have $|\,\|s\| - \|t\|\,| \leq \|s-t\| < \delta$, as required.

For *Part 3*, fix $r \in S$ and consider the translation map $\sigma^r : S \to S$. Now translation maps are strictly order-preserving (by the translation-invariance property of partially-ordered abelian groups), hence they are continuous w.r.t. the order topology, since taking pre-images, we have $(\sigma^r)^{-1}(\,(s,t)\,) = (s-r, t-r)$, and also $(\sigma^r)^{-1}(\,[s,t]\,) = [s-r, t-r]$. To show that the translation maps are continuous w.r.t. the order topology, consider a basic open norm ball $B(s,\delta)$. Then its pre-image evaluates as the translated norm-ball, as follows:

$$\begin{aligned}
(\sigma^r)^{-1}(\,B(s,\delta)\,) &= \{\, t \in S \mid t - r \in B(s,\delta) \,\} \\
&= \{\, t \in S \mid \|s - (t-r)\| < \delta \,\} \\
&= \{\, t \in S \mid \|(s+r) - t\| < \delta \,\} \\
&= B(s+r, \delta)\,.
\end{aligned}$$

For *Part 4*, because S is archimedean, the verification that each of the order sets $[s,t]$, $[s\uparrow)$, and $(s\downarrow]$, are closed in the norm topology, is immediate from the following result from [21], Lemma 7.17 [CHECK]:
If S is archimedean, and $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences converging in the norm to elements s and t, respectively, and for all $n \in \mathbb{N}$, we have $s_n \leqslant t_n$, then in the limit, $s \leqslant t$. Since $(s\bot) = S - (\,[s\uparrow) \cup (s\downarrow]\,)$,

it follows that $(s\perp)$ is open in the norm topology.

For *Part 5*, continuity w.r.t. to the order topology is immediate, and for the norm topology and addition, it suffices to show that for all $s, s' \in S$ and for all real $\varepsilon > 0$, there exists $\delta, \delta' > 0$ such that for all $t, t' \in S$, if $\| s - t \| < \delta$ and $\| s' - t' \| < \delta'$, then $\| (s + s') - (t + t') \| < \varepsilon$. Taking $\delta = \delta' = \frac{1}{2}\varepsilon$ and using the triangle inequality, we have $\| (s + s') - (t + t') \| \leq \| (s - t) \| + \| (s' - t') \| < \varepsilon$, as required. For the norm topology and the additive inverse, it suffices to show that for all $s \in S$ and for all real $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \in S$, if $\| s - t \| < \delta$ then $\| (-s) - (-t) \| < \varepsilon$. Taking $\delta = \varepsilon$ and using the homogeneity property for norms, we have $\| (-s) - (-t) \| = \| t - s \| = \| s - t \| < \varepsilon$, as required.

For *Part 6*, suppose S is linearly-ordered. Then by *Part 4*, we know all order sets of the form $[s, t]$, $[s\uparrow)$, or $(s\downarrow]$ are closed in $\mathcal{T}_{\text{norm}}$. Now since S is linearly-ordered, we have the incompatability set $(s\perp) = \varnothing$ for all $s \in S$. Hence for any $s < t$, we have $(s, t) = S - ((s\downarrow] \cup [t\uparrow))$, and hence the strict interval (s, t) is open in $\mathcal{T}_{\text{norm}}$. Since the strict intervals also form a basis for the order topology $\mathcal{T}_{\leqslant}$, and we already have $\mathcal{T}_{\text{norm}} \subseteq \mathcal{T}_{\leqslant}$, we can conclude that $\mathcal{T}_{\text{norm}} = \mathcal{T}_{\leqslant}$.

For *Parts 7, 8 and 9*, we suppose that $\dim(S) = n$, so S is time-isomorphic with an ordered sub-group of $(\mathbb{R}^n, \mathbf{1}^n)$. Then S is lattice-ordered and for any element $s \in S$, we have a coordinate representation $s = (s_1, \ldots, s_n)$ via the time-isomorphism between S and an ordered sub-group of $(\mathbb{R}^n, \mathbf{1}^n)$, and the order-unit norm evaluates as $\|s\| = \max\{ |s_i| \mid 1 \leq i \leq n \}$. Now fix $s \in S$ and a real $\delta \in \mathbb{R}^{>0}$, and suppose that $\mathrm{gr}(S, u) < \delta$. Then for any $v \in S^{++}$ such that $\|v\| < \delta$, we will have $[s - v, s + v] \subseteq B(s, \delta)$ and $\mathrm{gr}(S, u) \leq \|v\| < \delta$, hence the right-to-left inclusion must hold. Now for the converse, left-to-right inclusion, suppose $t \in B(s, \delta)$, and thus $\|t - s\| < \delta$. If $t = s$, then choose any $v \in S^{++}$ such that $\mathrm{gr}(S, u) \leq \|v\| < \delta$ (and there are some, since $\mathrm{gr}(S, u) < \delta$); then we have $t \in [s - v, s + v]$, as required. Otherwise, $t \neq s$, and in this case, we break into sub-cases, depending on the order relationship between t and s.

Case I: $s < t$. Then $0 < t - s$, so pick $v = t - s$, so $v > 0$ and $\|v\| = \|t - s\| < \delta$ and $t \in [s - v, s + v]$.

Case II: $t < s$. Then $0 < s - t$, so pick $v = s - t$, so $v > 0$ and $\|v\| = \|s - t\| < \delta$ and $t \in [s - v, s + v]$.

Case III: $t \in (s\perp) \cap B(s, \delta)$, say $t = (t_1, \ldots, t_n)$ and $s = (s_1, \ldots, s_n)$, since S is time-isomorphic with a sub-group of $(\mathbb{R}^n, \mathbf{1}^n)$. So there must exist $i, j, k \in \{1, \ldots, n\}$ with $i \neq j$ such that $s_i \leq t_i$ and $s_j \geq t_j$ and $s_k \neq t_k$. In this case, pick $v = (|t_1 - s_1|, \ldots, |t_n - s_n|)$, so that $v > 0$ and $\|v\| = \max\{ |t_m - s_m| \mid 1 \leq m \leq n \} = \|t - s\| < \delta$. Now for each $m \in \{1, \ldots, n\}$, if $s_m \leq t_m$ (e.g. $m = i$), then $s_m - v_m = t_m - 2v_m$ and $s_m + v_m = t_m$, while if $s_m \geq t_m$ (e.g. $m = j$), then $s_m - v_m = t_m$ and $s_m + v_m = t_m + 2v_m$. Hence we can conclude that $t \in [s - v, s + v]$, as required.

For the remaining assertion within *Part 6*, suppose $\mathrm{gr}(S, u) \geq \delta$. Now suppose, for a contradiction, that there exists $t \in B(s, \delta)$ with $t \neq s$. Then set $v = (|t_1 - s_1|, \ldots, |t_n - s_n|)$, so that $v > 0$ and $\|v\| = \|t - s\| < \delta$. But since $\mathrm{gr}(S, u)$ is the infimum of all norms $\|w\|$ for $w \in (0, u]$, the conclusion that $\|v\| < \delta$ while $\mathrm{gr}(S, u) \geq \delta$ gives a contradiction, as required.

For *Part 8*, again suppose that $\dim(S) = n$, as above. Now fix any subset $A \subseteq S$. First suppose that A is norm-bounded. So $A \subseteq B(0, r)$ for some real $r > 0$; moreover, we can choose $r > \mathrm{gr}(S, u)$. Now for each $i \in \{1, \ldots, n\}$, choose $t_i \in \mathbb{R}^{>0}$ such that $t_i > r$ and the n-tuple $t = (t_1, \ldots, t_n) \in S^{>0}$. Then

$\| t \| > r$ and, applying *Part 7*, we can conclude that $A \subseteq B(0, r) \subset [-t, t]$, and hence $-t < s < t$ for all $s \in A$, so A is order-bounded. Conversely, suppose that A is order-bounded. So for some strictly positive $t \in S^{>0}$, we have $-t \leqslant s \leqslant t$ for all $s \in A$. Hence $\| s \| \leq \| t \|$ for all $s \in A$, so A is norm-bounded.

Finally, for *Part 9*, again suppose that $\dim(S) = n$, as above and also suppose that S is Dedekind-complete. Now for each $i \in \{1, \ldots, n\}$, the projection $S_i = \pi_i(S) := \{ s_i \mid s = (s_1, \ldots, s_i, \ldots, s_n) \in S \}$ is a Dedekind-complete linearly-ordered subgroup of $\mathbb{R}$. Hence by the BolzanoWeierstrass theorem, we can conclude that a set $A \subseteq S$ is compact in $\mathcal{T}_{\text{norm}}$ iff A is sequentially-compact in $\mathcal{T}_{\text{norm}}$ iff A is closed and bounded in $\mathcal{T}_{\text{norm}}$. ■

For a counter-example to the converse of *Part 2*, consider the hybrid time structure $S = \mathbb{Z} \times \mathbb{R}$, for an arbitrary hybrid point $s = (i, t) \in \mathbb{Z} \times \mathbb{R}$, and for a strictly positive element $v = (k, r) \in \mathbb{H}^{>0}$, the strict symmetric interval $(s - v, s + v)$ around s evaluates as a union of linearly-ordered intervals, most including end-points, and only the first and last without some endpoints; if $k \leq 1$ then: $(s - v, s + v) = \{i\} \times (t - r, t + r)$, while if $k > 1$ then:

$$(s - v, s + v) = \{i - k\} \times (t - r, t + r] \cup \left(\bigcup_{j=i-k+1}^{i+k-1} \{j\} \times [t - r, t + r] \right) \cup \{i + k\} \times [t - r, t + r) .$$

In what follows, we will be particularly interested in subsets of time structures that are linearly-ordered and are closed or compact in $\mathcal{T}_{\text{norm}}$ (and hence also closed in $\mathcal{T}_{\leqslant}$). For the hybrid time structure $S = \mathbb{Z}\times\mathbb{R}$, it is clear that linearly-ordered intervals of the form $\{i\} \times [s, s']$ or $\{i\} \times [s, \infty)$ are closed in $\mathcal{T}_{\text{norm}}$, and those of the form $\{i\} \times [s, s']$ are also compact in $\mathcal{T}_{\text{norm}}$. Moreover, countable unions of such intervals, of the form:

$$L = \bigcup_{i \in \mathbb{N}} \{i\} \times [s_i, s_{i+1}] \tag{3}$$

where $s_i \leq s_{i+1}$ and as $i \to \infty$, either $s_i \to \infty$ or $s_i \to s < \infty$, are also closed sets in $\mathcal{T}_{\text{norm}}$; such sets will arise as the time domains of maximally-extended hybrid trajectories with infinitely many discrete jumps.

3 Compact Paths and Their Maximal Extensions

For the hybrid future time structure $T = \mathbb{H}$, the entities we will call *regular hybrid paths* are functions typically taking values in a space $X \subseteq Q \times \mathbb{R}^n$, with Q a finite set, and their time domains are finite disjoint unions of linearly-ordered and norm-compact intervals:

$$L = \bigcup_{i<N} \{i\} \times [s_i, s_i + \Delta_i] = \bigcup_{i<N} [\,(i, s_i),\ (i, s_{i+1})\,] \tag{4}$$

where $s_0 := 0$ and $s_{i+1} := s_i + \Delta_i$ and $(\Delta_0, \Delta_1, \ldots, \Delta_{N-1})$ is a finite sequence (of length N) of *interval durations* $\Delta_i \in \mathbb{R}^+$ for $i < N$, and $s_{i+1} \in \mathbb{R}^+$ for $i < N - 1$ are the real-valued *discrete transition times* along L. Along a regular hybrid path, for each $i < N - 1$, the time position $(i + 1, s_{i+1})$ is the immediate *discrete successor* of the transition time position (i, s_{i+1}) within the domain D. Applying *Part 4* of Theorem 2.2, we know that set $L \subset \mathbb{H}$ of the form (4) are *compact* in the norm topology $\mathcal{T}_{\text{norm}}$ on $\mathbb{H}$, since they are norm-closed subsets of the norm-compact set $[0, s_N]$; we shall subsequently refer to them as *regular compact hybrid time domains*.

The framework here, developed from [13, 14], allows not only for regular hybrid paths in the sense above, but also for hybrid paths γ that explicitly model some bounded real time delay (and temporal gap), with $s_{i+1} - t_i \leq \delta$, between a switching position (i, t_i), when a sensor detects that a discrete reset is enabled, and its discrete successor position $(i + 1, s_{i+1})$, when an actuator effects the required switch in dynamics and continuous evolution begins again from the new state $\gamma(i+1, s_{i+1}) \in G(\gamma(i, t_i))$, where $G\colon X \rightsquigarrow X$ is the discrete reset map. For hybrid times (j, s) with $j \in \{i, i + 1\}$ and $t - i < s < s_{i+1}$, we can treat such a hybrid path γ as being undefined, so in this case $\mathrm{dom}(\gamma)$ will be of the form $\bigcup_{i<N} \{i\} \times [s_i, t_i]$, where $s_i \leq t_i \leq s_{i+1}$ and $s_{i+1} - t_i \leq \delta$.

A further advantage of the general way we formulate paths is that it allows us to deal with *samplings* of hybrid paths in the same framework as the original paths. For example, take a regular compact hybrid time domain L of the form (4) and a fixed real-time sampling period $\Delta > 0$. Then a "time-driven" Δ-sampling of L will be a finite set of hybrid time points, of the form:

$$L' \;=\; \bigcup_{i<N} \{i\} \times \{\, (m_i + 1)\Delta,\; (m_i + 2)\Delta,\; \ldots\; m_{i+1}\Delta \,\}$$

where $m_0 := 0$ and for each $i \in \{1, \ldots, N\}$, $m_i := \lfloor \frac{s_i}{\Delta} \rfloor$ using the integer-floor function $\lfloor \cdot \rfloor$. In contrast, "time+event-driven" Δ-sampling of L will be a finite set of hybrid time points, of the form:

$$L'' \;=\; L' \;\cup\; \{\, (i, s_{i+1}) \mid i < N \,\}$$

where the hybrid time points (i, s_{i+1}) are the switching times when a discrete event is detected (and the sampling device knows to increment the discrete time counter i).

In earlier work [13, 14], we developed a notion of paths $\gamma : T \dashrightarrow X$ and their maximal extensions without assuming any structure on the value space or signal space X, as the focus in that work was on developing a sematics for branching or non-deterministic temporal logic which can express complex dynamic properties, but which is not equipped to express any topological or other space-structural concepts. Here, since we are examining topological structure on path spaces, we assume the minimal structure of a metric space (X, d_X), and further restrict our attention to paths that, as partial functions, are continuous on their domain with respect to the norm topology on T and the metric topology on X.

Definition 3.1 [Compact time domains]
Given a time structure S with future time T, a compact time domain *in T is a linearly-ordered subset*

$L \subseteq T$ with minimum element 0 *and a maximum element* $b_L := \max(L)$ *such that L is compact in the norm topology $\mathcal{T}_{\text{norm}}$ on T. Let* $\mathsf{CoTD}(T)$ *denote the set of all compact time domains in T.*

The partial-ordering $\leqslant$ on the future time structure T induces a partial-ordering on $\mathsf{CoTD}(T)$*: for $L, L' \in$* $\mathsf{CoTD}(T)$*, we say L' is a (proper)* ordered extension *of L, and (re-using notation) we write $L < L'$, if $L \subset L'$ and $t < t'$ for all $t \in L$ and all $t' \in L' - L$; (as usual) $L \leqslant L'$ iff $L < L'$ or $L = L'$.*

Applying *Part 9* of Theorem 2.2, if S is finite dimensional and Dedekind-complete, then a linearly-ordered subset $L \subseteq T$ will be in $\mathsf{CoTD}(T)$ iff it contains 0 and a maximum element b_L and is closed $\mathcal{T}_{\text{norm}}$. The Zeno continuation time structure $\mathbb{Q}_{\mathbb{B}} \times \mathbb{R}$ fails to be Dedekind-complete, but if L is of the form:

$$L \;=\; \bigcup_{n\in\mathbb{N}} \{\frac{2^n-1}{2^n}\} \times [s_n, s_{n+1}]$$

where the real-valued sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} s_n = s_* < \infty$, then the set $L \cup \{(1, s_*)\}$ will be in $\mathsf{CoTD}(\mathbb{Q}_{\mathbb{B}} \times \mathbb{R})$, while the set L is bounded but neither closed nor compact.

From the standard definitions, a partial function $\eta : T \dashrightarrow X$ is continuous on its domain iff for all $t \in \operatorname{dom}(\eta)$ and for every real $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $s \in \operatorname{dom}(\eta)$, if $\|t - s\|_T < \delta$ then $d_X(\eta(t), \eta(s)) < \varepsilon$. In contrast, a partial function $\eta : T \dashrightarrow X$ is *uniformly continuous* on its domain iff there exists a total function $u : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ such that for every $\varepsilon \in \mathbb{R}^{>0}$, and for all $t, s \in \operatorname{dom}(\eta)$, if $\|t - s\|_T < u(\varepsilon)$ then $d_X(\eta(t), \eta(s)) < \varepsilon$. When $\operatorname{dom}(\eta)$ is compact in T, a partial function $\eta : T \dashrightarrow X$ is uniformly continuous iff it is continuous. Note that if T has the discrete topology, such as $T = \mathbb{N}$ or $T = \mathbb{N} \times \mathbb{N}$, then all partial functions are uniformly continuous, while if $T = \mathbb{H}$ the basic hybrid time structure, and $\operatorname{dom}(\eta)$ is a disjoint union of countably many linearly-ordered norm-compact intervals of the form $\{i\}\times[s_i, s_{i+1}]$, with transition time sequence $\{(i, s_{i+1})\}_{i\in\mathbb{N}}$, then since $\|\,(i+1, s_{i+1}) - (i, s_{i+1})\,\|_{\mathbb{H}} = 1$, η will be (uniformly) continuous on its domain iff for each i, the restriction of η to the set $\{i\}\times[s_i, s_{i+1}]$ is (uniformly) continuous with time-bound witness $\delta < 1$ ($u(\varepsilon) < 1$) for each ε.

Definition 3.2 [Compact continuous paths]
Given a time structure S with future time T, let the signal value space be a non-empty metric space (X, d_X). We define the set of compact continuous T-paths in X *as:*

$$\mathsf{CPath}(T, X) \;:=\; \{\, \gamma \colon T \dashrightarrow X \mid \operatorname{dom}(\gamma) \in \mathsf{CoTD}(T) \;\wedge\; \gamma \textit{ is continuous on } \operatorname{dom}(\eta)\,\}\,.$$

For $\gamma \in$ $\mathsf{CPath}(T, X)$*, define the* length *of γ by* $\operatorname{len}(\gamma) := \|\max(\operatorname{dom}(\gamma))\|$.
Define a partial order on $\mathsf{CPath}(T, X)$ *from the underlying order on T that is a sub-ordering of the subset relation; (again re-using notation) define: $\gamma < \gamma'$ if $\gamma \subset \gamma'$ and* $\operatorname{dom}(\gamma) < \operatorname{dom}(\gamma')$*, in which case we say the path γ' is a (proper)* extension *of γ, and γ is a (proper)* prefix *of γ'. As usual, $\gamma \leqslant \gamma'$ iff $\gamma < \gamma'$ or $\gamma = \gamma'$.*

Since the domain of a continuous path in $\mathsf{CPath}(T, X)$ is compact, it follows that all $\gamma \in \mathsf{CPath}(T, X)$ are in fact uniformly continuous.

When $T = \mathbb{R}^+$ or $T = \mathbb{N}$, the paths $\gamma \in \mathsf{CPath}(T, X)$ such that $\mathrm{dom}(\gamma) = [0, b]$ are just finite length signals in the usual sense.

Proposition 3.3 [Operations on paths]
The following three operations on paths are well-defined partial functions on the set $\mathsf{CPath}(T, X)$ *of compact continuous paths: for* $\gamma \in \mathsf{CPath}(T, X)$*,* $t \in T$ *and* $b_\gamma := \max(\mathrm{dom}(\gamma))$*:*

- *the* t-prefix *(or* restriction*)* $\gamma|_t$*, with* $\mathrm{dom}(\gamma|_t) := [0, t] \cap \mathrm{dom}(\gamma)$ *and* $\gamma|_t(s) := \gamma(s)$ *for all* $s \in \mathrm{dom}(\gamma|_t)$*;*
- *the* t-suffix *(or* translation*)* ${}_t|\gamma$*, which is defined only when* $t \in \mathrm{dom}(\gamma)$*, with* $\mathrm{dom}({}_t|\gamma) := [0, b_\gamma - t] \cap \sigma^{-t}(\mathrm{dom}(\gamma))$ *where* ${}_t|\gamma(s) := \gamma(s + t)$ *for all* $s \in \mathrm{dom}({}_t|\gamma)$*; and*
- *the* t-fusion *(or* point-concatenation*)* $\gamma *_t \gamma'$*, which is defined only when* $t \in \mathrm{dom}(\gamma)$ *and* $\gamma(t) = \gamma'(0)$*, and which has* $\mathrm{dom}(\gamma *_t \gamma') = \mathrm{dom}(\gamma|_t) \cup \sigma^{+t}(\mathrm{dom}(\gamma'))$ *and* $(\gamma *_t \gamma')(s) := \gamma(s)$ *if* $s \in \mathrm{dom}(\gamma|_t)$ *and* $(\gamma *_t \gamma')(s) := \gamma'(s - t)$ *if* $s \in \sigma^{+t}(\mathrm{dom}(\gamma'))$*.*

Note that the prefix operation is well-defined for all $t \in T$, not just $t \in \mathrm{dom}(\gamma)$, and that $\gamma|_t \leqslant \gamma$ for all $t \in T$; in particular, $\gamma|_t < \gamma$ if $t \ngeqslant b_\gamma$, while $\gamma|_t = \gamma$ if $t \geqslant b_\gamma$. In contrast, for the suffix and fusion operations, if $t \notin \mathrm{dom}(\gamma)$, then ${}_t|\gamma = \epsilon$ and $\gamma *_t \gamma' = \epsilon$, where ϵ is the empty map which is not in $\mathsf{CPath}(T, X)$. Under the assumption that T is Dedekind-complete, we have:

$$\mathrm{dom}(\gamma|_t) = [0, t_1] \cap \mathrm{dom}(\gamma) \qquad \text{where} \quad t_1 = \sup\{s \in \mathrm{dom}(\gamma) \mid s \leqslant t\}.$$

To see this, observe that being norm-closed, the set $\mathrm{dom}(\gamma)$ is also closed in the order topology on T, so must contain the supremum of any upper-bounded subset. A set $P \subseteq \mathsf{CPath}(T, X)$ is called *prefix-closed* if for all $\gamma \in P$ and all $t \in T$, the path $\gamma|_t \in P$.

A basic property of sets of compact paths is that of *extendibility* under the extension partial order. For any set of paths $P \subseteq \mathsf{CPath}(T, X)$, we say that P is *deadlock-free* if for all $\gamma \in P$, there exists $\gamma' \in P$ such that $\gamma < \gamma'$. When P represents the behaviour of a dynamical system, deadlock-freedom means that further non-trivial motion is possible from every reachable state. From our earlier work on non-deterministic temporal logic with semantics over paths of a system, we have a generic model of dynamical systems which uniformly covers discrete-time state machines, continuous-time differential systems, and hybrid-time systems.

Definition 3.4 [General flow systems]
A general flow system [13, 14] *is a set-valued map* $\Phi \colon X \rightsquigarrow \mathsf{CPath}(T, X)$ *such that for all* $x \in \mathrm{dom}(\Phi)$*, for all* $\gamma \in \Phi(x)$*, and all* $t \in \mathrm{dom}(\gamma)$*:*

(GF0) $x = \gamma(0)$*;*

(GF1) ${}_t|\gamma \in \Phi(\gamma(t))$*; and*

(GF2) $(\gamma *_t \gamma') \in \Phi(x)$ *for all* $\gamma' \in \Phi(\gamma(t))$*.*

For the assymptotic analysis of dynamical systems, as well as for the semantics of temporal logics of such systems, we need to determine the *maximal extensions* of finite-length paths. Infinitary extensions or asymptotic limits of paths are formed by taking unions of strictly extending sequences of compact paths. Given a future time structure T, let $|T|^+$ denote the successor cardinal of the cardinality of the largest linearly-ordered subset of T, and the initial *ordinal* of that cardinality, and let $\mathrm{LO}(T)$ be the set of all limit ordinals $\nu < |T|^+$ with $\omega \leq \nu$, where ω is the ordinal length of $\mathbb{N}$. Given any set of compact paths $P \subseteq \mathsf{CPath}(T, X)$, and a $\nu \in \mathrm{LO}(T)$, a ν-length sequence $\{\gamma_m\}_{m<\nu}$ is called a *P-chain* if for all $m, m' < \nu$, $\gamma_m \in P$ and $m < m'$ implies $\gamma_m < \gamma_{m'}$. The *limit* of a P-chain is the partial function $\eta\colon T \dashrightarrow X$ such that $\eta = \bigcup_{m<\nu} \gamma_m$, with the *length* $\mathrm{len}(\eta) := \sup_{m<\nu} \mathrm{len}(\gamma_m)$, possibly infinite. Since a P-chain is a strictly extending sequence of continuous partial functions, it is easy to see that the union η must also be continuous everywhere on its domain[3]. To see this, fix any $t \in \mathrm{dom}(\eta)$, and within the strictly extending sequence of compact paths $\{\gamma_m\}_{m<\nu}$, choose an ordinal index $m < \nu$ such that $t \in \mathrm{dom}(\gamma_m)$ and there is a $\delta_m > 0$ such that for all $s \in \mathrm{dom}(\eta)$, if $\| s - t \|_T < \delta_m$ then $s \in \mathrm{dom}(\gamma_m)$. Then γ_m is continuous at t, so for each $\varepsilon > 0$, let $\delta \leq \delta_m$ be such that for all $s \in \mathrm{dom}(\gamma_m)$ if $\| s - t \|_T < \delta$ then $d_X(\gamma_m(s), \gamma_m(t)) < \varepsilon$. Thus we have that $s \in \mathrm{dom}(\eta)$ if $\| s - t \|_T < \delta$ then $d_X(\eta(s), \eta(t)) < \varepsilon$, as required. The extension partial order on compact paths can also be applied to limit paths: $\eta < \eta'$ iff $\eta \subset \eta'$ and $t < t'$ for all $t \in \mathrm{dom}(\eta)$ and $t' \in \mathrm{dom}(\eta') \setminus \mathrm{dom}(\eta)$. The prefix, suffix and fusion operations also extend to limit paths in the straight-forward way.

Definition 3.5 [Limit extension and maximal extension of path sets]
Let T be a future time structure. For any set $P \subseteq \mathsf{CPath}(T, X)$ of compact paths, define the limit extension $\mathsf{Ext}(P)$*, the* maximal extension $\mathsf{M}(P) \subseteq \mathsf{Ext}(P)$*, and the* maximal length-unbounded extension $\mathsf{MU}(P) \subseteq \mathsf{M}(P)$*, as follows:*

$$\begin{aligned}
\mathsf{Ext}(P) \;&:=\; \{\eta \in [T \dashrightarrow X] \mid (\exists \nu \in \mathrm{LO}(T)) \big(\exists \overline{\gamma} \in [\nu \to \mathsf{CPath}(T, X)]\big) (\forall m < \nu) \\
&\qquad \gamma_m := \overline{\gamma}(m) \in P \;\wedge\; (\forall m' < \nu)\, (m < m' \;\Rightarrow\; \gamma_m < \gamma_{m'}) \;\wedge\; \eta = \textstyle\bigcup_{m<\nu} \gamma_m \};\\
\mathsf{M}(P) \;&:=\; \{\eta \in \mathsf{Ext}(P) \mid (\forall \gamma \in P)\; \eta \not< \gamma\} \quad \textit{and} \quad \mathsf{MU}(P) \;:=\; \{\eta \in \mathsf{M}(P) \mid \mathrm{len}(\eta) = \infty\}.
\end{aligned}$$

A set of compact paths P is called maximally extendible *if for all $\gamma \in P$, there exists $\eta \in \mathsf{M}(P)$ such that $\gamma < \eta$, and P is called* forward complete *if P is maximally extendible and $\mathsf{M}(P) = \mathsf{MU}(P)$.*
Given a general flow system $\Phi\colon X \rightsquigarrow \mathsf{CPath}(T, X)$, the maximal extension *of Φ is the set-valued map* $\mathsf{M}\Phi\colon X \rightsquigarrow \mathsf{LPath}(T, X)$ *given by* $\mathsf{M}\Phi(x) := \mathsf{M}(\Phi(x))$*, and a general flow Φ is* maximally extendible *(*forward complete*) if for all $x \in \mathrm{dom}(\Phi)$, the path set $\Phi(x)$ is maximally extendible (forward complete).*

From [13, 14], a core result (requiring the Axiom of Choice) is that a set of paths $P \subseteq \mathsf{CPath}(T, X)$ (or a general flow $\Phi\colon X \rightsquigarrow \mathsf{CPath}(T, X)$) is maximally extendible iff P is deadlock-free (or for each $x \in \mathrm{dom}(\Phi)$, the path set $\Phi(x)$ is deadlock-free). Thus the infinitary and time-global property of being maximally-extendible – which is a minimal requirement for asymptotic analysis – is equivalent to the finitary and time-local property of being deadlock-free.

[3]Note, in contrast, that the union of a strictly extending sequence of uniformly-continuous partial functions can fail to be uniformly continuous; a condition sufficient to guarantee that the union will be uniformly-continous is that the chain of compact partial functions are *uniformly equicontinuous*, which means there is a single common uniform witness function $u : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ for the whole chain.

We will subsequently be interested in the set $\mathsf{CPath}^\infty(T,X) := \mathsf{CPath}(T,X) \cup \mathsf{Ext}(\mathsf{CPath}(T,X))$ of all continuous paths, of finite or infinite length, and also the subsets:

$$\begin{aligned}
\mathsf{CPath}^\infty_{\mathrm{cl}}(T,X) &:= \mathsf{CPath}(T,X) \ \cup\ \{\, \eta \in \mathsf{CPath}^\infty(T,X) \mid \mathrm{dom}(\eta) \textit{ is norm-closed in } T \,\} \\
\mathsf{CPath}^\infty_{\mathrm{bd}}(T,X) &:= \mathsf{CPath}(T,X) \ \cup\ (\mathsf{CPath}^\infty(T,X) \setminus \mathsf{CPath}^\infty_{\mathrm{cl}}(T,X))\,.
\end{aligned}$$

Thus $\mathsf{CPath}^\infty_{\mathrm{cl}}(T,X)$ is the set of all continuous paths with norm-closed time domains, while the set $\mathsf{CPath}^\infty_{\mathrm{bd}}(T,X)$ consists of all continous paths with norm-bounded time domains, and the set $\mathsf{CPath}(T,X)$ of compact paths is the intersection of $\mathsf{CPath}^\infty_{\mathrm{cl}}(T,X)$ and $\mathsf{CPath}^\infty_{\mathrm{bd}}(T,X)$. The basic fact being used here is that if $\eta \in (\mathsf{CPath}^\infty(T,X) \setminus \mathsf{CPath}^\infty_{\mathrm{cl}}(T,X))$ then η is a limit path whose time domain fails to be norm-closed, and hence $\mathrm{dom}(\eta)$ must be norm-bounded with finite length (for if η had infinite length, then $\mathrm{dom}(\eta)$ would be norm-closed).

An important case requiring extra care with the prefix operation is when $\eta \in \mathsf{CPath}^\infty_{\mathrm{bd}}(T,X)$ but $\eta \notin \mathsf{CPath}(T,X)$, which will be the case when $\mathrm{len}(\eta) < \infty$ and $t_0 := \sup(\mathrm{dom}(\eta)) \notin \mathrm{dom}(\eta)$. In this case, the sole point of failure for $\mathrm{dom}(\eta)$ being norm-closed is due to $t_0 \notin \mathrm{dom}(\eta)$: for all $t \not\geqslant t_0$, we will still have $\eta|_t < \eta$ with $\mathrm{dom}(\eta|_t)$ in $\mathsf{CoTD}(T)$, but for all $t \geqslant t_0$, we will have $\eta|_t = \eta$ and so $\mathrm{dom}(\eta|_t)$ will fail to be in the set $\mathsf{CoTD}(T)$ of norm-compact time domains.

Fundamental relationships between the extension partial order, the path operations, and deadlock-free path sets $P \subseteq \mathsf{CPath}(T,X)$, are expressed in the following.

Proposition 3.6 *Let S be a finite-dimensional time structure. Then for all $\eta, \eta' \in \mathsf{CPath}^\infty(T,X)$, the following are equivalent:*

(1) $\eta < \eta'$;

(2) $\eta' = \eta *_t ({}_t|\eta')$ *for some* $t \in \mathrm{dom}(\eta)$ *such that* ${}_t|\eta < {}_t|\eta'$.

If $\eta, \eta' \in (P \cup \mathsf{M}(P))$ for some deadlock-free set $P \subseteq \mathsf{CPath}(T,X)$, then **(1)** *and* **(2)** *are also equivalent to the following:*

(3) $\eta \in P$ *and* $\eta = \eta'|_t$ *where* $t = \max(\mathrm{dom}(\eta))$;

(4) $\eta' = \eta *_t ({}_t|\eta')$ *where* $t = \max(\mathrm{dom}(\eta))$.

Given a set of compact paths $P \subseteq \mathsf{CPath}(T,X)$, a limit path $\eta \in \mathsf{Ext}(P)$ exhibits *finite escape time* w.r.t. P if $\eta \in \mathsf{M}(P)$ but $\eta \notin \mathsf{MU}(P)$, so $\mathrm{dom}(\eta)$ is norm-bounded in T but will fail to be norm-closed, since $\sup(\mathrm{dom}(\eta))$ will not be in $\mathrm{dom}(\eta)$.

If $P \subseteq \mathsf{CPath}(\mathbb{R}^+,X)$ is a deadlock-free set of real time interval-domain paths, then for a maximal limit path, $\eta \in \mathsf{M}(P) \setminus \mathsf{MU}(P)$ iff the time domain $\mathrm{dom}(\eta) = [0,c)$ for some $c < \infty$, which means η exhibits finite escape time w.r.t. P

If $P \subset \mathsf{CPath}(\mathbb{H}, X)$ is a deadlock-free set of regular compact hybrid paths, of the form (4), then similarly, $\eta \in \mathsf{M}(P) \setminus \mathsf{MU}(P)$ iff $\mathrm{dom}(\eta)$ fails to be norm-closed and $\mathrm{dom}(\eta) \subset [\mathbf{0}, (i, c))$ for some $i, c < \infty$, which will be the case exactly when the last continuous time evolution exhibits finite escape time. A hybrid limit path η is *Zeno* iff $\mathrm{len}(\eta) = \infty$ and $\mathrm{dom}(\eta) \subset \mathbb{N} \times [0, c)$ for some $c < \infty$, in which case the length of η is infinite but the total real-time duration is finite and bounded by c. A hybrid limit path η is called *instantaneously Zeno* [3, 5] if $\mathrm{dom}(\eta) = \mathbb{N} \times \{0\}$, in which case the total real-time duration is 0. For both general Zeno and instantaneously Zeno limit paths η, the time domain $\mathrm{dom}(\eta)$ is always a closed set in $\mathcal{T}_{\text{norm}}$.

Cast in the framework of differential and difference inclusions [1, 3], an *hybrid system* is a structure $H = (X, F, G, C, D)$ where $X \subseteq \mathbb{R}^n$, $F \colon X \rightsquigarrow \mathbb{R}^n$, $G \colon X \rightsquigarrow X$, $C := \mathrm{dom}(F)$ and $D := \mathrm{dom}(G)$. The paths or trajectories of H determine a prefix-closed general flow system $\Phi^H \colon X \rightsquigarrow \mathsf{CPath}(\mathbb{H}, X)$ such that a finite hybrid path $\gamma \in \Phi^H(x)$ iff $\mathrm{dom}(\gamma)$ is a regular hybrid time domain, of the form (4), $x \in C \cup D$ and $\gamma(0, 0) = x$, and for each $(i, t) \in \mathrm{dom}(\gamma)$, if $(i + 1, t) \in \mathrm{dom}(\gamma)$ (so that $t = s_{i+1}$, a switching time), then $\gamma(i, t) \in D$ and $\gamma(i + 1, t) \in G(\gamma(i, t))$, while if $s_i \leqslant t < s_{i+1}$ then $\gamma(i, t) \in C$ and $\frac{d}{d\tau}\gamma(i, \tau) \in F(\gamma(i, \tau))$ for almost all $\tau \in [s_i, t]$, where the curve segment $\xi \colon [s_i, s_{i+1}] \to X$ given by $\xi(\tau) := \eta(i, \tau)$ for all $\tau \in [s_i, s_{i+1}]$ is required to be absolutely continuous on the interval $[s_i, s_{i+1}]$. If one of the vector coordinates x_i of $x \in X$ (w.r.t. a fixed basis) is designated *discrete*, then the i-th component $F_i \colon X \rightsquigarrow \mathbb{R}$ obtained by projection from F is such that $F_i(x) = \{0\}$ for all $x \in C$ and $x_i \in Q$ for all $x \in C \cup D$, with Q a finite set.

In [3], Proposition 2.4, sufficient conditions are identified for the continuous and discrete components of an impulse differential inclusion hybrid system H under which the compact path flow map Φ^H is maximally extendible, and every maximal path is either of infinite length, or eventually leaves every compact subset of the signal space X (so finite escape time is possible). In [1], Corollary 2 and Assumption 1, slightly stronger conditions are identified on the components of H under which the flow map Φ^H is forward complete, and consequently, for every maximal path $\eta \in \mathsf{M}\Phi^H(x) = \mathsf{MU}\Phi^H(x)$, the time domain $\mathrm{dom}(\eta)$ is closed in the norm topology on $\mathbb{H}$.

4 Uniform Topologies on Path Spaces

Given a time structure $(S, \leqslant, 0, +, -, u, \|\cdot\|)$ and its future time T, we have available both the order topology $\mathcal{T}_{\leqslant}$ with basic opens the strict intervals (s, t), and the norm topology $\mathcal{T}_{\text{norm}}$ with basic opens the norm balls $B(s, \delta)$. By Theorem 2.2, we in general have $\mathcal{T}_{\text{norm}} \subseteq \mathcal{T}_{\leqslant}$, with equality in special cases such as S linearly-ordered like $\mathbb{R}$ and $\mathbb{Z}$. In developing topological structure on path sets $Z \subseteq \mathsf{CPath}(T, X)$, we take the norm topology $\mathcal{T}_{\text{norm}}$ on S and T as primary, since it gives a quantitative measure on time points, as well as suitably respecting the order topology $\mathcal{T}_{\leqslant}$, as in Theorem 2.2. Since we want to make use of additional properties identified in Theorem 2.2, such as a set $A \subset T$ being norm-bounded iff it is order-bounded, we will henceforth always assume that the time structure S is finite dimensional, and hence lattice-ordered.

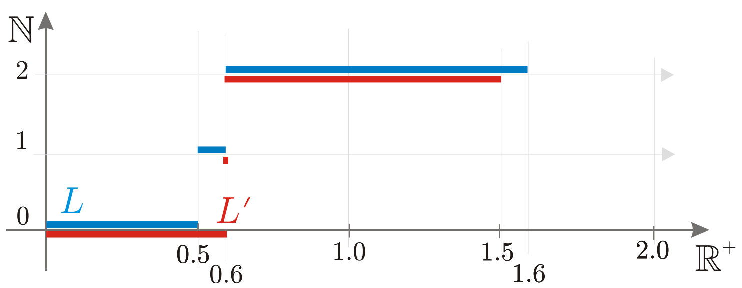
When two paths γ and γ' in $\mathsf{CPath}(T, X)$ have *the same time domain*, we can use the metric d_X on the signal space X to determine whether they are spatially ε-close, meaning the supremum over t in the common time domain of $d_X(\gamma(t), \gamma'(t))$ is less than ε. In order to allow the quantitative comparison of paths with *different time domains*, we need a suitable notion of *retiming maps* between the time domains of paths, and then make use of a second parameter δ to bound the *deviation* of the retimings.

The *Skorokhod metric* (considered for infinite non-Zeno hybrid trajectories in [17, 18, 19]) allows for the comparison of piecewise-continuous signals with differing points of discontinuity by making use of *retiming maps* which are bijective, strictly order-preserving functions between the time domains of the signals. Specifically, for two signals $\eta, \eta' : \mathbb{R}^+ \to X$, the Skorokhod metric distance between them is defined by:

$$d_{\mathrm{Skor}}(\eta, \eta') := \inf \left\{ \varepsilon > 0 \mid \exists \rho \in \mathsf{BRet}(\mathbb{R}^+),\ \sup_{t \in \mathbb{R}^+} \|t - \rho(t)\| < \varepsilon \ \wedge\ \sup_{t \in \mathbb{R}^+} d_X(\eta(t), \eta(\rho(t))) < \varepsilon \right\}.$$

where $\mathsf{BRet}(\mathbb{R}^+)$ is the set of all functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ that are bijective and strictly order-preserving, with *deviation* $\mathrm{dev}(\rho) = \sup_{t \in \mathbb{R}^+} \|t - \rho(t)\|$ for $\rho \in \mathsf{BRet}(\mathbb{R}^+)$.

Within $T = \mathbb{N} \times \mathbb{R}^+$, consider two compact hybrid time domains $L = \{0\} \times [0, 0.5] \cup \{1\} \times [0.5, 0.6] \cup \{2\} \times [0.6, 1.6]$, and $L' = \{0\} \times [0, 0.6] \cup \{1\} \times [0.6, 0.6] \cup \{2\} \times [0.6, 1.5]$, illustrated below.



For the time domains L and L', there are *no* strictly order-preserving single-valued functions between them, yet we are inclined to say that they are "close". It will not work relax to single-valued maps that are order-preserving but not strictly so, because the strictness is needed for invertibility and symmetry. This motivates our relaxation to retiming maps that are order-preserving in a set-valued sense.

Definition 4.1 [Retimings] *Given a time structure S with future time T, define:*

$$\mathsf{Lin}(T) := \{ L \subseteq T \mid L \text{ is linearly-ordered} \ \wedge\ L \neq \varnothing \}.$$

Define the earlier-than *relation* $\trianglelefteq$ *on* $\mathsf{Lin}(T)$ *as follows: for all* $L, L' \in \mathsf{Lin}(T)$*,*

$$L \trianglelefteq L' \quad \Leftrightarrow \quad \forall t \in (L \smallsetminus L'), \forall t' \in L', \ \ t < t' \ \ \wedge \ \ \forall t \in L, \forall t' \in (L' \smallsetminus L), \quad t < t' .$$

A set-valued map $\rho\colon T \rightsquigarrow T$ *will be called* order-preserving *if for all* $t_1, t_2 \in \mathrm{dom}(\rho)$*, if* $t_1 < t_2$ *then* $\rho(t_1) \trianglelefteq \rho(t_2)$*. Given sets* $L, L' \in \mathsf{Lin}(T)$*, a map* $\rho\colon T \rightsquigarrow T$ *will be called a* retiming *from* L *to* L' *if the following conditions are satisfied:*

(i) $\mathrm{dom}(\rho) = L$ *and* $\mathrm{ran}(\rho) = L'$*;*

(ii) *for all* $t \in L$*,* $\rho(t) \in \mathsf{Lin}(T)$*, and for all* $t' \in L'$*,* $\rho^{-1}(t') \in \mathsf{Lin}(T)$*;*

(iii) ρ *and* ρ^{-1} *are both order-preserving.*

For a retiming $\rho\colon L \rightsquigarrow L'$*, define the* deviation $\mathrm{dev}(\rho) \in \mathbb{R}^{+\infty}$ *as follows:*

$$\mathrm{dev}(\rho) \ := \ \sup \{ \| t - s \| \in \mathbb{R}^{+} \mid t \in \mathrm{dom}(\rho) \ \wedge \ s \in \rho(t) \} .$$

Let $\mathsf{Ret}(L, L')$ *denote the set of all retimings* $\rho\colon L \rightsquigarrow L'$ *together with all retimings* $\rho'\colon L' \rightsquigarrow L$*, so that* $\mathsf{Ret}(L, L') = \mathsf{Ret}(L', L)$*.*

Clearly, the compact time domains $\mathsf{CoTD}(T) \subset \mathsf{Lin}(T)$, since $\mathsf{Lin}(T)$ contains arbitrary linearly-ordered subsets; in particular, every singleton set $\{t\}$ is in $\mathsf{Lin}(T)$. For all limit paths $\eta \in \mathsf{LPath}(T, X)$ and all compact paths $\eta \in \mathsf{CPath}(T, X)$, $\mathrm{dom}(\eta) \in \mathsf{Lin}(T)$.

Proposition 4.2 *The* earlier-than *relation* $\trianglelefteq$ *is a* partial order *on* $\mathsf{Lin}(T)$*: it is reflexive, transitive and anti-symmetric, and has least element* $\{0\}$*.*

Proof: For reflexivity, we trivially have $L \trianglelefteq L$ for all $L \in \mathsf{Ext}(T)$, since then $L \smallsetminus L = \varnothing$, so the universal quantification is vacuous. For transitivity, suppose $L \trianglelefteq L'$ and $L' \trianglelefteq L''$. Then for arbitrary points $t \in (L \smallsetminus L'')$ and $t'' \in L''$, we want to show that $t < t''$. Now we have the disjoint unions $L \smallsetminus L'' = (L \smallsetminus L') \cup ((L \cap L') \smallsetminus L'')$ and $L'' = (L'' \cap L') \cup (L'' \smallsetminus L')$, so we can proceed by cases.
Case I: $t \in (L \smallsetminus L')$ and $t'' \in (L'' \cap L')$; then $t < t''$ since $L \trianglelefteq L'$.
Case II: $t \in (L \smallsetminus L')$ and $t'' \in (L'' \smallsetminus L')$; then since $L \trianglelefteq L'$ and $L' \trianglelefteq L''$, for any $t' \in L'$ such that $t' \leqslant t''$, we have $t < t'$ and hence $t < t''$.
Case III: $t \in ((L \cap L') \smallsetminus L''$ and $t'' \in (L'' \cap L')$; then $t < t''$ since $L' \trianglelefteq L''$ and $t \in (L' \smallsetminus L'')$ and $t'' \in L''$.
Case IV: $t \in (L \cap L') \smallsetminus L''$ and $t'' \in (L'' \smallsetminus L')$; then again we get $t < t''$ since $L' \trianglelefteq L''$ and $t \in (L' \smallsetminus L'')$ and $t'' \in L''$.
Conversely, consider arbitrary points $t'' \in (L'' \smallsetminus L)$ and $t \in L$, and we want to show that $t < t''$. This time, using the disjoint unions $L'' \smallsetminus L = (L'' \smallsetminus L') \cup ((L'' \cap L') \smallsetminus L)$ and $L = (L \cap L') \cup (L \smallsetminus L')$, we can again proceed by four cases, to derive the conclusion that $t < t''$. Hence $L \trianglelefteq L''$, as required.

For anti-symmetry, suppose that $L \trianglelefteq L'$ and $L \neq L'$; we want to show that $L' \ntrianglelefteq L$. Since $L \neq L'$, we have either $(L \setminus L') \neq \emptyset$ or $(L' \setminus L) \neq \emptyset$. In the first case, pick any $t \in (L \setminus L')$ and any $t' \in L'$; then since $L \trianglelefteq L'$, we have $t < t'$, and hence $t' \nless t$, and thus $L' \ntrianglelefteq L$. In the second case, pick any $t' \in (L' \setminus L)$ and any $t \in L$; then since $L \trianglelefteq L'$, we have $t < t'$, and hence $t' \nless t$, and thus $L' \ntrianglelefteq L$. ■.

The earlier-than partial-order is distinct from, but a relative of, the extension partial-order on compact time domains and the domains of limit paths, which has $L < L'$ iff $L \subset L'$ and for all $t \in L$ and all $t' \in L' \setminus L$, we have $t < t'$. Hence we have $L < L'$ iff $L \subset L'$ and $L \trianglelefteq L'$.

Retiming maps $\rho\colon L \rightsquigarrow L'$ are total and surjective (condition (i)), whose set-images $\rho(t)$ and $\rho^{-1}(t')$ are non-empty and linearly-ordered (condition (ii)), and it and its inverse satisfy a set-valued order-preservation property (condition (iii)).

If a retiming ρ is in fact single-valued, with $\rho\colon L \to L'$, then condition (iii) requires that ρ is order-preserving, but not necessarily *strictly* order-preserving (as in [24]), since if $t_1, t_2 \in L$ with $t_1 < t_2$, and $\rho(t_1) = \{t'_1\}$ and $\rho(t_2) = \{t'_2\}$, then $\rho(t_1) \trianglelefteq \rho(t_2)$ implies that $t'_1 \leqslant t'_2$ (rather than the strict order relationship $t'_1 < t'_2$). Note that, in general, ρ^{-1} is set-valued even when ρ is single-valued; indeed, ρ^{-1} is single-valued iff ρ is single-valued and strictly order-preserving (and hence injective).

When the deviation of a retiming ρ is bounded, then each of the linearly-ordered image-sets $\rho(t)$ within $\mathrm{ran}(\rho)$ are bounded in the norm. More precisely, for real $\delta > 0$, if $\mathrm{dev}(\rho) < \delta$ then $\|t\| - \delta < \|s\| < \|t\| + \delta$ for all $t \in \mathrm{dom}(\rho)$ and all $s \in \rho(t)$. The notion of set-valued retimings has been formulated to ensure closure under relational inverse; a further highly desirable property of set-valued retimings is that they are also closed under relational composition.

Proposition 4.3 [Inverses and compositions of retimings]
Given $L, L', L'' \in \mathsf{Lin}(T)$,
if $\rho \in \mathsf{Ret}(L, L')$, then $\rho^{-1} \in \mathsf{Ret}(L, L')$ and $\mathrm{dev}(\rho^{-1}) = \mathrm{dev}(\rho)$; and
if $\rho_1 \in \mathsf{Ret}(L, L')$ and $\rho_2 \in \mathsf{Ret}(L', L'')$, then $(\rho_1 \circ \rho_2) \in \mathsf{Ret}(L, L'')$, and $\mathrm{dev}(\rho_1 \circ \rho_2) \leq \mathrm{dev}(\rho_1) + \mathrm{dev}(\rho_2)$.

Proof: Fix $\rho \in \mathsf{Ret}(L, L')$. Since $\mathrm{dom}(\rho^{-1}) = \mathrm{ran}(\rho) = L'$ and $\mathrm{ran}(\rho^{-1}) = \mathrm{dom}(\rho) = L$, it is clear that ρ^{-1} satisfies condition (i). It is also immediate that ρ^{-1} satisfies conditions (ii) and (iii), since both properties are symmetric across inverses. The equation $\mathrm{dev}(\rho^{-1}) = \mathrm{dev}(\rho)$ also follows by symmetry across inverses of the definition of deviation.

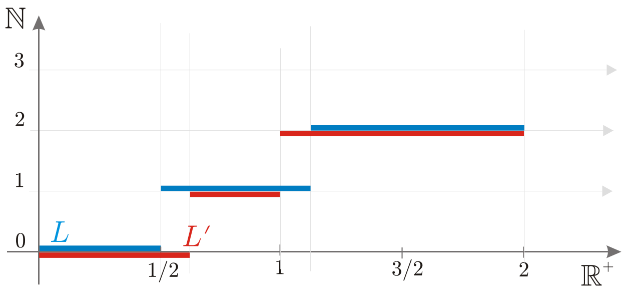
For the second part, fix $\rho_1 \in \mathsf{Ret}(L, L')$ and $\rho_2 \in \mathsf{Ret}(L', L'')$, and consider the relational composition $\rho_1 \circ \rho_2$. By replacing ρ_1 with ρ_1^{-1} if need be, and likewise possibly also replacing ρ_2 with ρ_2^{-1}, we can assume that $\mathrm{ran}(\rho_1) = \mathrm{dom}(\rho_2) = L'$. Then by condition (i) for ρ_1 and ρ_2, we can conclude that $\mathrm{dom}(\rho_1 \circ \rho_2) = \mathrm{dom}(\rho_1) = L$ and $\mathrm{ran}(\rho_1 \circ \rho_2) = \mathrm{ran}(\rho_2) = L''$, so condition (i) is satisfied for $\rho_1 \circ \rho_2$.

For each $t \in \mathrm{dom}(\rho_1 \circ \rho_2)$, we have $(\rho_1 \circ \rho_2)(t) = \bigcup\{\rho_2(s) \mid s \in \rho_1(t)\}$; to show that $(\rho_1 \circ \rho_2)(t)$ is linearly-ordered, consider two elements $t'_1, t'_2 \in (\rho_1 \circ \rho_2)(t)$. we want to show that either $t'_1 < t'_2$ or else $t'_2 \leqslant t'_1$. Now there must exist $s_1 \in \rho_1(t)$ and $s_2 \in \rho_1(t)$ such that $t'_1 \in \rho_2(s_1)$ and $t'_2 \in \rho_2(s_2)$. Since $\rho_1(t)$ is linearly-ordered, we know that either $s_1 < s_2$ or $s_1 = s_2$ or $s_2 < s_1$.

In *Case I*: $s_1 < s_2$, we have $\rho_2(s_1) \trianglelefteq \rho_2(s_2)$, and then break into further sub-cases, depending on t'_1 and t'_2. Either we have $t'_1 \in \rho_2(s_1) - \rho_2(s_2)$, in which case $t'_1 < t'_2$, or else $t'_1 \in \rho_2(s_1) \cap \rho_2(s_2)$ and $t'_2 \in \rho_2(s_2) \smallsetminus \rho_2(s_1)$, in which case $t'_1 < t'_2$, or else both $t'_1 \in \rho_2(s_1) \cap \rho_2(s_2)$ and $t'_2 \in \rho_2(s_1) \cap \rho_2(s_2)$, in which case either $t'_1 < t'_2$ or $t'_2 \leqslant t'_1$, since the set $\rho_2(s_1) \cap \rho_2(s_2)$ is linearly-ordered.
In *Case II*: $s_1 = s_2$, we have $\rho_2(s_1) = \rho_2(s_2)$, and hence either $t'_1 < t'_2$ or $t'_2 \leqslant t'_1$, since the set $\rho_2(s_1)$ is linearly-ordered.
In *Case III*: $s_2 < s_1$, we have $\rho_2(s_2) \trianglelefteq \rho_2(s_1)$, and then break into further sub-cases, depending on t'_1 and t'_2. Either we have $t'_2 \in \rho_2(s_2) \smallsetminus \rho_2(s_1)$, in which case $t'_2 < t'_1$, or else $t'_2 \in \rho_2(s_2) \cap \rho_2(s_1)$ and $t'_1 \in \rho_2(s_1) \smallsetminus \rho_2(s_2)$, in which case $t'_2 < t'_1$, or else both $t'_2 \in \rho_2(s_1) \cap \rho_2(s_2)$ and $t'_1 \in \rho_2(s_1) \cap \rho_2(s_2)$, in which case either $t'_2 < t'_1$ or $t'_1 \leqslant t'_2$, since the set $\rho_2(s_1) \cap \rho_2(s_2)$ is linearly-ordered.
Hence $(\rho_1 \circ \rho_2)(t)$ is linearly-ordered, as required. The argument for $(\rho_1 \circ \rho_2)^{-1}(t') = \bigcup\{\rho_1^{-1}(s) \mid s \in \rho_2^{-1}(t')\}$, for each $t' \in \mathrm{ran}(\rho_1 \circ \rho_2)$, proceeds symmetrically. Hence condition (ii) is satisfied for $\rho_1 \circ \rho_2$.

For condition (iii), suppose $t_1, t_2 \in \mathrm{dom}(\rho_1 \circ \rho_2) = \mathrm{dom}(\rho_1) = L$ are such that $t_1 < t_2$; we want to show that $(\rho_1 \circ \rho_2)(t_1) \trianglelefteq (\rho_1 \circ \rho_2)(t_2)$. First, consider arbitrary points $t'_1 \in (\rho_1 \circ \rho_2)(t_1) \smallsetminus (\rho_1 \circ \rho_2)(t_2)$ and $t'_2 \in (\rho_1 \circ \rho_2)(t_2)$. We want to show that $t'_1 < t'_2$. Since $t'_2 \in (\rho_1 \circ \rho_2)(t_2)$, there exists $s_2 \in \rho_1(t_2)$ such that $t'_2 \in \rho_2(s_2)$. Since $t'_1 \in (\rho_1 \circ \rho_2)(t_1) \smallsetminus (\rho_1 \circ \rho_2)(t_2)$, there exists $s_1 \in \rho_1(t_1)$ such that $t'_1 \in \rho_2(s_1)$, but there does not exist any $s_3 \in \rho_1(t_2)$ such that $t'_1 \in \rho_2(s_3)$. Hence we can conclude that $s_1 \neq s_2$. Moreover, since $t_1 < t_2$, we know that $\rho_1(t_1) \trianglelefteq \rho_1(t_2)$, so we can deduce order relationships between $s_1 \in \rho_1(t_1)$ and $s_2 \in \rho_1(t_2)$. Indeed, we know that $s_1 \in \rho_1(t_1) \smallsetminus \rho_1(t_2)$ because $s_1 \in \rho_1(t_1) \cap \rho_2(t_2)$ would contradict $t'_1 \in (\rho_1 \circ \rho_1)(t_1) \smallsetminus (\rho_1 \circ \rho_2)(t_2)$. Hence we have $s_1 < s_2$, and thus $\rho_2(s_1) \trianglelefteq \rho_2(s_2)$. Now $t'_1 \in \rho_2(s_1) \smallsetminus \rho_2(s_2)$ and $t'_2 \in \rho_2(s_2)$, hence we can conclude that $t'_1 < t'_2$, as required. Next, consider arbitrary points $t'_2 \in (\rho_1 \circ \rho_2)(t_2) \smallsetminus (\rho_1 \circ \rho_2)(t_1)$ and $t'_1 \in (\rho_1 \circ \rho_2)(t_1)$. We want to show that $t'_1 < t'_2$. Since $t'_1 \in (\rho_1 \circ \rho_2)(t_1)$, there exists $s_1 \in \rho_1(t_1)$ such that $t'_1 \in \rho_2(s_1)$. Since $t'_2 \in (\rho_1 \circ \rho_2)(t_2) \smallsetminus (\rho_1 \circ \rho_2)(t_1)$, there exists $s_2 \in \rho_1(t_2)$ such that $t'_2 \in \rho_2(s_2)$, but there does not exist any $s_4 \in \rho_1(t_1)$ such that $t'_2 \in \rho_2(s_4)$. Hence we can conclude that $s_1 \neq s_2$. Moreover, since $t_1 < t_2$, we know that $\rho_1(t_1) \trianglelefteq \rho_1(t_2)$, so we can deduce order relationships between $s_1 \in \rho_1(t_1)$ and $s_2 \in \rho_1(t_2)$. Indeed, we know that $s_2 \in \rho_1(t_2) \smallsetminus \rho_1(t_1)$ because $s_2 \in \rho_1(t_1) \cap \rho_1(t_2)$ would contradict $t'_2 \in (\rho_1 \circ \rho_2)(t_2) \smallsetminus (\rho_1 \circ \rho_2)(t_1)$. Hence we have $s_1 < s_2$, and thus $\rho_2(s_1) \trianglelefteq \rho_2(s_2)$. Now $t'_2 \in \rho_2(s_2) \smallsetminus \rho_2(s_1)$ and $t'_1 \in \rho_2(s_1)$, hence we can conclude that $t'_1 < t'_2$, as required. We have thus established that $t_1 < t_2$ implies $(\rho_1 \circ \rho_2)(t_1) \trianglelefteq (\rho_1 \circ \rho_2)(t_2)$. The argument that, for $t'_1, t'_2 \in \mathrm{ran}(\rho_1 \circ \rho_2) = \mathrm{ran}(\rho_2) = L''$, that t'_1, t'_2 implies $(\rho_1 \circ \rho_2)^{-1}(t'_1) \trianglelefteq (\rho_1 \circ \rho_2)^{-1}(t'_2)$, proceeds symmetrically, using the fact that $(\rho_1 \circ \rho_2)^{-1} = \rho_2^{-1} \circ \rho_1^{-1}$. Hence condition (iii) is satisfied, to conclude that proof that $(\rho_1 \circ \rho_2) \in \mathsf{Ret}(L, L'')$. ■

For some concrete examples of retimings, consider hybrid future time $\mathbb{H}$, and two bounded hybrid time domains: $L := \{0\}\times[0, \frac{1}{2}] \cup \{1\}\times[\frac{1}{2}, \frac{9}{8}] \cup \{2\}\times[\frac{9}{8}, 2]$, and $L' := \{0\}\times[0, \frac{5}{8}] \cup \{1\}\times[\frac{5}{8}, 1] \cup \{2\}\times[1, 2]$, as illustrated below.



There are numerous possible retimings ρ between L and L', some single-valued and some set-valued, although all have $\mathrm{dev}(\rho) \geq \frac{1}{8}$. As one example (which illustrates several different patterns of mapping relationships), consider the map $\rho_1 \colon L \rightsquigarrow L'$ defined for all $(i,t) \in L$ as follows:

$$\rho_1(i,t) \;:=\; \begin{cases} \{\,(i,t)\,\} & \text{if } (i,t) \in \{0\} \times [0,\frac{1}{2}) \,\cup\, \{2\} \times \big((\frac{9}{8},\frac{5}{4}] \cup (\frac{15}{8},2]\big) \\ L' \cap \big(\{i\} \times (t-\frac{1}{8}, t+\frac{1}{8})\big) & \text{if } (i,t) \in \{1\} \times (\frac{5}{8},\frac{9}{8}) \cup \{2\} \times (\frac{5}{4},\frac{15}{8}] \\ \{0\} \times [\frac{1}{2},\frac{5}{8}] & \text{if } (i,t) = (0,\frac{1}{2}) \\ \{\,(1,\frac{5}{8})\,\} & \text{if } (i,t) \in \{1\} \times [\frac{1}{2},\frac{5}{8}] \\ \{\,(1,1)\,\} & \text{if } (i,t) \in \{1\} \times [1,\frac{9}{8}] \\ \{2\} \times [1,\frac{9}{8}] & \text{if } (i,t) = (2,\frac{9}{8})\,. \end{cases}$$

It is readily seen that ρ_1 is a retiming, and $\mathrm{dev}(\rho_1) = \frac{1}{8}$. This example illustrates the possibility of $s < s'$ and $\rho_1(s) \trianglelefteq \rho_1(s')$ and the intersection $\rho_1(s) \cap \rho_1(s')$ containing a non-trivial interval. For example, for $s = (1,t)$ and $s' = (1,t')$ with $\frac{5}{8} < t < t' < t+\frac{1}{8} < 1$, we have $\rho_1(s) \cap \rho_1(s') = L' \cap (\{1\} \times (t'-\frac{1}{8}, t+\frac{1}{8}))$. For the point $s'' = (1,\frac{5}{8})$, we have $s'' < s$ with $\rho_1(s'') = \{\,(1,\frac{5}{8})\,\}$ and $\rho_1(s) = L' \cap (\{1\} \times (t-\frac{1}{8}, t+\frac{1}{8}))$, so either $\rho_1(s) = \{1\} \times [\frac{5}{8}, t+\frac{1}{8})$ (when $t < \frac{3}{4}$), or else $\rho_1(s) = \{1\} \times (t-\frac{1}{8}, t+\frac{1}{8})$ (when $\frac{3}{4} \leqslant t \leqslant \frac{7}{8}$); in either case, it is clear that $\rho_1(s'') \trianglelefteq \rho_1(s)$.

For another example, consider the single-valued and strictly order-preserving (hence injective) function $\rho_2 \colon L \to L'$ defined for all $(i,t) \in L$ as follows:

$$\rho_2(i,t) \;:=\; \begin{cases} (0, \frac{5}{4}\,t) & \text{if } (i,t) \in \{0\} \times [0,\frac{1}{2}] \\ (1, \frac{5}{8} + \frac{3}{5}(t-\frac{1}{2})) & \text{if } (i,t) \in \{1\} \times [\frac{1}{2},\frac{9}{8}] \\ (2, 1 + \frac{8}{7}(t-\frac{9}{8})) & \text{if } (i,t) \in \{2\} \times [\frac{9}{8},2]\,. \end{cases}$$

In this case, we also have $\mathrm{dev}(\rho_2) = \frac{1}{8}$. As one further example, with a different deviation, consider the singled-valued map $\rho_3 \colon L \to L'$ that is the identity on the real-coordinate t: $\rho_3(i,t) := (i,t)$ if $(i,t) \in L \cap L'$; $\rho_3(i,t) := (0,t)$ if $(i,t) \in \{1\} \times [\frac{1}{2},\frac{5}{8})$; and $\rho_3(i,t) := (2,t)$ if $(i,t) \in \{1\} \times (1,\frac{9}{8}]$. Again, it is readily seen that ρ_3 is a retiming because it is order-preserving, and in this case, $\mathrm{dev}(\rho_3) = 1$.

To illustrate retimings in the context of *samplings* of hybrid paths, suppose a "time+event-driven" Δ-sampling of L' is taken, for $\Delta = \frac{1}{5}$, resulting in a compact time domain L'' such that: $L'' = \{0\} \times \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{5}{8}\} \cup \{1\} \times \{\frac{4}{5}, 1\} \cup \{2\} \times \{\frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2\}$. One possible retiming map $\rho_4 : L' \rightsquigarrow L''$ does the time-driven sampling by mapping intervals of length $\frac{1}{5}$ to sample points, and does the event-driven sampling by adding further sample points at switching times and mapping intervals of length at most $\frac{1}{5}$ to these switching times:

$$\rho_4(i,t) \;:=\; \begin{cases} \{(0,0)\} & \text{if } (i,t) = (0,0) \\ \{(0,\frac{k+1}{5})\} & \text{if } i = 0 \text{ and } t \in [\frac{k}{5},\frac{k+1}{5}] \text{ for } k \in \{0,1,2\} \\ \{(0,\frac{5}{8})\} & \text{if } i = 0 \text{ and } t \in [\frac{3}{5},\frac{5}{8}] \\ \{(1,\frac{4}{5})\} & \text{if } i = 1 \text{ and } t \in [\frac{5}{8},\frac{4}{5}] \\ \{(1,1)\} & \text{if } i = 1 \text{ and } t \in [\frac{4}{5},1] \\ \{(2,\frac{k+1}{5})\} & \text{if } i = 2 \text{ and } t \in [\frac{k}{5},\frac{k+1}{5}] \text{ for } k \in \{5,6,7,8,9\} \end{cases} .$$

Then we have $\mathrm{dev}(\rho_4) = \frac{1}{5}$. By composability of retimings, the map $\rho_5 := \rho_1 \circ \rho_4$ is a retiming $\rho_5 : L \rightsquigarrow L''$ of deviation $\mathrm{dev}(\rho_5) \leq \mathrm{dev}(\rho_1) + \mathrm{dev}(\rho_4) = \frac{13}{40}$.

We will first identify topological structure on the set $\mathsf{CPath}^{\infty}_{\mathrm{bd}}(T,X)$ of paths with bounded time domains (which includes the compact paths in $\mathsf{CPath}(T,X)$ as well as those $\eta \in \mathsf{CPath}^{\infty}(T,X)$ with bounded but not closed time domains within the norm topology on T). In a second stage, we then lift that structure to the set $\mathsf{CPath}^{\infty}(T,X)$ of all paths, finite or limit.

We develop a natural topology on $\mathsf{CPath}^{\infty}_{\mathrm{bd}}(T,X)$ (and thus on any set $Z \subseteq \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T,X)$ via the standard subspace topology) by constructing a *uniformity* or *uniform structure* [25] on the path set $\mathsf{CPath}^{\infty}_{\mathrm{bd}}(T,X)$, which gives quantitative measures of closeness of paths that utilizes both spatial closeness under the metric d_X on X and temporal closeness under the norm $\|\cdot\|_T$ on T.

From [25], on a set Z, a family of binary relations or set-valued maps $\mathcal{V} \subseteq [Z \rightsquigarrow Z]$ constitutes a *basis for a uniformity* on the set Z if the following conditions are satisfied:

(i) every $V \in \mathcal{V}$ is a reflexive and total binary relation (i.e. $(z,z) \in V$ for all $z \in Z$);

(ii) $\mathcal{V}$ is closed under relation-inverse (i.e. $V^{-1} \in \mathcal{V}$ for all $V \in \mathcal{V}$);

(iii) $\mathcal{V}$ is downward-closed under intersection, in the sense that for all $V_1, V_2 \in \mathcal{V}$, there exists $W \in \mathcal{V}$ such that $W \subseteq (V_1 \cap V_2)$; and

(iv) for all $V \in \mathcal{V}$, there exists $W \in \mathcal{V}$ such that $(W \circ W) \subseteq V$ (*generalized triangle inequality*).

The relations/maps $V \in \mathcal{V}$ are called the *basic entourages*, and the *uniformity* or uniform structure generated by the basis $\mathcal{V}$ is the *filter* generated by $\mathcal{V}$, namely:

$$\mathcal{U}_{\mathcal{V}} \;:=\; \{\, U \in [Z \rightsquigarrow Z] \mid \exists V \in \mathcal{V},\; V \subseteq U \,\},$$

the family of all supersets of the basic entourages $V \in \mathcal{V}$. For each $z \in Z$ and basic entourage $V \in \mathcal{V}$, let $V(z) = \{z' \in Z \mid (z,z') \in V\}$ be the set-image of the map V on z. The *uniform topology* on Z generated by $\mathcal{V}$ is the topology $\mathcal{T}_{\mathcal{V}}$ which has as a basis the family of all sets:

$$\mathcal{B}_{\mathcal{V}} \;:=\; \{\, V(z) \mid z \in Z \;\wedge\; V \in \mathcal{V} \,\}.$$

For any set $P \subseteq Z$ and basic entourage $V \in \mathcal{V}$, the set-image $V(P) = \bigcup_{z \in P} V(z)$ is the union of all the V-basic opens with center at $z \in P$. Assume the topology $\mathcal{T}_{\mathcal{V}}$ is *first-countable*, which requires there exists a countable subset of $\mathcal{V}_0 \subseteq \mathcal{V}$ such that $\mathcal{V}_0$ constitutes a basis for a uniformity, and the topology $\mathcal{T}_{\mathcal{V}_0}$ generated by $\mathcal{V}_0$ is equal to $\mathcal{T}_{\mathcal{V}}$. Further assume that the $\mathcal{T}_{\mathcal{V}}$ is *Hausdorff*; this requires that $(z, z') \in \bigcap \mathcal{V}$ iff $z = z'$, where $\bigcap \mathcal{V}$ is the intersection of all the basic entourages. In particular, every *metric space* is a Hausdorff uniform space. In a first-countable and Hausdorff uniform space, given $z \in Z$ and a sequence of elements $(z_n)_{n \in \mathbb{N}}$ in Z, the sequence *converges* to a limit z within the uniformity $\mathcal{U}_{\mathcal{V}}$, and the limit is *unique* and we write $\lim\limits_{n \to \infty} z_n = z$, if for every basic open $V(z)$, there exists an $m \in \mathbb{N}$ such that $z_n \in V(z)$ for all $n \geq m$.

We also need a further result (a special case of [25], Proposition 4.12) that in a first-countable Hausdorff uniform space $(Z, \mathcal{V})$, a set $K \subseteq Z$ is *compact* in Z iff K is *Cauchy-sequence-complete* and *totally-bounded*. A set K is Cauchy-sequence-complete if every Cauchy sequence in K converges to a limit in K, where a sequence $\{z_n\}_{n \in \mathbb{N}}$ is Cauchy if for every entourage $V \in \mathcal{V}$, there exists an integer k such that $(z_n, z_m) \in V$ for all $n, m > k$. A set K is totally-bounded if for every entourage $V \in \mathcal{V}$, there exists a finite set $F \subset K$ such that $K \subseteq \bigcup_{z \in F} V(z)$. The result of [25], Proposition 4.12, is for general uniform spaces, where the notions of limits, convergence, and the Cauchy property must be formulated in terms of filters rather than countable sequences; the result states that a set is compact iff it is Cauchy-filter-complete and totally-bounded.

With this background, we can now set out the 2-parameter uniform structure on compact path spaces.

Definition 4.4 [Retimings and relative distance on spaces of bounded time paths]
Let S be a finite dimensional time structure with future time T, let (X, d_X) be a metric space, and let $Z \subseteq \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T, X)$ be any set of paths with bounded time domains. For each pair $(\gamma, \gamma') \in Z \times Z$, define $\mathsf{Ret}(\gamma, \gamma') := \mathsf{Ret}(\mathrm{dom}(\gamma), \mathrm{dom}(\gamma'))$, and define the set $\mathsf{Ret}(Z)$ of retimings for Z as:

$$\mathsf{Ret}(Z) \; := \; \bigcup \{ \, \mathsf{Ret}(\gamma, \gamma') \mid \gamma \in Z \; \wedge \; \gamma' \in Z \, \} \, .$$

Then define an extended-real-valued 3-argument metric-like function $d_{X\sup} \colon \, (Z \times Z \times \mathsf{Ret}(Z)) \to \mathbb{R}^{+\infty}$ by setting $d_{X\sup}(\gamma, \gamma', \rho) := \infty$ if $\rho \notin \mathsf{Ret}(\gamma, \gamma')$, and otherwise, assuming $\mathrm{dom}(\rho) = \mathrm{dom}(\gamma)$ and $\mathrm{ran}(\rho) = \mathrm{dom}(\gamma')$ (and this can always be arranged when $\rho \in \mathsf{Ret}(\gamma, \gamma')$, by replacing ρ with its inverse if need be), we have:

$$d_{X\sup}(\gamma, \gamma', \rho) \; := \; \sup \{ \, d_X(\, \gamma(t), \, \gamma'(t') \,) \mid t \in \mathrm{dom}(\gamma) \; \wedge \; t' \in \mathrm{dom}(\gamma') \; \wedge \; (t, t') \in \rho \, \} \, .$$

Define the parameter set $R_2 := \mathbb{R}^{>0} \times \mathbb{R}^{>0}$, and for each real pair $(\delta, \varepsilon) \in R_2$, define the relation $V_{\delta,\varepsilon} \colon Z \rightsquigarrow Z$ as follows:

$$V_{\delta,\varepsilon} \; := \; \{ \, (\gamma, \gamma') \in Z \times Z \mid \exists \rho \in \mathsf{Ret}(\gamma, \gamma'), \; \mathrm{dev}(\rho) < \delta \; \wedge \; d_{X\sup}(\gamma, \gamma', \rho) < \varepsilon \, \} \, .$$

The first task is to verify that the family of all relations $V_{\delta,\varepsilon}$ for $(\delta,\varepsilon) \in R_2$ constitutes a basis for a uniformity. It is clear that each $V_{\delta,\varepsilon}$ is total and reflexive, since for any path γ, the identity map *id* on $\mathrm{dom}(\gamma)$ is a retiming with deviation 0. Symmetry is also clear, with the inverse of a retiming a retiming of the same deviation, hence $(V_{\delta,\varepsilon})^{-1} = V_{\delta,\varepsilon}$. Observe that for $\rho \in \mathsf{Ret}(\gamma,\gamma')$, if $\mathrm{dev}(\rho) < \delta$, then $\|t - t'\| < \delta$ for all $(t,t') \in \rho$, and if $d_{X\mathrm{sup}}(\gamma,\gamma',\rho) < \varepsilon$, then (independent of δ), we will have the initial states such that $d_X(\gamma(0),\gamma'(0)) < \varepsilon$, since $(0,0) \in \rho$ for all retimings ρ.

The further, more challenging task, is to demarcate sets $Z \subseteq \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T,X)$ of paths with bounded time domains for which the 2-parameter uniform topology is Hausdorff. The condition on Z we identify is that for some deadlock-free and prefix-closed set of compact paths $P \subseteq \mathsf{CPath}(T,X)$, we have $P \subseteq Z$ and $Z \smallsetminus \mathsf{CPath}(T,X) \subseteq \mathsf{M}(P) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T,X)$. The key idea is that if we are comparing distinct paths where at least one of them has a compact time domain, then we can quantify the discrepancy between the paths by looking at compact prefixes. Maximality also allows us to rule out any compact paths properly extending a maximal path.

Theorem 4.5 [2-parameter uniform topology on spaces of compact paths]
Let S be a finite dimensional time structure with future time T, let (X,d_X) be a metric space, let $Z \subseteq \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T,X)$ be any set of paths with bounded time domains, and for each $(\delta,\varepsilon) \in R_2$, let $V_{\delta,\varepsilon}\colon Z \rightsquigarrow Z$ be the relation as in Definition 4.4. Then for all parameter pairs $(\delta,\varepsilon),(\delta_1,\varepsilon_1),(\delta_2,\varepsilon_2) \in R_2$, we have:

$$\begin{aligned}
V_{\delta_1,\varepsilon_1} &\subseteq V_{\delta_2,\varepsilon_2} && \textit{when } (\delta_1,\varepsilon_1) \leqslant (\delta_2,\varepsilon_2)\\
V_{\delta,\varepsilon} &\subseteq V_{\delta_1,\varepsilon_1} \cap V_{\delta_2,\varepsilon_2} && \textit{when } (\delta,\varepsilon) \leqslant (\delta_1,\varepsilon_1) \wedge (\delta_2,\varepsilon_2)\\
V_{\delta_1,\varepsilon_1} \circ V_{\delta_2,\varepsilon_2} &\subseteq V_{\delta,\varepsilon} && \textit{when } (\delta_1,\varepsilon_1) + (\delta_2,\varepsilon_2) \leqslant (\delta,\varepsilon)\\
V_{\delta,\varepsilon} \circ V_{\delta,\varepsilon} &\subseteq V_{\delta_1,\varepsilon_1} && \textit{when } (\delta,\varepsilon) \leqslant \tfrac{1}{2}(\delta_1,\varepsilon_1)\,.
\end{aligned}$$

*Hence the family $\mathcal{V}_2 := \{\, V_{\delta,\varepsilon}\colon Z \rightsquigarrow Z \mid (\delta,\varepsilon) \in R_2 \,\}$ constitutes a basis for a uniformity on the path set Z, and additionally, the basic entourages are inclusion-monotone w.r.t. the product partial-ordering on the parameter space $R_2 = \mathbb{R}^{>0} \times \mathbb{R}^{>0}$. The basic open sets in the 2-parameter uniform topology $\mathcal{T}_2$ on Z are the (δ,ε)-*tubes *$V_{\delta,\varepsilon}(\gamma)$ around a path $\gamma \in Z$.*
Moreover, the uniform topology $\mathcal{T}_2$ is always first-countable, and $\mathcal{T}_2$ will be Hausdorff if the path set $Z \subseteq \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T,X)$ is such that for some deadlock-free and prefix-closed set of compact paths $P \subseteq \mathsf{CPath}(T,X)$, we have $P \subseteq Z$ and $Z \smallsetminus \mathsf{CPath}(T,X) \subseteq \mathsf{M}(P) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T,X)$. When $\mathcal{T}_2$ is Hausdorff, the topology $\mathcal{T}_2$ is metrizable, and we can use the metric $d_2 : Z \times Z \to \mathbb{R}^+$ given by:

$$d_2(\gamma,\gamma') := \max\{\,\delta_0,\ \varepsilon_0\,\} \quad \textit{where} \quad (\delta_0,\varepsilon_0) := \inf\{\,(\delta,\varepsilon) \in \mathbb{R}^{>0} \times \mathbb{R}^{>0} \mid (\gamma,\gamma') \in V_{\delta,\varepsilon}\,\}\ .$$

In particular, for a sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ in Z and a path $\gamma \in Z$, we have $\gamma = \lim_{k\to\infty}\gamma_k$ in the uniformity $\mathcal{V}_2$ iff for every $(\delta,\varepsilon) \in R_2$, there exists an index m such that $\gamma_k \in V_{\delta,\varepsilon}(\gamma)$ for all $k \geq m$.

Proof: The first inclusion is trivial, and for the second, suppose paths γ and γ' are such that $(\gamma,\gamma') \in$

$V_{\delta_1,\varepsilon_1} \circ V_{\delta_2,\varepsilon_2}$. Hence there exists a third path $\gamma_1 \in Z$ such that $(\gamma, \gamma_1) \in V_{\delta_1,\varepsilon_1}$ and $(\gamma_1, \gamma') \in V_{\delta_2,\varepsilon_2}$. Now $(\gamma, \gamma_1) \in V_{\delta_1,\varepsilon_1}$ implies that there exists a retiming $\rho_1 \in \mathsf{Ret}(\mathrm{dom}(\gamma), \mathrm{dom}(\gamma_1))$, such that $\mathrm{dev}(\rho_1) < \delta_1$ and $d_{X\mathrm{sup}}(\gamma, \gamma_1, \rho_1) < \varepsilon_1$, and without loss of generality, we may suppose that $\mathrm{dom}(\rho_1) = \mathrm{dom}(\gamma)$ and $\mathrm{ran}(\rho_1) = \mathrm{dom}(\gamma_1)$. On the other side, $(\gamma_1, \gamma') \in V_{\delta_2,\varepsilon_2}$ implies that there exists a retiming $\rho_2 \in \mathsf{Ret}(\mathrm{dom}(\gamma_1), \mathrm{dom}(\gamma'))$, such that $\mathrm{dev}(\rho_2) < \delta_2$ and $d_{X\mathrm{sup}}(\gamma_1, \gamma', \rho_2) < \varepsilon_2$, and without loss of generality, we may suppose that $\mathrm{dom}(\rho_2) = \mathrm{dom}(\gamma_1)$ and $\mathrm{ran}(\rho_2) = \mathrm{dom}(\gamma')$. Now set $\delta' := \delta_1 + \delta_2$ and $\varepsilon' := \varepsilon_1 + \varepsilon_2$. By Proposition 4.3, the composite map $\rho := \rho_1 \circ \rho_2$ is a retiming $\rho \colon \mathrm{dom}(\gamma) \rightsquigarrow \mathrm{dom}(\gamma')$ with $\mathrm{dev}(\rho) \le \mathrm{dev}(\rho_1) + \mathrm{dev}(\rho_2) < \delta_1 + \delta_2 = \delta'$. It remains to show that $d_{X\mathrm{sup}}(\gamma, \gamma', \rho) < \varepsilon'$. To see this, fix arbitrary points $t \in \mathrm{dom}(\rho) = \mathrm{dom}(\gamma)$ and $t' \in \mathrm{ran}(\rho) = \mathrm{dom}(\gamma')$ such that $(t, t') \in \rho$. Since ρ is the composition $\rho_1 \circ \rho_2$, there exists $t_1 \in \mathrm{ran}(\rho_1) = \mathrm{dom}(\rho_2)$ such that $(t, t_1) \in \rho_1$ and $(t_1, t') \in \rho_2$. Then since $\rho_1 \in \mathsf{Ret}(\gamma, \gamma_1)$ and $d_{X\mathrm{sup}}(\gamma, \gamma_1, \rho_1) < \varepsilon_1$, we know that $d_X(\gamma(t), \gamma_1(t_1)) < \varepsilon_1$. Additionally, since $\rho_2 \in \mathsf{Ret}(\gamma_1, \gamma')$ and $d_{X\mathrm{sup}}(\gamma_1, \gamma', \rho_2) < \varepsilon_2$, we know that $d_X(\gamma_1(t_1), \gamma'(t')) < \varepsilon_2$. Then by the triangle inequality for the metric d_X on X, we have: $d_X(\gamma(t), \gamma'(t')) \le d_X(\gamma(t), \gamma_1(t_1)) + d_X(\gamma_1(t_1), \gamma'(t'))) < \varepsilon_1 + \varepsilon_2 =< \varepsilon'$, so $d_X(\gamma(t), \gamma'(t')) < \varepsilon'$, as required. The third inclusion, that $V_{\delta'',\varepsilon''} \circ V_{\delta'',\varepsilon''} \subseteq V_{\delta_1,\varepsilon_1}$ when $\varepsilon'' := \frac{1}{2}\varepsilon_1$ and $\delta'' := \frac{1}{2}\delta_1$, is a particular case of the second inclusion.

For first-countability of a uniform space, we only need show that there exists a countable subset of entourages $\mathcal{V}_0 \subseteq \mathcal{V}_2$ such that $\mathcal{V}_0$ constitutes a basis for a uniformity, and the topology $\mathcal{T}_{\mathcal{V}_0}$ generated by $\mathcal{V}_0$ is equal to $\mathcal{T}_2$. So take $\mathcal{V}_0 := \{V_{\delta,\varepsilon} \mid \varepsilon \in \mathbb{Q}^{>0} \wedge \delta \in \mathbb{Q}^{>0}\}$, the family of all entourages parameterized by positive rationals. The verifications are then straightforward.

Now suppose that the path set $Z \subseteq \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T, X)$ is such that for some deadlock-free and prefix-closed set of compact paths $P \subseteq \mathsf{CPath}(T, X)$, we have $P \subseteq Z$ and $Z \setminus \mathsf{CPath}(T, X) \subseteq \mathsf{M}(P) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T, X)$. Hence for all $\gamma \in Z$, either $\gamma \in \mathsf{CPath}(T, X)$ or $\gamma \in \mathsf{M}(P) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T, X)$. To prove that the topology $\mathcal{T}_2$ is Hausdorff, it suffices to prove the topology satisfies $\mathbf{T_0}$ separation, which for a uniform space is equivalent to the condition that $(\gamma, \gamma') \in \bigcap \mathcal{V}_2$ iff $\gamma = \gamma'$. The ($\Leftarrow$) direction holds for all uniform spaces. For the converse ($\Rightarrow$) implication, suppose a pair of paths (γ, γ') is such that $\gamma \neq \gamma'$; we need to exhibit a parameter pair $(\delta_1, \varepsilon_1) \in R_2$ such that $(\gamma, \gamma') \notin V_{\delta_1,\varepsilon_1}$, which means that for all retimings $\rho \in \mathsf{Ret}(\gamma, \gamma')$ such that $\mathrm{dev}(\rho) < \delta_1$, there exists $(t_1, t_2) \in \rho$ such that $d_X(\gamma(t_1), \gamma'(t_2)) \ge \varepsilon_1$. Since $\gamma \neq \gamma'$, there are two cases to consider: *Case I*: $\mathrm{dom}(\gamma) = \mathrm{dom}(\gamma')$ but there exists $t_1 \in \mathrm{dom}(\gamma)$ such that $\gamma(t_1) \neq \gamma'(t_1)$; and *Case II*: $\mathrm{dom}(\gamma) \neq \mathrm{dom}(\gamma')$.

In *Case I*, set $\varepsilon_1 := \frac{1}{2} d_X(\gamma(t_1), \gamma'(t_1))$, so $\varepsilon_1 > 0$ since d_X is a metric. Since γ' is continuous at the time point t_1, there exists a real $\delta_0 > 0$ such that for all $s \in \mathrm{dom}(\gamma')$, if $\| t_1 - s \| < \delta_0$ then $d_X(\gamma'(t_1), \gamma'(s)) < \varepsilon_1$. Then set:

$$\delta_1 \ := \ \sup \{ \delta \in (0, \delta_0] \mid \forall s \in \mathrm{dom}(\gamma') : \ \| t_1 - s \| < \delta \ \Rightarrow \ d_X(\gamma'(t_1), \gamma'(s)) < \varepsilon_1 \}$$

So δ_1 is well-defined (since the reals are Dedkind-complete) and $\delta_1 > 0$. Now consider any retiming $\rho \in \mathsf{Ret}(\gamma, \gamma')$ such that $\mathrm{dev}(\rho) < \delta_1$, and choose any time point $t_2 \in \rho(t_1)$. Then we have $t_2 \in \mathrm{dom}(\gamma') = \mathrm{dom}(\gamma)$, and $\| t_1 - t_2 \| < \delta_1$, hence $d_X(\gamma'(t_1), \gamma'(t_2)) < \varepsilon_1$. By the triangle inequality for the metric d_X, we have $2\varepsilon_1 = d_X(\gamma(t_1), \gamma'(t_1)) \le d_X(\gamma(t_1), \gamma'(t_2)) + d_X(\gamma'(t_1), \gamma'(t_2))$, and hence:

$$d_X(\gamma(t_1), \gamma'(t_2)) \;\geq\; 2\varepsilon_1 - d_X(\gamma'(t_1), \gamma'(t_2)) \;>\; \varepsilon_1 \,.$$

Hence $(\gamma, \gamma') \notin V_{\delta_1, \varepsilon_1}$, as required.

In *Case II*, when $\mathrm{dom}(\gamma) \neq \mathrm{dom}(\gamma')$, we either have $(\mathrm{dom}(\gamma) \setminus \mathrm{dom}(\gamma')) \neq \varnothing$. or $(\mathrm{dom}(\gamma') \setminus \mathrm{dom}(\gamma)) \neq \varnothing$; by symmetry, we can consider only the first of these. We then break into further sub-cases, depending upon whether (a) $\gamma' \in \mathsf{CPath}(T, X)$, or else (b) $\gamma' \in \mathsf{M}(P) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T, X)$ for some deadlock-free and prefix-closed set of compact paths $P \subseteq \mathsf{CPath}(T, X)$.

In *Case II.a*, when $\mathrm{dom}(\gamma')$ is compact, choose any element $t_1 \in (\mathrm{dom}(\gamma) \setminus \mathrm{dom}(\gamma'))$, and then set:

$$\delta_1 \;:=\; \inf \{ \| t_1 - s \|_T \;\mid\; s \in \mathrm{dom}(\gamma') \} \,.$$

Since $\mathrm{dom}(\gamma')$ is compact, and hence closed, in the norm topology on T, and $t_1 \notin \mathrm{dom}(\gamma')$, we can conclude that $\delta_1 > 0$. This means that for every retiming $\rho \in \mathsf{Ret}(\gamma, \gamma')$ and for every $t_2 \in \rho(t_1) \subseteq \mathrm{dom}(\gamma')$, we have $\| t_1 - t_2 \|_T \geq \delta_1$ and hence $\mathrm{dev}(\rho) \geq \delta_1$. This now means that for *any choice* of $\varepsilon_1 > 0$, it is the case that for all retimings $\rho \in \mathsf{Ret}(\gamma, \gamma')$, if $d_{X\mathrm{sup}}(\gamma, \gamma', \rho) < \varepsilon_1$, then $\mathrm{dev}(\rho) \geq \delta_1$. Hence $(\gamma, \gamma') \notin V_{\delta_1, \varepsilon_1}$, as required.

In *Case II.b*, when $\gamma' \in \mathsf{M}(P) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T, X)$ for P deadlock-free and prefix-closed, we have that $\mathrm{dom}(\gamma')$ is bounded but not closed in the norm topology; set $t_0 := \sup(\mathrm{dom}(\gamma'))$, so $t_0 \notin \mathrm{dom}(\gamma')$. We now break into three exhaustive sub-sub-cases, to complete the proof.

• *Case II.b.1*: there exists $t_1 \in (\mathrm{dom}(\gamma) \setminus \mathrm{dom}(\gamma'))$ such that $t_1 \not\geqslant t_0$. Then we know that $\mathrm{dom}(\gamma'|_{t_1})$ is compact, and $\gamma'|_{t_1} \in P$ since P is prefix-closed, and $\max(\mathrm{dom}(\gamma'|_{t_1})) < t_1$. Now set $\delta_1 := \| t_1 - \max(\mathrm{dom}(\gamma'|_{t_1})) \|_T > 0$. This means that for every retiming $\rho \in \mathsf{Ret}(\gamma|_{t_1}, \gamma'|_{t_1})$ and for every $t_2 \in \rho(t_1) \subseteq \mathrm{dom}(\gamma'|_{t_1})$, we have $\| t_1 - t_2 \|_T \geq \delta_1$ and hence $\mathrm{dev}(\rho) \geq \delta_1$. This now means that for *any choice* of $\varepsilon_1 > 0$, it is the case that for all retimings $\rho \in \mathsf{Ret}(\gamma|_{t_1}, \gamma'|_{t_1})$, if $d_{X\mathrm{sup}}(\gamma|_{t_1}, \gamma'|_{t_1}, \rho) < \varepsilon_1$, then $\mathrm{dev}(\rho) \geq \delta_1$. Hence $(\gamma|_{t_1}, \gamma'|_{t_1}) \notin V_{\delta_1, \varepsilon_1}$. In this case, we must also have $(\gamma, \gamma') \notin V_{\delta_1, \varepsilon_1}$, as required.

Note that the negation of the condition for *Case II.b.1* is that for all $t_1 \in (\mathrm{dom}(\gamma) \setminus \mathrm{dom}(\gamma'))$, we have $t_1 \geqslant t_0$, which is equivalent to the condition that for all $t_1 \in (\mathrm{dom}(\gamma)$, if $t_1 \not\geqslant t_0$ then $t_1 \in \mathrm{dom}(\gamma'))$; given that time domains of paths are linearly-ordered, so $\mathrm{dom}(\gamma) \cap (t_0 \perp) = \varnothing$, this latter condition is in turn equivalent to the property that $[0, t_0) \cap \mathrm{dom}(\gamma) \subseteq \mathrm{dom}(\gamma')$.

• *Case II.b.2*: $[0, t_0) \cap \mathrm{dom}(\gamma) \subseteq \mathrm{dom}(\gamma')$ and $(\mathrm{dom}(\gamma) \setminus \mathrm{dom}(\gamma')) = \{ t_0 \}$. In this case, we have $\mathrm{dom}(\gamma') \subsetneq \mathrm{dom}(\gamma) = [0, t_0] \cap \mathrm{dom}(\gamma)$, and thus $\gamma = \gamma|_{t_0} \in P$ is a compact path. Now by the maximality of γ' with respect to P, with $\gamma \in Z = (P \cup \mathsf{M}(P)) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T, X)$, we know that $\gamma' \not\subset \gamma|_{t_0}$, for otherwise we would have $\gamma' < \gamma$, contradicting the maximality of γ'. Thus there must exist a time $t_1 \in \mathrm{dom}(\gamma')$ such that $\gamma'(t_1) \neq \gamma(t_1)$. Now set $\varepsilon_1 := \frac{1}{2} d_X(\gamma(t_1), \gamma'(t_1))$, so $\varepsilon_1 > 0$ since d_X is a metric, and then proceed as in *Case I* to determine a parameter $\delta_1 > 0$ such that $(\gamma, \gamma') \notin V_{\delta_1, \varepsilon_1}$, as required.

• *Case II.b.3*: $[0, t_0) \cap \mathrm{dom}(\gamma) \subseteq \mathrm{dom}(\gamma')$ and there exists $t_1 \in (\mathrm{dom}(\gamma) \setminus \mathrm{dom}(\gamma'))$ such that $t_1 > t_0$, and thus $\gamma|_{t_1}$ is a compact path and $\gamma|_{t_1} \in P$, since P is prefix-closed. By the maximality of γ' with respect to P, we know that $\gamma' \not\leq \gamma|_{t_1}$, and hence either $\mathrm{dom}(\gamma') \not\leq \mathrm{dom}(\gamma|_{t_1})$, or else $\mathrm{dom}(\gamma') < \mathrm{dom}(\gamma|_{t_1})$ and $\gamma' \not\subset \gamma|_{t_1}$. To finish the proof, we break into these two exhaustive sub-sub-sub-cases.

⋆ *Case II.b.3.i*: $\mathrm{dom}(\gamma') \not\leq \mathrm{dom}(\gamma|_{t_1})$, and hence there exists a time $t_3 \in \mathrm{dom}(\gamma|_{t_1}) \setminus \mathrm{dom}(\gamma')$ and a time $s_3 \in \mathrm{dom}(\gamma')$ such that such that $t_3 \not> s_3$. But by the linear ordering on time domains of paths, we have $t_1 \geqslant t_3 \geqslant t_0$, and $t_0 > s_3$, and thus we must have $t_3 > s_3$, so this case cannot happen.

⋆ *Case II.b.3.ii*: $\mathrm{dom}(\gamma') < \mathrm{dom}(\gamma|_{t_1})$ and $\gamma' \not\subset \gamma|_{t_1}$. In this case, there must exist a time point $s_3 \in \mathrm{dom}(\gamma')$ such that $\gamma'(s_3) \neq \gamma(s_3)$. Now set $\varepsilon_1 := \frac{1}{2} d_X(\gamma(s_3), \gamma'(s_3))$, so $\varepsilon_1 > 0$ since d_X is a metric, and then proceed as in *Case I* to determine a parameter $\delta_1 > 0$ such that $(\gamma', \gamma|_{t_1}) \notin V_{\delta_1,\varepsilon_1}$. We can then conclude that $(\gamma', \gamma) \notin V_{\delta_1,\varepsilon_1}$, as required. ■

For example, if $H = (X, F, G, C, D)$ is a hybrid system satisfying the conditions of [3], Proposition 2.4, with prefix-closed general flow map $\Phi^H : X \rightsquigarrow \mathsf{CPath}(\mathbb{H}, X)$, then the set $P := \mathrm{ran}(\Phi^H)$ of compact paths is deadlock-free and prefix-closed, and thus the set $Z = (P \cup \mathsf{M}(P)) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(\mathbb{H}, X)$ is a metrizable space under the 2-parameter uniform topology $\mathcal{T}_2$.

More generally, for any finite dimensional time structure, if $\Phi : X \rightsquigarrow \mathsf{CPath}(T, X)$ is deadlock-free and $P := \mathrm{ran}(\Phi^H)$ is prefix-closed, then the set $Z = (P \cup \mathsf{M}(P)) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T, X)$ with the uniform topology $\mathcal{T}_2$ is a metrizable space.

The following result confirms that the basic entourages $V_{\delta,\varepsilon} \in \mathcal{V}_2$ of the uniformity on $\mathsf{CPath}(T, X)$ are closed under the operations of taking prefixes and suffixes of paths, and of taking fusions of paths; this is close to, but actually weaker than, these partial functions on the path space being uniformly continuous w.r.t. the uniformity generated by $\mathcal{V}_2$ on the path space[4].

Theorem 4.6 [Basic operations on compact paths within 2-parameter uniform topology]
Let S be a finite dimensional time structure with future time T, and let (X, d_X) be a metric space. For any paths $\gamma, \gamma', \gamma_1, \gamma'_1 \in \mathsf{CPath}(T, X)$ and for any parameter pair $(\delta, \varepsilon) \in R_2$, if $(\gamma, \gamma') \in V_{\delta,\varepsilon}$ and the retiming $\rho \in \mathsf{Ret}(\gamma, \gamma')$ witnesses this, with $\mathrm{dev}(\rho) < \delta$ and $d_{X\mathrm{sup}}(\gamma, \gamma', \rho) < \varepsilon$, then for all $(s, s') \in \rho$, we have:

1. *the s/s'-prefixes: $(\gamma|_s, \gamma'|_{s'}) \in V_{\delta,\varepsilon}$;*
2. *the s/s'-suffixes: $({}_s|\gamma, {}_{s'}|\gamma') \in V_{\delta,\varepsilon}$; and*
3. *the s/s'-fusions: if $(\gamma_1, \gamma'_1) \in V_{\delta,\varepsilon}$ and $\gamma(s) = \gamma_1(0)$ and $\gamma'(s') = \gamma'_1(0)$, then $(\gamma *_s \gamma_1, \gamma' *_{s'} \gamma'_1) \in V_{\delta,\varepsilon}$.*

[4] A partial function $f : \mathsf{CPath}(T, X) \dashrightarrow \mathsf{CPath}(T, X)$ is *uniformly continuous* w.r.t. $\mathcal{V}_2$ if there exists a function $u : R_2 \to R_2$ such that for every parameter pair $(\delta, \varepsilon) \in R_2$, and for $(\delta', \varepsilon') = u(\delta, \varepsilon)$, and for all $\gamma, \gamma' \in \mathrm{dom}(f)$, if $(\gamma, \gamma') \in V_{\delta',\varepsilon'}$ then $(f(\gamma), f(\gamma')) \in V_{\delta,\varepsilon}$, and hence $\mathrm{dom}(f) \cap V_{\delta',\varepsilon'}(\gamma) \subseteq f^{-1}(V_{\delta,\varepsilon}(f(\gamma))$. In contrast, a partial function $f : \mathsf{CPath}(T, X) \dashrightarrow \mathsf{CPath}(T, X)$ is (merely) *continuous* w.r.t. the topology $\mathcal{T}_2$ if for each $\gamma \in \mathrm{dom}(f)$ and for every parameter pair $(\delta, \varepsilon) \in R_2$, there exists a pair $(\delta', \varepsilon') \in R_2$ such that for all $\gamma' \in \mathrm{dom}(f)$, if $\gamma' \in V_{\delta',\varepsilon'}(\gamma)$ then $f(\gamma') \in V_{\delta,\varepsilon}(f(\gamma))$; that is, if γ' is in the (δ', ε')-tube around γ, then $f(\gamma')$ is in the (δ, ε)-tube around $f(\gamma)$.

Proof: Fix $(s,s') \in \rho$, where $\mathrm{dom}(\rho) = \mathrm{dom}(\gamma)$ and $\mathrm{ran}(\rho) = \mathrm{dom}(\gamma')$. For closure under prefixes, consider the map $\rho_0 \colon T \rightsquigarrow T$ such that $\rho_0(t) := [0,s'] \cap \rho(t)$ for all $t \in \mathrm{dom}(\rho_0) := \mathrm{dom}(\gamma) \cap [0,s] = \mathrm{dom}(\gamma|_s)$. Hence $\mathrm{ran}(\rho_0) = \mathrm{dom}(\gamma') \cap [0,s'] = \mathrm{dom}(\gamma'|_{s'})$. It is readily verified that $\rho_0 \in \mathsf{Ret}(\gamma|_s, \gamma'|_{s'})$, and it is immediate that $\mathrm{dev}(\rho_0) \leq \mathrm{dev}(\rho) < \delta$. It is also clear that $d_{X\mathrm{sup}}(\gamma|_s, \gamma'|_{s'}, \rho_0) \leq d_{X\mathrm{sup}}(\gamma,\gamma',\rho) < \varepsilon$. Hence $(\gamma|_s, \gamma'|_{s'}) \in V_{\delta,\varepsilon}$, as required.

For closure under suffixes, consider the paths ${}_s|\gamma$ and ${}_{s'}|\gamma'$ with $\mathrm{dom}({}_s|\gamma) = \sigma^{-s}(\mathrm{dom}(\gamma)) \cap T$ and $\mathrm{dom}({}_{s'}|\gamma') = \sigma^{-s'}(\mathrm{dom}(\gamma')) \cap T$. Now for each $t' \in \mathrm{dom}({}_{s'}|\gamma')$ with $t' > 0$, we have $s' + t' \in \mathrm{dom}(\gamma')$ with $s' < s' + t'$, hence $\rho^{-1}(s') \trianglelefteq \rho^{-1}(s'+t')$; in particular, $s \in \rho^{-1}(s')$, so either $s \in \rho^{-1}(s'+t')$ or else $s < w$ for all $w \in \rho^{-1}(s'+t')$. Going the other way, for each $t \in \mathrm{dom}({}_s|\gamma)$ with $t > 0$, we have $s + t \in \mathrm{dom}(\gamma)$ with $s < s+t$, hence $\rho(s) \trianglelefteq \rho(s+t)$; in particular, $s' \in \rho(s)$, so either $s' \in \rho(s+t)$ or else $s' < v$ for all $v \in \rho(s+t)$. Now let $\rho_0 \colon T \rightsquigarrow T$ be the map such that:

$$\rho_0 \ := \ \{\, (t,t') \in \mathrm{dom}({}_s|\gamma) \times \mathrm{dom}({}_{s'}|\gamma') \ \mid \ (s+t, s'+t') \in \rho \,\}\,.$$

Then $\mathrm{dom}(\rho_0) = \mathrm{dom}({}_s|\gamma)$ and $\mathrm{ran}(\rho_0) = \mathrm{dom}({}_{s'}|\gamma')$, and it is readily established that $\rho_0 \in \mathsf{Ret}({}_s|\gamma, {}_{s'}|\gamma')$. It is immediate that $\mathrm{dev}(\rho_0) \leq \mathsf{dev}(\rho) < \delta$. It is also clear that $d_{X\mathrm{sup}}({}_s|\gamma, {}_{s'}|\gamma', \rho_0) \leq d_{X\mathrm{sup}}(\gamma,\gamma',\rho) < \varepsilon$. Hence $({}_s|\gamma, {}_{s'}|\gamma') \in V_{\delta,\varepsilon}$, as required.

Finally, for closure under the fusion operation, suppose $(s,s') \in \rho$ and suppose that $(\gamma_1, \gamma_1') \in V_{\delta,\varepsilon}$ and $\gamma(s) = \gamma_1(0)$ and $\gamma'(s') = \gamma_1'(0)$. Let $\rho_1 \in \mathsf{Ret}(\gamma_1,\gamma_1')$ be a witness retiming, such that with $\mathrm{dev}(\rho_1) < \delta$ and $d_{X\mathrm{sup}}(\gamma_1,\gamma_1',\rho_1) < \varepsilon$, and suppose $\mathrm{dom}(\rho_1) = \mathrm{dom}(\gamma_1)$ and $\mathrm{ran}(\rho_1) = \mathrm{dom}(\gamma_1')$. Now consider the paths $\gamma_2 := \gamma *_s \gamma_1$ and $\gamma_2' := \gamma' *_{s'} \gamma_1'$. Then we have $\mathrm{dom}(\gamma_2) = \mathrm{dom}(\gamma|_s) \cup \sigma^s(\mathrm{dom}(\gamma_1))$ and $\mathrm{dom}(\gamma_2') = \mathrm{dom}(\gamma'|_{s'}) \cup \sigma^{s'}(\mathrm{dom}(\gamma_1'))$ Now let $\rho_2 \colon T \rightsquigarrow T$ be the map such that $\mathrm{dom}(\rho_2) = \mathrm{dom}(\gamma_2)$, and for all $t \in \mathrm{dom}(\rho_2)$,

$$\rho_2(t) \ := \ \begin{cases} [0,s'] \cap \rho(t) & \text{if } \ t \in \mathrm{dom}(\gamma) \cap [0,s) \\ \{\, t' \in T \ \mid \ (t-s, t'-s') \in \rho_1 \,\} & \text{if } \ s+t \in \mathrm{dom}(\gamma_1)\,. \end{cases}$$

So ρ_2 is formed by glueing together the restriction of the retiming ρ to $[0,s] \times [0,s']$ with the s/s'-translation of the retiming ρ_1 from $\mathrm{dom}(\gamma_1)$ to $\mathrm{dom}(\gamma_1')$. One then verifies that $\mathrm{ran}(\rho_2) = \mathrm{dom}(\gamma_2')$, and that $\rho_2 \in \mathsf{Ret}(\gamma_2, \gamma_2')$. Now we have $\mathrm{dev}(\rho) < \delta$ and $\mathrm{dev}(\rho_1) < \delta$, with ρ_2 formed from a disjoint union of a restriction of ρ and a translated version of ρ_1. It then follows that we have $\mathrm{dev}(\rho_2) < \delta$ as well. Moreover, since $d_{X\mathrm{sup}}(\gamma,\gamma',\rho) < \varepsilon$ and $d_{X\mathrm{sup}}(\gamma_1,\gamma_1',\rho_1) < \varepsilon$, we can also conclude that $d_{X\mathrm{sup}}(\gamma_2,\gamma_2',\rho_2) < \varepsilon$, as required. ∎

We now turn to spaces $Z \subseteq \mathsf{CPath}^\infty(T,X) = \mathsf{CPath}(T,X) \cup \mathsf{Ext}(\mathsf{CPath}(T,X))$ containing both compact paths and limit continuous paths, and the subspace $\mathsf{CPath}^\infty_{\mathrm{cl}}(T,X)$. Starting from the the 2-parameter uniform structure on bounded paths, the key idea is that since a limit path is just the union of a chain of longer and longer compact prefixes, we should look at closeness of longer and longer compact prefixes, and thus we should introduce a third parameter which references the *time position* up to which two limit paths are required to be (δ, ε)-close.

Theorem 4.7 [3-parameter uniform topology on path spaces]
Let S be a finite dimensional time structure with future time T, let (X, d_X) be a metric space, and let $Z \subseteq \mathsf{CPath}^\infty(T, X)$ be any set of paths, compact or limit. For the parameter space $R_2 \times T := \mathbb{R}^{>0} \times \mathbb{R}^{>0} \times T$, define for each triple $(\delta, \varepsilon, t) \in R_2 \times T$ the relation $U_{\delta,\varepsilon,t} \colon Z \rightsquigarrow Z$ as follows:

$$U_{\delta,\varepsilon,t} \;:=\; \{ (\eta, \eta') \in Z \times Z \mid (\eta|_t, \eta'|_t) \in V_{\delta,\varepsilon} \} \;.$$

Then for all $(\delta, \varepsilon) \in R_2$, for all compact paths $\gamma \in Z \cap \mathsf{CPath}(T, X)$ with $b_\gamma = \max(\mathrm{dom}(\gamma))$, and for all paths $\eta \in Z$,

$$V_{\delta,\varepsilon}(\gamma) \subseteq \bigcap_{t \geqslant b_\gamma} U_{\delta,\varepsilon,t}(\gamma) \quad \textit{and} \quad U_{\delta,\varepsilon,0}(\eta) = \{\eta' \in Z \mid d_X(\eta(0), \eta'(0)) < \varepsilon\} \subseteq \bigcap_{t \in T} U_{\delta,\varepsilon,t}(\eta)$$

and for all $(\delta, \varepsilon, t), (\delta_1, \varepsilon_1, t_1), (\delta_2, \varepsilon_2, t_2) \in R_2 \times T$:

$$\begin{array}{rcll}
U_{\delta_1,\varepsilon_1,t_1} & \subseteq & U_{\delta_2,\varepsilon_2,t_2} & \textit{when } (\delta_1, \varepsilon_1) \leqslant (\delta_2, \varepsilon_2) \textit{ and } t_1 \geqslant t_2 \\
U_{\delta,\varepsilon,t} & \subseteq & U_{\delta_1,\varepsilon_1,t_1} \cap U_{\delta_2,\varepsilon_2,t_2} & \textit{when } (\delta, \varepsilon) \leqslant (\delta_1, \varepsilon_1) \wedge (\delta_2, \varepsilon_2) \textit{ and } t \geqslant (t_1 \vee t_2) \\
U_{\delta_1,\varepsilon_1,t_1} \circ U_{\delta_2,\varepsilon_2,t_2} & \subseteq & U_{\delta,\varepsilon,t} & \textit{when } (\delta_1, \varepsilon_1) + (\delta_2, \varepsilon_2) \leqslant (\delta, \varepsilon) \textit{ and } t \leqslant (t_1 \wedge t_2) \,. \\
U_{\delta_1,\varepsilon_1,t_1} \circ U_{\delta_1,\varepsilon_1,t_1} & \subseteq & U_{\delta,\varepsilon,t} & \textit{when } (\delta_1, \varepsilon_1) \leqslant \tfrac{1}{2}(\delta, \varepsilon) \textit{ and } t \leqslant t_1 \,.
\end{array}$$

Hence the family $\mathcal{U}_3 := \{ U_{\delta,\varepsilon,t} \colon Z \rightsquigarrow Z \mid (\delta, \varepsilon, t) \in R_2 \times T \}$ constitutes a basis for a uniformity on the path set Z. The 3-parameter uniform topology $\mathcal{T}_3$ on Z will be first-countable if the time structure T is finite-dimensional, and it will be Hausdorff if $Z \subseteq \mathsf{CPath}^\infty(T, X)$ is such that for some deadlock-free and prefix-closed set of compact paths $P \subseteq \mathsf{CPath}(T, X)$, we have $P \subseteq Z$ and $Z \setminus \mathsf{CPath}^\infty_{\mathrm{cl}}(T, X) \subseteq \mathsf{M}(P) \cap \mathsf{CPath}^\infty_{\mathrm{bd}}(T, X)$. When the topology $\mathcal{T}_3$ is first-countable and Hausdorff, it is metrizable, and in this case we can use the metric $d_3 : Z \times Z \to \mathbb{R}^+$ given by:

$$d_3(\eta, \eta') \;:=\; \max\{ \delta_0, \varepsilon_0, r_0 \}$$

where

$$(\delta_0, \varepsilon_0, r_0) \;:=\; \inf\Big\{ (\delta, \varepsilon, r) \in (\mathbb{R}^{>0})^3 \mid \exists\, t \in T,\ r = \exp(-\|t\|_T) \wedge (\eta, \eta') \in U_{\delta,\varepsilon,t} \Big\} \,.$$

Proof: The verifications are straight-forward adaptations of the proof of Theorem 4.5, with the adjustment that for the 3-parameter uniformity, the basic entourages are inclusion-anti-monotone in the third parameter $t \in T$, while being inclusion-monotone in the first two parameters. For the metric d_3, we use the mapping $r = \exp(-\|t\|_T)$ to give a positive real-valued parameter that is also anti-monotone in the third parameter $t \in T$, since $t \geqslant t'$ implies $\exp(-\|t\|_T) \leq \exp(-\|t'\|_T)$.

Now suppose that $Z \subseteq \mathsf{CPath}^\infty(T, X)$ is such that for some deadlock-free and prefix-closed set of compact paths $P \subseteq \mathsf{CPath}(T, X)$, we have $P \subseteq Z$ and $Z \setminus \mathsf{CPath}^\infty_{\mathrm{cl}}(T, X) \subseteq \mathsf{M}(P) \cap \mathsf{CPath}^\infty_{\mathrm{bd}}(T, X)$.

To prove that the topology $\mathcal{T}_3$ is Hausdorff, it suffices to prove of $\eta, \eta' \in Z$, that if $\eta \neq \eta'$, then there exists a parameter triple $(\delta_1, \varepsilon_1, t_1) \in R_2 \times T$ such that $(\eta, \eta') \notin U_{\delta_1, \varepsilon_1, t_1}$, which means that for all retimings $\rho \in \mathsf{Ret}(\eta|_{t_1}, \eta'|_{t_1})$ such that $\mathrm{dev}(\rho) < \delta_1$, there exists $(s_1, s_2) \in \rho$ such that $d_X(\eta(s_1), \eta'(s_2)) \geq \varepsilon_1$. As in the proof of the Hausdorff property for Theorem 4.5, there are two cases to consider: *Case I*: $\mathrm{dom}(\eta) = \mathrm{dom}(\eta')$ but there exists $s_1 \in \mathrm{dom}(\eta)$ such that $\eta(s_1) \neq \eta'(s_1)$; and *Case II*: $\mathrm{dom}(\eta) \neq \mathrm{dom}(\eta')$.

In *Case I*, set $\varepsilon_1 := \frac{1}{2} d_X(\eta(s_1), \eta'(s_1))$, so $\varepsilon_1 > 0$ since d_X is a metric. Since η' is continuous at the time point s_1, there exists a real $\delta_0 > 0$ such that for all $t \in \mathrm{dom}(\gamma')$, if $\| s_1 - t \| < \delta_0$ then $d_X(\gamma'(s_1), \gamma'(t)) < \varepsilon_1$. Then set:

$$\delta_1 := \sup \{ \delta \in (0, \delta_0] \mid \forall t \in \mathrm{dom}(\eta') : \| s_1 - t \| < \delta \Rightarrow d_X(\eta'(s_1), \eta'(t)) < \varepsilon_1 \} .$$

Now set $t_1 := s_1$ and set $\gamma := \eta|_{t_1}$ and $\gamma' := \eta'|_{t_1}$. Then proceed as in *Case I* of the proof of the Hausdorff property for Theorem 4.5, to conclude that $(\gamma, \gamma') \notin V_{\delta_1, \varepsilon_1}$, and hence $(\eta, \eta') \notin U_{\delta_1, \varepsilon_1, t_1}$, as required.

In *Case II*, when $\mathrm{dom}(\eta) \neq \mathrm{dom}(\eta')$, we either have $(\mathrm{dom}(\eta) \setminus \mathrm{dom}(\eta')) \neq \emptyset$. or $(\mathrm{dom}(\eta') \setminus \mathrm{dom}(\eta)) \neq \emptyset$; by symmetry, we can consider only the first of these. We then break into further sub-cases, depending upon whether (a) $\eta' \in \mathsf{CPath}^{\infty}_{\mathrm{cl}}(T, X)$, or (b) $\eta' \in \mathsf{M}(P) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T, X)$.

In *Case II.a*, when $\mathrm{dom}(\eta')$ is norm-closed, choose any element $t_1 \in (\mathrm{dom}(\eta) \setminus \mathrm{dom}(\eta'))$, and then set:

$$\delta_1 := \inf \{ \| t_1 - s \|_T \mid s \in \mathrm{dom}(\gamma') \} .$$

Since $\mathrm{dom}(\eta')$ is closed in the norm topology on T, and $t_1 \notin \mathrm{dom}(\eta')$, we can conclude that $\delta_1 > 0$. Then proceed as in *Case II.a* of the proof of the Hausdorff property for Theorem 4.5, with $\gamma := \eta|_{t_1}$ and $\gamma' := \eta'|_{t_1}$, to derive the conclusion that $(\gamma, \gamma') \notin V_{\delta_1, \varepsilon_1}$ for suitable $\varepsilon_1 > 0$, and hence $(\eta, \eta') \notin U_{\delta_1, \varepsilon_1, t_1}$.

In *Case II.b*, when $\eta' \in \mathsf{M}(P) \cap \mathsf{CPath}^{\infty}_{\mathrm{bd}}(T, X)$, we know that $\mathrm{dom}(\eta')$ is bounded but not closed in the norm topology; set $t_0 := \sup(\mathrm{dom}(\eta'))$, so $t_0 \notin \mathrm{dom}(\eta')$, and set $s_0 := \sup([t_0, 2t_0] \cap \mathrm{dom}(\eta))$. Then proceed as in *Case II.b* of the proof of the Hausdorff property for Theorem 4.5, with $\gamma := \eta|_{s_0}$ and $\gamma' := \eta'|_{s_0} = \eta'$, to derive (via three sub-sub-cases) the conclusion that $(\gamma, \gamma') \notin V_{\delta_1, \varepsilon_1}$ for suitable $\varepsilon_1 > 0$, and hence $(\eta, \eta') \notin U_{\delta_1, \varepsilon_1, s_0}$, to complete the proof. ∎

In comparison, the natural extension of the Skorokhod metric topology [17, 18, 19] on path spaces $Z \subseteq \mathsf{CPath}^{\infty}(T, X)$ for finite-dimensional T can be described by a uniformity with basic entourages $U^{\mathrm{Skor}}_{\delta, \varepsilon, t}$ such that $(\eta, \eta') \in U^{\mathrm{Skor}}_{\delta, \varepsilon, t}$ iff there exists a *bijective*, single-valued retiming $\rho \in \mathsf{BRet}(\eta|_t, \eta'|_t)$ such that $\mathrm{dev}(\rho) < \delta$ and $d_{X\mathrm{sup}}(\eta|_t, \eta'|_t, \rho) < \varepsilon$. It follows that the resulting topology $\mathcal{T}_{\mathrm{Skor}}$ is finer than the topology $\mathcal{T}_3$ considered here; i.e. $\mathcal{T}_3 \subseteq \mathcal{T}_{\mathrm{Skor}}$.

Given a deadlock-free and prefix-closed set of compact paths $P \subseteq \mathsf{CPath}(T, X)$, if $Z = P \cup \mathsf{M}(P)$ then the uniform topology $\mathcal{T}_3$ on Z will be Hausdorff and thus metrizable.

5 Equivalence with Other Topological Structures on Path Spaces

Goebel and Teel in [3] develop a notion of convergence for sequences of hybrid paths (compact or limit) for the case of Euclidean space $X \subseteq \mathbb{R}^n$ and $T = \mathbb{H} \subset \mathbb{R}^2$ by employing the machinery of *set-convergence* for sequences of subsets of $\mathbb{R}^n$, applied to paths $\eta \in \mathsf{CPath}^\infty(T, X)$ considered via their graphs as subsets of $T \times X \subset \mathbb{R}^{n+2}$. (The text by Rockafellar and Wets [20] is a standard reference on the set-convergence approach to set-valued anaylsis.) We generalize the Goebel and Teel notion of graph convergence to Dedekind-complete and finite-dimensional future time structures T and metric spaces X, and establish that for spaces $Z \subseteq \mathsf{CPath}^\infty(T, X)$ of paths with norm-closed time domains, graph convergence is equivalent to convergence in the 3-parameter uniformity. The topology on the product space $T \times X$ inherited from the norm topology on T and the metric topology on X is equivalent to the topology generated by the 2-parameter uniform topology $\mathcal{W}$ whose basic entourages are:

$$W_{\delta,\varepsilon} \;:=\; \{\, ((t,x),(t',x')) \in (T \times X) \times (T \times X) \;\mid\; \|\, t - t' \,\|_T < \delta \;\wedge\; d_X(x,x') < \varepsilon \,\}$$

for each parameter pair $(\delta, \varepsilon) \in R_2$. Being the product of a normed space and a metric space, the topology on $T \times X$ is also metrizable; one such metric takes the maximum of the temporal and spatial distances, so that for all $(t,x),(t',x') \in T \times X$, we have:

$$d_{T\times X}((t,x),(t',x')) \;:=\; \max\{\, \|\, t - t' \,\|_T,\; d_X(x,x') \,\} \;.$$

In [3], and also in [5, 7], this metric is used to describe the topology on $T \times X$.

We extend the notion of set-convergence to first-countable Hausdorff uniform spaces $(Z, \mathcal{V})$, with $\mathcal{V}$ a basis for the uniformity. Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of subsets of Z. The *inner limit* (*limit inferior*) and the *outer limit* (*limit superior*), of the sequence of sets $\{A_k\}_{k\in\mathbb{N}}$, are defined as follows:

$$\begin{aligned}
\liminf_{k\to\infty} A_k \;&:=\; \{\, z \in Z \mid \exists \textit{seq.}\, \{z_k\}_{k\in\mathbb{N}},\; \forall k \in \mathbb{N},\; z_k \in A_k \;\wedge\; z = \lim_{k\to\infty} z_k \,\} \\
&=\; \{\, z \in Z \mid \forall V \in \mathcal{V},\; \exists m \in \mathbb{N},\; \forall k > m,\; A_k \cap V(z) \neq \varnothing \,\} \\
\limsup_{k\to\infty} A_k \;&:=\; \{\, z \in Z \mid \exists \textit{seq.}\, \{z_k\}_{k\in\mathbb{N}},\; \forall k \in \mathbb{N},\; z_k \in A_k \;\wedge\; \exists \textit{sub-seq.}\, \{z_{k_i}\}_{i\in\mathbb{N}},\; z = \lim_{i\to\infty} z_{k_i} \,\} \\
&=\; \{\, z \in Z \mid \forall V \in \mathcal{V},\; \exists\, \textit{infinite set } K \subseteq \mathbb{N},\; \forall k \in K,\; A_k \cap V(z) \neq \varnothing \,\} \,.
\end{aligned}$$

Both the inner and outer limit sets always exist, and are always closed sets in the uniform topology $\mathcal{T}_\mathcal{V}$ (although possibly empty), and $\liminf_{k\to\infty} A_k \subseteq \limsup_{k\to\infty} A_k$. The sequence $\{A_k\}_{k\in\mathbb{N}}$ *converges* to a set A if $\limsup_{k\to\infty} A_k = A = \liminf_{k\to\infty} A_k$ in which case A must be closed in the topology $\mathcal{T}_\mathcal{V}$ on Z, and we write $A = \operatorname{setlim}_{k\to\infty} A_k$. When $\operatorname{setlim}_{k\to\infty} A_k = \varnothing$, the sequence is said to *escape to the horizon*.

In [3, 4, 8], in taking graphical limits, there is a restriction to sequences $\{\eta_k\}_{k\in\mathbb{N}}$ of regular hybrid paths (finite or limit) that are *locally eventually bounded* in X, which means that for all $(i,t) \in \mathbb{H}$, there exists $m \in \mathbb{N}$ and a compact set $K \subset X$ such that for all $k \geq m$ and all $(j,s) \in \mathrm{dom}(\eta_k)$, if $(j,s) \leqslant (i,t)$

then $\eta_k(i,t) \in K$; i.e. $\mathrm{ran}(\eta_k|_{(i,t)}) \subseteq K$. More generally, for finite-dimensional future time structures T, a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ in $\mathsf{CPath}^\infty(T,X)$ is *locally eventually bounded* in X if for all $t \in T$, there exists $m \in \mathbb{N}$ and a compact set $K \subset X$ such that $\mathrm{ran}(\eta_k|_t) \subseteq K$ for all $k \geq m$. In the papers [3, 4, 8], the restriction to locally eventually bounded sequences of regular hybrid paths is there for multiple purposes: (a) to ensure that the set-limit of a sequence of graphs of hybrid paths is indeed the graph of a hybrid path (and neither empty nor set-valued, for instance); (b) to ensure that none of the paths η_k "blow up" to infinity with finite escape time; and (c) to give a uniform bound on the sets $\mathrm{ran}(\eta_k|_t)$ for an infinite tail of the whole sequence (in the form of the compact set $K \subset X$). In our framework, reason (a) is a non-issue as we consider convergence only when, for some $\eta \in \mathsf{CPath}^\infty(T,X)$, we have $\eta = \underset{k\to\infty}{\mathrm{setlim}}\ \eta_k$, so the set-convergence limit being empty or set-valued does not arise. Reasons (b) and (c) are not so pressing in our framework, as we can accommodate both paths with norm-closed time domains (which excludes maximal paths with finite escape time) and also paths that are maximal relative to a set of compact paths (which allows the possibility of finite escape time).

We are able to establish a weaker property as a consequence of convergence in the 3-parameter uniform topology. For finite-dimensional future time structures T, and time points $t_* \in T$, we call a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ in $\mathsf{CPath}^\infty(T,X)$ *t_*-locally eventually bounded* in X if there exists $m \in \mathbb{N}$ and a compact set $K \subset X$ such that for all $t \in T$ with $t \not\geq t_*$, we have $\mathrm{ran}(\eta_k|_t) \subseteq K$ for all $k \geq m$.

Proposition 5.1 [Local eventual boundedness of sequences]
Suppose S is finite-dimensional and Dedekind-complete, and $Z \subseteq \mathsf{CPath}^\infty(T,X)$ is such that either $Z \subseteq \mathsf{CPath}^\infty_{\mathrm{cl}}(T,X)$ or $Z = P \cup \mathsf{M}(P)$ for some deadlock-free and prefix-closed path set $P \subseteq \mathsf{CPath}(T,X)$. Let $\{\eta_k\}_{k\in\mathbb{N}}$ be any sequence in Z, and let $\eta \in Z$.

(i) *If $\eta = \lim_{k\to\infty} \eta_k$ in the uniform topology $\mathcal{T}_3$ on Z, and $\eta \in Z \cap \mathsf{CPath}^\infty_{\mathrm{cl}}(T,X)$, then the sequence $\{\eta_k\}_{k\in\mathbb{N}}$ is locally eventually bounded in X.*

(ii) *If $\eta = \lim_{k\to\infty} \eta_k$ in the uniform topology $\mathcal{T}_3$ on Z, and $\eta \notin \mathsf{CPath}^\infty_{\mathrm{cl}}(T,X)$, then for some $t_* \in T$, the sequence $\{\eta_k\}_{k\in\mathbb{N}}$ is t_*-locally eventually bounded in X.*

Proof: For (i), suppose that $\eta = \lim_{k\to\infty} \eta_k$ in the topology $\mathcal{T}_3$, and that $\mathrm{dom}(\eta)$ is norm-closed. Then fix $t \in T$, arbitrary, and choose any $(\delta,\varepsilon) \in R_2$; by the convergence in the uniform topology, there exists an $m \in \mathbb{N}$ such that $\eta_k|_t \in V_{\delta,\varepsilon}(\eta|_t)$ for all $k \geq m$. Since $\mathrm{dom}(\eta)$ is norm-closed, we know that $\mathrm{dom}(\eta|_t)$ is norm-compact in T. Then set:

$$K := \{\, x \in X \mid \exists s \in \mathrm{dom}(\eta),\ s \leq t \ \wedge\ d_X(x, \eta(s)) \leq \varepsilon \,\}.$$

Then K is compact in X since η is continuous and $\mathrm{dom}(\eta|_t)$ is compact in T. We can now conclude that $\mathrm{ran}(\eta_k|_t) \subseteq K$ for all $k \geq m$. Hence the sequence $\{\eta_k\}_{k\in\mathbb{N}}$ is locally eventually bounded in X.

For (ii), suppose that $\eta = \lim_{k\to\infty} \eta_k$ in the topology $\mathcal{T}_3$, and that $\mathrm{dom}(\eta)$ is not norm-closed. Then $\eta \in \mathsf{M}(P) \cap \mathsf{CPath}^\infty_{\mathrm{bd}}(T,X)$ for some deadlock-free and prefix-closed path set $P \subseteq \mathsf{CPath}(T,X)$. Then there must exist an integer $n_* \geq 1$ such that $\eta_k \in \mathsf{M}(P) \cap \mathsf{CPath}^{\infty}_{\text{bd}}(T, X)$ for all $k \geq n_*$, for otherwise, we could not have $\eta = \lim_{k\to\infty} \eta_k$ in $\mathcal{T}_3$. So η and all but finitely many of the paths η_k are maximal with $\text{dom}(\eta_k)$ norm-bounded but not norm-closed. Set $t_0 := \sup(\text{dom}(\eta))$, so $t_0 \notin \text{dom}(\eta)$, and for each $k \geq n_*$, set $t_k := \sup(\text{dom}(\eta_k))$, so $t_k \notin \text{dom}(\eta_k)$. Then set $t_* := \inf\{\, t_k \mid k = 0 \vee k \geq n_* \geq 1 \,\}$, which will exist, since T is assumed to be Dedekind-complete. Now for all $t \in T$ such that $t \not\geqslant t_*$, we have $t \not\geqslant t_0$ and thus $\eta|_t < \eta$ and $\text{dom}(\eta|_t)$ is norm-compact in T. Then choose any $(\delta, \varepsilon) \in R_2$ and take K as the compact set as in *Case I*. By the convergence in the uniform topology, can then conclude that there exists an $m \in \mathbb{N}$ such that for all $t \not\geqslant t_*$, we have $\text{ran}(\eta_k|_t) \subseteq K$ for all $k \geq m$. Thus the sequence $\{\eta_k\}_{k\in\mathbb{N}}$ is t_*-locally eventually bounded in X. ■

Collins, in [5, 7], proposes the *compact-open* topology $\mathcal{T}_{\text{co}}$ on spaces of *prefix-free* or maximal paths; in our framework, this means sets of paths $Z = \mathsf{M}(P)$ for some $P \subseteq \mathsf{CPath}(T, X)$. Since we restrict to time structures S that are finite dimensional, *Part 4* of Theorem 2.2, gives that for each $t \in T$, the closed interval $[0, t]$ is compact in $\mathcal{T}_{\text{norm}}$. The topology $\mathcal{T}_{\text{c.o.}}$ has as a basis the family of all sets:

$$B(\eta,\, t, \delta, \varepsilon) \;:=\; \{\, \eta' \in Z \mid \eta|_t \subseteq W_{\delta,\varepsilon}(\eta') \;\wedge\; \eta'|_t \subseteq W_{\delta,\varepsilon}(\eta) \,\}$$

for $\eta \in Z$, $t \in T$, and pairs $(\delta, \varepsilon) \in R_2$. Then $\eta = \lim_{k\to\infty} \eta_k$ in the compact-open topology $\mathcal{T}_{\text{c.o.}}$ iff for all $t \in T$ and $(\delta, \varepsilon) \in R_2$, there exists an integer $m \in \mathbb{N}$ such that for all $k \geq m$, we have $\eta_k \in B(\eta, t, \delta, \varepsilon)$. Note that this is a modification of the standard compact-open topology on spaces of *total* continuous functions $\eta : T \to X$, which declares η and η' to be (K, ε)-close if the range of η restricted to the compact subset K of T is contained in an ε-neighbourhood of η' within X, and symmetrically, the range of η' restricted to K is contained in an ε-neighbourhood of η within X. The modification is required to accommodate partial functions with differing time domains, and the non-total-ness of paths as functions means that the usual concept of *uniform convergence* (for sequences of total functions on a common domain) does not straight-forwardly apply.

In proving the equivalence of the various notions of convergence for sequences of continuous paths, we need a condition on time structures T and path sets $Z \subseteq \mathsf{CPath}^{\infty}_{\text{cl}}(T, X)$ that guarantees the existence of certain supremums and infimums, but is weaker than the assumption that T is Dedekind-complete, since Dedekind-completeness excludes the Zeno time structure $\mathbb{Q}_{\mathbb{B}} \times \mathbb{R}$. We will say that a path set $Z \subseteq \mathsf{CPath}^{\infty}_{\text{cl}}(T, X)$ is *rich enough over time T* when the following condition holds:

$$\begin{aligned}
&\text{for all paths } \eta \in Z, \text{ the following points exist in } \text{dom}(\eta) : \\
&\text{for each } t \in T, \text{ the point:} \\
&\qquad r_t \;:=\; \sup\{s \in \text{dom}(\eta) \mid s \leqslant t\} \\
&\text{and for each } \delta > 0 \text{ such that } \|\, s \,\|_T \geq \delta \text{ for some } s \in \text{dom}(\eta), \text{ the point:} \\
&\qquad r_\delta \;:=\; \inf\{s \in \text{dom}(\eta) \mid \|\, s \,\|_T \geq \delta\}\,.
\end{aligned}$$

This weaker condition is satisfied by sets of continuations of Zeno trajectories when modelled as paths with time $\mathbb{Q}_{\mathbb{B}} \times \mathbb{R}$ as discussed in Section 2.

Theorem 5.2 [Equivalence of concepts of convergence for sequences of continuous paths]
Let S be a finite-dimensional time structure with future time T, and suppose the time granularity $\mathrm{gr}(S, u) = 0$. Let (X, d_X) be a metric space with a distinguished state x_0, so that $(0, x_0)$ is a reference point in $T \times X$, and further suppose that the space granularity of X is 0, in the sense that for all $x \in X$, we have $\inf\{d_X(x, y) \mid y \neq x\} = 0$. Let $Z \subseteq \mathsf{CPath}^{\infty}_{\mathrm{cl}}(T, X)$ be a path set such that Z is rich enough over time T. Then for all paths $\eta \in Z$ and for all sequences of paths $\{\eta_k\}_{k \in \mathbb{N}}$ within Z, the following five conditions are equivalent:

(1) $\eta = \lim\limits_{k\to\infty} \eta_k$ *in the 3-parameter Hausdorff uniform topology $\mathcal{T}_3$ on Z;*

(2) $\eta = \operatorname*{setlim}\limits_{k\to\infty} \eta_k$ *as graphs in the product topology on $T \times X$;*

(3) $\eta = \lim\limits_{k\to\infty} \eta_k$ *in the compact-open topology $\mathcal{T}_{\mathrm{c.o.}}$ on Z.*

(4) $\forall$ *open sets O in $T \times X$, if $\eta \cap O \neq \emptyset$ then $\exists m_1 \in \mathbb{N}$, $\forall k \geq m_1$, $\eta_k \cap O \neq \emptyset$, and*
$\forall$ *compact sets K in $T \times X$, if $\eta \cap K = \emptyset$ then $\exists m_2 \in \mathbb{N}$, $\forall k \geq m_2$, $\eta_k \cap K = \emptyset$;*

(5) $\forall(\delta, \varepsilon) \in R_2$, $\forall(\alpha, \beta) \in R_2$, $\exists m \in \mathbb{N}$, $\forall k \geq m$, *the following two set-inclusions hold in $T \times X$:*

$$\eta \cap \mathrm{cl}(W_{\alpha,\beta}(0, x_0)) \subseteq \mathrm{cl}(W_{\delta,\varepsilon}(\eta_k)) \quad \text{and} \quad \eta_k \cap \mathrm{cl}(W_{\alpha,\beta}(0, x_0)) \subseteq \mathrm{cl}(W_{\delta,\varepsilon}(\eta)).$$

Since topologies on first-countable Hausdorff spaces are uniqely determined by the notion of convergence of sequences, the topologies $\mathcal{T}_3$ and $\mathcal{T}_{\mathrm{c.o.}}$ on Z coincide.

Proof: We will first prove the equivalence of **(2)** and **(4)**, which is the open set/compact set "hit-and-miss criteria" for set-convergence, generalizing Theorem 4.5 of the Rockafellar and Wets text [20]. We will then prove the equivalence of **(2)** and **(5)**, which generalizes Theorem 4.10 of [20] and Lemma 4.2 of [3], and then finally prove the equivalence of **(1)** and **(5)**. The equivalence of **(3)** and **(5)** is straightforward, and will be omitted. We require the assumptions that both T and X have granularity 0 in only some of the proofs, namely **(2)** $\Rightarrow$ **(5)**, **(5)** $\Rightarrow$ **(2)**, and **(5)** $\Rightarrow$ **(1)**.

[**(2)** $\Rightarrow$ **(4)**] Suppose that **(2)** holds, and thus $\limsup_{k\to\infty} \eta_k \subseteq \eta \subseteq \liminf_{k\to\infty} \eta_k$ in $T \times X$, with $\eta \neq \emptyset$ and η a closed set in $T \times X$. Now fix an open set O in $T \times X$, and suppose that $\eta \cap O \neq \emptyset$. Pick any $(t, x) \in \eta \cap O$, and then there must exist an entourage $W_{\delta,\varepsilon} \in \mathcal{W}$ such that the basic open $W_{\delta,\varepsilon}(t, x) \subseteq O$. But then by the definition of the inner limit, there exists an $m_1 \in \mathbb{N}$ such that for all $k \geq m_1$, we have $\eta_k \cap W_{\delta,\varepsilon}(t, x) \neq \emptyset$, and hence $\eta_k \cap O \neq \emptyset$, as required. Then fix a compact set K in $T \times X$, and suppose that $\eta \cap K = \emptyset$. Now consider an arbitrary element $(t, x) \in K$; we claim there is some basic open $W_{\delta,\varepsilon}(t, x)$ such that the set $F(t, x) := \{k \in \mathbb{N} \mid \eta_k \cap W_{\delta,\varepsilon}(t, x) \neq \emptyset\}$ is of finite cardinality.

To see this, suppose the claim did not hold; then by the definition of the outer limit, we would have $(t,x) \in \limsup_{k\to\infty} \eta_k$ and hence $(t,x) \in \eta \cap K$, contradicting the supposition that $\eta \cap K = \varnothing$. Now the family of basic opens $\{W_{\delta,\varepsilon}(t,x)\}_{(t,x)\in K}$ forms an open cover of K and thus there exists a finite subcover given by $\{W_{\delta,\varepsilon}(t,x)\}_{(t,x)\in A}$ where $A \subseteq K$ is finite. Then let $F = \cup_{(t,x)\in A} F(t,x)$, which is also finite. Now choose $m_2 \in \mathbb{N}$ such that m_2 is strictly greater than all elements in F. Hence for all $k \geq m_2$, we have $\eta_k \cap K = \varnothing$, as required.

[**(4)** $\Rightarrow$ **(2)**] Suppose that **(4)** holds; we will then establish the two inclusions $\eta \subseteq \liminf_{k\to\infty} \eta_k$ and $\limsup_{k\to\infty} \eta_k \subseteq \eta$ in $T \times X$. For the first of these inclusions, suppose that $(t,x) \in \eta$ and thus $x = \eta(t)$; we want to show that $(t,x) \in \liminf_{k\to\infty} \eta_k$. Now consider an arbitrary basic open neighbourhood $W_{\delta,\varepsilon}(t,x)$ of (t,x), for some parameter pair $(\delta,\varepsilon) \in R_2$. Then since $(t,x) \in \eta \cap W_{\delta,\varepsilon}(t,x)$ and **(4)** holds, there exists some $m_1 \in \mathbb{N}$ such that $\eta_k \cap W_{\delta,\varepsilon}(t,x) \neq \varnothing$ for all $k \geq m_1$. By the definition of the inner limit, we can then conclude that $(t,x) \in \liminf_{k\to\infty} \eta_k$, as required. For the second inclusion, we prove the contrapositive: if $(t,x) \notin \eta$ then $(t,x) \notin \limsup_{k\to\infty} \eta_k$. So suppose that $(t,x) \notin \eta$, and also suppose, toward a contradiction, that $(t,x) \in \limsup_{k\to\infty} \eta_k$. Then by the definition of the outer limit, there exists a sequence $\{(t_k,x_k)\}_{k\in\mathbb{N}}$ such that $(t_k,x_k) \in \eta_k$ for all $k \in \mathbb{N}$ (so $x_k = \eta_k(t_k)$) and there is a sub-sequence $\{(t_{k_i},x_{k_i})\}_{i\in\mathbb{N}}$ such that $(t,x) = \lim_{i\to\infty}(t_{k_i},x_{k_i})$. Since η is closed set in $T \times X$ and $(t,x) \notin \eta$, we claim there is some integer $m_0 \in \mathbb{N}$ such that $(t_{k_i},x_{k_i}) \notin \eta$ for all $i \geq m_0$. To see this, suppose the claim did not hold; then we would have $(t_{k_i},x_{k_i}) \in \eta$ for infinitely many $i \in \mathbb{N}$, and thus η would contain a sequence converging to (t,x), to give $(t,x) \in C$. Now consider the set $K := \{(t,x)\} \cup \{\, (t_{k_i},x_{k_i}) \mid i \geq m_0 \,\}$. This set K is compact in $T \times X$ because it is Cauchy-complete and totally-bounded, within a metricizable space. The Cauchy-completeness of K is trivial, and to verify the totally-bounded property for K, we use the convergence of the sub-sequence $\{(t_{k_i},x_{k_i})\}_{i\in\mathbb{N}}$ to (t,x) in the uniformity: for every parameter pair $(\delta,\varepsilon) \in R_2$, there is an $n \geq m_0$ such that $(t_{k_i},x_{k_i}) \in W_{\delta,\varepsilon}(t,x)$ for all $i \geq n$, so we can find a finite set of indices $I \subseteq \{m_0, m_0+1, \ldots, n\}$ such that the family $\{\, W_{\delta,\varepsilon}(t,x) \,\} \cup \{\, W_{\delta,\varepsilon}(t_{k_i},x_{k_i}) \mid i \in I \,\}$ is a finite cover of $W_{\delta,\varepsilon}$ basic opens. Then since K is compact and $\eta \cap K = \varnothing$, we can conclude from **(3)** that there exists some integer m_2 such that $\eta_k \cap K = \varnothing$ for all $k \geq m_2$, which is a clear contradiction with $(t_{k_i},x_{k_i}) \in \eta_{k_i} \cap K$. Thus, it must be the case that $(t,x) \notin \limsup_{k\to\infty} \eta_k$, as required.

[**(2)** $\Rightarrow$ **(5)**] Suppose that **(2)** holds, and thus so does **(4)**, and we have $\limsup_{k\to\infty} \eta_k \subseteq \eta \subseteq \liminf_{k\to\infty} \eta_k$, with $\eta \neq \varnothing$ and η a closed set in $T \times X$. Fix two parameter pairs $(\delta,\varepsilon) \in R_2$ and $(\alpha,\beta) \in R_2$. To prove the first inclusion for **(5)**, suppose, for a contradiction, that for all $m \in \mathbb{N}$, there exists $k_m \geq m$ such that $\eta \cap \mathrm{cl}(W_{\alpha,\beta}(0,x_0)) \setminus \mathrm{cl}(W_{\delta,\varepsilon}(\eta_{k_m})) \neq \varnothing$. Thus we can choose a sequence of time points $\{t_m\}_{m\in\mathbb{N}}$ in $\mathrm{dom}(\eta)$ such that for all $m \in \mathbb{N}$, we have $(t_m, \eta(t_m)) \in \mathrm{cl}(W_{\alpha,\beta}(0,x_0))$ and $(t_m,\eta(t_m)) \notin \mathrm{cl}(W_{\delta,\varepsilon}(\eta_{k_m}))$, and we also have $k_{m+1} > k_m$. Moreover, the sequence $\{t_m\}_{m\in\mathbb{N}}$ can be chosen so that it converges to a limit, with $(\hat{t},\hat{x}) = \lim_{k\to\infty}(t_m,\eta(t_m))$ in the uniform topology on $T \times X$. Since η is a closed set in $T \times X$, we can conclude that $(\hat{t},\hat{x}) \in \eta$, and thus $\hat{x} = \eta(\hat{t}) = \lim_{m\to\infty} \eta(t_m)$. Since **(2)** holds, we have $\eta \subseteq \liminf_{k\to\infty} \eta_k$ and thus $(\hat{t},\hat{x}) \in \liminf_{k\to\infty} \eta_k$. Since both T and X have granularity 0, we can set $\delta_\star := \frac{1}{2}\min(\delta,\alpha)$ and $\varepsilon_\star := \frac{1}{2}\min(\varepsilon,\beta)$. Applying condition **(3)** to the open neighbourhood $O = W_{\delta_\star,\varepsilon_\star}(\hat{t},\hat{x}))$ of $(\hat{t},\hat{x})$ in $T \times X$, as well as using the convergence $(\hat{t},\hat{x}) = \lim_{k\to\infty}(t_m,\eta(t_m))$, we can find a sufficiently large integer $m_\star \in \mathbb{N}$ such that for $k_\star := k_{m_\star}$, we have both $\eta_{k_\star} \cap W_{\delta_\star,\varepsilon_\star}(\hat{t},\hat{x})) \neq \varnothing$ and $(t_{m_\star}, \eta(t_{m_\star})) \in W_{\delta_\star,\varepsilon_\star}(\hat{t},\hat{x}))$. From the second condition, we can conclude that $\|\hat{t} - t_{m_\star}\| < \delta_\star$ and $d_X(\hat{x}, \eta(t_{m_\star}) < \varepsilon_\star$, while from the first condition, we can conclude that there exists a time point $s_\star \in \mathrm{dom}(\eta_{k_\star})$ such that $\|\hat{t} - s_\star\| < \delta_\star$ and $d_X(\hat{x}, \eta_{k_\star}(s_\star) < \varepsilon_\star$. Applying the triangle inequalities for norms and metrics, we then have $\|t_{m_\star} - s_\star\| < 2\delta_\star < \delta$ and that $d_X(\eta(t_{m_\star}), \eta_{k_\star}(s_\star) < 2\varepsilon_\star < \varepsilon$. But now we have $(t_{m_\star}, \eta(t_{m_\star})) \in W_{\delta,\varepsilon}(s_\star, \eta_{k_\star}(s_\star))$, contradicting the fact that $(t_{m_\star}, \eta(t_{m_\star})) \notin \mathrm{cl}(W_{\delta,\varepsilon}(\eta_{k_\star}))$. Thus there exists an integer $n_1 \in \mathbb{N}$ such that $\eta \cap \mathrm{cl}(W_{\alpha,\beta}(0, x_0)) \subseteq \mathrm{cl}(W_{\delta,\varepsilon}(\eta_k))$ for all $k \geq n_1$.

To prove the second inclusion for **(5)**, suppose, for a contradiction, that for all $m \in \mathbb{N}$, there exists $k_m \geq m$ such that $\eta_{k_m} \cap \mathrm{cl}(W_{\alpha,\beta}(0, x_0)) \setminus \mathrm{cl}(W_{\delta,\varepsilon}(\eta)) \neq \varnothing$. In this case, we get a sub-sequence of paths $\{\eta_{k_m}\}_{m\in\mathbb{N}}$ with $k_{m+1} > k_m$ and a sequence of points $\{(t_m, \eta_{k_m}(t_m))\}_{m\in\mathbb{N}}$ in $T \times X$ such that for all $m \in \mathbb{N}$, we have $(t_m, \eta_{k_m}(t_m)) \in \mathrm{cl}(W_{\alpha,\beta}(0, x_0))$ and $(t_m, \eta_{k_m}(t_m)) \notin \mathrm{cl}(W_{\delta,\varepsilon}(\eta))$. Moreover, since the sequence $\{(t_m, \eta_{k_m}(t_m))\}_{m\in\mathbb{N}}$ is bounded by the compact set $\mathrm{cl}(W_{\alpha,\beta}(0, x_0))$ in $T \times X$, it contains a sub-sequence convergent in the uniform topology on $T \times X$; let $(\bar{t}, \bar{x})$ be the limit of this convergent sub-sequence. Then by the definition of the outer limit, we have $(\bar{t}, \bar{x}) \in \limsup_{k\to\infty}$. Since **(2)** holds, we have $\limsup_{k\to\infty} \eta_k \subseteq \eta$, and thus $(\bar{t}, \bar{x}) \in \eta$, and hence $\bar{x} = \eta(\bar{t})$. But since $(t_m, \eta_{k_m}(t_m)) \notin \mathrm{cl}(W_{\delta,\varepsilon}(\eta))$ for all $m \in \mathbb{N}$, we can also conclude of the accumulation point $(\bar{t}, \bar{x})$ that $(\bar{t}, \bar{x}) \notin W_{\delta,\varepsilon}(\eta)$, in contradiction with the fact that $(\bar{t}, \bar{x}) \in \eta$. Thus there exists an integer $n_2 \in \mathbb{N}$ such that $\eta_k \cap \mathrm{cl}(W_{\alpha,\beta}(0, x_0)) \subseteq \mathrm{cl}(W_{\delta,\varepsilon}(\eta))$ for all $k \geq n_2$. To complete the proof of **(5)**, take $m := \max(n_1, n_2)$, and we are done.

[**(5)** $\Rightarrow$ **(2)**] Suppose that **(5)** holds. To prove **(2)**, we will establish the two inclusions $\eta \subseteq \liminf_{k\to\infty} \eta_k$ and $\limsup_{k\to\infty} \eta_k \subseteq \eta$ in $T \times X$. For the first of these inclusions, suppose that $(t, x) \in \eta$ and thus $x = \eta(t)$; we want to show that $(t, x) \in \liminf_{k\to\infty} \eta_k$. From the definition of the inner limit, it suffices to show that for an arbitrary basic open neighbourhood $W_{\delta,\varepsilon}(t, x)$ of (t, x), there exists an $m \in \mathbb{N}$ such that $\eta_k \cap W_{\delta,\varepsilon}(t, x) \neq \varnothing$ for all $k \geq m$. Since both T and X have granularity 0, consider any parameter pairs $(\delta_1, \varepsilon_1) \in R_2$ and $(\delta_2, \varepsilon_2) \in R_2$ such that $\delta_1 < \delta$, $\varepsilon_1 < \varepsilon$, $\delta_2 > \|t\|_T$ and $\varepsilon_2 > d_X(x_0, x)$, and thus $(t, x) \in \eta \cap \mathrm{cl}(W_{\delta_2,\varepsilon_2}(0, x_0))$. Since **(5)** holds, there exists an integer $m \in \mathbb{N}$ such that $\eta \cap \mathrm{cl}(W_{\delta_2,\varepsilon_2}(0, x_0)) \subseteq \mathrm{cl}(W_{\delta_1,\varepsilon_1}(\eta_k)) \subseteq W_{\delta,\varepsilon}(\eta_k)$ for all $k \geq m$. Hence $(t, x) \in W_{\delta,\varepsilon}(\eta_k)$ for all $k \geq m$, as required to establish that $(t, x) \in \liminf_{k\to\infty} \eta_k$.

For the second inclusion for **(2)**, fix an arbitrary element $(t, x) \in \limsup_{k\to\infty} \eta_k$; we want to show that $(t, x) \in \eta$. It suffices to show that $(t, x) \in W_{\delta,\varepsilon}(\eta)$ for an arbitrary basic entourage $W_{\delta,\varepsilon}$ in $T \times X$; since η is a closed set, it will then follow that $(t, x) \in \eta$. Again, since both T and X have granularity 0, we can choose parameter pairs $(\delta_1, \varepsilon_1) \in R_2$ and $(\delta_2, \varepsilon_2) \in R_2$ such that $\mathrm{cl}(W_{\delta_1,\varepsilon_1}(\eta)) \subseteq W_{\delta,\varepsilon}(\eta)$, $\delta_2 > \|t\|_T$ and $\varepsilon_2 > d_X(x_0, x)$. Since $(t, x) \in \limsup_{k\to\infty} \eta_k$, there exists a sub-sequence of paths $\{\eta_{k_i}\}_{i\in\mathbb{N}}$ such that $(t, x) \in \eta_{k_i} \cap \mathrm{cl}(W_{\delta_2,\varepsilon_2}(0, x_0))$ for all $i \in \mathbb{N}$. Since **(5)** holds, there exists an integer $m \in \mathbb{N}$ such that $\eta_k \cap \mathrm{cl}(W_{\delta_2,\varepsilon_2}(0, x_0)) \subseteq \mathrm{cl}(W_{\delta_1,\varepsilon_1}(\eta)) \subseteq W_{\delta,\varepsilon}(\eta)$ for all $k \geq m$. So choosing an i such that $k_i \geq m$, we can conclude that $(t, x) \in W_{\delta,\varepsilon}(\eta)$, as required.

[**(1)** $\Rightarrow$ **(5)**] Suppose that **(1)** holds, hence for all $(\delta_1, \varepsilon_1, t_1) \in\in R_2 \times T$, there exists an integer $m_1 \in \mathbb{N}$ such that $\eta_k \in U_{\delta_1,\varepsilon_1,t_1}(\eta)$ for all $k \geq m_1$. To prove **(5)**, fix parameter pairs $(\delta, \varepsilon) \in R_2$ and $(\alpha,\beta) \in R_2$. Now set $\delta_1 := \min(\delta, \alpha)$, $\varepsilon_1 := \min(\varepsilon,\beta)$ and $t_1 := \inf\{s \in \mathrm{dom}(\eta) \mid \|s\|_T \geq 2\alpha\}$;

then t_1 exists and $t_1 \in \mathrm{dom}(\eta)$, since Z is rich enough over T. Then let the integer m_1 be such that $\eta_k \in U_{\delta_1,\varepsilon_1,t_1}(\eta)$ for all $k \geq m_1$. We claim that this m_1 will also provide a witness to establish the two inclusions for **(5)** for (δ,ε) and (α,β). So fix an arbitrary index $k \geq m_1$, and since $\eta_k \in U_{\delta_1,\varepsilon_1,t_1}(\eta)$, there exists compact prefixes $\gamma \leqslant \eta$ and $\gamma_k \leqslant \eta_k$ such that $\gamma = \eta|_{t_1}$ and $\gamma_k = \eta_k|_{t_1}$ and $(\gamma,\gamma_k) \in V_{\delta,\varepsilon}$. Thus there exists a retiming $\rho\colon \mathrm{dom}(\gamma) \rightsquigarrow \mathrm{dom}(\gamma_k)$ such that $\mathrm{dev}(\rho) < \delta_1$ and $d_{X\sup}(\gamma,\gamma_k,\rho) < \varepsilon_1$. To establish the first inclusion $\eta \cap \mathrm{cl}(W_{\alpha,\beta}(0,x_0)) \subseteq \mathrm{cl}(W_{\delta,\varepsilon}(\eta_k))$, fix $(s,x) \in \eta \cap \mathrm{cl}(W_{\alpha,\beta}(0,x_0))$. Now choose a time point $s_k \in \rho(s)$, so $s_k \in \mathrm{dom}(\gamma_k)$. Then we have $\| s \| \leq \alpha$ and $\| s - s_k \|_T < \delta_1 \leq \delta$. Hence by the traingle inequality, we have $\| s_k \|_T \leq \| s \|_T + \| s_k - s \|_T \leq \alpha + \delta_1 \leq 2\alpha \leq \| t_1 \|_T$, and so we can conclude that $s_k \leqslant t_1$. Hence we have $d_X(x,\eta_k(s_k)) = d_X(\gamma(s),\gamma_k(s_k)) < \varepsilon_1 \leq \varepsilon$. We can thus conclude that $(s,x) \in W_{\delta,\varepsilon}(s_k,\eta_k(s_k))$ and hence $(s,x) \in \mathrm{cl}(W_{\delta,\varepsilon}(\eta_k))$, as required. To establish the second inclusion $\eta_k \cap \mathrm{cl}(W_{\alpha,\beta}(0,x_0)) \subseteq \mathrm{cl}(W_{\delta,\varepsilon}(\eta))$, fix $(s_k,x) \in \eta_k \cap \mathrm{cl}(W_{\alpha,\beta}(0,x_0))$. Hence $s_k \in \mathrm{dom}(\eta_k)$ and $x = \eta_k(s_k)$ and $\| s_k \|_T \leq \alpha < \| t_1 \|_T$, so that $s_k \in \mathrm{dom}(\gamma_k)$ and $x = \gamma_k(s_k)$. Now choose any time point $s \in \rho^{-1}(s_k)$, so that $s \in \mathrm{dom}(\gamma)$, $\| s - s_k \|_T < \delta$, and $\| s \|_T \leq \| s_k \|_T + \| s - s_k \|_T \leq \| t_1 \|_T$. We then have $d_X(x,\eta(s)) = d_X(\gamma_k(s_k),\gamma(s)) < \varepsilon_1 \leq \varepsilon$. Thus we can conclude that $(s_k,x) \in W_{\delta,\varepsilon}(s,\eta(s))$, and hence $(s_k,x) \in \mathrm{cl}(W_{\delta,\varepsilon}(\eta))$, as required.

[**(5)** $\Rightarrow$ **(1)**] Suppose that **(5)** holds. To establish the convergence **(1)** in the 3-parameter uniformity $\mathcal{U}$, fix an arbitrary triple $(\delta_1,\varepsilon_1,t_1) \in R_2 \times T$; we need to exhibit an integer $m \in \mathbb{N}$ such that $\eta_k \in U_{\delta_1,\varepsilon_1,t_1}(\eta)$ for all $k \geq m_1$. Since both T and X have granularity 0, we can choose parameter pairs $(\delta,\varepsilon),(\alpha,\beta) \in R_2$ such that $\delta < \delta_1$, $\varepsilon < \varepsilon_1$, $\alpha := \| t_1 \|_T + 2\delta$ and $\beta := d_X(x_0,\eta(t_0)) + 2\varepsilon$ where $t_0 := \sup\{s \in \mathrm{dom}(\eta) \mid s \leqslant t_1\} = \max\{s \in \mathrm{dom}(\eta) \mid s \leqslant t_1\}$, since Z is rich enough over T. Applying **(5)** to these parameter pairs, which have $\delta \leq \frac{1}{2}\alpha$ and $\varepsilon \leq \frac{1}{2}\beta$, we can conclude that there exists an $m_1 \in \mathbb{N}$ such that for $k \geq m_1$, we have:

$$\eta \cap \mathrm{cl}(W_{\alpha,\beta}(0,x_0)) \subseteq \mathrm{cl}(W_{\delta,\varepsilon}(\eta_k)) \quad \textit{and} \quad \eta_k \cap \mathrm{cl}(W_{\alpha,\beta}(0,x_0)) \subseteq \mathrm{cl}(W_{\delta,\varepsilon}(\eta)) .$$

We will prove that the integer $m_1 + 1$ witnesses convergence for the triple $(\delta_1,\varepsilon_1,t_1)$. So fix $k > m_1$, so $k \geq 1$; we want to prove that $(\gamma,\gamma_k) \in V_{\delta_1,\varepsilon_1}$, where $\gamma := \eta|_{t_1}$ and $\gamma_k := \eta_k|_{t_1}$. First, set $t_k := \sup\{s \in \mathrm{dom}(\eta_k) \mid s \leqslant t_1\}$; then t_k exists and $t_k \in \mathrm{dom}(\eta_k)$ since Z is rich enough over T. Hence $t_k \geqslant 0$ and $\mathrm{dom}(\gamma) = [0,t_0] \cap \mathrm{dom}(\eta)$ and $\mathrm{dom}(\gamma_k) = [0,t_k] \cap \mathrm{dom}(\eta_k)$, and both $\| t_0 \|_T \leq \| t_1 \|_T \leq \alpha$ and $\| t_k \|_T \leq \| t_1 \|_T \leq \alpha$. Next, we seek to define a retiming map $\rho\colon \mathrm{dom}(\gamma) \rightsquigarrow \mathrm{dom}(\gamma_k)$ such that for all $s \in \mathrm{dom}(\gamma)$ and $s_k \in \mathrm{dom}(\gamma_k)$, if $(s,s_k) \in \rho$ then $\|s - s_k\|_T \leq \delta < \delta_1$ and $d_X(\gamma(s),\gamma_k(s_k)) \leq \varepsilon < \varepsilon_1$. Now, we have $\gamma \subseteq \eta \cap \mathrm{cl}(W_{\alpha,\beta}(0,x_0))$ and $\gamma_k \subseteq \eta_k \cap \mathrm{cl}(W_{\alpha,\beta}(0,x_0))$, hence $\gamma \subseteq \mathrm{cl}(W_{\delta,\varepsilon}(\eta_k))$ and $\gamma_k \subseteq \mathrm{cl}(W_{\delta,\varepsilon}(\eta))$. Thus we have:

$$\begin{aligned} & \forall s \in \mathrm{dom}(\gamma),\ \exists s_k \in \mathrm{dom}(\eta_k) : \ \| s - s_k \|_T \leq \delta \ \wedge\ d_X(\gamma(s),\eta_k(s_k)) \leq \varepsilon \\ \text{and} \quad & \forall s_k \in \mathrm{dom}(\gamma_k),\ \exists s \in \mathrm{dom}(\eta) : \ \| s - s_k \|_T \leq \delta \ \wedge\ d_X(\eta(s),\gamma_k(s_k)) \leq \varepsilon . \end{aligned}$$

Further observe that if $s \in \mathrm{dom}(\gamma)$ and $s_k \in \mathrm{dom}(\eta_k)$ and $\| s - s_k \|_T \leq \delta$, then we have $\| s_k \|_T \leq \| s \|_T + \| s - s_k \|_T \leq \| t_1 \|_T + \delta < \alpha$ and thus $s_k \in \mathrm{dom}(\gamma_k)$, and likewise, if $s_k \in \mathrm{dom}(\gamma_k)$ and $s \in \mathrm{dom}(\eta)$

and $\| s - s_k \|_T \leq \delta$, then we have $\| s \|_T < \alpha$ and thus $s \in \mathrm{dom}(\gamma)$. Hence we have:

$$\begin{aligned} &\forall s \in \mathrm{dom}(\gamma),\ \exists s_k \in \mathrm{dom}(\gamma_k) :\ \| s - s_k \|_T \leq \delta\ \wedge\ d_X(\gamma(s), \eta_k(s_k)) \leq \varepsilon \\ \text{and} \qquad &\forall s_k \in \mathrm{dom}(\gamma_k),\ \exists s \in \mathrm{dom}(\gamma) :\ \| s - s_k \|_T \leq \delta\ \wedge\ d_X(\eta(s), \gamma_k(s_k)) \leq \varepsilon\ . \end{aligned}$$

It is then straight-forward to construct a total and surjective set-valued map $\rho\colon \mathrm{dom}(\gamma) \rightsquigarrow \mathrm{dom}(\gamma_k)$ that enforces the non-strict (δ, ε) closeness constraint and also meets the conditions of a being retiming. The idea is to break up the bounded time domains $\mathrm{dom}(\gamma)$ and $\mathrm{dom}(\gamma_k)$ into a finite union of linearly-ordered closed intervals with overlapping end-points, each of length at most δ, and define ρ piecewise. This completes the proof. ∎

In [8], an *abstract hybrid system* over a state space $X \subseteq \mathbb{R}^n$, with X open in $\mathbb{R}^n$, is a set of hybrid paths $S \subseteq \mathsf{CPath}^\infty(\mathbb{H}, X)$ satisfying the following three conditions:

(B1) for all $\eta \in S$, $\mathrm{ran}(\eta) \subset X$;

(B2) for all $\eta \in S$ and all $(i, t) \in \mathrm{dom}(\eta)$, ${}_{(i,t)}|\eta \in S$; and

(B3) for all sequences $\{\eta_k\}_{k \in \mathbb{N}}$ within S that are locally eventually bounded in X, and for all $\eta \in \mathsf{CPath}^\infty(\mathbb{H}, X)$, if $\eta = \operatorname*{setlim}_{k\to\infty} \eta_k$ as graphs in $T \times X$, then $\eta \in S$.

In [8], the suffix- or translation-invariance condition **(B2)** is claimed to "... reduce to the standard semi-group property under further existence and uniqueness conditions"; what is most likely intended there is closure under fusion or point-concatenation, as in the third condition for general flows:

(B2+) for all $\eta \in S$, for all $(i, t) \in \mathrm{dom}(\eta)$, and for all $\eta' \in S$, if $\eta'(0, 0) = \eta(i, t)$ then $\eta *_{(i,t)} \eta' \in S$.

In the light of Theorem 5.2 and Proposition 5.1, we then have the following.

Proposition 5.3 *Given a state space $X \subseteq \mathbb{R}^n$, with X open in $\mathbb{R}^n$, and a set $S \subseteq \mathsf{CPath}^\infty(\mathbb{H}, X)$, if S is closed in the uniform topology $\mathcal{T}_3$ on $\mathsf{CPath}^\infty(\mathbb{H}, X)$ and $S = \mathrm{ran}(\Phi) \cup \mathrm{ran}(\mathsf{M}\Phi)$ for some general flow $\Phi : X \rightsquigarrow \mathsf{CPath}(\mathbb{H}, X)$ such that $\mathrm{dom}(\gamma)$ is regular for all $\gamma \in \mathrm{ran}(\Phi)$, then S is an abstract hybrid system in the sense of* [8].

6 Conclusion

This paper makes several contributions:

- formulating a quite general notion of a time structure that accommodates all the "pathologies" of hybrid dynamics;
- developing a uniform topology on general hybrid path spaces that (a) generalizes Skorokhod-type topologies by allowing set-valued retiming maps instead of bijective single-valued retiming

maps, and (b) gives explicit quantitative measures of closeness of paths with respect to three parameters: one temporal for signal domains, one spatial for signal values, and a third quantifying the duration for which finite prefixes of the signals are close with respect to the first two parameters;

- proving that the path operations of prefix, suffix and fusion all respect the uniformity structure; and
- proving the equivalence of this uniform topology with a compact-open topology on path spaces associated with graph-convergence for hybrid paths, for finite-dimensional time and path spaces that have norm-closed time domains.

Directions for further enquiry include the following:

- semi-continuity and compactness properties of general flows $\Phi\colon X \rightsquigarrow \mathsf{CPath}(T,X)$ and their maximal extensions $\mathsf{M}\Phi\colon X \rightsquigarrow \mathsf{Ext}(\mathsf{CPath}(T,X))$, relating to [3, 4, 7, 8];
- enriching the syntax and semantics of the temporal logic $\mathbf{GFL}^\star$ [13, 14] in order to express in the logic both topological and dynamical properties of general flow systems;
- pursuing in a topological extension of $\mathbf{GFL}^\star$ the idea put forth in [26, 27] that for robust satisfaction of a temporal logic formula, the path denotation set for the formula should be open in a topology on the path space, so the set contains an open tube around each of the paths within it.

References

[1] J.-P. Aubin, J. Lygeros, M. Quincampoix, S. Sastry, and N. Seube. Impulse differential inclusions: A viability approach to hybrid systems. *IEEE Trans. on Automatic Control*, 47:2–20, 2002.

[2] R. Goebel, J. Hespanha, A.R. Teel, C. Cai, and R. Sanfelice. Hybrid systems: Generalized solutions and robust stability. In *IFAC Symp. Nonlin.ear Control Systems*, page 112, 2004.

[3] R. Goebel and A.R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42(4):596–613, 2006.

[4] C. Cai, A.R. Teel, and R. Goebel. Smooth lyapunov functions for hybrid systems part i: Existence is equivalent to robustness. *IEEE Trans. Automatic Control*, 52(7):1264–77, 2007.

[5] P.J. Collins. Generalized hybrid trajectory spaces. In *Proc. of 17th Int. Symp. on Math. Theory of Networks and Systems (MTNS'06)*, pages 2101–2109, 2006.

[6] P.J. Collins. A trajectory-space approach to hybrid systems. In *Proc. of 16th Int. Symp. on Math. Theory of Networks and Systems (MTNS'04)*, 2004.

[7] P.J. Collins. Hybrid trajectory spaces. Technical Report MAS-R0501, Centrum voor Wiskundeen Informatica, Amsterdam, 2005.

[8] R. Sanfelice, R. Goebel, and A.R. Teel. Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Trans. Automatic Control*, 52(12):2282–97, 2007.

[9] J. Lygeros, K.H. Johansson, S.N. Simic, J. Zhang, and S.S. Sastry. Dynamical properties of hybrid automata. *IEEE Trans. Automatic Control*, 48:2–16, 2003.

[10] R. Alur, C. Courcoubetis, N. Halbwachs, T.A. Henzinger, P.-H. Ho, X. Nicollin, A. Olivero, J. Sifakis, and S. Yovine. The algorithmic analysis of hybrid systems. *Theoretical Computer Science*, 138:3–34, 1995.

[11] R. Alur, T.A. Henzinger, and P.-H. Ho. Automatic symbolic verification of embedded systems. *IEEE Trans. on Software Engineering*, 22:181–201, 1996.

[12] J.M. Davoren and A. Nerode. Logics for hybrid systems. *Proceedings of the IEEE*, 88:985–1010, July 2000.

[13] J.M. Davoren, V. Coulthard, N. Markey, and T. Moor. Non-deterministic temporal logics for general flow systems. In *Hybrid Systems: Computation and Control (HSCC'04)*, LNCS 2993, pages 280–295. Springer-Verlag, 2004.

[14] J.M. Davoren and P. Tabuada. On simulations and bisimulations of general flow systems. In *Hybrid Systems: Computation and Control (HSCC'07)*, LNCS 4416, pages 145–158. Springer-Verlag, 2007.

[15] A.J. Van der Schaft and J.M. Schumacher. *An Introduction to Hybrid Dynamical Systems*. Springer-Verlag, 2000.

[16] A.A. Julius. *On Interconnection and Equivalences of Continuous and Discrete Systems: A Behavioural Perspective*. The University of Twente, 2005. PhD thesis.

[17] M.E. Broucke. Regularity of solutions and homotopic equivalence for hybrid systems. In *37th IEEE Conference on Decision and Control (CDC'1998)*, pages 4283–8, 1998.

[18] M.E. Broucke and A. Arapostathis. Continuous selections of trajectories of hybrid systems. *Systems and Control Letters*, 47:149–157, 2002.

[19] C. Kossentini and P. Caspi. Mixed delay and threshold voters in critical real-time systems. In *Formal Techniques, Modelling and Analysis of Timed and Fault-Tolerant Systems (FORMATS/FTRTFT 2004)*, LNCS 3253, pages 21–35. Springer-Verlag, 2004.

[20] R. Rockafellar and R.J. Wets. *Variational Analysis*. Springer-Verlag, Berlin, 1998.

[21] K.R. Goodearl. *Partially Ordered Abelian Groups With Interpolation*. Mathematical Surveys and Monographs. American Mathematical Society, Providence, 1986.

[22] F.D. Torrisi and A. Bemporad. Hysdel – a tool for generating computational hybrid models for analysis and synthesis. *IEEE Trans. Control Systems Technology*, 12:235–249, 2004.

[23] J.M. Davoren. On hybrid systems and the modal mu-calculus. In *Hybrid Systems V*, LNCS 1567, pages 38–69. Springer-Verlag, 1999.

[24] C. Kossentini and P. Caspi. Approximation, sampling and voting in hybrid computing systems. In *Hybrid Systems: Computation and Control (HSCC'06)*, LNCS 3927, pages 363–376. Springer-Verlag, 2006.

[25] I.M. James. *Introduction to Uniform Spaces*. London Mathematical Society Lecture Notes. Cambridge University Press, 1990.

[26] V.Gupta, T.A. Henzinger, and R. Jagadeesan. Robust timed automata. In *Proc. of International Workshop on Hybrid and Real-Time Systems (HART)*, LNCS 1201, pages 331–345. Springer-Verlag, 1997.

[27] T.A. Henzinger and J.-F. Raskin. Robust undecidability of timed and hybrid systems. In *Hybrid Systems: Computation and Control (HSCC'00)*, LNCS 1790, pages 145–159. Springer-Verlag, 2000.