Topological Semantics and Bisimulations for Intuitionistic Modal Logics and Their Classical Companion Logics*

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Abstract. We take the well-known intuitionistic modal logic of Fischer Servi with semantics in bi-relational Kripke frames, and give the natural extension to topological Kripke frames. Fischer Servi's two interaction conditions relating the intuitionistic pre-order (or partial-order) with the modal accessibility relation generalise to the requirement that the relation and its inverse be lower semi-continuous with respect to the topology. We then investigate the notion of topological bisimulation relations between topological Kripke frames, as introduced by Aiello and van Benthem, and show that their topology-preserving conditions are equivalent to the properties that the inverse-relation and the relation are lower semi-continuous with respect to the topologies on the two models. Our first main result is that this notion of topological bisimulation yields semantic preservation w.r.t. topological Kripke models for both intuitionistic tense logics, and for their classical companion multi-modal logics in the setting of the Gödel translation. After giving canonical topological Kripke models for the Hilbert-style axiomatizations of the Fischer Servi logic and its classical multi-modal companion logic, we show that the syntactic Gödel translation induces a natural semantic map from the intuitionistic canonical model into the canonical model of the classical companion logic, and this map is itself a topological bisimulation.

1 Introduction

Topological semantics for intuitionistic logic and for the classical modal logic S4 have a long history going back to Tarski and co-workers in the 1930s and 40s, predating the relational Kripke semantics for both [25,31]. A little earlier again is the 1933 Gödel translation GT [21] of intuitionistic logic into classical S4. The translation makes perfect sense within the topological semantics: where $\Box$ is interpreted by topological interior, the translation $\mathrm{GT}(\neg\varphi) = \Box\neg\,\mathrm{GT}(\varphi)$

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says that intuitionistic negation calls for the *interior* of the complement, and not just the complement. In the topological semantics, a basic semantic object is the *denotation set* $[\![\varphi]\!]^{\mathcal{M}}$ of a formula φ, consisting of the set of all states/worlds of the model $\mathcal{M}$ at which the formula is true, and the semantic clauses of the logic are given in terms of operations on sets of states. The intuitionistic requirement on the semantics is that all formulas must denote open sets: that is, sets that are equal to their own interior. Any formula φ partitions the state space X into three disjoint sets: the two open sets $[\![\varphi]\!]^{\mathcal{M}}$ and $[\![\neg\varphi]\!]^{\mathcal{M}}$, and the closed set $bd([\![\varphi]\!]^{\mathcal{M}})$, with the points in the topological boundary set $bd([\![\varphi]\!]^{\mathcal{M}})$ falsifying the law of excluded middle, since they neither satisfy nor falsify φ.

For the extension from intuitionistic propositional logics to intuitionistic modal logics, Fischer Servi in the 1970s [16,17,18] developed semantics over bi-relational Kripke frames, and this work has generated a good deal of research [15,20,22,29,32,36,37]. In bi-relational frames $(X, \preccurlyeq, R)$ where $\preccurlyeq$ is a pre-order (quasi-order) for the intuitionistic semantics, and R is a binary accessibility relation on X for the modal operators, the two Fischer Servi conditions are equivalent to the following relation inclusions [18,29,32]:

$$(R^{-1} \circ \preccurlyeq) \subseteq (\preccurlyeq \circ R^{-1}) \qquad \text{and} \qquad (R \circ \preccurlyeq) \subseteq (\preccurlyeq \circ R) \tag{1}$$

where $\circ$ is relational/sequential composition, and $(\cdot)^{-1}$ is relational inverse. Axiomatically, the base Fischer Servi modal logic **IK** has normality axioms for both the modal box $\boxdot$ and the diamond $\Diamond$, as well as the additional two axiom schemes:

$$\mathbf{FS1}: \Diamond(\varphi \to \psi) \to (\boxdot\varphi \to \Diamond\psi) \text{ and } \mathbf{FS2}: (\Diamond\varphi \to \boxdot\psi) \to \boxdot(\varphi \to \psi) \tag{2}$$

A study of various normal extensions of **IK** is given in [32]. Earlier, starting from the 1950s, the intuitionistic S5 logic **MIPC** [30,8] was given algebraic semantics in the form of *monadic Heyting algebras* [4,27,28,34,35][1] and later as bi-relational frames with an equivalence relation for the S5 modality [5,14,28,34]. This line of work has focused on $\mathbf{MIPC} = \mathbf{IK} \oplus \mathbf{T}\boxdot\Diamond \oplus \mathbf{5}\boxdot\Diamond$ and its normal extensions[2], and translations into intuitionistic and intermediate predicate logics. Within algebraic semantics, topological spaces arise in the context of Stone duality, and in [4,5,14], the focus restricts to Stone spaces (compact, Hausdorff and having as a basis the Boolean algebra of closed-and-open sets).

In this paper, following [12], we give semantics for intuitionistic modal logic over topological Kripke frames $\mathcal{F} = (X, \mathcal{T}, R)$, where $(X, \mathcal{T})$ is a topological space and $R \subseteq X \times X$ is an accesibility relation for the modalities; the Fisher Servi bi-relational semantics are straight-forwardly extended from pre-orders $\preccurlyeq$ on X and their associated *Alexandrov topology* $\mathcal{T}_{\preccurlyeq}$, to arbitrary topological spaces

[1] The additional *monadic* operators are $\forall$ and $\exists$ unary operators behaving as S5 box and diamond modalities, and come from Halmos' work on monadic Boolean algebras.

[2] Here, $\mathbf{T}\boxdot\Diamond$ is the conjunction of the separate $\boxdot$ and $\Diamond$ characteristic schemes for reflexivity, and likewise $\mathbf{5}\boxdot\Diamond$ for Euclideanness, so together they characterize equivalence relations.

$(X, \mathcal{T})$[3]. Over topological Kripke frames, the two Fischer Servi bi-relational conditions on the interaction between modal and intuitionistic semantics ((1) above) generalize to *semi-continuity* properties of the relation R, and of its inverse R^{-1}, with respect to the topology. As for the base logic, Fischer Servi's extension of the Gödel translation reads as a direct transcription of the topological semantics. The translation $\mathrm{GT}(\boxdot\varphi) = \Box\boxdot\,\mathrm{GT}(\varphi)$ says that the intuitionistic box requires the interior of the classical box operator, since the latter is defined by an intersection and may fail to preserve open sets. In contrast, the translation clause $\mathrm{GT}(\diamonddot\varphi) = \diamonddot\,\mathrm{GT}(\varphi)$ says that, semantically, the operator $\diamonddot$ preserves open sets. This condition is exactly the lower semi-continuity (l.s.c.) condition on the accessibility relation, and corresponds to the first Fischer Servi bi-relational inclusion $R^{-1}\circ \preccurlyeq\, \subseteq\, \preccurlyeq \circ R^{-1}$ in (1), and it is this condition that is required to verify topological soundness of the axiom scheme **FS1** in (2)[4]. Similarly, Fischer Servi's second bi-relational inclusion $R \circ \preccurlyeq\, \subseteq\, \preccurlyeq \circ R$ generalizes to the l.s.c. property of the R^{-1} relation, where the latter is required to verify topological soundness of the axiom scheme **FS2** in (2).

The symmetry of the interaction conditions on the modal relation R and its inverse R^{-1} means that we can – with no additional semantic assumptions – lift the topological semantics to intuitionistic tense logics extending Fischer Servi's modal logic (introduced by Ewald in [15]), with modalities in pairs $\diamonddot$, $\boxdot$, and $\blacklozenge$, $\blacksquare$, for future and past along the accessibility relation. It soon becomes clear that the resulting semantics and metatheoretic results such as completeness come out *cleaner and simpler* for the tense logic than they do for the modal logic. We can often streamline arguments involving the box modality $\boxdot$ by using its adjoint diamond $\blacklozenge$, which like $\diamonddot$, preserves open sets. Furthermore, with regard to applications of interest, the flexibility of having both forwards and backwards modalities is advantageous. For example, if $X \subseteq \mathbb{R}^n$ is equipped with the Euclidean topology, and $R \subseteq X \times X$ is the *reachability relation* of a continuous or hybrid dynamical system [2,3,11], then the formula $\blacklozenge p$ denotes the set of states *reachable from* the p states, with p considered as a source or initial state set, while the forward modal diamond formula $\diamonddot p$ denotes the set of states *from which* p states *can be reached*, here p denoting a target or goal state set. Under some standard regularity assumptions on the differential inclusions or equations, [2,3], the reachability relation R and its inverse will be l.s.c. (as

[3] Other work giving topological semantics for intuitionistic modal logics is [36], further investigated in [23]. This logic is properly weaker than Fischer Servi's as its intuitionistic diamond is not required to distribute over disjunction (hence is sub-normal). Both the bi-relational and topological semantics in [36] and the *relational spaces* in [23] have *no* conditions on the interaction of the intuitionistic and modal semantic structures, and the semantic clauses for both box and diamond require application of the interior operator to guarantee open sets.

[4] In the algebraic setting of Monteiro and Varsavsky's work [27] w.r.t. the logic MIPC, a special case of the l.s.c. property is anticipated: the lattice of open sets of a topological space is a complete Heyting algebra, and the structure yields a monadic Heyting algebra when the space is further equipped with an equivalence relation R with the property that the "R-saturation" or R-expansion of an open set is open.

well as reflexive and transitive), while further assumptions are required for the u.s.c. property (e.g. R is a closed set in the product topology on $X \times X$).

We continue on the theme of semi-continuity properties of relations in our second topic of investigation, namely that of *topological bisimulations* between topological Kripke models. A bisimulation notion for topological spaces $(X, \mathcal{T})$ has recently been developed by Aiello and van Benthem (e.g. [1], Def. 2.1). We show below that their forth and back topology-preserving conditions are equivalent to the lower semi-continuity of the inverse relation and of the relation, respectively. The first main result of the paper is that this notion of topological bisimulation yields the semantic preservation property w.r.t. topological Kripke models for both intuitionistic tense logics, and for their classical companion multi-modal logics in the setting of the Gödel translation.

In the last part of the paper, we give canonical topological Kripke models for the Hilbert-style axiomatizations of the Fischer Servi logics and their classical companions logics – over the set of prime theories of the intuitionistic logic and the set of ultrafilters of the companion classical logic, respectively, with topologies on the spaces that are neither Alexandrov nor Stone. We conclude the paper with the second main result: the syntactic Gödel translation induces a natural semantic map from the intuitionistic canonical model to a sub-model of the canonical model of the classical companion logic, and this map is itself a topological bisimulation.

2 Preliminaries from General Topology

We adopt the notation from set-valued analysis [2] in writing $r : X \rightsquigarrow Y$ to mean both that $r : X \to 2^Y$ is a *set-valued map*, with (possibly empty) set-values $r(x) \subseteq Y$ for each $x \in X$, and equivalently, that $r \subseteq X \times Y$ is a *relation*. The expressions $y \in r(x)$, $(x, y) \in r$ and $x\,r\,y$ are synonymous. For a map $r : X \rightsquigarrow Y$, the *inverse* $r^{-1} : Y \rightsquigarrow X$ given by: $x \in r^{-1}(y)$ iff $y \in r(x)$; the *domain* is $dom(r) := \{x \in X \mid r(x) \neq \varnothing\}$, and the *range* is $ran(r) := dom(r^{-1}) \subseteq Y$. A map $r : X \rightsquigarrow Y$ is *total on* X if $dom(r) = X$, and *surjective on* Y if $ran(r) = Y$. We write (as usual) $r : X \to Y$ to mean r is a *function*, i.e. a single-valued map total on X with values written $r(x) = y$ (rather than $r(x) = \{y\}$). For $r_1 : X \rightsquigarrow Y$ and $r_2 : Y \rightsquigarrow Z$, we write their relational composition as $r_1 \circ r_2 : X \rightsquigarrow Z$ given by $(r_1 \circ r_2)(x) := \{z \in Z \mid (\exists y \in Y)\,[(x, y) \in r_1 \wedge (y, z) \in r_2]\}$. Recall that $(r_1 \circ r_2)^{-1} = r_2^{-1} \circ r_1^{-1}$. A *pre-order* (*quasi-order*) is a reflexive and transitive binary relation, and a *partial-order* is a pre-order that is also anti-symmetric.

A relation $r : X \rightsquigarrow Y$ determines two *pre-image operators* (predicate transformers). The *existential* (or *lower*) pre-image is of type $r^{-\exists} : 2^Y \to 2^X$ and the *universal* (or *upper*) pre-image $r^{-\forall} : 2^Y \to 2^X$ is its dual w.r.t. set-complement:

$$\begin{aligned} r^{-\exists}(W) &:= \{x \in X \mid (\exists y \in Y)[\,(x, y) \in r \quad \wedge \quad y \in W]\} \\ &= \{x \in X \mid W \cap r(x) \neq \varnothing\} \\ r^{-\forall}(W) &:= X - r^{-\exists}(Y - W) \;=\; \{x \in X \mid r(x) \subseteq W\} \end{aligned}$$

for all $W \subseteq Y$. The operator $r^{-\exists}$ distributes over arbitrary unions, while $r^{-\forall}$ distributes over arbitrary intersections: $r^{-\exists}(\varnothing) = \varnothing$, $r^{-\exists}(Y) = dom(r)$, $r^{-\forall}(\varnothing) = X - dom(r)$, and $r^{-\forall}(Y) = X$. Note that when $r : X \to Y$ is a function, the pre-image operators reduce to the standard *inverse-image* operator; i.e. $r^{-\exists}(W) = r^{-\forall}(W) = r^{-1}(W)$. The pre-image operators respect relational inclusions: if $r_1 \subseteq r_2 \subseteq X \times Y$, then for all $W \subseteq Y$, we have $r_1^{-\exists}(W) \subseteq r_2^{-\exists}(W)$, but reversing to $r_2^{-\forall}(W) \subseteq r_1^{-\forall}(W)$. For the case of binary relations $r\colon X \rightsquigarrow X$ on a space X, the pre-images express in operator form the standard relational Kripke semantics for the (future) diamond and box modal operators determined by r. The operators on sets derived from the inverse relation r^{-1} are usually called the *post-image operators* $r^{\exists}$, $r^{\forall} : 2^X \to 2^Y$ defined by $r^{\exists} := (r^{-1})^{-\exists}$ and $r^{\forall} := (r^{-1})^{-\forall}$; these arise in the relational Kripke semantics for the *past* diamond and box operators in tense and temporal logics. The fundamental relationship between pre- and post-images is the *adjoint property*:

$$\forall W_1 \subseteq X,\ \forall W_2 \subseteq Y, \quad W_1 \subseteq r^{-\forall}(W_2) \quad \text{iff} \quad r^{\exists}(W_1) \subseteq W_2\,. \tag{3}$$

A *topology* $\mathcal{T} \subseteq 2^X$ on a set X is a family of subsets of X closed under arbitrary unions and finite intersections. The extreme cases are the *discrete* topology $\mathcal{T}_D = 2^X$, and the *trivial* (or *indiscrete*) topology $\mathcal{T}_\varnothing = \{\varnothing, X\}$. The *interior operator* $int_{\mathcal{T}} : 2^X \to 2^X$ determined by $\mathcal{T}$ is given by $int_{\mathcal{T}}(W) := \bigcup\{U \in \mathcal{T} \mid U \subseteq W\}$. Sets $W \in \mathcal{T}$ are called *open* w.r.t. $\mathcal{T}$, and this is so iff $W = int_{\mathcal{T}}(W)$. Let $-\mathcal{T} := \{W \subseteq X \mid (X - W) \in \mathcal{T}\}$ denote the dual lattice under set-complement. Sets $W \in -\mathcal{T}$ are called *closed* w.r.t. $\mathcal{T}$, and this is so iff $W = cl_{\mathcal{T}}(W)$, where the dual *closure operator* $cl_{\mathcal{T}} : 2^X \to 2^X$ is given by $cl_{\mathcal{T}}(W) := X - cl_{\mathcal{T}}(X - W)$, and the topological *boundary* is $bd_{\mathcal{T}}(W) := cl_{\mathcal{T}}(W) - int_{\mathcal{T}}(W)$. A family of open sets $\mathcal{B} \subseteq \mathcal{T}$ constitutes a *basis* for a topology $\mathcal{T}$ on X if every open set $W \in \mathcal{T}$ is a union of basic opens in $\mathcal{B}$, and for every $x \in X$ and every pair of basic opens $U_1, U_2 \in \mathcal{B}$ such that $x \in U_1 \cap U_2$, there exists $U_3 \in \mathcal{B}$ such that $x \in U_3 \subseteq (U_1 \cap U_2)$.

The purely topological notion of *continuity* for a function $f\colon X \to Y$ is that the inverse image $f^{-1}(U)$ is open whenever U is open. Analogous notions for relations/set-valued maps were introduced by Kuratowski and Bouligand in the 1920s. Given two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$, a map $R : X \rightsquigarrow Y$ is called: *lower semi-continuous* (l.s.c.) if for every $\mathcal{S}$-open set U in Y, $R^{-\exists}(U)$ is $\mathcal{T}$-open in X; *upper semi-continuous* (u.s.c.) if for every $\mathcal{S}$-open set U in Y, $R^{-\forall}(U)$ is $\mathcal{T}$-open in X; and (Vietoris) *continuous* if it is both l.s.c. and u.s.c. [2,7,24,33]. The u.s.c. condition is equivalent to $R^{-\exists}(V)$ is $\mathcal{T}$-closed in X whenever V is $\mathcal{S}$-closed in Y. Moreover, we have: $R : X \rightsquigarrow Y$ is l.s.c. iff $R^{-\exists}(int_{\mathcal{S}}(W)) \subseteq int_{\mathcal{T}}(R^{-\exists}(W))$ for all $W \subseteq Y$; and $R : X \rightsquigarrow Y$ is u.s.c. iff $R^{-\forall}(int_{\mathcal{S}}(W)) \subseteq int_{\mathcal{T}}(R^{-\forall}(W))$ for all $W \subseteq Y$ ([24], Vol. I, §18.I, p.173). The two semi-continuity properties reduce to the standard notion of continuity for functions $R : X \to Y$. Both semi-continuity properties are preserved under relational composition, and also under finite unions of relations.

We note the subclass of *Alexandrov topologies* because of their correspondence with Kripke relational semantics for classical S4 and intuitionistic logics. e.g. [1,26]. A topological space $(X, \mathcal{T})$ is called *Alexandrov* if for every $x \in X$, there is a *smallest* open set $U \in \mathcal{T}$ such that $x \in U$. In particular, every *finite* topology (i.e. only finitely many open sets) is Alexandrov. There is a one-to-one correspondence between pre-orders on X and Alexandrov topologies on X. Any pre-order $\preccurlyeq$ on X induces an Alexandrov topology $\mathcal{T}_{\preccurlyeq}$ by taking $int_{\mathcal{T}_{\preccurlyeq}}(W) := (\preccurlyeq)^{-\forall}(W)$, which means $U \in \mathcal{T}_{\preccurlyeq}$ iff U is upwards-$\preccurlyeq$-closed. In particular, $\mathcal{T}_{\preccurlyeq}$ is closed under arbitrary intersections as well as arbitrary unions, and $-\mathcal{T}_{\preccurlyeq} = \mathcal{T}_{\succcurlyeq}$. Conversely, for any topology, define a pre-order $\preccurlyeq_{\mathcal{T}}$ on X, known as the *specialisation pre-order*: $x \preccurlyeq_{\mathcal{T}} y$ iff $(\forall U \in \mathcal{T})\,[x \in U \Rightarrow y \in U]$. For any pre-order, $\preccurlyeq_{\mathcal{T}_{\preccurlyeq}} = \preccurlyeq$, and for any topology, $\mathcal{T}_{\preccurlyeq_{\mathcal{T}}} = \mathcal{T}$ iff $\mathcal{T}$ is Alexandrov (e.g. see [1], pp. 893-894). For further concepts in topology, see, e.g. [33].

3 Intuitionistic Modal and Tense Logics, and Their Classical Companion Logics: Syntax and Topological Semantics

Fix a countably infinite set AP of atomic propositions. The propositional language $\mathcal{L}_0$ is generated from $p \in AP$ using the connectives $\vee$, $\wedge$, $\to$ and the constant $\bot$. As usual, define further connectives: $\neg\,\varphi := \varphi \to \bot$ and $\varphi_1 \leftrightarrow \varphi_2 := (\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$, and $\top := \bot \to \bot$. Let $\mathcal{L}_{0,\Box}$ be the mono-modal language extending $\mathcal{L}_0$ with the addition of the unary modal operator $\Box$. A further modal operator $\Diamond$ can be defined as the classical dual: $\Diamond\varphi := \neg\Box\neg\varphi$.

For the intuitionistic modal and tense languages, let $\mathcal{L}^{\mathbf{m}}$ ($\mathcal{L}^{\mathbf{t}}$) be the modal (tense) language extending $\mathcal{L}_0$ with the addition of two (four) modal operators $\Diamonddot$ and $\boxdot$ (and $\blacklozenge$ and $\blacksquare$), generated by the grammar:

$$\varphi \ ::= \ p \mid \bot \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \to \varphi_2 \mid \Diamonddot\varphi \mid \boxdot\varphi \quad (\mid \blacklozenge\varphi \mid \blacksquare\varphi\,)$$

for $p \in AP$. Likewise, for the classical topological modal and tense logics, let $\mathcal{L}^{\mathbf{m}}_{\Box}$ ($\mathcal{L}^{\mathbf{t}}_{\Box}$) be the modal (tense) language extending $\mathcal{L}_{0,\Box}$ with the addition of $\Diamonddot$ and $\boxdot$ (and $\blacklozenge$ and $\blacksquare$).

The original Gödel translation [21], as a function $\mathrm{GT} : \mathcal{L}_0 \to \mathcal{L}_{0,\Box}$, simply prefixes $\Box$ to *every* subformula of a propositional formula. Reading the S4 $\Box$ as topological interior, this means we force every propositional formula to intuitionistically denote an open set. In Fischer Servi's extension of the Gödel translation [18,16], the clauses for the propositional fragment are from a variant translation used by Fitting [19], who shows it to be equivalent to Gödel's original ([19], Ch. 9, # 20). Define the function $\mathrm{GT} : \mathcal{L}^{\mathbf{t}} \to \mathcal{L}^{\mathbf{t}}_{\Box}$ by induction on formulas as follows:

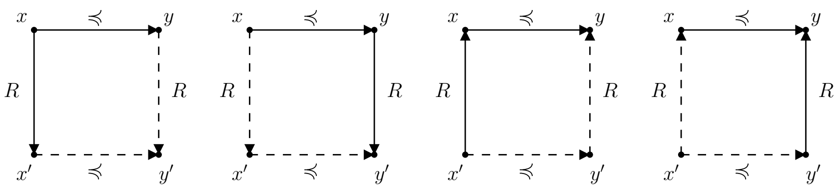
$$\begin{array}{rlrl}
\mathrm{GT}(p) & := \Box p \ \text{ for } p \in AP & & \\
\mathrm{GT}(\varphi_1 \to \varphi_2) & := \Box\,(\mathrm{GT}(\varphi_1) \to \mathrm{GT}(\varphi_2)) & \mathrm{GT}(\bot) & := \bot \\
\mathrm{GT}(\varphi_1 \vee \varphi_2) & := \mathrm{GT}(\varphi_1) \vee \mathrm{GT}(\varphi_2) & \mathrm{GT}(\varphi_1 \wedge \varphi_2) & := \mathrm{GT}(\varphi_1) \wedge \mathrm{GT}(\varphi_2) \\
\mathrm{GT}(\Diamonddot\varphi) & := \Diamonddot\,\mathrm{GT}(\varphi) & \mathrm{GT}(\blacklozenge\varphi) & := \blacklozenge\,\mathrm{GT}(\varphi) \\
\mathrm{GT}(\boxdot\varphi) & := \Box\boxdot\,\mathrm{GT}(\varphi) & \mathrm{GT}(\blacksquare\varphi) & := \Box\blacksquare\,\mathrm{GT}(\varphi)
\end{array}$$

In topological terms, the only clauses in the translation where it is essential to have an explicit $\Box$ to guarantee openness of denotation sets are for atomic propositions, for implication $\rightarrow$, and for the box modalties $\boxdot$ and $\blacksquare$. There is no such need in the clauses for $\vee$ and $\wedge$ because finite unions and finite intersections of open sets are open. For the diamond modalties, the semi-continuity conditions that R and its inverse R^{-1} are both l.s.c. ensure that the semantic operators $R^{-\exists}$ and $R^{\exists}$ interpretting $\diamonddot$ and $\blacklozenge$ must preserve open sets. We now explain this generalization, which was first presented in [12].

The bi-relational semantics of Fischer Servi [16,17], and Plotkin and Stirling [29,32] are over Kripke frames $\mathcal{F} = (X, \preccurlyeq, R)$, where $\preccurlyeq$ is a pre-order on X and $R : X \rightsquigarrow X$ is the modal accessibility relation. Using the induced Alexandrov topology $\mathcal{T}_{\preccurlyeq}$, a bi-relational Kripke frame $\mathcal{F}$ is equivalent to the topological frame $(X, \mathcal{T}_{\preccurlyeq}, R)$. A set is open in $\mathcal{T}_{\preccurlyeq}$ exactly when it is $\preccurlyeq$-*persistent* or upward-$\preccurlyeq$-closed. The four bi-relational conditions identified in [29], and also familiar as the forth ("Zig") and back ("Zag") conditions for *bisimulations* (e.g. [6], Ch. 2), can be cleanly transcribed as *semi-continuity conditions* on the relations $R : X \rightsquigarrow X$ and $R^{-1} : X \rightsquigarrow X$ with respect to the topology $\mathcal{T}_{\preccurlyeq}$.

Definition 1. *Let $\mathcal{F} = (X, \preccurlyeq, R)$ be a bi-relational frame. Four conditions expressing interaction between $\preccurlyeq$ and R are identified as follows:*

$$
\begin{array}{llllllll}
\mathbf{Zig}(\preccurlyeq, R): & \textit{if} & x \preccurlyeq y & \textit{and} & x\,R\,x' & \textit{then } (\exists y' \in X) & [\; y\,R\,y' & \textit{and} \quad x' \preccurlyeq y' \;] \\
\mathbf{Zag}(\preccurlyeq, R): & \textit{if} & x \preccurlyeq y & \textit{and} & y\,R\,y' & \textit{then } (\exists x' \in X) & [\; x\,R\,x' & \textit{and} \quad x' \preccurlyeq y' \;] \\
\mathbf{Zig}(\preccurlyeq, R^{-1}): & \textit{if} & x \preccurlyeq y & \textit{and} & x'\,R\,x & \textit{then } (\exists y' \in X) & [\; y'\,R\,y & \textit{and} \quad x' \preccurlyeq y' \;] \\
\mathbf{Zag}(\preccurlyeq, R^{-1}): & \textit{if} & x \preccurlyeq y & \textit{and} & y'\,R\,y & \textit{then } (\exists x' \in X) & [\; x'\,R\,x & \textit{and} \quad x' \preccurlyeq y' \;]
\end{array}
$$



$\mathbf{Zig}(\preccurlyeq, R)$ $\mathbf{Zag}(\preccurlyeq, R)$ $\mathbf{Zig}(\preccurlyeq, R^{-1})$ $\mathbf{Zag}(\preccurlyeq, R^{-1})$

From earlier work [9], we know these bi-relational conditions correspond to semi-continuity properties of R with respect to the Alexandrov topology $\mathcal{T}_{\preccurlyeq}$.

Proposition 1. ([9]) *Let $\mathcal{F} = (X, \preccurlyeq, R)$ be a bi-relational frame, with $\mathcal{T}_{\preccurlyeq}$ it induced topology. The conditions in each row below are equivalent.*

1.	$\mathbf{Zig}(\preccurlyeq, R)$	$(R^{-1} \circ \preccurlyeq) \subseteq (\preccurlyeq \circ R^{-1})$	R *is* l.s.c. *in* $\mathcal{T}_{\preccurlyeq}$
2.	$\mathbf{Zag}(\preccurlyeq, R)$	$(\preccurlyeq \circ R) \subseteq (R \circ \preccurlyeq)$	R *is* u.s.c. *in* $\mathcal{T}_{\preccurlyeq}$
3.	$\mathbf{Zig}(\preccurlyeq, R^{-1})$	$(R \circ \preccurlyeq) \subseteq (\preccurlyeq \circ R)$	R^{-1} *is* l.s.c. *in* $\mathcal{T}_{\preccurlyeq}$
4.	$\mathbf{Zag}(\preccurlyeq, R^{-1})$	$(\preccurlyeq \circ R^{-1}) \subseteq (R^{-1} \circ \preccurlyeq)$	R^{-1} *is* u.s.c. *in* $\mathcal{T}_{\preccurlyeq}$

The Fischer Servi interaction conditions between the intuitionistic and modal relations, introduced in [17] and used in [15,18,22,29,32], are the first and third bi-relational conditions $\mathbf{Zig}(\preccurlyeq, R)$ and $\mathbf{Zig}(\preccurlyeq, R^{-1})$. In Kripke frames meeting these conditions, one can give semantic clauses for the diamond and box that are natural under the intuitionistic reading of the restricted $\exists$ and $\forall$ quantification with respect to R-successors. More precisely, the resulting logic is faithfully embedded into intuitionistic first-order logic by the standard modal to first-order translation, and a natural extension of the Gödel translation faithfully embeds it into the classical bi-modal logic combining **S4**$\Box$ with **K** or extensions.

Since the Fischer Servi interaction conditions for the *forward* or *future* modal operators $\diamonddot$ and $\boxdot$ for R require the same l.s.c. property of both R and R^{-1}, this means that, *at no extra cost* in semantic assumptions, we can add on the *backward* or *past* modal operators $\blacklozenge$ and $\blacksquare$ for R^{-1}, and obtain the desired interaction condition for R^{-1} *for free.*

Definition 2. *A* topological frame *is a structure* $\mathcal{F} = (X, \mathcal{T}, R)$ *where* $(X, \mathcal{T})$ *is a topological space and* $R : X \rightsquigarrow X$ *is a binary relation.* $\mathcal{F}$ *is an* l.s.c. topological frame *if both* R *and* R^{-1} *are l.s.c. in* $\mathcal{T}$*. A* model *over* $\mathcal{F}$ *is a structure* $\mathcal{M} = (\mathcal{F}, v)$ *where* $v : AP \rightsquigarrow X$ *is an atomic valuation relation. A model* $\mathcal{M}$ *is an* open model *if for each* $p \in AP$*, the denotation set* $v(p)$ *is open in* $\mathcal{T}$*. For open models* $\mathcal{M}$ *over l.s.c. frames* $\mathcal{F}$*, the intuitionistic denotation map* $[\![\cdot]\!]^{\mathcal{M}}_{\mathbf{I}} : \mathcal{L}^{\mathbf{t}} \rightsquigarrow X$ *(or* $[\![\cdot]\!]^{\mathcal{M}}_{\mathbf{I}} : \mathcal{L}^{\mathbf{m}} \rightsquigarrow X$*) is defined by:*

$$
\begin{aligned}
[\![p]\!]^{\mathcal{M}}_{\mathbf{I}} &:= v(p) \quad \text{for } p \in AP & [\![\bot]\!]^{\mathcal{M}}_{\mathbf{I}} &:= \varnothing \\
[\![\varphi_1 \to \varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}} &:= int_{\mathcal{T}}((X - [\![\varphi_1]\!]^{\mathcal{M}}_{\mathbf{I}}) \cup [\![\varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}}) & & \\
[\![\varphi_1 \vee \varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}} &:= [\![\varphi_1]\!]^{\mathcal{M}}_{\mathbf{I}} \cup [\![\varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}} & [\![\varphi_1 \wedge \varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}} &:= [\![\varphi_1]\!]^{\mathcal{M}}_{\mathbf{I}} \cap [\![\varphi_2]\!]^{\mathcal{M}}_{\mathbf{I}} \\
[\![\diamonddot\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} &:= R^{-\exists}([\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}}) & [\![\boxdot\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} &:= int_{\mathcal{T}}\big(R^{-\forall}([\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}})\big) \\
[\![\blacklozenge\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} &:= R^{\exists}([\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}}) & [\![\blacksquare\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} &:= int_{\mathcal{T}}\big(R^{\forall}([\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}})\big)
\end{aligned}
$$

A formula $\varphi \in \mathcal{L}^{\mathbf{t}}$ *(or* $\varphi \in \mathcal{L}^{\mathbf{m}}$*) is* int-modal-top valid *in an open model* $\mathcal{M}$*, written* $\mathcal{M} \Vdash \varphi$*, if* $[\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} = X$*, and is* int-modal-top valid *in an l.s.c. frame* $\mathcal{F} = (X, \mathcal{T}, R)$*, written* $\mathcal{F} \Vdash \varphi$*, if* $\mathcal{M} \Vdash \varphi$ *for all open models* $\mathcal{M}$ *over* $\mathcal{F}$*. Formula* φ *is* satisfiable *in* $\mathcal{M}$ *if* $[\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} \neq \varnothing$*, and* φ *is* falsifiable *in* $\mathcal{M}$ *if* $[\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}} \neq X$*. Let* $\mathbb{IK}^{\mathbf{t}}\mathbb{T}$ *(*$\mathbb{IK}^{\mathbf{m}}\mathbb{T}$*) be the set of all* $\varphi \in \mathcal{L}^{\mathbf{t}}$ *(*$\varphi \in \mathcal{L}^{\mathbf{m}}$*) such that* $\mathcal{F} \Vdash \varphi$ *in every l.s.c. topological frame* $\mathcal{F}$*.*

The property that every denotation set $[\![\varphi]\!]^{\mathcal{M}}_{\mathbf{I}}$ is open in $\mathcal{T}$ follows immediately from the openness condition on $v(p)$, the l.s.c. properties of $R^{-\exists}$ and $R^{\exists}$, and the extra interior operation in the semantics for $\to$, $\boxdot$ and $\blacksquare$.

Definition 3. *For the tense (modal) language* $\mathcal{L}^{\mathbf{t}}_{\Box}$ *(and* $\mathcal{L}^{\mathbf{m}}_{\Box}$ *), we define the classical denotation map* $[\![\cdot]\!]^{\mathcal{M}} : \mathcal{L}^{\mathbf{t}}_{\Box} \rightsquigarrow X$ *(* $[\![\cdot]\!]^{\mathcal{M}} : \mathcal{L}^{\mathbf{m}}_{\Box} \rightsquigarrow X$ *) with respect to arbitrary topological models* $\mathcal{M} = (X, \mathcal{T}, R, v)$*, where* $v : AP \rightsquigarrow X$ *is unrestricted. The map* $[\![\cdot]\!]^{\mathcal{M}}$ *is defined the same way as* $[\![\cdot]\!]^{\mathcal{M}}_{\mathbf{I}}$ *for atomic* $p \in AP$*,* $\bot$*,* $\vee$*,* $\wedge$*,* $\diamonddot$ *and* $\blacklozenge$*, but differs on the following clauses:*

$$
\begin{aligned}
[\![\varphi_1 \to \varphi_2]\!]^{\mathcal{M}} &:= (X - [\![\varphi_1]\!]^{\mathcal{M}}) \cup [\![\varphi_2]\!]^{\mathcal{M}} & [\![\Box\varphi]\!]^{\mathcal{M}} &:= int_{\mathcal{T}}([\![\varphi]\!]^{\mathcal{M}}) \\
[\![\boxdot\varphi]\!]^{\mathcal{M}} &:= R^{-\forall}([\![\varphi]\!]^{\mathcal{M}}) & [\![\blacksquare\varphi]\!]^{\mathcal{M}} &:= R^{\forall}([\![\varphi]\!]^{\mathcal{M}})
\end{aligned}
$$

A formula $\varphi \in \mathcal{L}^{\mathbf{t}}_{\Box}$ *(or* $\varphi \in \mathcal{L}^{\mathbf{m}}_{\Box}$*) is* modal-top valid *in* $\mathcal{M}$*, written* $\mathcal{M} \models \varphi$*, if* $[\![\varphi]\!]^{\mathcal{M}} = X$*, and is* modal-top valid *in a topological frame* $\mathcal{F} = (X, \mathcal{T}, R)$*, written*

$\mathcal{F} \models \varphi$, if $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$ over $\mathcal{F}$. Let $\mathbb{K}^{\mathbf{t}}\mathbb{T}$ ($\mathbb{K}^{\mathbf{m}}\mathbb{T}$) be the set of all $\varphi \in \mathcal{L}^{\mathbf{t}}_{\Box}$ ($\varphi \in \mathcal{L}^{\mathbf{m}}_{\Box}$) such that $\mathcal{F} \models \varphi$ for every topological frame $\mathcal{F}$. Let $\mathbb{K}^{\mathbf{t}}\mathbb{LSC}$ ($\mathbb{K}^{\mathbf{m}}\mathbb{LSC}$) be the set of all $\varphi \in \mathcal{L}^{\mathbf{t}}_{\Box}$ ($\varphi \in \mathcal{L}^{\mathbf{m}}_{\Box}$) such that $\mathcal{F} \models \varphi$ in every l.s.c. topological frame $\mathcal{F}$.

For Fischer Servi's extension of Gödel's translation, Definitions 2 and 3 imply that for any model $\mathcal{M} = (\mathcal{F}, v)$ over an l.s.c. topological frame $\mathcal{F}$, if $\mathcal{M}' = (\mathcal{F}, v')$ is the variant open model with $v'(p) := int_{\mathcal{T}}(v(p))$, then $\forall \varphi \in \mathcal{L}^{\mathbf{t}}$:

$$[\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}'} = [\![\,\mathrm{GT}(\varphi)\,]\!]^{\mathcal{M}} = [\![\,\Box\mathrm{GT}(\varphi)\,]\!]^{\mathcal{M}}. \tag{4}$$

Consequently, we have semantic faithfulness, as well as the openness property: for all $\varphi \in \mathcal{L}^{\mathbf{t}}$, the formula $\mathrm{GT}(\varphi) \leftrightarrow \Box\mathrm{GT}(\varphi)$ is in $\mathbb{K}^{\mathbf{t}}\mathbb{LSC}$.

Proposition 2. [Extended Gödel translation: semantic faithfulness]
For all $\varphi \in \mathcal{L}^{\mathbf{t}}$, $\varphi \in \mathbb{IK}^{\mathbf{t}}\mathbb{T}$ iff $\mathrm{GT}(\varphi) \in \mathbb{K}^{\mathbf{t}}\mathbb{LSC}$.

The semi-continuity conditions can be cleanly characterized in the companion classical multi-modal logics, as given in [13].

Proposition 3. [[13] Modal characterization of semi-continuity conditions]
Let $\mathcal{F} = (X, \mathcal{T}, R)$ be a topological frame and let $p \in AP$. In the following table, the conditions listed across each row are equivalent.

(1.)	*R is l.s.c. in $\mathcal{T}$*	$\mathcal{F} \models \Diamonddot\Box p \to \Box\Diamonddot p$	$\mathcal{F} \vDash \Diamonddot\Box p \leftrightarrow \Box\Diamonddot\Box p$
(2.)	*R is u.s.c. in $\mathcal{T}$*	$\mathcal{F} \models \boxdot\Box p \to \Box\boxdot p$	
(3.)	*R^{-1} is l.s.c. in $\mathcal{T}$*	$\mathcal{F} \models \Box\boxdot p \to \boxdot\Box p$	$\mathcal{F} \vDash \blacklozenge\Box p \leftrightarrow \Box\blacklozenge\Box p$
(4.)	*R^{-1} is u.s.c. in $\mathcal{T}$*	$\mathcal{F} \models \Box\Diamonddot p \to \Diamonddot\Box p$	

4 Topological Bisimulations

Aiello and van Benthem's notions of topological simulation and bisimulation between classical S4 topological models are as follows.

Definition 4. [[1], Definition 2.1] *Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces, let $v_1 : AP \rightsquigarrow X_1$ and $v_2 : AP \rightsquigarrow X_2$ be valuations of atomic propositions, and let $\mathcal{M}_1 = (X_1, \mathcal{T}_1, v_1)$ and $\mathcal{M}_2 = (X_2, \mathcal{T}_2, v_2)$ be topological models.*
A relation $B : X_1 \rightsquigarrow X_2$ is a topo-bisimulation *between $\mathcal{M}_1$ and $\mathcal{M}_2$ if*
(i.a) $\forall x \in X_1,\ \forall y \in X_2,\ \forall p \in AP,$ if $x\,B\,y$ and $x \in v_1(p)$ then $y \in v_2(p)$;
(i.b) $\forall x \in X_1,\ \forall y \in X_2,\ \forall p \in AP,$ if $x\,B\,y$ and $y \in v_2(p)$ then $x \in v_1(p)$;
(ii.a) $\forall x \in X_1,\ \forall y \in X_2,\ \forall U \in \mathcal{T}_1,$ if $x\,B\,y$ and $x \in U$
then $\exists V \in \mathcal{T}_2$ with $y \in V$ and $\forall y' \in V,\ \exists x' \in U$ such that $x'\,B\,y'$;
(ii.b) $\forall x \in X_1,\ \forall y \in X_2,\ \forall V \in \mathcal{T}_2,$ if $x\,B\,y$ and $y \in V$
then $\exists U \in \mathcal{T}_1$ with $x \in U$ and $\forall x' \in U,\ \exists y' \in V$ such that $x'\,B\,y'$.
If only conditions (i.a) *and* (ii.a) *hold of a relation $B : X_1 \rightsquigarrow X_2$, then B is called a* topo-simulation *of $\mathcal{M}_1$ by $\mathcal{M}_2$.*

Proposition 4. *Given a map* $B : X_1 \rightsquigarrow X_2$ *between* $(X_1, \mathcal{T}_1)$ *and* $(X_2, \mathcal{T}_2)$*,*
(1.) *B satisfies condition* (ii.a) *of Definition 4* *iff* B^{-1} *is* l.s.c.*;*
(2.) *B satisfies condition* (ii.b) *of Definition 4* *iff* B *is* l.s.c..

Proof. By rewriting in terms of the pre- and post-image set operators, it is easy to show that conditions (ii.a) and (ii.b) are equivalent to the following:

$$\begin{array}{ll} (\text{ii.a}^\sharp) & \forall U \in \mathcal{T}_1, \quad B^{\exists}(U) \subseteq int_{\mathcal{T}_2}\big(B^{\exists}(U)\big) \\ (\text{ii.b}^\sharp) & \forall V \in \mathcal{T}_2, \ B^{-\exists}(V) \subseteq int_{\mathcal{T}_1}\big(B^{-\exists}(V)\big) \end{array}$$

Clearly, (ii.a$^\sharp$) says that $B^{\exists}(U)$ is open in X_2 whenever U open in X_1, while (ii.b$^\sharp$) says that $B^{-\exists}(V)$ is open in X_1 whenever V open in X_2. ⊣

For the appropriate notion of topological bisimulation between topological Kripke models for the intuitionistic and classical companion modal and tense logics under study here, we need to put together the topology-preserving conditions (ii.a) and (ii.b) above with the standard clauses for preservation of the modal/tense semantic structure.

Definition 5. *Let* $\mathcal{M}_1 = (X_1, \mathcal{T}_1, R_1, v_1)$ *and* $\mathcal{M}_2 = (X_2, \mathcal{T}_2, R_2, v_2)$ *be two topological models. A map* $B : X_1 \rightsquigarrow X_2$ *will be called a* tense topo-bisimulation *between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *if for all atomic* $p \in AP$*:*

(i.a) $B^{\exists}(v_1(p)) \subseteq v_2(p)$	(i.b) $B^{-\exists}(v_2(p)) \subseteq v_1(p)$
(ii.a) $B^{-1} : X_2 \rightsquigarrow X_1$ *is l.s.c.*	(ii.b) $B : X_1 \rightsquigarrow X_2$ *is l.s.c.*
(iii.a) $(B^{-1} \circ R_1) \subseteq (R_2 \circ B^{-1})$	(iii.b) $(B \circ R_2) \subseteq (R_1 \circ B)$
(iv.a) $(B^{-1} \circ R_1^{-1}) \subseteq (R_2^{-1} \circ B^{-1})$	(iv.b) $(B \circ R_2^{-1}) \subseteq (R_1^{-1} \circ B)$

If only conditions (i.a), (ii.a) *and* (iii.a) *hold of the map* $B : X_1 \rightsquigarrow X_2$*, then B is called a* modal topo-simulation *of* $\mathcal{M}_1$ *by* $\mathcal{M}_2$*; if all but conditions* (iv.a) *and* (iv.b) *hold, then B is a* modal topo-bisimulation *between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*.*

Combining all the conditions (iii) and (iv), one obtains two equalities:
$(R_1 \circ B) = (B \circ R_2)$ and $(R_2 \circ B^{-1}) = (B^{-1} \circ R_1)$. The set-operator form of the semantic preservation conditions are:

$$B^{\exists}([\![\varphi]\!]^{\mathcal{M}_1}_{\mathrm{I}}) \subseteq [\![\varphi]\!]^{\mathcal{M}_2}_{\mathrm{I}} \quad \text{and} \quad B^{-\exists}([\![\varphi]\!]^{\mathcal{M}_2}_{\mathrm{I}}) \subseteq [\![\varphi]\!]^{\mathcal{M}_1}_{\mathrm{I}} \tag{5}$$

and likewise for classical denotation maps $[\![\varphi]\!]^{\mathcal{M}_i}$. We will also use the dual versions under the adjoint equivalence (3). These are:

$$[\![\varphi]\!]^{\mathcal{M}_1}_{\mathrm{I}} \subseteq B^{-\forall}([\![\varphi]\!]^{\mathcal{M}_2}_{\mathrm{I}}) \quad \text{and} \quad [\![\varphi]\!]^{\mathcal{M}_2}_{\mathrm{I}} \subseteq B^{\forall}([\![\varphi]\!]^{\mathcal{M}_1}_{\mathrm{I}}) \tag{6}$$

and likewise for $[\![\varphi]\!]^{\mathcal{M}_i}$. Note also that $B^{-1} : X_2 \rightsquigarrow X_1$ being l.s.c. has a further equivalent characterization: $int_{\mathcal{T}_1}(B^{-\forall}(W)) \subseteq B^{-\forall}(int_{\mathcal{T}_2}(W))$, for all $W \subseteq X_2$; this is a generalization of the characterization for binary relations on a single space X that is formalized in Proposition 3, Row (*3.*).

What we discover is that *exactly the same* notion of a bisimulation between models yields the same semantic preservation property for *both* the intuitionistic and the classical semantics. Otherwise put, the specifically *topological* requirement that the operators $B^{\exists}$ and $B^{-\exists}$ preserve open sets is enough to push through the result for intuitionistic modal and tense logics.

Theorem 1. [Semantic preservation for tense topo-bisimulations] *Let* $\mathcal{M}_1 = (X_1, \mathcal{T}_1, R_1, v_1)$ *and* $\mathcal{M}_2 = (X_2, \mathcal{T}_2, R_2, v_2)$ *be any two topological models, and let* $B : X_1 \rightsquigarrow X_2$ *be a tense topo-bisimulation between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$.

(1.) *If* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are open and l.s.c., then for all* $x \in X_1$ *and* $y \in X_2$*:*

$$x\,B\,y \qquad \textit{implies} \qquad (\forall \varphi \in \mathcal{L}^{\mathbf{t}})\,[\; x \in [\![\,\varphi\,]\!]_{\mathtt{I}}^{\mathcal{M}_1} \;\Leftrightarrow\; y \in [\![\,\varphi\,]\!]_{\mathtt{I}}^{\mathcal{M}_2} \;]$$

(2.) *For all* $x \in X_1$ *and* $y \in X_2$*:*

$$x\,B\,y \qquad \textit{implies} \qquad (\forall \varphi \in \mathcal{L}^{\mathbf{t}}_{\Box})\,[\; x \in [\![\,\varphi\,]\!]^{\mathcal{M}_1} \;\Leftrightarrow\; y \in [\![\,\varphi\,]\!]^{\mathcal{M}_2} \;]$$

Proof. The proof proceeds as usual, by induction on the structure of formulas, to establish the two inclusions displayed in (5), or their analogs for the classical denotation maps. The base case for atomic propositions is given by conditions (i.a) and (i.b). For the classical semantics in Part (2.), the argument is completely standard for the propositional and modal/tense operators, and the case for topological $\Box$ is made in [1]. For the intuitionistic semantics in Part (1.), we give the cases for implication $\rightarrow$ and for box $\boxdot$. Assume the result holds for φ_1 and φ_2 in $\mathcal{L}^{\mathbf{t}}$. In particular, from Assertions (5) and (6), we have:

$(X_1 - [\![\varphi_1]\!]_{\mathtt{I}}^{\mathcal{M}_1}) \subseteq (X_1 - B^{-\exists}([\![\varphi_1]\!]_{\mathtt{I}}^{\mathcal{M}_2}))$, and $[\![\varphi_2]\!]_{\mathtt{I}}^{\mathcal{M}_1} \subseteq B^{-\forall}([\![\,\varphi_2\,]\!]_{\mathtt{I}}^{\mathcal{M}_2})$. Now:

$$\begin{array}{ll}
\quad B^{\exists}\,([\![\,\varphi_1 \rightarrow \varphi_2\,]\!]_{\mathtt{I}}^{\mathcal{M}_1}) & \\
= B^{\exists}\,(int_{\mathcal{T}_1}\,(\,(\,X_1 - [\![\varphi_1]\!]_{\mathtt{I}}^{\mathcal{M}_1}\,) \,\cup\, [\![\varphi_2]\!]_{\mathtt{I}}^{\mathcal{M}_1})) & \\
\subseteq B^{\exists}\left(int_{\mathcal{T}_1}\left(X_1 - B^{-\exists}([\![\varphi_1]\!]_{\mathtt{I}}^{\mathcal{M}_2}) \,\cup\, B^{-\forall}([\![\varphi_2]\!]_{\mathtt{I}}^{\mathcal{M}_2})\right)\right) & \text{by induction hypothesis} \\
= B^{\exists}\left(int_{\mathcal{T}_1}\left(B^{-\forall}(X_2 - [\![\varphi_1]\!]_{\mathtt{I}}^{\mathcal{M}_2}) \,\cup\, B^{-\forall}([\![\varphi_2]\!]_{\mathtt{I}}^{\mathcal{M}_2})\right)\right) & \text{by duality } B^{-\forall}\,/\,B^{-\exists} \\
\subseteq int_{\mathcal{T}_2}\left(B^{\exists}\left(B^{-\forall}(X_2 - [\![\varphi_1]\!]_{\mathtt{I}}^{\mathcal{M}_2}) \,\cup\, B^{-\forall}([\![\varphi_2]\!]_{\mathtt{I}}^{\mathcal{M}_2})\right)\right) & \text{by } B^{-1} \text{ being l.s.c.} \\
\subseteq int_{\mathcal{T}_2}\left(B^{\exists}\left(B^{-\forall}\,(\,(X_2 - [\![\varphi_1]\!]_{\mathtt{I}}^{\mathcal{M}_2}) \,\cup\, [\![\varphi_2]\!]_{\mathtt{I}}^{\mathcal{M}_2}\,)\right)\right) & \text{by monotonicity of } B^{-\forall} \\
\subseteq int_{\mathcal{T}_2}\,(\,(\,X_2 - [\![\varphi_1]\!]_{\mathtt{I}}^{\mathcal{M}_2}\,) \,\cup\, [\![\varphi_2]\!]_{\mathtt{I}}^{\mathcal{M}_2}\,) & \text{by adjoint property} \\
= [\![\,\varphi_1 \rightarrow \varphi_2\,]\!]_{\mathtt{I}}^{\mathcal{M}_2} &
\end{array}$$

Verifying that $B^{-\exists}\,([\![\,\varphi_1 \rightarrow \varphi_2\,]\!]_{\mathtt{I}}^{\mathcal{M}_2}) \subseteq [\![\,\varphi_1 \rightarrow \varphi_2\,]\!]_{\mathtt{I}}^{\mathcal{M}_1}$ proceeds similarly, using from the induction hypothesis: $(X_2 - [\![\varphi_1]\!]_{\mathtt{I}}^{\mathcal{M}_2}) \subseteq (X_2 - B^{\exists}([\![\varphi_1]\!]_{\mathtt{I}}^{\mathcal{M}_1}))$, and $[\![\varphi_2]\!]_{\mathtt{I}}^{\mathcal{M}_2} \subseteq B^{\forall}([\![\,\varphi_2\,]\!]_{\mathtt{I}}^{\mathcal{M}_1})$.
For the $\boxdot$ case:

$$\begin{array}{ll}
\quad [\![\,\boxdot\varphi\,]\!]_{\mathtt{I}}^{\mathcal{M}_1} & \\
= int_{\mathcal{T}_1}\left(R_1^{-\forall}\,([\![\,\varphi\,]\!]_{\mathtt{I}}^{\mathcal{M}_1})\right) & \\
\subseteq int_{\mathcal{T}_1}\left(R_1^{-\forall}\left(B^{-\forall}\,([\![\,\varphi\,]\!]_{\mathtt{I}}^{\mathcal{M}_2})\right)\right) & \text{by induction hypothesis} \\
\subseteq int_{\mathcal{T}_1}\left(B^{-\forall}\left(R_2^{-\forall}\,([\![\,\varphi\,]\!]_{\mathtt{I}}^{\mathcal{M}_2})\right)\right) & \text{since } R_1 \circ B = B \circ R_2 \\
\subseteq B^{-\forall}\left(int_{\mathcal{T}_2}\left(R_2^{-\forall}\,([\![\,\varphi\,]\!]_{\mathtt{I}}^{\mathcal{M}_2})\right)\right) & \text{by } B^{-1} \text{ being l.s.c. (dual } B^{-\forall} \text{ form)} \\
= B^{-\forall}\,([\![\,\varphi\,]\!]_{\mathtt{I}}^{\mathcal{M}_2}) &
\end{array}$$

The argument for $\blacksquare$ symmetrically appeals to B being l.s.c. (dual $B^{\forall}$ form). $\dashv$

In a sequel paper, [10], we give a partial converse (Hennessy-Milner type result) by proving that a certain class of open l.s.c. models has the property that for any two models $\mathcal{M}_1$ and $\mathcal{M}_2$ in the class, there is a total and surjective tense topo-bisimulation B between them that maximally preserves the intuitionistic semantics, in the sense that for all $x \in X_1$ and $y \in X_2$:

$$x\,B\,y \qquad \text{iff} \qquad (\forall \varphi \in \mathcal{L}^{\mathbf{t}})\,[\ x \in [\![\,\varphi\,]\!]^{\mathcal{M}_1}_{\mathbf{I}} \ \Leftrightarrow\ y \in [\![\,\varphi\,]\!]^{\mathcal{M}_2}_{\mathbf{I}}\ ]\,.$$

5 Axiomatizations and Canonical Models

Let **IPC** $\subseteq \mathcal{L}_0$ be the set of intuitionistic propositional theorems, and abusing notation, let **IPC** also denote a standard axiomatisation for that logic. Likewise, let **S4**$\Box \subseteq \mathcal{L}_{0,\Box}$ be the set of theorems of classical S4, and let **S4**$\Box$ also denote any standard axiomatisation of classical S4. To be concrete, let **S4**$\Box$ contain all instances of classical propositional tautologies in the language $\mathcal{L}_{0,\Box}$, and the axiom schemes:

N$\Box$: $\Box\top$	**T**$\Box$: $\Box\varphi \to \varphi$
R$\Box$: $\Box(\varphi_1 \wedge \varphi_2) \leftrightarrow \Box\varphi_1 \wedge \Box\varphi_2$	**4**$\Box$: $\Box\varphi \to \Box\Box\varphi$

and be closed under the inference rules of *modus ponens* (**MP**), *uniform substitution* (**Subst**) (of formulas for atomic propositions), and $\Box$-monotonicity (**Mono**$\Box$): from $\varphi_1 \to \varphi_2$ infer $\Box\varphi_1 \to \Box\varphi_2$.

On notation, for any axiomatically presented logic Λ in a language $\mathcal{L}$, set of formulas $\mathcal{A} \subseteq \mathcal{L}$ and formula $\varphi \in \mathcal{L}$, we write $\mathcal{A} \vdash_\Lambda \varphi$ to mean that there exists a finite set $\{\psi_1, \ldots, \psi_n\} \subseteq \mathcal{A}$ of formulas such that $(\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$ is a theorem of Λ (allowing $n = 0$ and φ is a theorem of Λ). The relation $\vdash_\Lambda \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ is the consequence relation of Λ. We will abuse notation (as we have with **IPC** and **S4**$\Box$) and identify Λ with its set of theorems: i.e. $\Lambda = \{\varphi \in \mathcal{L} \mid \varnothing \vdash_\Lambda \varphi\}$.

Let **IK** be the axiomatic system of Fischer Servi [18,15,22], which is equivalent to an alternative axiomatisation given in [29,32]; **IK** also goes by the name **FS** in [22] and [20,37,38]. **IK** has as axioms all instances in the language $\mathcal{L}^{\mathbf{m}}$ of the axiom schemes of **IPC**, and further axiom schemes:

R$\diamondsuit$: $\diamondsuit(\varphi \vee \psi) \leftrightarrow (\diamondsuit\varphi \vee \diamondsuit\psi)$	**N**$\diamondsuit$: $\neg\diamondsuit\bot$
R$\boxdot$: $\boxdot(\varphi \wedge \psi) \leftrightarrow (\boxdot\varphi \wedge \boxdot\psi)$	**N**$\boxdot$: $\boxdot\top$
F1$\boxdot\diamondsuit$: $\diamondsuit(\varphi \to \psi) \to (\boxdot\varphi \to \diamondsuit\psi)$	**F2**$\boxdot\diamondsuit$: $(\diamondsuit\varphi \to \boxdot\psi) \to \boxdot(\varphi \to \psi)$

and is closed under the inference rules (**MP**) and (**Subst**), and the rule (**Mono**$\diamondsuit$): from $\varphi_1 \to \varphi_2$ infer $\diamondsuit\varphi_1 \to \diamondsuit\varphi_2$, and likewise (**Mono**$\boxdot$).

With regard to notation for combinations of modal logics, we follow that of [20]. If Λ_1 and Λ_2 are axiomatically presented modal logics in languages $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then the *fusion* $\Lambda_1 \otimes \Lambda_2$ is the smallest multi-modal logic in the language $\mathcal{L}_1 \otimes \mathcal{L}_2$ containing Λ_1 and Λ_2, and closed under all the inference rules of Λ_1 and Λ_2, where $\mathcal{L}_1 \otimes \mathcal{L}_2$ denotes the least common extension of the languages $\mathcal{L}_1$ and $\mathcal{L}_2$. If Λ is a logic in language $\mathcal{L}$, and Γ is a finite list of schemes in $\mathcal{L}$, then the *extension* $\Lambda \oplus \Gamma$ is the smallest logic in $\mathcal{L}$ extending Λ, containing the schemes in Γ as additional axioms, and closed under the rules of Λ. The basic system in

[37], under the name **IntK**, is such that: $\mathbf{IK} = \mathbf{IntK} \oplus \mathbf{F1}\boxdot\Diamonddot \oplus \mathbf{F2}\boxdot\Diamonddot$. The latter two schemes were identified by Fischer Servi in [18][5].

For the extension to tense logics with forwards and backwards modalities, let $\mathbf{IK^t}$ be Ewald's [15] deductive system, which is the fusion of $\mathbf{IK}\Diamonddot\boxdot := \mathbf{IK}$ with the "mirror" system $\mathbf{IK}\blacklozenge\blacksquare$ having axiom schemes $\mathbf{R}\blacklozenge$, $\mathbf{N}\blacklozenge$, $\mathbf{R}\blacksquare$, $\mathbf{N}\blacksquare$, $\mathbf{F1}\blacksquare\blacklozenge$ and $\mathbf{F2}\blacksquare\blacklozenge$, and inference rules (**Mono**$\blacklozenge$) and (**Mono**$\blacksquare$), which is then further extended with four axiom schemes expressing the *adjoint property* (Assertion (3)) of the operators interpreting the tense modalities:

$$\mathbf{Ad1} : \varphi \to \boxdot\blacklozenge\varphi \quad \mathbf{Ad2} : \varphi \to \blacksquare\Diamonddot\varphi \quad \mathbf{Ad3} : \Diamonddot\blacksquare\varphi \to \varphi \quad \mathbf{Ad4} : \blacklozenge\boxdot\varphi \to \varphi$$

Thus $\mathbf{IK^t} := (\mathbf{IK}\Diamonddot\boxdot \otimes \mathbf{IK}\blacklozenge\blacksquare) \oplus \mathbf{Ad1} \oplus \mathbf{Ad2} \oplus \mathbf{Ad3} \oplus \mathbf{Ad4}$.

We now identify the companion classical logics. Let $\mathbf{K}\boxdot$ be the minimal normal modal logic (over a classical propositional base), and let $(\mathbf{S4}\Box \otimes \mathbf{K}\boxdot)$ be the bi-modal fusion of $\mathbf{S4}\Box$ and $\mathbf{K}\boxdot$, and let $\mathbf{K^mLSC} := (\mathbf{S4}\Box \otimes \mathbf{K}\boxdot) \oplus (\Diamonddot\Box\varphi \to \Box\Diamonddot\varphi) \oplus (\Box\boxdot\varphi \to \boxdot\Box\varphi)$ be the extension of $(\mathbf{S4}\Box \otimes \mathbf{K}\boxdot)$ with characteristic modal schemes for the R-l.s.c. and R^{-1}-l.s.c. frame conditions, from Proposition 3 (and as identified in [16]). Likewise, $\mathbf{K^t} := (\mathbf{K}\boxdot \otimes \mathbf{K}\blacksquare) \oplus \mathbf{Ad1} \oplus \mathbf{Ad2}$ is the minimal normal tense logic, and $\mathbf{K^tLSC} := (\mathbf{S4}\Box \otimes \mathbf{K^t}) \oplus (\Diamonddot\Box\varphi \to \Box\Diamonddot\varphi) \oplus (\blacklozenge\Box\varphi \to \Box\blacklozenge\varphi)$, here using instead the tense scheme for R^{-1}-l.s.c. from Proposition 3.

In what follows, we will deal generically with extensions $\mathbf{IK} \oplus \Gamma$ or $\mathbf{IK^t} \oplus \Gamma$ for subsets Γ of the five axiom schemes below or their $\blacksquare$-$\blacklozenge$ mirror images:

$$\begin{array}{ll}
\mathbf{T}\boxdot\Diamonddot : (\boxdot\varphi \to \varphi) \wedge (\varphi \to \Diamonddot\varphi) & \mathbf{B}\boxdot\Diamonddot : (\varphi \to \boxdot\Diamonddot\varphi) \wedge (\Diamonddot\boxdot\varphi \to \varphi) \\
\mathbf{D}\Diamonddot : \Diamonddot\top & \\
\mathbf{4}\boxdot\Diamonddot : (\boxdot\varphi \to \boxdot\boxdot\varphi) \wedge (\Diamonddot\Diamonddot\varphi \to \Diamonddot\varphi) & \mathbf{5}\boxdot\Diamonddot : (\Diamonddot\boxdot\varphi \to \boxdot\varphi) \wedge (\Diamonddot\varphi \to \boxdot\Diamonddot\varphi)
\end{array} \tag{7}$$

where the schemes characterize, in turn, the properties of relations $R : X \rightsquigarrow X$ of reflexivity, symmetry, totality (seriality), transitivity and Euclideanness, and the mirror image scheme characterize relations R such that R^{-1} has the property[6]. For a set Γ of schemes, let $\mathbb{C}(\Gamma)$ be the set of all formulas $\varphi \in \mathcal{L}^{\mathbf{t}}$ that are int-modal-top valid in every l.s.c. topological frame whose relation R has the properties corresponding to the schemes in Γ, and let $\mathbb{C}_\Box(\Gamma)$ be the set of all formulas $\varphi \in \mathcal{L}^{\mathbf{t}}_\Box$ that are modal-top valid in every topological frame whose relation R has the properties corresponding to the schemes in Γ.

The topological soundness of $\mathbf{IK^t}$ and of $\mathbf{K^tLSC}$ are easy verifications. For example, the soundness of the Fischer Servi scheme $\mathbf{F1}\boxdot\Diamonddot$ is equivalent to the assertion that, for all open sets $U, V \in \mathcal{T}$:

$$R^{-\exists}\left(int_{\mathcal{T}}(-U \cup V)\right) \;\subseteq\; int_{\mathcal{T}}\big(-int_{\mathcal{T}}(R^{-\forall}(U)) \cup R^{-\exists}(V)\big)\,.$$

[5] The intuitionistic modal logics considered in [36] and [23] are yet weaker sub-systems: they have the normality schemes $\mathbf{R}\boxdot$ and $\mathbf{N}\boxdot$ for $\boxdot$, but $\Diamonddot$ is sub-normal – they include the scheme $\mathbf{N}\Diamonddot$, but $\mathbf{R}\Diamonddot$ is replaced by $(\boxdot\varphi \wedge \Diamonddot\psi) \to \Diamonddot(\varphi \wedge \psi)$.

[6] Note that R has reflexivity, symmetry or transitivity iff R^{-1} has the same property, so the mirrored tense schemes $\mathbf{T}\blacksquare\blacklozenge$, $\mathbf{B}\blacksquare\blacklozenge$ and $\mathbf{4}\blacksquare\blacklozenge$ are semantically equivalent to their un-mirrored modal versions.

The inclusion $R^{-\exists}\left(int_{\mathcal{T}}(-U\cup V)\right)\subseteq int_{\mathcal{T}}\left(R^{-\exists}(-U\cup V)\right)$ follows from R being l.s.c. Applying distribution over unions, duality, and monotonicity, we can get $int_{\mathcal{T}}\left(R^{-\exists}(-U\cup V)\right)\subseteq int_{\mathcal{T}}\left(-int_{\mathcal{T}}(R^{-\forall}(U))\cup R^{-\exists}(V)\right)$, so we are done. R being l.s.c. is also used for soundness of the adjoint axioms **Ad2** and **Ad3**.

From Proposition 2 and topological completeness in Proposition 6 below, we can derive deductive faithfulness of the extended Gödel translation.

Proposition 5. [Extended Gödel translation: deductive faithfulness]
Let Γ be any finite set of schemes in $\mathcal{L}^{\mathbf{t}}$ from the list in (7) *above.*
For all $\varphi\in\mathcal{L}^{\mathbf{t}}$, $\varphi\in\mathbf{IK^t}\oplus\Gamma$ *iff* $\mathrm{GT}(\varphi)\in\mathbf{K^tLSC}\oplus\Gamma$.

This result can also be derived from a general result for (an equivalent) Gödel translation given in [38], Theorem 8, on the faithful embedding of modal logics $L=\mathbf{IntK}\oplus\Gamma_1$ (including $\mathbf{IK}\oplus\Gamma=\mathbf{IntK}\oplus\mathbf{F1}\boxdot\diamondsuit\oplus\mathbf{F2}\boxdot\diamondsuit\oplus\Gamma$) into bimodal logics in the interval between $(\mathbf{S4}\Box\otimes\mathbf{K}\boxdot)\oplus\mathrm{GT}(\Gamma_1)$ and $(\mathbf{Grz}\Box\otimes\mathbf{K}\boxdot)\oplus\mathrm{GT}(\Gamma_1)\oplus\mathbf{mix}$, where $\mathbf{Grz}\Box=\mathbf{S4}\Box\oplus\Box\left(\Box(\varphi\to\Box\varphi)\to\varphi\right)\to\varphi$ and $\mathbf{mix}=(\Box\boxdot\varphi\leftrightarrow\boxdot\varphi)\wedge(\boxdot\Box\varphi\leftrightarrow\Box\varphi)$. We have restricted the schemes in Γ to those from a "safe" list of relational properties that *don't* require translating, since the schemes characterize the same relations in the intuitionistic and classical semantics.

Recall that for a logic Λ in a language $\mathcal{L}$ with deductive consequence relation $\vdash_\Lambda$, a set of formulas $x\subseteq\mathcal{L}$ is said to be Λ-*consistent* if $x\nvdash_\Lambda\bot$; x is Λ-*deductively closed* if $x\vdash_\Lambda\varphi$ implies $\varphi\in x$ for all formulas $\varphi\in\mathcal{L}$; and x is *maximal* Λ-*consistent* if x is Λ-consistent, and no proper superset of x is Λ-consistent. A set $x\subseteq\mathcal{L}$ is a *prime theory* of Λ if $\Lambda\subseteq x$, and x has the disjunction property, and is Λ-consistent, and Λ-deductively closed.

Completeness w.r.t. bi-relational frames for **IK** and $\mathbf{IK^t}$ is proved in [18,32] and [15] by building a canonical model over the state space X_{ip} defined to be the set of all sets of formulas $x\subseteq\mathcal{L}^{\mathbf{t}}$ that are prime theories of $\mathbf{IK^t}$. The space X_{ip} is partially ordered by inclusion, so we have available an Alexandrov topology $\mathcal{T}_\subseteq$. One then defines the modal accessibility relation R_0 in an "almost classical" way, the only concession to intuitionistic semantics being clauses in the definition for both $\diamondsuit$ and $\boxdot$. As verified in [18] and [32] for the modal logic, and [15] for the tense logic, the relations R_0 and R_0^{-1} satisfy the frame conditions $\mathbf{Zig}(\subseteq,R_0)$ and $\mathbf{Zig}(\subseteq,R_0^{-1})$. So we get an l.s.c. topological frame $\mathcal{F}_0=(X_{\mathrm{ip}},\mathcal{T}_\subseteq,R_0)$, and with the canonical valuation $u:AP\rightsquigarrow X_{\mathrm{ip}}$ given by $u(p)=\{x\in X_{\mathrm{ip}}\mid p\in x\}$; one then proves of the model $\mathcal{M}_0=(\mathcal{F}_0,u)$ the "Truth Lemma": for all $\varphi\in\mathcal{L}^{\mathbf{t}}$ and $x\in X_{\mathrm{ip}}$, $x\in[\![\varphi]\!]_{\mathrm{I}}^{\mathcal{M}_0}$ iff $\varphi\in x$.

Adapting [1], Sec. 3, on classical **S4**, to the classical companion logics here, we can go beyond pre-orders by equipping the space of maximal consistent sets of formulas with a topology that is neither Alexandrov nor Stone, but rather is the intersection of those two topologies.

Proposition 6. [Topological soundness and completeness]
Let Γ be any finite set of axiom schemes from $\mathcal{L}^{\mathbf{t}}$ from the list in (7) *above.*
(1.) *For all $\psi\in\mathcal{L}^{\mathbf{t}}_{\Box}$, ψ is a theorem of* $\mathbf{K^tLSC}\oplus\Gamma$ *iff* $\psi\in\mathbb{K}^{\mathbf{t}}\mathbb{LSC}\cap\mathbb{C}_{\Box}(\Gamma)$.
(2.) *For all $\varphi\in\mathcal{L}^{\mathbf{t}}$, φ is a theorem of* $\mathbf{IK^t}\oplus\Gamma$ *iff* $\varphi\in\mathbb{IK}^{\mathbf{t}}\mathbb{T}\cap\mathbb{C}(\Gamma)$.

In what follows, we use **IL** and $\mathbf{L}_\Box$, respectively, as abbreviations for the axiomatically presented logics $\mathbf{IK^t} \oplus \Gamma$ and $\mathbf{K^tLSC} \oplus \Gamma$. Taking soundness as established, we sketch completeness by describing the canonical models.

For the classical companion $\mathbf{L}_\Box$, define a model $\mathcal{M}_\Box = (Y_{\Box\mathrm{m}}, \mathcal{S}_\Box, Q_\Box, v_\Box)$:

$Y_{\Box\mathrm{m}} := \{ y \subseteq \mathcal{L}^{\mathbf{t}}_\Box \mid y \text{ is a maximal } \mathbf{L}_\Box\text{-consistent set of formulas} \}$;

$\mathcal{S}_\Box$ is the topology on $Y_{\Box\mathrm{m}}$ which has as a basis the family
$\{ V(\Box\psi) \mid \psi \in \mathcal{L}^{\mathbf{t}}_\Box \}$ where $V(\Box\psi) := \{ y \in Y_{\Box\mathrm{m}} \mid \Box\psi \in y \}$;

$Q_\Box : Y_{\Box\mathrm{m}} \rightsquigarrow Y_{\Box\mathrm{m}}$ defined for all $y \in Y_{\Box\mathrm{m}}$ by
$Q_\Box(y) := \{ y' \in Y_{\Box\mathrm{m}} \mid \{ \diamonddot\psi \mid \psi \in y' \} \subseteq y \text{ and } \{ \blacklozenge\psi \mid \psi \in y \} \subseteq y' \}$;

$v_\Box : AP \rightsquigarrow Y_{\Box\mathrm{m}}$ defined for all $p \in AP$ by $v_\Box(p) := \{ y \in Y_{\Box\mathrm{m}} \mid p \in y \}$.

As noted in [1], the topology $\mathcal{S}_\Box$ on $Y_{\Box\mathrm{m}}$ is the intersection the "default" Alexandrov topology from the canonical relational Kripke model, and the standard Stone topology on $Y_{\Box\mathrm{m}}$ which has as a basis all sets of the form $V(\psi)$ for all formulas $\psi \in \mathcal{L}^{\mathbf{t}}_\Box$, not just the $V(\Box\psi)$ ones. Moreover, the space $(Y_{\Box\mathrm{m}}, \mathcal{S}_\Box)$ is compact and dense-in-itself (has no isolated points). Verification that $Q_\Box$ and $Q_\Box^{-1}$ are l.s.c. reduces to establishing that for all $\psi \in \mathcal{L}^{\mathbf{t}}_\Box$:

$$Q_\Box^{-\exists}(V(\Box\psi)) = V(\Box\diamonddot\Box\psi) \quad \text{and} \quad Q_\Box^{\exists}(V(\Box\psi)) = V(\Box\blacklozenge\Box\psi).$$

The "Truth Lemma" is $y \in [\![\varphi]\!]^{\mathcal{M}_\Box}$ iff $\psi \in y$, for all $\psi \in \mathcal{L}^{\mathbf{t}}_\Box$ and $y \in Y_{\Box\mathrm{m}}$.

For the intuitionistic logic **IL**, define an open model $\mathcal{M}_\star = (X_{\mathrm{ip}}, \mathcal{T}_{\mathrm{sp}}, R_\star, u_\star)$:

$X_{\mathrm{ip}} := \{ x \subseteq \mathcal{L}^{\mathbf{t}} \mid x \text{ is a prime } \mathbf{IL}\text{-theory} \}$;

$\mathcal{T}_{\mathrm{sp}}$ is the topology on X_{ip} which has as a basis the family
$\{ U(\varphi) \mid \varphi \in \mathcal{L}^{\mathbf{t}} \}$ where $U(\varphi) := \{ x \in X_{\mathrm{ip}} \mid \varphi \in x \}$;

$R_\star : X_{\mathrm{ip}} \rightsquigarrow X_{\mathrm{ip}}$ defined for all $x, x' \in X_{\mathrm{ip}}$ by $R_\star := R_0$; i.e. $x \, R_\star \, x'$ iff
$\{ \diamonddot\psi \mid \psi \in x' \} \subseteq x$ and $\{ \psi \mid \boxdot\psi \in x \} \subseteq x'$ and
$\{ \blacklozenge\psi \mid \psi \in x \} \subseteq x'$ and $\{ \psi \mid \blacksquare\psi \in x' \} \subseteq x$;

$u_\star : AP \rightsquigarrow X_{\mathrm{ip}}$ defined for all $p \in AP$ by $u_\star(p) := U(p)$.

Here, the toplogical space $(X_{\mathrm{ip}}, \mathcal{T}_{\mathrm{sp}})$ has a *spectral topology* (e.g. [33], Sec.4), which means it is compact and T_0; the family of compact and open sets in $\mathcal{T}_{\mathrm{sp}}$ gives a basis for the toplogy; and $\mathcal{T}_{\mathrm{sp}}$ is sober, i.e. for every completely prime filter $\mathcal{F}$ of $\mathcal{T}_{\mathrm{sp}}$, there exists a (unique) point $x \in X_{\mathrm{ip}}$ such that $\mathcal{F} = \mathcal{F}_x := \{ U \in \mathcal{T}_{\mathrm{sp}} \mid x \in U \}$, the filter of neighbourhoods of x. The hardest parts of the verification are the l.s.c. properties for $R_\star$ and $R_\star^{-1}$, and the task reduces to establishing that for all $\varphi \in \mathcal{L}^{\mathbf{t}}$:

$$R_\star^{-\exists}(U(\varphi)) = U(\diamonddot\varphi) \qquad \text{and} \qquad R_\star^{\exists}(U(\varphi)) = U(\blacklozenge\varphi)$$

To prove the right-to-left inclusions, a recursive Henkin-style construction can be used to produce a prime **IL**-theory x' such that $x \, R_\star \, x'$ and $\varphi \in x'$, to derive $x \in R_\star^{-\exists}(U(\varphi))$ given $x \in U(\diamonddot\varphi)$, and symmetrically for the $R_\star^{\exists}(U(\varphi))$ inclusion. The required "Truth Lemma" is $x \in [\![\varphi]\!]_{\mathbf{I}}^{\mathcal{M}_\star}$ iff $\varphi \in x$ for all $\varphi \in \mathcal{L}^{\mathbf{t}}$ and $x \in X_{\mathrm{ip}}$.

6 Topological Bisimulation Between Canonical Models

The Gödel translation is a syntactic function $\mathrm{GT} : \mathcal{L}^{\mathbf{t}} \to \mathcal{L}^{\mathbf{t}}_{\Box}$, which naturally gives rise to a semantic relationship between the canonical model spaces X_{ip} and $Y_{\Box\mathrm{m}}$. Define a set-valued map $G : X_{\mathrm{ip}} \rightsquigarrow Y_{\Box\mathrm{m}}$ by:

$$G(x) \;:=\; \{\, y \in Y_{\Box\mathrm{m}} \;\mid\; \mathrm{GT}(x) \subseteq y \,\}$$

Note that the image $\mathrm{GT}(x)$ of an intuitionistic prime theory $x \in X_{\mathrm{ip}}$ will in general have many classical maximal consistent extensions $y \in Y_{\Box\mathrm{m}}$.

Let $\mathcal{M}^*_{\Box} = (Y_{\Box\mathrm{m}}, \mathcal{S}^*_{\Box}, Q_{\Box}, v^*_{\Box})$ be the open and l.s.c. model obtained from $\mathcal{M}_{\Box}$ by taking $\mathcal{S}^*_{\Box}$ to be the proper sub-topology of $\mathcal{S}_{\Box}$ having as a basis the open sets $\{\, V(\Box\mathrm{GT}(\varphi)) \mid \varphi \in \mathcal{L}^{\mathbf{t}} \,\}$[7], with valuation $v^*_{\Box}(p) := int_{\mathcal{S}^*_{\Box}}(v_{\Box}(p)) = V(\Box p)$.

Theorem 2. *The maps $G : X_{\mathrm{ip}} \rightsquigarrow Y_{\Box\mathrm{m}}$ and $G^{-1} : Y_{\Box\mathrm{m}} \rightsquigarrow X_{\mathrm{ip}}$ are such that:*
(1.) *both G and G^{-1} are l.s.c. with respect to $\mathcal{T}_{\mathrm{sp}}$ and $\mathcal{S}^*_{\Box}$;*
(2.) *both G and G^{-1} are total and surjective;*
(3.) *$R_\star \circ G = G \circ Q_{\Box}$ and $Q_{\Box} \circ G^{-1} = G^{-1} \circ R_\star$; and*
(4.) *$G^{\exists}(u_\star(p)) \subseteq v_{\Box}(p)$ and $G^{-\exists}(v_{\Box}(p)) \subseteq u_\star(p)$ for all atomic $p \in AP$.*
Hence G is a tense topo-bisimulation between $\mathcal{M}_\star$ and $\mathcal{M}^*_{\Box}$.

Proof. For Part (1.), the l.s.c. properties, we need only look at the basic opens in $\mathcal{T}_{\mathrm{sp}}$ and $\mathcal{S}^*_{\Box}$. Using the openness theorem $\Box\mathrm{GT}(\varphi) \leftrightarrow \mathrm{GT}(\varphi)$, it is readily established that for all $\varphi \in \mathcal{L}^{\mathbf{t}}$:

$$G^{-\exists}(V(\Box\mathrm{GT}(\varphi))) \;=\; U(\varphi) \qquad \text{and} \qquad G^{\exists}(U(\varphi)) \;=\; V(\Box\mathrm{GT}(\varphi)).$$

For Part (2.), the totality of G, note that every prime theory $x \in X_{\mathrm{ip}}$ is **IL**-consistent, hence the image $\mathrm{GT}(x) \subseteq \mathcal{L}_{\Box}$ is $\mathbf{L}_{\Box}$-consistent, and so has a maximal $\mathbf{L}_{\Box}$-consistent superset $y \supseteq \mathrm{GT}(x)$ with $y \in Y_{\Box\mathrm{m}}$, by Lindenbaum's Lemma. For the surjectivity of G (equivalently, the totality of G^{-1}), define as follows the (proper) subset $\mathcal{G}^*$ of formulas $\mathbf{L}_{\Box}$-equivalent to the image under GT of some $\Box$-free formula: $\mathcal{G}^* := \{\psi \in \mathcal{L}^{\mathrm{t}}_{\Box} \mid (\exists\varphi \in \mathcal{L}^{\mathrm{t}}) \;\vdash_{\mathbf{L}_{\Box}} \psi \leftrightarrow \mathrm{GT}(\varphi)\,\}$. Now for any maximal $\mathbf{L}_{\Box}$-consistent theory $y \in Y_{\Box\mathrm{m}}$, define the subset $y^* := y \cap \mathcal{G}^*$. Define $X^{\mathrm{m}}_{\mathrm{ip}} := \{x_0 \in X_{\mathrm{ip}} \mid (\forall x \in X_{\mathrm{ip}})\; x_0 \nsubseteq x\,\}$ to be the (proper) subset of prime **IL** theories that are $\subseteq$-maximal. Then every $x_0 \in X^{\mathrm{m}}_{\mathrm{ip}}$ is a maximal **IL**-consistent theory, and is also a classical $\mathbf{L}_{\Box}$-consistent theory that is maximal within the $\Box$-free language $\mathcal{L}^{\mathrm{t}}$. So by the deductive faithfulness of the Gödel translation, for every $y \in Y_{\Box\mathrm{m}}$, there is a maximal $x_0 \in X^{\mathrm{m}}_{\mathrm{ip}}$ such that $\mathrm{GT}(x_0) = y^*$, and hence $\mathrm{GT}(x_0) \subseteq y$. Hence G is surjective. The verifications for the remaining Parts (3.) and (4.) are somewhat lengthy, but straight-forward. ⊣

In a sequel [10], we return to the intuitionistic canonical model $\mathcal{M}_\star$, and use it to give a Hennessy-Milner type result on maximal topological bisimulations preserving the intuitionistic semantics. For both the intuitionistic and classical

[7] $\mathcal{M}^*_{\Box}$ will be an l.s.c. model, as $Q_{\Box}$ and $Q^{-1}_{\Box}$ will still be l.s.c. w.r.t. the sub-topology $\mathcal{S}^*_{\Box}$; using $Q^{-\exists}_{\Box}(V(\Box\psi)) = V(\Box\diamondsuit\Box\psi)$ and $Q^{\exists}_{\Box}(V(\Box\psi)) = V(\Box\blacklozenge\Box\psi)$ and the openness property $\mathrm{GT}(\varphi) \leftrightarrow \Box\mathrm{GT}(\varphi)$, we have $Q^{-\exists}_{\Box}(V(\Box\mathrm{GT}(\varphi))) = V(\Box\mathrm{GT}(\diamondsuit\varphi))$ and $Q^{\exists}_{\Box}(V(\Box\mathrm{GT}(\varphi))) = V(\Box\mathrm{GT}(\blacklozenge\varphi))$.

semantics, the classes of models identified have suitable 'saturation' properties w.r.t. the semantics, and the maximal topo-bisimulations are constructed via natural maps into the canonical models. Within these Hennessy-Milner classes, we identify some subclasses of models of continuous, discrete and hybrid dynamical systems.

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