On the formulation and solution of robust performance problems ⋆

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Abstract

A new approach to robust performance problems is proposed in this paper. The approach involves the optimisation of so-called performance weights subject to a constraint, formulated in terms of the structured singular value, which ensures the existence of a stabilising feedback compensator that achieves robust performance with respect to the optimised performance weights and an uncertain plant set. Optimising over the performance weights in this way gives rise to an algorithm for systematically trading-off desired performance against specified plant uncertainty and performance limitations due to plant dynamics. The algorithm also yields a robust controller. The designer is only required to specify the uncertain plant set and an optimisation directionality, which is used to reflect desired closed-loop performance over frequency in terms of a corresponding cost function. Design of this directionality appears to be simpler than designing the performance weights directly.

Key words: optimising performance, performance weight synthesis, robust performance, μ-synthesis, D-K iterations, skewed-μ, $\mathscr{H}_\infty$-control.

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1 Introduction

Optimisation-based frameworks for control system design typically involve two steps: (*i*) the specification of weights to reflect desired performance and robustness requirements; and (*ii*) the synthesis of a controller via the solution of a correspondingly weighted optimisation problem. It is well-known that the design of such weights is a non-trivial task. In particular, the desired performance, specified robustness requirements and/or fundamental performance limitations (e.g. due to right-half plane poles and zeros) may be incompatible. As such, suitable weights are typically obtained via a trial and error process, based in large part on engineering judgement and intuition [18,10]. In this paper, a new optimisation-based approach to weight selection is proposed for a class of robust performance problems.

A feedback compensator is said to achieve robust performance if a certain level of closed-loop performance is achieved for all plants in a specified set. Use of the structured singular value (*i.e.* μ) to convert the problem of synthesising a controller that achieves robust performance into one of synthesising a compensator that achieves robust stability, in the face of a structured uncertainty, is well-known in the case that the measure of closed-loop performance can be expressed in terms of the $\mathscr{H}_\infty$-norm of possibly weighted closed-loop transfer functions [21,17]. Central to this is the linear fractional transformation (LFT)

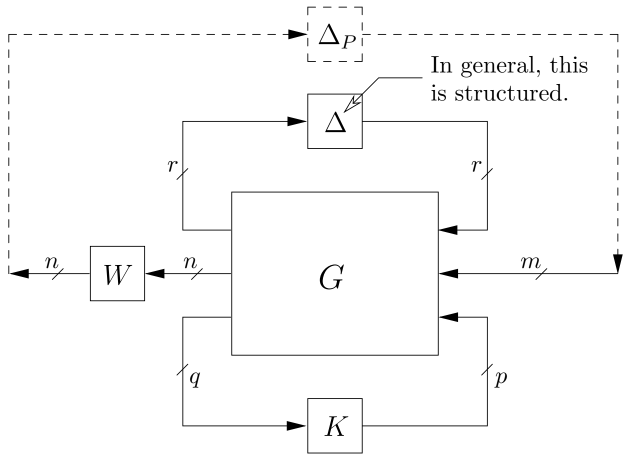

Fig. 1. Typical μ-synthesis LFT framework

configuration shown in Figure 1. Here, the linear time-invariant (LTI) system

$$G = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}$$

is a generalised plant. This system is constructed from a nominal model and appropriate weights, so that:

(1) The upper LFT

$$\mathcal{F}_u\left(\left(\begin{smallmatrix} G_{11} & G_{13} \\ G_{31} & G_{33} \end{smallmatrix}\right), \Delta\right) := G_{33} + G_{31}(I - G_{11}\Delta)^{-1}G_{13}$$

describes the uncertain plant set as Δ varies over some structured set $\boldsymbol{\Delta}$; and

(2) For a given compensator K, the lower LFT

$$\mathcal{F}_l\left(\left(\begin{smallmatrix} G_{22} & G_{23} \\ G_{32} & G_{33} \end{smallmatrix}\right), K\right) := G_{22} + G_{23}(I - G_{33}K)^{-1}G_{32}$$

corresponds to all nominal closed-loop transfer functions by which performance is to be gauged in terms of the $\mathscr{H}_\infty$-norm.

The unstructured block Δ_P is used to convert the corresponding robust performance problem into a robust stabilisation problem; *i.e.* to convert the problem of synthesising K so that

$$\begin{gathered} \|W \cdot \mathcal{F}_u\left(\mathcal{F}_l\left(G, K\right), \Delta\right)\|_\infty < 1 \\ \text{for all } \Delta \in \boldsymbol{\Delta} \text{ satisfying } \|\Delta\|_\infty \leq 1 \end{gathered}$$

into the problem of synthesising K so that the structured singular value $\mu_{\boldsymbol{\Delta}_{TOT}}((\begin{smallmatrix} I & 0 \\ 0 & W \end{smallmatrix})\mathcal{F}_l\left(G, K\right))$, taken with respect to the uncertainty structure $\boldsymbol{\Delta}_{TOT} := \{\text{diag}(\Delta, \Delta_P) : \Delta \in \boldsymbol{\Delta},\ \Delta_P \in \mathbb{C}^{m \times n}\}$, is strictly less than unity over all frequency. Usually, the so-called performance weight W would be specified directly by the designer. In this case, it would be absorbed into G and hence, not be explicitly visible in the expressions given immediately above. By contrast, it is proposed here to also synthesise W via an optimisation problem in which the cost reflects desired performance.

The rationale behind the new approach to robust performance problems developed herein, is related to that of skewed-μ, whereby worst-case performance in the presence of uncertainty satisfying a fixed bound on its norm, and with structure $\boldsymbol{\Delta}$, is determined via a μ-based calculation involving a (constant across frequency) scaling in the channels corresponding to the performance block Δ_P [3,17]. In particular, the new approach involves optimisation (in an appropriate sense) of the performance weight W (which is akin to the constant scaling in skewed-μ), over frequency and in each performance channel direction, so that for a fixed uncertain plant set, a prescribed level of robust performance is achieved by some controller K. The performance weight W is optimised with respect to a cost that reflects the desired performance. In this way, there is a systematic trading-off of desired performance [1] against the robustness requirements, as specified through the structure of and bound on Δ, and/or fundamental performance limitations due to plant dynamics.

[1] which may in part also include desirable robustness properties not captured through the uncertainty block Δ.

In addition to this, potential incompatibilities between the objectives represented by the n performance channels (cf. Figure 1) are resolved in frequency regions of relevance. Such conflicts could arise, for example, if one of the performance channels were to correspond to the standard closed-loop sensitivity function and another to the standard complementary sensitivity function, as these cannot be simultaneously small at any frequency [11].

In the proposed optimisation-based procedure, the designer is required to specify the uncertain plant set by prescribing the generalised plant G, the uncertainty block structure $\boldsymbol{\Delta}$ and an associated norm bound. In addition to this, the designer must specify an optimisation directionality, which appears in the cost associated with the optimisation problem by which W and K are synthesised. The optimisation directionality qualitatively reflects desired performance over all frequency, and should be specified as small (resp. large) at frequencies and in directions where the performance weight would be required to be small (resp. large) in order to capture the desired performance objectives. It would appear to be much easier to specify the optimisation directionality than the performance weight directly itself, since incompatible specifications will be resolved through the constrained optimisation. In this sense, the proposed procedure provides an indication of achievable performance, in addition to a controller achieving robust performance with respect to the suitably optimised performance weights and the uncertain plant set.

It is worth noting that it is not a requirement of the new method that specification of the weight W be left entirely up to the optimisation procedure and judicious selection of the optimisation directionality. If the designer knows that a certain level of performance is achievable and compatible with the robustness requirements, then the corresponding weight can be absorbed into the generalised plant G. The optimisation-based synthesis of W can then be thought of as a systematic mechanism for 'tuning' the design.

Briefly, the paper is structured as follows. First, the optimisation problem, by which both the performance weight W and a robust controller K are synthesised, is formulated. This includes discussion of approximations made to facilitate computation (e.g. replacement of μ with an upper bound). A state-space algorithm based on LMIs is then given for solving the resultant optimisation problem in a D-K style iteration. Finally, numerical examples are presented to help illustrate the new approach to robust performance problems presented in this paper.

2 Formulation of an Optimisation Problem for Synthesis

Recall the linear time-invariant (LTI) LFT configuration shown in Figure 1, where G is a generalised plant, Δ represents the uncertainty in the system and Δ_P is the fictitious uncertainty used only to transform the robust performance problem into an equivalent robust stability problem, as outlined in the previous section. Motivated by the discussion therein, the objective here is to formulate a problem in which the performance weight W is to be optimised in terms of a cost that reflects desired performance, subject to the existence of a feedback compensator K achieving robust performance with respect to the optimised weight and prescribed uncertain plant set. First the following definitions are made to facilitate a precise mathematical formulation. For notational convenience, all uncertainty blocks, except those for performance, are assumed to be square. This can be achieved without loss of generality by adding dummy inputs or outputs [8].

Definition 1 *The sets of allowable perturbations are defined by:*

$$\begin{aligned}
\mathbf{\Delta} &:= \left\{ \operatorname*{diag}_{i=1}^{f} \left(I_{\alpha_i} \otimes \Delta_i \right) : \Delta_i \in \mathbb{C}^{\beta_i \times \beta_i}, \sum_{i=1}^{f} \alpha_i \beta_i = r \right\} \\
\mathbf{B\Delta}^{TF} &:= \left\{ \Delta \in \mathscr{RH}_\infty : \Delta(s_o) \in \mathbf{\Delta} \ \forall s_o \in \overline{\mathbb{C}}_+, \|\Delta\|_\infty \leq 1 \right\}, \\
\mathbf{\Delta}_{TOT} &:= \left\{ \operatorname{diag}\left(\Delta, \Delta_P \right) : \Delta \in \mathbf{\Delta}, \Delta_P \in \mathbb{C}^{m \times n} \right\}
\end{aligned}$$

where $\otimes$ denotes the standard Kronecker product $A \otimes B := [a_{ij}B]$, $\mathscr{RH}_\infty$ the Hardy space of functions analytic and bounded in the open right-half plane $\mathbb{C}_+$ of $\mathbb{C}$, and $\overline{\mathbb{C}}_+$ the closure of $\mathbb{C}_+$.

Definition 2 *The sets of performance weights and directionality matrices are defined by:*

$$\begin{aligned}
\boldsymbol{\mathcal{W}}^{TF} &:= \left\{ W \in \mathscr{RH}_\infty : W^{-1} \in \mathscr{RH}_\infty, W(s_o) \in \mathbf{\Lambda} \ \forall s_o \in \overline{\mathbb{C}}_+ \right\} \\
\mathbf{\Upsilon}^{TF} &:= \left\{ \Upsilon \in \mathscr{RH}_\infty : \Upsilon(\infty) = 0, \Upsilon(s_o) \in \mathbf{\Lambda} \ \forall s_o \in \overline{\mathbb{C}}_+ \right\},
\end{aligned}$$

where $\mathbf{\Lambda} := \left\{ \operatorname{diag}_{i=1}^{n} \left(\ell_i \right) : \ell_i \in \mathbb{C} \right\}$ denotes the set of diagonal complex matrices.

Definition 3 *Given a generalised plant G with (state-space) realisation*

$$G(s) = C(sI - A)^{-1}B + D =: \left[\begin{array}{c|cc:c} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ \hdashline C_3 & D_{31} & D_{32} & D_{33} \end{array}\right]$$

where the partitioning here is conformable with Figure 1, the term "Standard Assumptions" refers to:

(A1) (A, B_3) *is stabilisable and* (C_3, A) *is detectable,*
(A2) $D_{33} = 0$.

Assumption (A1) is necessary and sufficient for the existence of an internally stabilising output-feedback controller [7, Appendix A.4], whereas assumption (A2) incurs no loss of generality but considerably simplifies calculations [6,16]. Given a generalised plant G, the set of internally stabilising output-feedback controllers $K \in \mathscr{R}^{p\times q}$ for the LFT interconnection $\mathcal{F}_l\,(G, K)$ is denoted by $\boldsymbol{\mathcal{K}}_G^{TF}$.

As a first formulation of the weight-optimisation/controller-synthesis problem, consider the following optimisation problem: Given a generalised plant $G(s)$ satisfying the standard assumptions in Definition 3 and a specified directionality transfer function matrix $\Upsilon(s) \in \boldsymbol{\Upsilon}^{TF}$,

$$\begin{gathered} \max_{W\in\boldsymbol{\mathcal{W}}^{TF}} \frac{1}{\|\Upsilon W^{-1}\|_2} \\ \text{subject to} \\ \min_{K\in\boldsymbol{\mathcal{K}}_G^{TF}} \sup_{\omega} \mu_{\boldsymbol{\Delta}_{TOT}} \left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega) \end{pmatrix} \mathcal{F}_l\,(G(j\omega), K(j\omega))\right] < 1, \end{gathered} \tag{1}$$

where $\|\cdot\|_2$ denotes the $\mathscr{H}_2$-norm and $\mu_{\boldsymbol{\Delta}_{TOT}}\left[\cdot\right]$ denotes the structured singular value taken with respect to uncertainty structure $\boldsymbol{\Delta}_{TOT}$. As formulated, this problem is not computationally tractable, since the μ constraint is difficult to handle. Before going on to discuss how this original formulation may be modified to facilitate computation, a few words of justification are in order.

First, observe that the condition

$$\sup_{\omega} \mu_{\boldsymbol{\Delta}_{TOT}} \left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega) \end{pmatrix} \mathcal{F}_l\,(G(j\omega), K(j\omega))\right] < 1$$

is equivalent to [14, Theorem 5.4]

$$\|W \cdot \mathcal{F}_u\left(\mathcal{F}_l\left(G, K\right), \Delta\right)\|_\infty < 1 \text{ for all } \Delta \in \mathbf{B\Delta}^{TF}.$$

As such, the constraint in optimisation problem (1) ensures that the maximisation of the performance weight $W(s)$ is limited by the requirement that there exist an internally stabilising controller $K(s)$ that achieves robust performance with respect to the optimised weight and uncertainty set $\mathbf{B\Delta}^{TF}$.

Secondly, note that

$$\left\|\Upsilon W^{-1}\right\|_2^2 = \int_{-\infty}^{\infty} \sum_{i=1}^{n} \frac{1}{\left|\frac{w_i(j\omega)}{v_i(j\omega)}\right|^2} \, d\omega,$$

where $w_i(j\omega)$ (resp. $v_i(j\omega)$) is the i-th diagonal element of $W(j\omega)$ (resp. $\Upsilon(j\omega)$). From this decomposition, it is clear that the cost function $1\big/\|\Upsilon W^{-1}\|_2$ is a cumulative measure of the frequency-dependent size of the performance weights $w_i(j\omega)$. Each performance weight $w_i(j\omega)$ is itself weighted across frequency by an optimisation directionality $v_i(j\omega)$, which reflects desired performance in that particular performance channel. Now, since the square of the cost function $1\big/\|\Upsilon W^{-1}\|_2$ has the form of the reciprocal of a weighted sum of reciprocals, at any particular frequency the direction of steepest ascent is that corresponding to the smallest ratio $\left|\frac{w_i(j\omega)}{v_i(j\omega)}\right|$. As such, the designer can direct the optimisation by choosing $v_i(j\omega)$ to be large (resp. small) where it would be desirable for the corresponding performance weight $w_i(j\omega)$ to be large (resp. small). Note that this does not make $\Upsilon(j\omega)$ a substitute for the performance weight $W(j\omega)$, since $\Upsilon(j\omega)$ only reflects *desired* performance. The absolute size of $\Upsilon(j\omega)$ is completely irrelevant as this only affects the value of the cost associated with the above optimisation problem. Only the shape across frequency and the relative sizes amongst the different diagonal entries of $\Upsilon(j\omega)$ are important. Since the optimisation directionality $\Upsilon(j\omega)$ only qualitatively reflects desired performance, inconsistencies between this and the specified robustness requirements and/or fundamental performance limitations, are resolved via the freedom in W and the μ constraint which ensures that robust performance is achieved by some controller. Of course, a sensible choice of $\Upsilon(j\omega)$ is still necessary in order that this approach yields a useful controller. However, the design of Υ would appear to be much easier than that of the actual performance weights directly.

In the remainder of this section, the optimisation problem (1) is progressively modified, to render it amenable to numerical solution.

As mentioned previously, the μ constraint above is not computationally tractable as it stands. As such, it is necessary to modify the original formulation of the optimisation problem (1). An approach is to replace $\mu_{\mathbf{\Delta}_{TOT}}$ with a computationally tractable upper bound. To this end, the following definition is required:

Definition 4 *Let the scaling sets $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{D}}^{TF}$ be defined by:*

$$\boldsymbol{\mathcal{D}} := \left\{ D = \operatorname*{diag}_{i=1}^{f} \left(D_i \otimes I_{\beta_i} \right) : \det D \neq 0, D_i \in \mathbb{C}^{\alpha_i \times \alpha_i}, \sum_{i=1}^{f} \alpha_i \beta_i = r \right\}$$
$$\boldsymbol{\mathcal{D}}^{TF} := \left\{ D \in \mathscr{RH}_\infty : D^{-1} \in \mathscr{RH}_\infty, D(s_o) \in \boldsymbol{\mathcal{D}} \; \forall s_o \in \overline{\mathbb{C}}_+ \right\}.$$

Note that elements of these sets commute with elements of $\mathbf{B\Delta}^{TF}$.

Then [14,21],

$$\sup_\omega \mu_{\mathbf{\Delta}_{TOT}} \left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega) \end{pmatrix} \mathcal{F}_l \left(G(j\omega), K(j\omega) \right) \right] \leq \inf_{D \in \boldsymbol{\mathcal{D}}^{TF}} \left\| \begin{pmatrix} D & 0 \\ 0 & W \end{pmatrix} \mathcal{F}_l \left(G, K \right) \begin{pmatrix} D^{-1} & 0 \\ 0 & I_m \end{pmatrix} \right\|_\infty .$$

Replacing $\mu_{\mathbf{\Delta}_{TOT}}$ with this upper bound, and replacing the maximisation of the original weight cost by minimisation of the inverse cost squared, yields the following modified optimisation problem for synthesis:

$$\min_{W \in \boldsymbol{\mathcal{W}}^{TF}} \left\| \Upsilon W^{-1} \right\|_2^2$$
$$\text{subject to}$$
$$\min_{K \in \boldsymbol{\mathcal{K}}_G^{TF}} \inf_{D \in \boldsymbol{\mathcal{D}}^{TF}} \left\| \begin{pmatrix} D & 0 \\ 0 & W \end{pmatrix} \mathcal{F}_l \left(G, K \right) \begin{pmatrix} D^{-1} & 0 \\ 0 & I_m \end{pmatrix} \right\|_\infty < 1.$$

Since $\left\|P\right\|_{2,(\infty)} = \left\|P^T\right\|_{2,(\infty)}$ and defining $\bar{D} := D^{-T} \in \boldsymbol{\mathcal{D}}^{TF}$ and $\bar{W} := W^{-T} \in \boldsymbol{\mathcal{W}}^{TF}$, this optimisation problem may be rewritten exactly as:

$$\begin{gathered} \min_{\bar{W}\in\boldsymbol{\mathcal{W}}^{TF}} \left\|\bar{W}\Upsilon\right\|_2^2 \\ \text{subject to} \\ \min_{K\in\boldsymbol{\mathcal{K}}_G^{TF}} \inf_{\bar{D}\in\boldsymbol{\mathcal{D}}^{TF}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\left(G,K\right)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1. \end{gathered} \tag{2}$$

The reasons for considering the dual systems inside the norms rather than the original systems will become apparent in the proofs of the main results of this paper. As an indication, note that this transpose operation moves the performance weights from the left of $\mathcal{F}_l\left(G,K\right)$ to the right of $\mathcal{F}_l\left(G,K\right)^T$ without altering the stability properties or size of the systems inside the norms.

2.2 Structural Properties of Realisations for $\bar{D}$ and $\bar{W}$

In what follows, state-space realisations are used to develop an algorithm for solving the optimisation problem (2). Towards this end, the structural properties required of state-space realisations for the scaling $\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}$ and weight $\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}$ are now characterised. This requires the following two technical lemmas:

Lemma 1 *Let A, B, P, S, R be real matrices of compatible dimensions such that $P = P^T$, $R = R^T$ and $\lambda_i(A) \neq -\lambda_j(A)$ $\forall i,j$. Define the parahermitian rational matrix function*

$$\Gamma(s) \;:=\; \begin{bmatrix} B^T(-sI - A^T)^{-1} & I \end{bmatrix} \begin{bmatrix} P & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix}.$$

Then, given an arbitrary real matrix $\hat{P} = \hat{P}^T$ of the same dimensions as P, there exists a real matrix $\hat{S}$ of the same dimensions as S such that

$$\Gamma(s) \;=\; \begin{bmatrix} B^T(-sI - A^T)^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{P} & \hat{S} \\ \hat{S}^T & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix}.$$

In fact, $\hat{S}$ is given by $\hat{S} = S + XB$, where the real matrix $X = X^T$ is the unique solution to the Lyapunov equation

$$XA + A^TX = (\hat{P} - P).$$

Proof A proof is given in [12]. □

Lemma 2 *Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with A Hurwitz.*

(i) For every $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ such that $T(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathscr{RH}_\infty$ satisfies $T^{-1} \in \mathscr{RH}_\infty$, there exist $Q_{12} \in \mathbb{R}^{n\times m}$ and $Q_{22} = Q_{22}^T \in \mathbb{R}^{m\times m}$ such that

$$T(j\omega)^* T(j\omega) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} 0 & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}.$$

(ii) For every $Q_{12} \in \mathbb{R}^{n\times m}$ and $Q_{22} = Q_{22}^T \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} 0 & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\},$$

there exist $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ such that $T(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathscr{RH}_\infty$ satisfies $T^{-1} \in \mathscr{RH}_\infty$ and

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} 0 & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} = T(j\omega)^* T(j\omega)$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$.

Proof A proof is given in [12]. □

Observe that Lemma 2 constitutes a complete parametrisation of frequency functions of the form $T(j\omega)^* T(j\omega)$ where $T, T^{-1} \in \mathscr{RH}_\infty$. Now, given any realisations

$$\bar{D} := \left[\begin{array}{c|c} A_{\bar{D}} & B_{\bar{D}} \\ \hline C_{\bar{D}} & D_{\bar{D}} \end{array}\right] \in \boldsymbol{\mathcal{D}}^{TF} \text{ and } \bar{W} := \left[\begin{array}{c|c} A_{\bar{W}} & B_{\bar{W}} \\ \hline C_{\bar{W}} & D_{\bar{W}} \end{array}\right] \in \boldsymbol{\mathcal{W}}^{TF},$$

with $A_{\bar{D}}$ and $A_{\bar{W}}$ Hurwitz, let

$$\begin{aligned} T^o_{\bar{D}}(s) &:= \begin{bmatrix} (sI_{s_D} - A_{\bar{D}})^{-1}B_{\bar{D}} \\ I_r \end{bmatrix} \\ \text{and} \quad T^o_{\bar{W}}(s) &:= \begin{bmatrix} (sI_{s_W} - A_{\bar{W}})^{-1}B_{\bar{W}} \\ I_n \end{bmatrix}. \end{aligned} \tag{3}$$

By Lemma 2, the frequency function:

1. $\bar{D}(j\omega)^*\bar{D}(j\omega)$ can be written as $T^o_{\bar{D}}(j\omega)^*\breve{D}T^o_{\bar{D}}(j\omega)$ for some $\breve{D} := \begin{bmatrix} 0 & \breve{D}_{12} \\ \breve{D}_{12}^T & \breve{D}_{22} \end{bmatrix}$ with $\breve{D}_{12} \in \mathbb{R}^{s_D \times r}$ and $\breve{D}_{22} = \breve{D}_{22}^T \in \mathbb{R}^{r \times r}$,
2. $\bar{W}(j\omega)^*\bar{W}(j\omega)$ can be written as $T^o_{\bar{W}}(j\omega)^*\breve{W}T^o_{\bar{W}}(j\omega)$ for some $\breve{W} := \begin{bmatrix} 0 & \breve{W}_{12} \\ \breve{W}_{12}^T & \breve{W}_{22} \end{bmatrix}$ with $\breve{W}_{12} \in \mathbb{R}^{s_W \times n}$ and $\breve{W}_{22} = \breve{W}_{22}^T \in \mathbb{R}^{n \times n}$.

Taking the arbitrary (1,1)-block to be zero in these parametrisations reduces the number of potential decision variables in an eventual optimisation. Furthermore, these particular parametrisations turn out to be of crucial importance in determining the sign definiteness of a Lyapunov variable in the proof of Theorem 5, which appears in Appendix C.

Since it is required that $\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}$ and $\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}$, it is clear that $T^o_{\bar{D}}(j\omega)^*\breve{D}T^o_{\bar{D}}(j\omega)$ should commute with $\boldsymbol{\Delta}$ and $T^o_{\bar{W}}(j\omega)^*\breve{W}T^o_{\bar{W}}(j\omega)$ should commute with $\boldsymbol{\Lambda}$. These commuting requirements determine the structure of each parameter in the above parametrisations. The top level structure is characterised in the definitions given below. In the interest of space, however, the full structure will not be stated explicitly here and the reader is referred to [12, Section 4.3] for further details (see also Section 8.3.4 in [8]).

Definition 5 *Define the following sets, which characterise the structure of* $\breve{D}$

and $\breve{W}$*:*

$$\Xi_{\breve{D}} := \left\{ \breve{D} = \begin{bmatrix} 0 & \breve{D}_{12} \\ \breve{D}_{12}^T & \breve{D}_{22} \end{bmatrix} : \breve{D}_{12} \in \mathbb{R}^{s_D \times r},\ \breve{D}_{22} = \breve{D}_{22}^T \in \mathbb{R}^{r\times r}, \text{ and } \breve{D}_{12}, \breve{D}_{22} \text{ have the appropriate structure} \right\},$$

$$\Xi_{\breve{W}} := \left\{ \breve{W} = \begin{bmatrix} 0 & \breve{W}_{12} \\ \breve{W}_{12}^T & \breve{W}_{22} \end{bmatrix} : \breve{W}_{12} \in \mathbb{R}^{s_W \times n},\ \breve{W}_{22} = \breve{W}_{22}^T \in \mathbb{R}^{n\times n}, \text{ and } \breve{W}_{12}, \breve{W}_{22} \text{ have the appropriate structure} \right\}.$$

Definition 6 *Define the following sets, which characterise the structure of* $(A_{\bar{D}}, B_{\bar{D}})$ *and* $(A_{\bar{W}}, B_{\bar{W}})$*:*

$$\Xi_{(A_{\bar{D}},B_{\bar{D}})} := \Big\{ (A_{\bar{D}}, B_{\bar{D}}) : A_{\bar{D}} \in \mathbb{R}^{s_D \times s_D}, B_{\bar{D}} \in \mathbb{R}^{s_D \times r}, A_{\bar{D}} \text{ is Hurwitz, and } A_{\bar{D}}, B_{\bar{D}} \text{ have the appropriate structure} \Big\},$$

$$\Xi_{(A_{\bar{W}},B_{\bar{W}})} := \Big\{ (A_{\bar{W}}, B_{\bar{W}}) : A_{\bar{W}} \in \mathbb{R}^{s_W \times s_W}, B_{\bar{W}} \in \mathbb{R}^{s_W \times n}, A_{\bar{W}} \text{ is Hurwitz, and } A_{\bar{W}}, B_{\bar{W}} \text{ have the appropriate structure} \Big\}.$$

2.3 A Convex Approximation

By appropriately restricting attention to a subclass of performance weights and D-scales in the optimisation problem (2), a convex approximation can be constructed in terms of state-space data. Since the frequency functions $\bar{D}(j\omega)^*\bar{D}(j\omega)$ and $\bar{W}(j\omega)^*\bar{W}(j\omega)$ are completely parametrised by $T^o_{\bar{D}}(j\omega)^*\breve{D}T^o_{\bar{D}}(j\omega)$ and $T^o_{\bar{W}}(j\omega)^*\breve{W}T^o_{\bar{W}}(j\omega)$ respectively, it seems natural to restrict these parametrisations by holding the basis functions $T^o_{\bar{D}}(j\omega)$ and $T^o_{\bar{W}}(j\omega)$ fixed. This amounts to keeping $(A_{\bar{D}}, B_{\bar{D}}) \in \Xi_{(A_{\bar{D}},B_{\bar{D}})}$ and $(A_{\bar{W}}, B_{\bar{W}}) \in \Xi_{(A_{\bar{W}},B_{\bar{W}})}$ fixed.

It is desirable to fix the values for $(A_{\bar{D}}, B_{\bar{D}}) \in \Xi_{(A_{\bar{D}},B_{\bar{D}})}$ and $(A_{\bar{W}}, B_{\bar{W}}) \in \Xi_{(A_{\bar{W}},B_{\bar{W}})}$ sufficiently close to the optimal values that would be obtained if these were free variables. Towards determining the "close to optimal" values, observe that for a given G satisfying the standard assumptions stated in Definition 3, a fixed $K \in \boldsymbol{\mathcal{K}}^{TF}_G$ and an optimisation directionality $\Upsilon \in \boldsymbol{\Upsilon}^{TF}$, the optimisation problem

$$\begin{gathered}\min_{\bar{W}\in\boldsymbol{\mathcal{W}}^{TF}} \left\|\bar{W}\Upsilon\right\|_2^2 \\ \text{subject to} \\ \inf_{\bar{D}\in\boldsymbol{\mathcal{D}}^{TF}} \left\| \begin{pmatrix}\bar{D} & 0\\ 0 & I_m\end{pmatrix} \mathcal{F}_l\left(G,K\right)^T \begin{pmatrix}\bar{D}^{-1} & 0\\ 0 & \bar{W}^{-1}\end{pmatrix}\right\|_\infty < 1,\end{gathered} \tag{4}$$

is convex when solved pointwise in frequency. To see this, define the following set and quantities.

Definition 7 *Let the set of strictly-positive vector valued functions be denoted by:*

$$\boldsymbol{\mathcal{V}} := \left\{f : \mathbb{R} \mapsto \mathbb{R}_+^n\right\}$$

For ease of notation, define the following vector functions:

$$\begin{aligned} v_W(\omega) &:= \begin{bmatrix}\frac{1}{|w_1(j\omega)|^2} & \frac{1}{|w_2(j\omega)|^2} & \cdots & \frac{1}{|w_n(j\omega)|^2}\end{bmatrix}^T \in \boldsymbol{\mathcal{V}}, \\ v_\Upsilon(\omega) &:= \begin{bmatrix}|v_1(j\omega)|^2 & |v_2(j\omega)|^2 & \cdots & |v_n(j\omega)|^2\end{bmatrix}^T. \end{aligned}$$

Using this notation, optimisation problem (4) can be rewritten as:

$$\begin{gathered}\min_{v_W\in\boldsymbol{\mathcal{V}}} \int_{-\infty}^{\infty} v_\Upsilon(\omega)^T v_W(\omega)\, d\omega \\ \text{subject to} \\ \forall\omega\in\mathbb{R}\cup\{\infty\}\ \ \exists \text{ continuous } \Theta_\omega\in\boldsymbol{\mathcal{D}} \text{ with } \Theta_\omega>0 \\ \text{satisfying} \\ \left[\mathcal{F}_l\left(G(j\omega),K(j\omega)\right)^T\right]^* \begin{pmatrix}\Theta_\omega & 0\\ 0 & I_m\end{pmatrix} \left[\mathcal{F}_l\left(G(j\omega),K(j\omega)\right)^T\right] \\ < \begin{pmatrix}\Theta_\omega & 0\\ 0 & \operatorname{diag}\left(v_W(\omega)\right)\end{pmatrix}.\end{gathered} \tag{5}$$

The optimisation problem is now clearly convex and can be solved pointwise in frequency over a finite grid using LMI techniques. Once Θ_ω and $v_W(\omega)$ have been determined as described in Section 4, $\bar{D}\in\boldsymbol{\mathcal{D}}^{TF}$ and $\bar{W}\in\boldsymbol{\mathcal{W}}^{TF}$ can be constructed via spectral factorisation. Then, $(A_{\bar{D}}, B_{\bar{D}})\in\boldsymbol{\Xi}_{(A_{\bar{D}},B_{\bar{D}})}$ and $(A_{\bar{W}}, B_{\bar{W}})\in\boldsymbol{\Xi}_{(A_{\bar{W}},B_{\bar{W}})}$ can be obtained from appropriate state-space realisations of $\bar{D}\in\boldsymbol{\mathcal{D}}^{TF}$ and $\bar{W}\in\boldsymbol{\mathcal{W}}^{TF}$.

Once "close to optimal" values for $(A_{\bar{D}}, B_{\bar{D}}) \in \Xi_{(A_{\bar{D}},B_{\bar{D}})}$ and $(A_{\bar{W}}, B_{\bar{W}}) \in \Xi_{(A_{\bar{W}},B_{\bar{W}})}$ have been found, the optimisation problem (2) can be modified, in order to focus on the remaining free parameters, as follows:

$$\begin{gathered}\min_{\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}},B_{\bar{W}})}} \left\| \bar{W}\Upsilon \right\|_2^2 \\ \text{such that} \\ \min_{K \in \boldsymbol{\mathcal{K}}^{TF}_G} \inf_{\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}},B_{\bar{D}})}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\left(G, K\right)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1,\end{gathered} \tag{6}$$

where

Definition 8 *Given* $(A_{\bar{D}}, B_{\bar{D}}) \in \Xi_{(A_{\bar{D}},B_{\bar{D}})}$ *and* $(A_{\bar{W}}, B_{\bar{W}}) \in \Xi_{(A_{\bar{W}},B_{\bar{W}})}$*, define*

$$\boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}},B_{\bar{D}})} := \left\{ \bar{D} = \left[\begin{array}{c|c} A_{\bar{D}} & B_{\bar{D}} \\ \hline C_{\bar{D}} & D_{\bar{D}} \end{array} \right] \ : \ \bar{D} \in \boldsymbol{\mathcal{D}}^{TF} \right\}$$

$$\boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}},B_{\bar{W}})} := \left\{ \bar{W} = \left[\begin{array}{c|c} A_{\bar{W}} & B_{\bar{W}} \\ \hline C_{\bar{W}} & D_{\bar{W}} \end{array} \right] \ : \ \bar{W} \in \boldsymbol{\mathcal{W}}^{TF} \right\}$$

3 Solving the Synthesis Problem

Results underpinning the algorithm that will be proposed for solving the optimisation problem (2), are established in this section. The results show that the related optimisation problem (6), although not simultaneously convex in all parameters, is simultaneously convex in D and W when K is fixed, and K and W when D is fixed. As such, a D-K style iteration can be formulated for synthesis.

3.1 The Cost Function

The following theorem establishes that the cost is a linear function of the weight parameters.

Theorem 3 *Given* $\Upsilon(s) = \left[\begin{array}{c|c} A_\Upsilon & B_\Upsilon \\ \hline C_\Upsilon & 0 \end{array} \right] \in \boldsymbol{\Upsilon}^{TF}$ *with* A_Υ *Hurwitz,* $(A_{\bar{W}}, B_{\bar{W}}) \in \Xi_{(A_{\bar{W}},B_{\bar{W}})}$ *and any* $\bar{W}(s) \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}},B_{\bar{W}})}$*, define* $T^o_{\bar{W}}(s)$ *as in equation* (3) *and let* $\breve{W} \in \Xi_{\breve{W}}$ *be such that* $T^o_{\bar{W}}(j\omega)^* \breve{W} T^o_{\bar{W}}(j\omega) = \bar{W}(j\omega)^* \bar{W}(j\omega)$ $\forall \omega \in \mathbb{R} \cup \{\infty\}$.

Then

$$\left\|\bar{W}\Upsilon\right\|_2^2 = c^T \mathrm{vec}\left(\breve{W}\right)$$

where

$$c := -\left(\begin{bmatrix} I_{s_W} & 0 \\ 0 & C_\Upsilon \end{bmatrix} \otimes \begin{bmatrix} I_{s_W} & 0 \\ 0 & C_\Upsilon \end{bmatrix}\right) \times \left(\begin{bmatrix} A_{\bar{W}} & B_{\bar{W}}C_\Upsilon \\ 0 & A_\Upsilon \end{bmatrix} \oplus \begin{bmatrix} A_{\bar{W}} & B_{\bar{W}}C_\Upsilon \\ 0 & A_\Upsilon \end{bmatrix}\right)^{-1} \times \left(\begin{bmatrix} 0 \\ B_\Upsilon \end{bmatrix} \otimes \begin{bmatrix} 0 \\ B_\Upsilon \end{bmatrix}\right) \mathrm{vec}(I_n).$$

Proof See Appendix A for proof. □

It is consequently noted that 'Minimising $\left\|\bar{W}\Upsilon\right\|_2^2$ over $\bar{W}(s) \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}},B_{\bar{W}})}$ subject to some constraint' is equivalent to 'Minimising $c^T \mathrm{vec}\left(\breve{W}\right)$ over $\breve{W} \in \boldsymbol{\Xi}_{\breve{W}}$ subject to the same constraint', provided that $T^o_{\breve{W}}(j\omega)^* \breve{W} T^o_{\breve{W}}(j\omega) > 0 \ \ \forall\omega \in \mathbb{R}\cup\{\infty\}$ is implicitly guaranteed by the constraint.

3.2 Holding K Fixed in the Constraint

The following theorem shows that for a fixed $K \in \boldsymbol{\mathcal{K}}^{TF}_G$, the constraint of optimisation problem (6) can be rewritten as a set of LMIs that are also simultaneously affine in $\breve{W}$.

Theorem 4 *Given $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}},B_{\bar{W}})}$ and any $\bar{W}(s) \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}},B_{\bar{W}})}$, define $T^o_{\breve{W}}(s)$ as in equation (3) and let $\breve{W} \in \boldsymbol{\Xi}_{\breve{W}}$ be such that $T^o_{\breve{W}}(j\omega)^* \breve{W} T^o_{\breve{W}}(j\omega) = \bar{W}(j\omega)^*\bar{W}(j\omega)$ $\forall\omega \in \mathbb{R}\cup\{\infty\}$. Then given also*

$$\mathcal{F}_l(G,K) = \left[\begin{array}{c|cc} A_{cl} & B_{1cl} & B_{2\,cl} \\ \hline C_{1cl} & D_{11cl} & D_{12\,cl} \\ C_{2\,cl} & D_{21cl} & D_{22\,cl} \end{array}\right],$$

where $A_{cl} \in \mathbb{R}^{s_{cl}\times s_{cl}}$ is Hurwitz and the partitioning is consistent with Figure 1, the following two statements are equivalent for any $(A_{\bar{D}}, B_{\bar{D}}) \in \boldsymbol{\Xi}_{(A_{\bar{D}},B_{\bar{D}})}$:

(i) $$\inf_{\bar{D}\in\boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}},B_{\bar{D}})}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l(G,K)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1.$$

(ii) $\exists\, \breve{D} \in \boldsymbol{\Xi}_{\breve{D}}$, $X = X^T \in \mathbb{R}^{s_D \times s_D}$ *and* $Y = Y^T \in \mathbb{R}^{(s_{cl}+2s_D+s_W)\times(s_{cl}+2s_D+s_W)}$ *such that*

$$\begin{bmatrix} XA_{\bar{D}} + A_{\bar{D}}^T X & XB_{\bar{D}} \\ B_{\bar{D}}^T X & 0 \end{bmatrix} + \breve{D} > 0,$$

$$\begin{bmatrix} Y\grave{A} + \grave{A}^T Y & Y\grave{B} \\ \grave{B}^T Y & 0 \end{bmatrix} + \begin{bmatrix} \grave{C}^T \\ \grave{D}^T \end{bmatrix} \grave{Q} \begin{bmatrix} \grave{C} & \grave{D} \end{bmatrix} < 0;$$

where $\grave{A}$, $\grave{B}$, $\grave{C}$, $\grave{D}$ *and* $\grave{Q}$ *are defined by*

$$\begin{bmatrix} \grave{A} & \grave{B} \\ \grave{C} & \grave{D} \end{bmatrix} := \left[\begin{array}{cccc|cc} A_{\bar{D}} & 0 & 0 & 0 & B_{\bar{D}} & 0 \\ 0 & A_{\bar{W}} & 0 & 0 & 0 & B_{\bar{W}} \\ 0 & 0 & A_{\bar{D}} & B_{\bar{D}} B_{1cl}^T & B_{\bar{D}} D_{11cl}^T & B_{\bar{D}} D_{21cl}^T \\ 0 & 0 & 0 & A_{cl}^T & C_{1cl}^T & C_{2\,cl}^T \\ \hline 0 & 0 & I_{s_D} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{1cl}^T & D_{11cl}^T & D_{21cl}^T \\ 0 & 0 & 0 & B_{2\,cl}^T & D_{12\,cl}^T & D_{22\,cl}^T \\ I_{s_D} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_r & 0 \\ 0 & I_{s_W} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n \end{array}\right],$$

$$\begin{aligned} \grave{Q} &:= \operatorname{diag}\left(\breve{D},\, I_m,\, -\breve{D},\, -\breve{W}\right) \\ &= \operatorname{diag}\left(\begin{bmatrix} 0 & \breve{D}_{12} \\ \breve{D}_{12}^T & \breve{D}_{22} \end{bmatrix},\, I_m,\, -\begin{bmatrix} 0 & \breve{D}_{12} \\ \breve{D}_{12}^T & \breve{D}_{22} \end{bmatrix},\, -\begin{bmatrix} 0 & \breve{W}_{12} \\ \breve{W}_{12}^T & \breve{W}_{22} \end{bmatrix} \right). \end{aligned}$$

Proof See Appendix B for proof. □

3.3 Holding $\bar{D}$ Fixed in the Constraint

The following theorem shows that for a fixed $\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}}, B_{\bar{D}})}$, the constraint appearing in optimisation problem (6) can be rewritten as a set of LMIs that are also simultaneously affine in $\breve{W}$.

Theorem 5 *Given scalings $\bar{D}(s) \in \boldsymbol{\mathcal{D}}^{TF}$ and a generalised plant $G(s)$ satis-*

fying the standard assumptions stated in Definition 3, define the scaled generalised plant $\tilde{G}(s)$ by

$$\tilde{G}(s) := \begin{bmatrix} \bar{D}(s)^{-T} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_q \end{bmatrix} G(s) \begin{bmatrix} \bar{D}(s)^{T} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_p \end{bmatrix},$$

and let

$$\left[\begin{array}{c|cc:c} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 & \tilde{B}_3 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} & \tilde{D}_{23} \\ \hdashline \tilde{C}_3 & \tilde{D}_{31} & \tilde{D}_{32} & \tilde{D}_{33} \end{array}\right]$$

be a stabilisable and detectable realisation for $\tilde{G}(s)$ with $\tilde{A} \in \mathbb{R}^{s_{\tilde{G}} \times s_{\tilde{G}}}$, $\tilde{D}_{11} \in \mathbb{R}^{r\times r}$, $\tilde{D}_{22} \in \mathbb{R}^{n\times m}$ and $\tilde{D}_{33} = 0 \in \mathbb{R}^{q\times p}$. Furthermore, given $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}},B_{\bar{W}})}$ and any $\bar{W}(s) \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}},B_{\bar{W}})}$, define $T^o_{\bar{W}}(s)$ as in equation (3) and let $\breve{W} \in \boldsymbol{\Xi}_{\breve{W}}$ be such that $T^o_{\bar{W}}(j\omega)^* \breve{W} T^o_{\bar{W}}(j\omega) = \bar{W}(j\omega)^* \bar{W}(j\omega)$ $\forall \omega \in \mathbb{R} \cup \{\infty\}$. Then the following two statements are equivalent:

(i) $\min_{K \in \boldsymbol{\mathcal{K}}^{TF}_G} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\left(G, K\right)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1.$

(ii) $\exists\, P = P^T \in \mathbb{R}^{(s_W + s_{\tilde{G}}) \times (s_W + s_{\tilde{G}})}$, $R = R^T \in \mathbb{R}^{s_W \times s_W}$, $S \in \mathbb{R}^{s_W \times s_{\tilde{G}}}$ *and* $T = T^T \in \mathbb{R}^{s_{\tilde{G}} \times s_{\tilde{G}}}$ *such that*

$$P > 0, \quad R > 0, \quad T > 0,$$

$$\begin{pmatrix} P & \begin{bmatrix} R & -S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} \\ \begin{bmatrix} R & 0 \\ -S^T & I_{s_{\tilde{G}}} \end{bmatrix} & \begin{bmatrix} R & 0 \\ 0 & T \end{bmatrix} \end{pmatrix} \geq 0,$$

$$
\Psi_P^T \cdot \begin{pmatrix} P\begin{bmatrix} A_{\bar{W}} & 0 \\ 0 & \tilde{A}^T \end{bmatrix} + \{\cdot\}^T & P\begin{bmatrix} 0 & B_{\bar{W}} \\ \tilde{C}_1^T & \tilde{C}_2^T \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ \tilde{B}_1 & \tilde{B}_2 \end{bmatrix} \\ * & \begin{bmatrix} -I_r & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \\ * & * & \begin{bmatrix} -I_r & 0 \\ 0 & -I_m \end{bmatrix} \end{pmatrix} \cdot \Psi_P
$$

$$
< \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & \psi_3^T \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix} \breve{W} \left(\begin{bmatrix} I_{s_W} & 0 \\ 0 & \psi_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),
$$

and

$$
\Psi_Q^T \cdot \begin{pmatrix} \begin{bmatrix} R & S \\ -S^T & T \end{bmatrix} \begin{bmatrix} A_{\bar{W}} & 0 \\ 0 & \tilde{A} \end{bmatrix} + \{\cdot\}^T & * & * \\ \begin{bmatrix} 0 & \tilde{B}_1^T \\ 0 & \tilde{B}_2^T \end{bmatrix} \begin{bmatrix} I_{s_W} & 0 \\ S^T & T \end{bmatrix} & \begin{bmatrix} -I_r & 0 \\ 0 & -I_m \end{bmatrix} & * \\ \begin{bmatrix} 0 & \tilde{C}_1 \\ B_{\bar{W}}^T & \tilde{C}_2 \end{bmatrix} \begin{bmatrix} R & -S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} & \begin{bmatrix} -I_r & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix} \cdot \Psi_Q
$$

$$
< \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \end{pmatrix} \breve{W} \left(\begin{bmatrix} I_{s_W} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \right),
$$

where

$$\Psi_P := \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & \psi_1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \psi_2 \\ 0 & \psi_3 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I_r & 0 \\ 0 & I_m \end{bmatrix} \end{pmatrix}, \qquad \Psi_Q := \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & \psi_4 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \psi_5 \\ 0 & \psi_6 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I_r & 0 \\ 0 & I_n \end{bmatrix} \end{pmatrix}$$

and the columns of $\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$ *(resp.* $\begin{bmatrix} \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix}$*) form bases for the null space of* $\begin{bmatrix} \tilde{B}_3^T & \tilde{D}_{13}^T & \tilde{D}_{23}^T \end{bmatrix}$ *(resp.* $\begin{bmatrix} \tilde{C}_3 & \tilde{D}_{31} & \tilde{D}_{32} \end{bmatrix}$*).*

Proof See Appendix C for proof. □

The following corollary gives a necessary and sufficient condition for the existence of controllers of order s_K in $\boldsymbol{\mathcal{K}}_G^{TF}$ that satisfy the norm constraint together with a procedure for constructing such controllers.

Corollary 6 *Let the suppositions of Theorem 5 hold. Then there exist controllers* $K \in \boldsymbol{\mathcal{K}}_G^{TF}$ *of order* s_K *satisfying*

$$\left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\left(G, K\right)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1$$

if and only if the LMI constraints given in Part (ii) *of Theorem 5 hold for some* $P = P^T \in \mathbb{R}^{(s_W + s_{\tilde{G}}) \times (s_W + s_{\tilde{G}})}$, $R = R^T \in \mathbb{R}^{s_W \times s_W}$, $S \in \mathbb{R}^{s_W \times s_{\tilde{G}}}$ *and* $T = T^T \in \mathbb{R}^{s_{\tilde{G}} \times s_{\tilde{G}}}$ *that further satisfy*

$$\operatorname{rank}\left(\begin{bmatrix} I_{s_W} & S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} P \begin{bmatrix} I_{s_W} & 0 \\ S^T & I_{s_{\tilde{G}}} \end{bmatrix} - \begin{bmatrix} R & 0 \\ 0 & T^{-1} \end{bmatrix} \right) \leq s_K.$$

Such controllers $K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *can then be constructed by solving*

$$F + U^T \Phi_K V + V^T \Phi_K^T U < 0$$

for the controller parameters $\Phi_K := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$, *where* F, U *and* V *are defined by:*

$$F := \begin{pmatrix} X \begin{bmatrix} A_{\bar{W}} & 0 & 0 \\ 0 & \tilde{A}^T & 0 \\ 0 & 0 & 0 \end{bmatrix} + \{\cdot\}^T & X \begin{bmatrix} 0 & B_{\bar{W}} \\ \tilde{C}_1^T & \tilde{C}_2^T \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ \tilde{B}_1 & \tilde{B}_2 \\ 0 & 0 \end{bmatrix} \\ * & \begin{bmatrix} -I_r & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \\ * & * & \begin{bmatrix} -I_r & 0 \\ 0 & -I_m \end{bmatrix} \end{pmatrix}$$

$$- \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix} \breve{W} \left(\begin{bmatrix} I_{s_W} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

$$U := \left(\begin{bmatrix} 0 & 0 & I_{s_K} \\ 0 & \tilde{B}_3^T & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \tilde{D}_{13}^T & \tilde{D}_{23}^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

$$V := \left(\begin{bmatrix} 0 & 0 & I_{s_K} \\ 0 & \tilde{C}_3 & 0 \end{bmatrix} X \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \tilde{D}_{31} & \tilde{D}_{32} \end{bmatrix} \right),$$

and X *is constructed as follows:*

- *Define* $Q := \begin{bmatrix} I_{s_W} & 0 \\ S^T & I_{s_{\tilde{G}}} \end{bmatrix} \begin{bmatrix} R^{-1} & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} I_{s_W} & S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix}$.
- *Factorise* $P - Q^{-1} = HH^T$ *with* $H \in \mathbb{R}^{(s_W + s_{\tilde{G}}) \times s_K}$.
- *Define* $X := \begin{bmatrix} P & H \\ H^T & I_{s_K} \end{bmatrix}$.

Proof See Appendix D for proof. □

4 The Synthesis Algorithm

This section serves to outline an iterative algorithm for solving optimisation problem (2). Recall that the convex approximation made in Section 2.3 relies on fixed "close to optimal" values for $(A_{\bar{W}}, B_{\bar{W}}) \in \mathbf{\Xi}_{(A_{\bar{W}},B_{\bar{W}})}$ and $(A_{\bar{D}}, B_{\bar{D}}) \in \mathbf{\Xi}_{(A_{\bar{D}},B_{\bar{D}})}$. These "close to optimal" values are constructed in Phase 1 of the algorithm (i.e. during the first few iterations) by solving optimisation problem (7) pointwise in frequency followed by a rational approximation of the results. In Phase 2 of the algorithm (i.e. during the last few iterations), the state-space characterisations developed in preceding sections are used together with these fixed "close to optimal" values to solve the weight and controller synthesis problem.

Observe that in Phase 1 of the algorithm, the problem is over-constrained (since $\gamma_i < 1$). This is to accommodate the introduction of error when reasonably low order rational approximations of the pointwise optimal D-scales and weights are constructed. In Phase 2, it is no longer necessary to over-constrain the problem (so γ_i can be set to 1), because the state-space characterisations with fixed $(A_{\bar{W}}, B_{\bar{W}})$ and $(A_{\bar{D}}, B_{\bar{D}})$ are exact (*i.e.* there are no approximation errors).

Inputs to the algorithm:

- Generalised plant $G(s)$ satisfying the standard assumptions stated in Definition 3,
- Optimisation directionality matrix transfer function $\Upsilon(s) \in \mathbf{\Upsilon}^{TF}$.

The solution algorithm:

Phase 1 of the algorithm involves Steps 3 and 5 but not Step 4 whereas Phase 2 of the algorithm involves Steps 4 and 5 but not Step 3.

1. First find a controller $K_0^\star$ which robustly stabilises the interconnection $\mathcal{F}_u\Big(\mathcal{F}_l\big(\hat{G}, K_0^\star\big), \Delta\Big)$ for all $\Delta \in \mathbf{B\Delta}^{TF}$, where

$$\hat{G} := \left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & 0 \end{array}\right].$$

Let γ_0 be some number in the interval $\left(\sup_{\omega} \inf_{D_\omega \in \boldsymbol{\mathcal{D}}} \overline{\sigma}\left[D_\omega \mathcal{F}_l(\hat{G}, K_0^\star) D_\omega^{-1}\right], 1\right)$. Such a $\gamma_0 < 1$ always exists if the robust stability problem above admits a solution. Now set $i = 0$ (where i denotes the iteration number) and $\eta_0^\star = \infty$.

2. Increment i by 1.
3. If $\gamma_{i-1} < 1$: (i.e. during the first few iterations)
   (a) Solve the following convex optimisation problem

$$\begin{aligned} &\min_{\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}} \left\| \bar{W} \Upsilon \right\|_2^2 \\ &\text{subject to} \\ &\inf_{\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\left(G, K_{i-1}^\star\right)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < \gamma_{i-1}, \end{aligned} \tag{7}$$

   pointwise in frequency and on a sufficiently dense, but finite, grid using the reformulation given in optimisation problem (5), after normalising the above constraint to unity. Let the optimal values of $v_W(\omega)$ and Θ_ω at each grid frequency $\omega = \omega_k$ (these are vector/matrix decision variables in optimisation problem (5)) be denoted by $v_{W,\omega_k}^\star$ and $\Theta_{\omega_k}^\star$ respectively.

   (b) Construct a low order [2] $\bar{W}^\star \in \boldsymbol{\mathcal{W}}^{TF}$ by fitting a stable minimum-phase transfer function to each magnitude function in $v_{W,\omega_k}^\star$.

   (c) Construct a low order[2] self-adjoint real-rational unit in $\mathscr{RL}_\infty$ which is positive at infinity by fitting real-rational functions to each element in $\Theta_{\omega_k}^\star$ and perhaps model reducing afterwards. Denote this unit by $\Theta^\star(s)$. Then compute a spectral factor [3] $\bar{D}_i^\star \in \boldsymbol{\mathcal{D}}^{TF}$ for this $\Theta^\star(s)$.

   (d) Let $\gamma_i := (1+\epsilon) \left\| \begin{pmatrix} \bar{D}_i^\star & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\left(G, K_{i-1}^\star\right)^T \begin{pmatrix} \bar{D}_i^\star & 0 \\ 0 & \bar{W}^\star \end{pmatrix}^{-1} \right\|_\infty$, for some $0 < \epsilon \ll 1$.

   (e) If $\gamma_i \le 1$, then let $(A_{\bar{W}}, B_{\bar{W}}) \in \Xi_{(A_{\bar{W}}, B_{\bar{W}})}$ and $(A_{\bar{D}}, B_{\bar{D}}) \in \Xi_{(A_{\bar{D}}, B_{\bar{D}})}$ be obtained from the appropriate state-space realisations of $\bar{W}^\star(s)$ and $\bar{D}_i^\star(s)$ respectively. Otherwise, go to Step 4(a).
4. If $\gamma_{i-1} = 1$: (i.e. during the last few iterations)
   (a) Let $\gamma_i := 1$.

[2] Low order approximations of the pointwise solutions are used since this is likely to result in a low complexity (in the sense of [19]) controller in the end. As noted in [19], given an appropriate bound on the complexity of the controller, significantly stronger robustness properties can be guaranteed for the corresponding feedback system. This is in line with the common engineering practice of employing the lowest complexity controller to do the job.

[3] Since spectral factors are not unique and the required spectral factor here has to have a block-diagonal structure, each individual diagonal block should be spectrally factored separately.

(b) Solve the following convex optimisation problem

$$\min_{\bar{W}\in\boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}},B_{\bar{W}})}} \left\|\bar{W}\Upsilon\right\|_2^2$$
subject to
$$\inf_{\bar{D}\in\boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}},B_{\bar{D}})}} \left\|\begin{pmatrix}\bar{D} & 0\\ 0 & I_m\end{pmatrix}\mathcal{F}_l\left(G,K^\star_{i-1}\right)^T\begin{pmatrix}\bar{D}^{-1} & 0\\ 0 & \bar{W}^{-1}\end{pmatrix}\right\|_\infty < 1$$

by making use of Theorem 3 and Theorem 4. Let the optimal value of $\breve{D}$ (a matrix decision variable in the LMI constraints of Theorem 4) be denoted by $\breve{D}^\star_i$.

(c) Using the values of $(A_{\bar{D}}, B_{\bar{D}}) \in \boldsymbol{\Xi}_{(A_{\bar{D}},B_{\bar{D}})}$ and the value of $\breve{D}^\star_i \in \boldsymbol{\Xi}_{\breve{D}}$ just obtained, define

$$\Theta^\star_i(s) := T^o_{\bar{D}}{}^\sim(s)\breve{D}^\star_i T^o_{\bar{D}}(s)$$

and compute a spectral factor[3] $\bar{D}^\star_i \in \boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}},B_{\bar{D}})}$ for $\Theta^\star_i(s)$.

5. (a) Solve the following convex optimisation problem

$$\min_{\bar{W}\in\boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}},B_{\bar{W}})}} \left\|\bar{W}\Upsilon\right\|_2^2$$
subject to
$$\min_{K\in\boldsymbol{\mathcal{K}}^{TF}_G} \left\|\begin{pmatrix}\bar{D}^\star_i & 0\\ 0 & I_m\end{pmatrix}\mathcal{F}_l\left(G,K\right)^T\begin{pmatrix}\bar{D}^{\star\,-1}_i & 0\\ 0 & \bar{W}^{-1}\end{pmatrix}\right\|_\infty < \gamma_i$$

by making use of Theorem 3 and Theorem 5 after normalising the above constraint to unity. Let the value of this minimum cost be denoted by $\eta^\star_i$ and let the optimal value of $\breve{W}$ (a matrix decision variable in the cost function/LMI constraints) be denoted by $\breve{W}^\star_i$.

(b) Using the values of $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}},B_{\bar{W}})}$ and the value of $\breve{W}^\star_i \in \boldsymbol{\Xi}_{\breve{W}}$ just obtained, define

$$\Pi^\star_i(s) := T^o_{\bar{W}}{}^\sim(s)\breve{W}^\star_i T^o_{\bar{W}}(s)$$

and compute a spectral factor[3] $\bar{W}^\star_i \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}},B_{\bar{W}})}$ for $\Pi^\star_i(s)$.

(c) Using Corollary 6, find a controller that satisfies the constraint of the above optimisation problem and denote this controller by $K^\star_i(s)$.

6. Evaluate $(\eta^\star_{i-1} - \eta^\star_i)$. If this difference (which is always positive) is very small and has remained very small for the last few iterations, then EXIT. Otherwise return to Step 2.

Outputs from the algorithm: (after i iterations)

- The inverse of the largest performance weights obtained by the algorithm in $\bar{W}_i^\star \in \boldsymbol{\mathcal{W}}^{TF}$,
- The controller $K_i^\star \in \boldsymbol{\mathcal{K}}_G^{TF}$ that achieves robust performance with respect to these weights,
- The final scalings $\bar{D}_i^\star \in \boldsymbol{\mathcal{D}}^{TF}$ used by the algorithm,
- The value of the minimum cost $\eta_i^\star$ obtained.

5 Numerical Examples

In this section, the algorithm proposed in the previous section is illustrated through several numerical examples. Two different examples demonstrating the applicability of the proposed algorithm are considered. The first example is discussed in three parts, each illustrating how the proposed algorithm can systematically handle particular situations of interest. The second example involves an experimental MIMO plant. It is used in order to compare our results with a standard μ-synthesis design which can be found in [1].

5.1 Sensitivity/Complementary Sensitivity reduction problem

Consider the block diagram shown in Figure 2, which captures a typical Sensitivity/Complementary Sensitivity reduction problem.

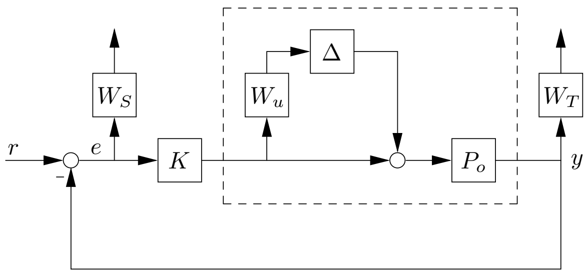

Fig. 2. Block diagram for a typical S/T problem

The plant is uncertain but known to belong to the set $\left\{P_o(1+\Delta W_u)\ :\ \Delta \in \mathscr{RH}_\infty, \|\Delta\|_\infty \leq 1\right\}$, parametrised by Δ (see the dashed box). Here, the nominal plant P_o and the uncertainty weight W_u are chosen as:

$$P_o = \frac{0.01(s/\omega_1 - 1)}{(s^2 + 0.06s + 0.01)} \quad \text{and} \quad W_u = \frac{0.1(s/\omega_2 + 1)^2}{(s/(10\omega_2) + 1)^2}.$$

Different values for ω_1 and ω_2 are used in each of the following subsections to illustrate different characteristics of the new approach. The uncertainty weight W_u represents any 'a priori' knowledge about the frequency dependent size of the uncertainty. In this example, the chosen W_u allows the magnitude of the actual plant to differ from that of the nominal plant by as much as 10% in the low-frequency region (say, below ω_2 `rad/s`) and by as much as 1000% in the high-frequency region (say, above $10\omega_2$ `rad/s`).

The objective here is to maximise the performance weights W_S and W_T according to some pre-specified directionality whilst ensuring that there exists an internally stabilising controller $K(s) \in \boldsymbol{\mathcal{K}}_G^{TF}$ that achieves robust performance with respect to these maximised weights and the uncertain plant set. To this end, the optimisation directionality

$$\Upsilon = \begin{bmatrix} \frac{100(s/3+1)^2}{(s/0.3+1)^2(s/10^5+1)} & 0 \\ 0 & \frac{(s/0.3+1)^2}{(s/3+1)^2(s/10^5+1)} \end{bmatrix}$$

is considered – see Figure 3. Note that Υ must be strictly proper in order that the cost function (a function of an $\mathscr{H}_2$ norm) in the proposed optimisation problem be finite. This directionality basically reflects that the algorithm

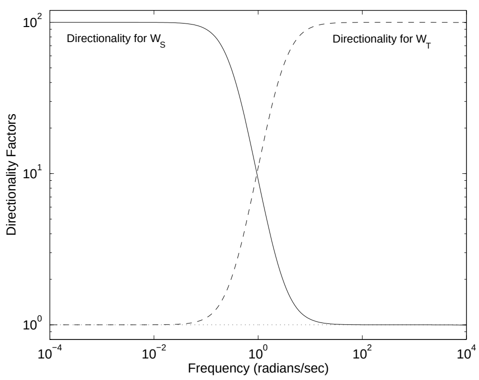

Fig. 3. Desired directionality for the optimisation

should maximise W_S (resp. W_T) in the low-frequency (resp. high-frequency) region and that it should not bother too much about maximising W_S (resp. W_T) in the remaining high-frequency (resp. low-frequency) region. The scale on the y-axis of this figure is unimportant as it only affects the cost associated with the optimisation. Only the relative sizes between the different curves and the shape of each curve across frequency is important. Note that the low-frequency value of the solid curve in Figure 3 is equal to the high-frequency value of the dashed curve. This reflects that the maximisation of W_S at low-frequency should be valued as much as the maximisation of W_T at high-frequency. At around 1 `rad/s`, the solid curve and the dashed curve are also equal. Again this indicates that around this frequency, the maximisation of W_S is as equally important as the maximisation of W_T. However, since $S + T = 1$, it is not possible to make both W_S and W_T large at this frequency. The proposed algorithm systematically determines how much each of these weights can be maximised at this frequency, trading-off desired performance against the specified plant uncertainty and performance limitations due to plant dynamics. Finally, note that the magnitude of the directionalities at 1 `rad/s` is about a decade less than the magnitude of the directionalities at low and high-frequency. As such, maximisation of W_S and W_T in this mid-frequency region is considered less important than maximisation of W_S and W_T at low and high-frequency, respectively. Correspondingly, 1 `rad/s` can be thought of as the desired bandwidth for the closed-loop.

5.1.1 Right half plane zero and high frequency uncertainty imposing no bandwidth limitations

In this subsection, $\omega_1 = 100$ `rad/s` and $\omega_2 = 30$ `rad/s`. This gives rise to the nominal plant P_o and uncertainty weight W_u shown in Figure 4. This partic-

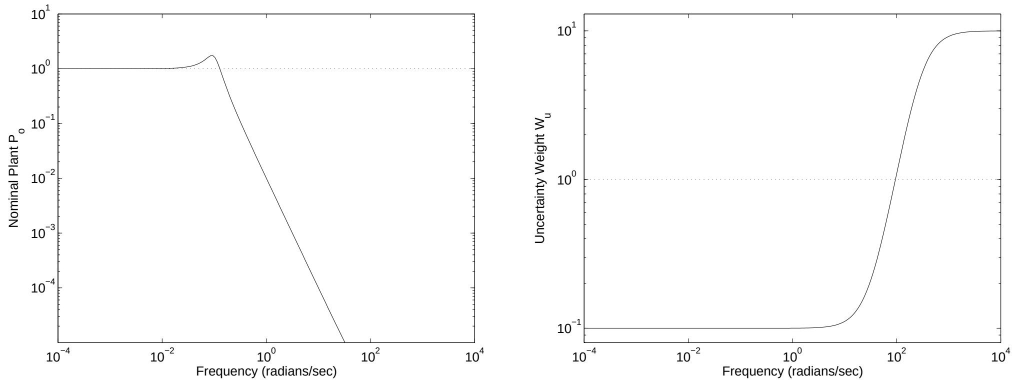

Fig. 4. Plant and uncertainty weight magnitude plots

ular choice of ω_1 and ω_2 imposes no bandwidth limitations, as the right half plane zero of the plant and the frequency region where uncertainty becomes significant lie far beyond the desired bandwidth of 1 `rad/s` (see Figure 3). Thus, it is expected that the algorithm will be able to synthesise optimised weights W_S and W_T (or rather their inverses) such that the bandwidth is 1 `rad/s`, as desired.

The proposed algorithm took 4 iterations to converge. The cost $\eta_i^\star$ associated with the proposed optimisation problem took the following sequence of values $\{2304, 543, 497, 484\}$, which is monotonically decreasing as expected. At each iteration, γ_i took values in the non-decreasing sequence $\{0.50, 0.85, 1, 1\}$, up to unity where the algorithm uses the exact state space solution, avoiding frequency data fitting and hence, the corresponding approximation errors. The results of the proposed algorithm are shown in Figure 5. Note that the magnitude plot of the sensitivity (resp. complementary sensitivity) function of any plant in the set $\left\{P_o(1 + \Delta W_u) : \Delta \in \mathscr{RH}_\infty, \|\Delta\|_\infty \leq 1\right\}$ will be below the $|W_S(j\omega)|^{-1}$ (resp. $|W_T(j\omega)|^{-1}$) plot, since robust performance is guaranteed.

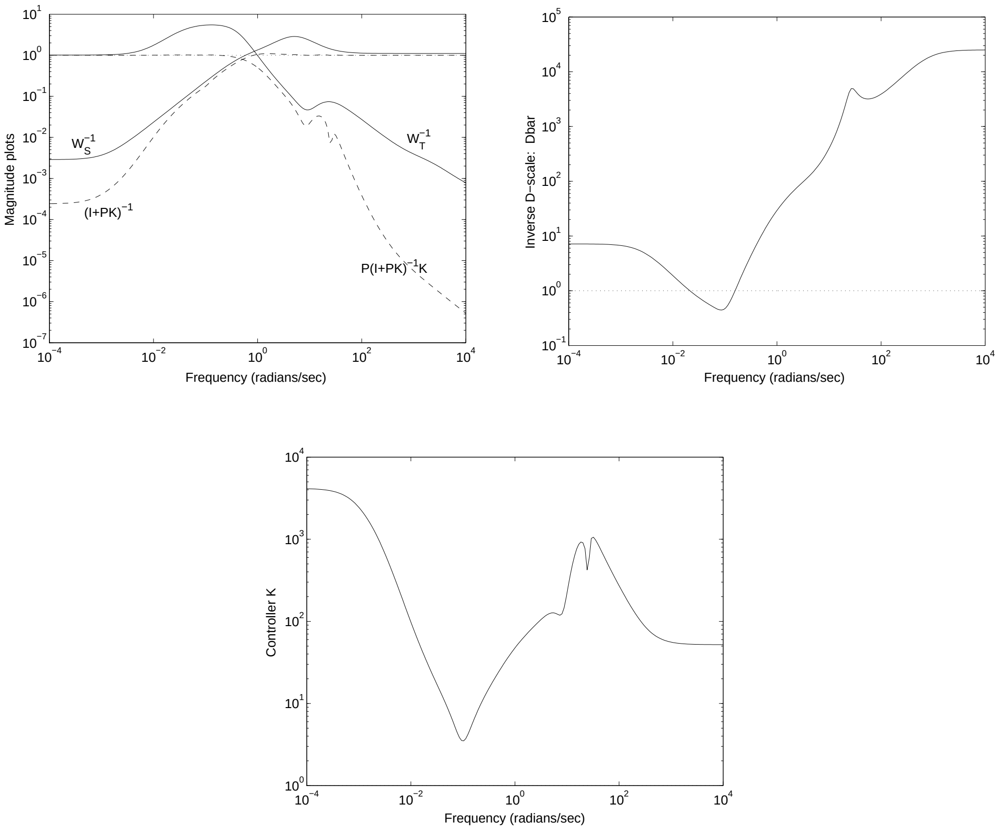

Fig. 5. Clockwise from top left: Magnitude plots of $W_S^{-1}(j\omega)$ and $W_T^{-1}(j\omega)$ in solid line and of $\frac{1}{1+PK}$ and $\frac{PK}{1+PK}$ in dashed line, Magnitude plot of $d^{-1}(j\omega)$, Magnitude plot of controller K

Also note that, as expected, the desired bandwidth of 1 `rad/s` was achieved, since the desired performance and uncertainty/plant-limitations were all compatible in this example. A controller achieving robust performance with respect to the maximised performance weights was simultaneously synthesised by the algorithm and is also shown in Figure 5.

5.1.2 Right half plane zero imposing bandwidth limitations

In this subsection, $\omega_1 = 0.1$ `rad/s` and $\omega_2 = 30$ `rad/s`, giving rise to the nominal plant P_o and uncertainty weight W_u shown in Figure 6. This particular choice of ω_2 imposes no bandwidth limitations, as the frequency region where uncertainty becomes significant lies far beyond the desired bandwidth of 1 `rad/s` (see Figure 3). However, the choice of ω_1 imposes a bandwidth limitation, since the right half plane zero of the plant now lies within the desired bandwidth. Consequently, the desired bandwidth may not be achievable. The proposed algorithm systematically resolves this by determining appropriate weights W_S and W_T, via the optimisation, to yield an appropriate closed-loop

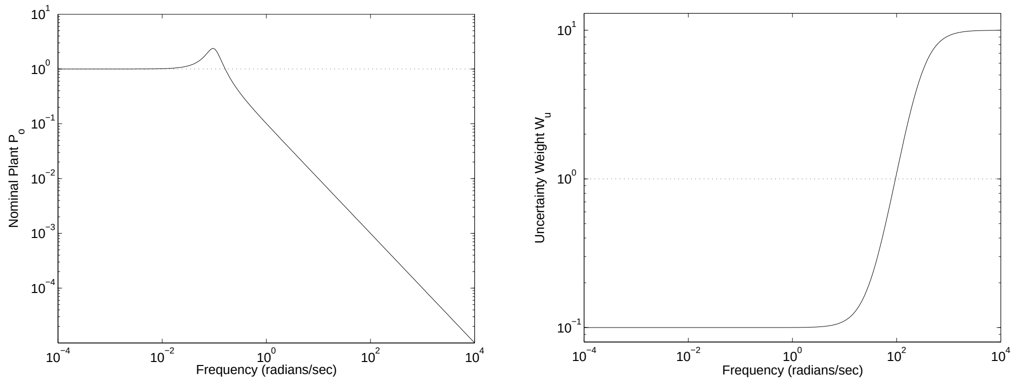

Fig. 6. Plant and uncertainty weight magnitude plots

bandwidth.

The algorithm took 8 iterations to converge. The cost $\eta_i^\star$ associated with the posed optimisation problem took values in the sequence {43290, 24890, 12923, 9811, 6479, 5251, 4649, 4531}, which is monotonically decreasing as expected. At each iteration, γ_i took values in the non-decreasing sequence {0.38, 0.45, 0.62, 0.69, 0.85, 0.93, 1, 1}, up to unity where the algorithm uses the exact state-space solutions, avoiding frequency data fitting and hence, the corresponding approximation errors. The results of the proposed algorithm are shown in Figure 7. Note that the weights W_S and W_T have been maximised in the appropriate frequency regions. Furthermore, the limitation arising due to the right half plane zero of the plant has been accounted for, with the achieved bandwidth being approximately 0.1 `rad/s`, despite the directionality factors reflecting a desired bandwidth of 1 `rad/s`. That is, the proposed algorithm has systematically traded-off desired performance against a fundamental limitation due to plant dynamics.

The magnitude plot of the sensitivity (resp. complementary sensitivity) function of any plant in the set $\left\{P_o(1+\Delta W_u) : \Delta \in \mathscr{RH}_\infty, \|\Delta\|_\infty \leq 1\right\}$ will be below the $|W_S(j\omega)|^{-1}$ (resp. $|W_T(j\omega)|^{-1}$) plot, since robust performance is guaranteed. A controller achieving robust performance with respect to the maximised performance weights was simultaneously synthesised by the algorithm and is also shown in Figure 7.

5.1.3 High frequency uncertainty imposing bandwidth limitations

In this subsection, $\omega_1 = 100$ `rad/s` and $\omega_2 = 0.03$ `rad/s`, giving rise to the nominal plant P_o and uncertainty weight W_u shown in Figure 8. This particular choice of ω_1 imposes no bandwidth limitations as the plant right half plane zero lies far beyond the desired bandwidth of 1 `rad/s` (see Figure 3). However, the choice of ω_2 imposes a bandwidth limitation as the frequency region where

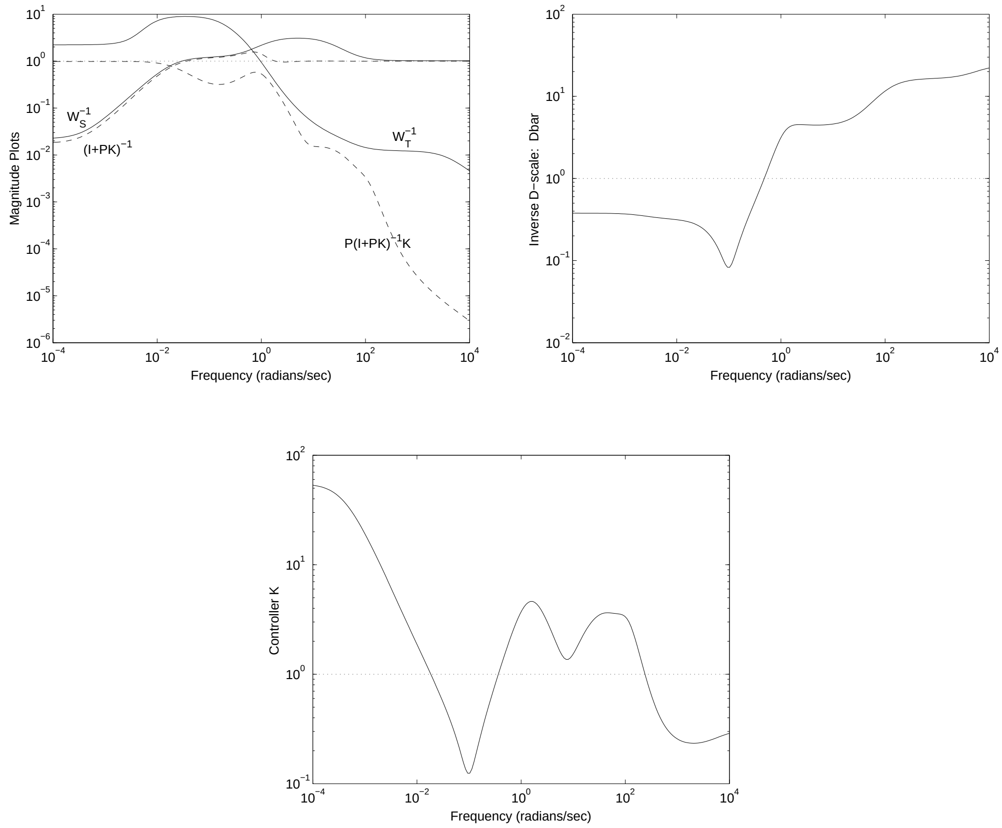

Fig. 7. Clockwise from top left: Magnitude plots of $W_S^{-1}(j\omega)$ and $W_T^{-1}(j\omega)$ in solid line and of $\frac{1}{1+PK}$ and $\frac{PK}{1+PK}$ in dashed line, Magnitude plot of $d^{-1}(j\omega)$, Magnitude plot of controller K

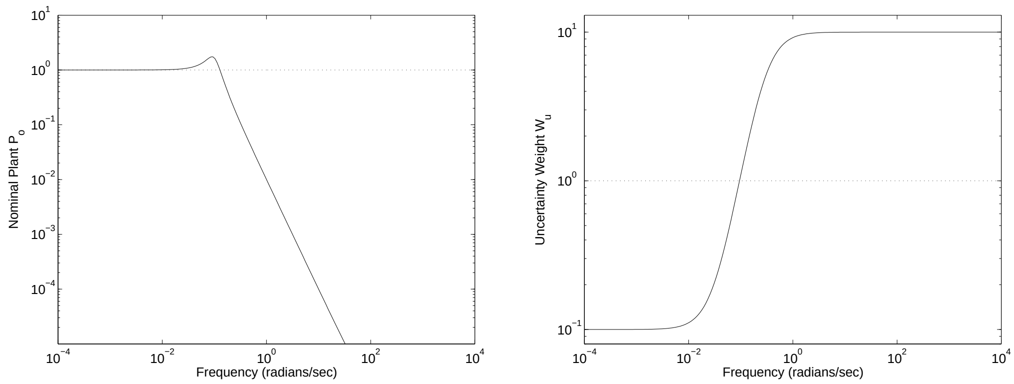

Fig. 8. Plant and uncertainty weight magnitude plots

uncertainty becomes significant now lies well within the desired bandwidth. Consequently, the desired bandwidth may not be achievable. This will be resolved by the algorithm by synthesising appropriate weights W_S and W_T (or rather their inverses) in order that a controller achieving robust performance with respect to these weights exist.

The algorithm took 5 iterations to converge. The cost $\eta_i^\star$ associated with the proposed optimisation problem took values in the sequence {444860, 104920, 9438, 4869, 4773}, which is monotonically decreasing as expected. At each iteration, γ_i took values in the non-decreasing sequence {0.35, 0.55, 0.82, 1, 1}, up to unity where the algorithm uses the exact state-space solutions, avoiding frequency data fitting and hence, the corresponding approximation errors. The results of the proposed algorithm are shown in Figure 9. Note that the weights

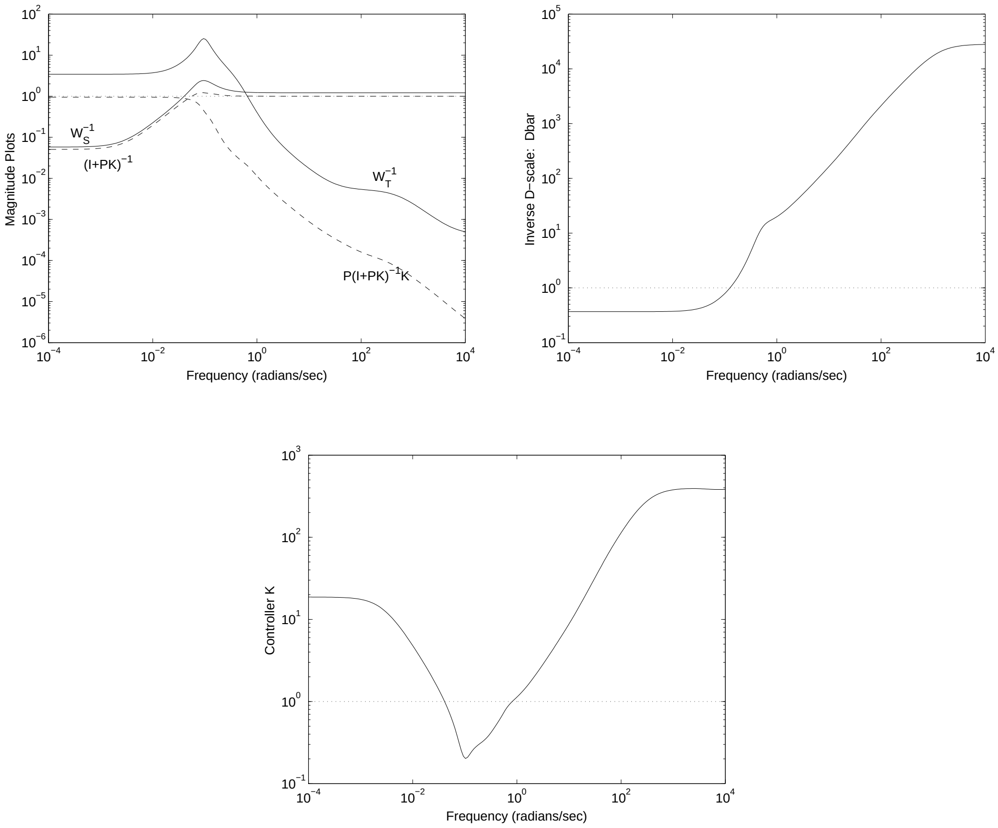

Fig. 9. Clockwise from top left: Magnitude plots of $W_S^{-1}(j\omega)$ and $W_T^{-1}(j\omega)$ in solid line and of $\frac{1}{1+PK}$ and $\frac{PK}{1+PK}$ in dashed line, Magnitude plot of $d^{-1}(j\omega)$, Magnitude plot of controller K

W_S and W_T have been maximised in the appropriate frequency regions. Furthermore, observe the trading-off of desired bandwidth (1 `rad/s`) against the specified plant uncertainty, as can be seen from the achieved bandwidth of approximately 0.1 `rad/s`.

The magnitude plot of the sensitivity (resp. complementary sensitivity) function of any plant in the set $\left\{P_o(1 + \Delta W_u) : \Delta \in \mathscr{RH}_\infty, \|\Delta\|_\infty \leq 1\right\}$ will be below the $|W_S(j\omega)|^{-1}$ (resp. $|W_T(j\omega)|^{-1}$) plot, since robust performance is guaranteed. A controller achieving robust performance with respect to the maximised performance weights was simultaneously synthesised by the algorithm and is also shown in Figure 9.

5.2 Pitch axis controller design for an experimental highly maneuverable aeroplane

An example from [1], used therein to illustrate the standard μ-synthesis approach to design, is now considered in order to facilitate comparison with the new approach to robust performance problems developed in this paper. The example involves the design of a pitch axis controller for an experimental highly maneuverable aeroplane, the HIMAT. A block diagram for the closed-loop system is shown in Figure 10. The state-space realisations of P_o, W_u and

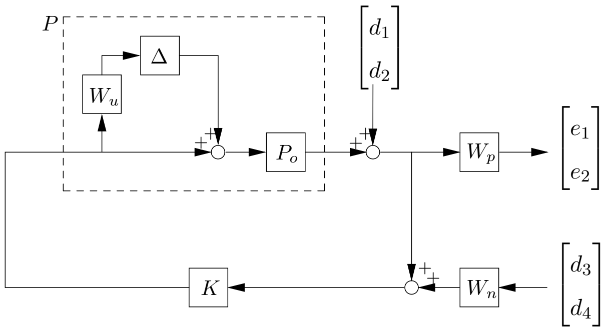

Fig. 10. Block diagram of HIMAT and required feedback structure

W_n can be found in [1]. The plant P is uncertain but known to belong to the set $\left\{P_o(I + \Delta W_u) : \Delta \in \mathscr{RH}_\infty, \|\Delta\|_\infty \leq 1\right\}$.

The objective in this design example is to maximise the performance weight W_p in appropriate frequency regions subject to the existence of an internally stabilising controller K guaranteeing robust performance with respect to this weight. In standard μ-synthesis based design W_p is fixed by the designer. By contrast, recall that the new approach proposed in this paper involves the synthesis of a suitable W_p via a constrained optimisation problem, with cost reflecting the desired performance through an appropriate directionality function. For a sensible control problem, W_p should be maximised in the low-frequency region, thereby yielding good disturbance rejection at the plant output at these frequencies. In order to capture this, the optimisation directionality $\Upsilon = \frac{5}{(s+0.005)} I_2$ is used here. This is consistent with the desired performance requirements specified in [1].

The results obtained by applying the proposed algorithm are shown in Figure 11, together with the results obtained via the design presented in [1], in order to facilitate comparison. Observe that unlike the μ-curve obtained in [1], the final μ-curve obtained via the new approach is flat across frequency and very close to unity. This reflects that robust performance has been optimised. In fact, it can be seen that the inverse performance weights synthesised by

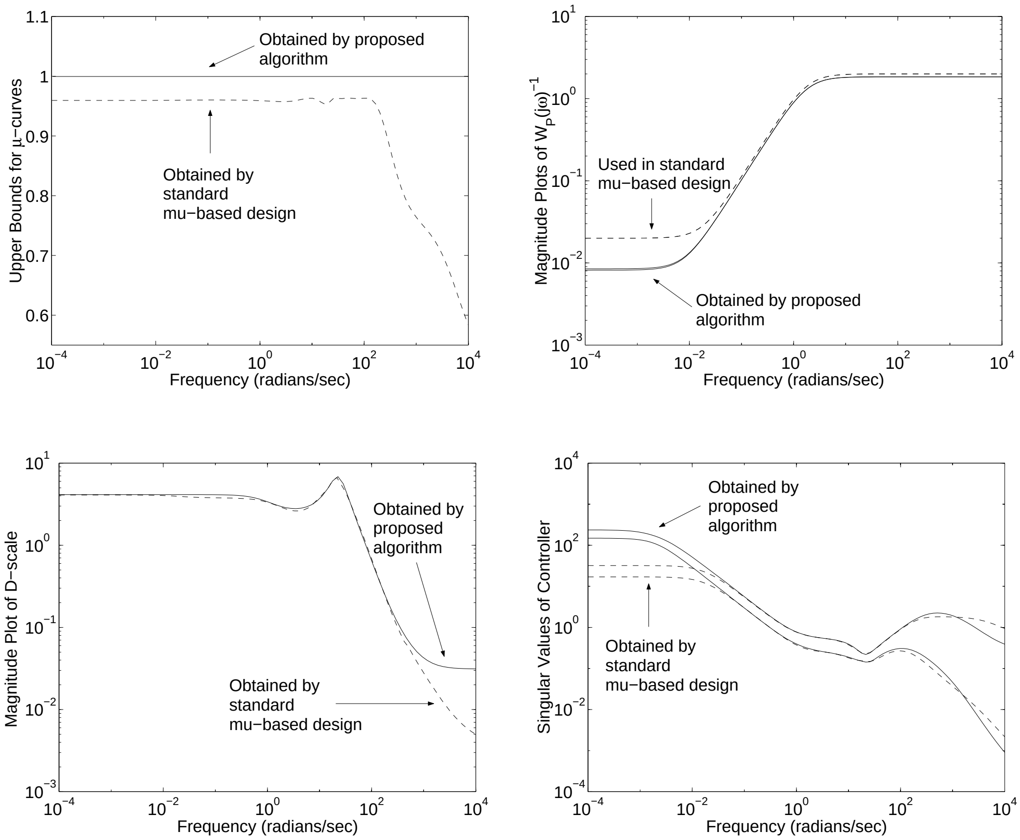

Fig. 11. Clockwise from top left: Upper bounds for μ-curves, Magnitude plots of $|w_{11}(j\omega)|^{-1}$ and $|w_{22}(j\omega)|^{-1}$, Magnitude plot of $|d(j\omega)|$, Singular values of controller K

the proposed algorithm are everywhere less than those used in [1]. That is, a higher level of robust performance is attained. The final controller synthesised by the proposed algorithm had 26 states and hence, was of the same order as the scaled generalised plant, as expected. The magnitude plot of the resulting controller is also shown in Figure 11.

6 Conclusions

A new approach to robust performance problems is developed in this paper. The rationale behind the new approach is related to that of using skewed-μ to determine worst-case performance in the face of prescribed uncertainty, in that it involves the optimisation of so-called performance weights, which scale over frequency the performance channels in a standard LFT setup for robust performance problems. The optimisation is constrained, in terms of the structured singular value, to ensure the existence of a stabilising feedback compensator that achieves robust performance with respect to the optimised performance weights and the specified uncertain plant set. Optimisation of the performance weights, with respect to a cost that reflects desired performance, in the way proposed here, gives rise to an algorithm for systematically trading-off the desired performance against specified plant uncertainty and performance limitations due to plant dynamics. In this sense, an indication of achievable performance is also provided.

As formulated, the optimisation problem admits a state-space solution in terms of LMIs. Further LMI constraints could be incorporated to capture closed-loop objectives in addition to the $\mathscr{H}_\infty$ performance criterion considered here, such as pole placement and $\mathscr{H}_2$-norm minimisation, for example.

Software has also been developed and can be requested via email from the first author.

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A Proof of Theorem 3

First note that

$$\Upsilon(j\omega)^*\bar{W}(j\omega)^*\bar{W}(j\omega)\Upsilon(j\omega) = \Upsilon(j\omega)^* T^o_{\bar{W}}(j\omega)^* \breve{W} T^o_{\bar{W}}(j\omega)\Upsilon(j\omega)$$
$$= \varphi(j\omega)^* \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & C_\Upsilon^T \end{bmatrix} \breve{W} \begin{bmatrix} I_{s_W} & 0 \\ 0 & C_\Upsilon \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & 0 \end{pmatrix} \varphi(j\omega),$$

where

$$\varphi(s) := \begin{bmatrix} \left(sI - \begin{bmatrix} A_{\bar{W}} & B_{\bar{W}}C_\Upsilon \\ 0 & A_\Upsilon \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 \\ B_\Upsilon \end{bmatrix} \\ I_n \end{bmatrix}.$$

Using the fact that parahermitian rational functions can be split into the sum of stable and anti-stable transfer functions [4,21,12], it follows that

$$\Upsilon(j\omega)^*\bar{W}(j\omega)^*\bar{W}(j\omega)\Upsilon(j\omega) = E(j\omega) + E(j\omega)^*$$

where

$$E(s) := \left[\begin{array}{cc|c} A_{\bar{W}} & B_{\bar{W}}C_\Upsilon & 0 \\ 0 & A_\Upsilon & B_\Upsilon \\ \hline \multicolumn{2}{c|}{\begin{pmatrix} 0 & B_\Upsilon^T \end{pmatrix} X} & 0 \end{array}\right]$$

and the real matrix $X = X^T$ is the unique solution to Lyapunov equation

$$X \begin{bmatrix} A_{\bar{W}} & B_{\bar{W}}C_\Upsilon \\ 0 & A_\Upsilon \end{bmatrix} + \begin{bmatrix} A_{\bar{W}} & B_{\bar{W}}C_\Upsilon \\ 0 & A_\Upsilon \end{bmatrix}^T X = - \begin{bmatrix} I_{s_W} & 0 \\ 0 & C_\Upsilon^T \end{bmatrix} \breve{W} \begin{bmatrix} I_{s_W} & 0 \\ 0 & C_\Upsilon \end{bmatrix}. \tag{A.1}$$

Consequently,

$$\left\|\bar{W}\Upsilon\right\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left\{\Upsilon(j\omega)^*\bar{W}(j\omega)^*\bar{W}(j\omega)\Upsilon(j\omega)\right\} d\omega$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \Re\left[\text{trace}\left\{E(j\omega)\right\}\right] d\omega$$
$$= \Re\left[\text{trace}\left\{\begin{bmatrix} 0 & B_\Upsilon^T \end{bmatrix} X \,.\, \frac{1}{\pi}\int_{-\infty}^{\infty} \left(j\omega I - \begin{bmatrix} A_{\bar{W}} & B_{\bar{W}}C_\Upsilon \\ 0 & A_\Upsilon \end{bmatrix}\right)^{-1} d\omega \,.\, \begin{bmatrix} 0 \\ B_\Upsilon \end{bmatrix}\right\}\right]$$

$$= \operatorname{trace}\left\{\begin{bmatrix}0 & B_\Upsilon^T\end{bmatrix} X \begin{bmatrix}0 \\ B_\Upsilon\end{bmatrix}\right\} \text{ by a standard calculus result}$$

$$= \operatorname{vec}(I_n)^T \left(\begin{bmatrix}0 \\ B_\Upsilon\end{bmatrix} \otimes \begin{bmatrix}0 \\ B_\Upsilon\end{bmatrix}\right)^T \operatorname{vec}(X). \tag{A.2}$$

Since the matrix $\begin{bmatrix}A_{\bar{W}} & B_{\bar{W}} C_\Upsilon \\ 0 & A_\Upsilon\end{bmatrix}$ is Hurwitz, Lyapunov equation (A.1) can be rewritten as

$$\operatorname{vec}(X) = -\left(\begin{bmatrix}A_{\bar{W}} & B_{\bar{W}} C_\Upsilon \\ 0 & A_\Upsilon\end{bmatrix} \oplus \begin{bmatrix}A_{\bar{W}} & B_{\bar{W}} C_\Upsilon \\ 0 & A_\Upsilon\end{bmatrix}\right)^{-T} \times \left(\begin{bmatrix}I_{s_W} & 0 \\ 0 & C_\Upsilon\end{bmatrix} \otimes \begin{bmatrix}I_{s_W} & 0 \\ 0 & C_\Upsilon\end{bmatrix}\right)^T \operatorname{vec}\left(\breve{W}\right). \tag{A.3}$$

The required result then follows directly from equations (A.2) and (A.3).

B Proof of Theorem 4

Statements (i) and (ii) in the Theorem will be connected by a sequence of equivalent reformulations.

(a) $\inf_{\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}}, B_{\bar{D}})}} \left\| \begin{pmatrix}\bar{D} & 0 \\ 0 & I_m\end{pmatrix} \mathcal{F}_l\left(G, K\right)^T \begin{pmatrix}\bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1}\end{pmatrix} \right\|_\infty < 1.$

(b) $\exists\, \bar{D} \in \boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}}, B_{\bar{D}})}$ such that

$$\left[\mathcal{F}_l\left(G(j\omega), K(j\omega)\right)^T\right]^* \begin{pmatrix}\bar{D}(j\omega)^* \bar{D}(j\omega) & 0 \\ 0 & I_m\end{pmatrix} \left[\mathcal{F}_l\left(G(j\omega), K(j\omega)\right)^T\right] < \begin{pmatrix}\bar{D}(j\omega)^* \bar{D}(j\omega) & 0 \\ 0 & \bar{W}(j\omega)^* \bar{W}(j\omega)\end{pmatrix}$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$.

(c) $\exists\, \breve{D} \in \mathbf{\Xi}_{\breve{D}}$ such that

$$T^o_{\bar{D}}(j\omega)^* \breve{D} T^o_{\bar{D}}(j\omega) > 0$$
$$\text{and } \varphi(j\omega)^* \grave{Q}\, \varphi(j\omega) < 0$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$, where $T^o_{\bar{D}}(s)$ is defined as in equation (3) and

$$\varphi(s) := \begin{pmatrix} \begin{bmatrix} T^o_{\bar{D}}(s) & 0 \\ 0 & I_m \end{bmatrix} \mathcal{F}_l\left(G(s), K(s)\right)^T \\ \\ \begin{bmatrix} T^o_{\bar{D}}(s) & 0 \\ 0 & T^o_{\bar{W}}(s) \end{bmatrix} \end{pmatrix},$$

$$\grave{Q} := \mathrm{diag}\left(\breve{D},\, I_m,\, -\breve{D},\, -\breve{W}\right).$$

Observe that (b) $\Leftrightarrow$ (c) follows from simple algebraic manipulations after replacing $\bar{D}(j\omega)^*\bar{D}(j\omega)$ with $T^o_{\bar{D}}(j\omega)^*\breve{D}T^o_{\bar{D}}(j\omega)$ and $\bar{W}(j\omega)^*\bar{W}(j\omega)$ with $T^o_{\bar{W}}(j\omega)^*\breve{W}T^o_{\bar{W}}(j\omega)$.

(d) $\exists\, \breve{D} \in \mathbf{\Xi}_{\breve{D}}$ such that

$$\begin{bmatrix} \left(j\omega I - A_{\bar{D}}\right)^{-1} B_{\bar{D}} \\ I_r \end{bmatrix}^* \breve{D} \begin{bmatrix} \left(j\omega I - A_{\bar{D}}\right)^{-1} B_{\bar{D}} \\ I_r \end{bmatrix} > 0$$

$$\text{and } \begin{bmatrix} \left(j\omega I - \grave{A}\right)^{-1} \grave{B} \\ I \end{bmatrix}^* \begin{bmatrix} \grave{C}^T \\ \grave{D}^T \end{bmatrix} \grave{Q} \begin{bmatrix} \grave{C} & \grave{D} \end{bmatrix} \begin{bmatrix} \left(j\omega I - \grave{A}\right)^{-1} \grave{B} \\ I \end{bmatrix} < 0$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$, where $\grave{A}$, $\grave{B}$, $\grave{C}$, $\grave{D}$ and $\grave{Q}$ are defined as in Part (ii) of the Theorem. Then (c) $\Leftrightarrow$ (d) easily follows by noting that $\varphi(s) = \left[\begin{array}{c|c} \grave{A} & \grave{B} \\ \hline \grave{C} & \grave{D} \end{array}\right]$.

(e) $\exists\, \breve{D} \in \mathbf{\Xi}_{\breve{D}}$, $X = X^T \in \mathbb{R}^{s_D \times s_D}$ and $Y = Y^T \in \mathbb{R}^{(s_{cl}+2s_D+s_W)\times(s_{cl}+2s_D+s_W)}$ such that

$$\begin{bmatrix} XA_{\bar{D}} + A^T_{\bar{D}}X & XB_{\bar{D}} \\ B^T_{\bar{D}}X & 0 \end{bmatrix} + \breve{D} > 0,$$

$$\begin{bmatrix} Y\grave{A} + \grave{A}^TY & Y\grave{B} \\ \grave{B}^TY & 0 \end{bmatrix} + \begin{bmatrix} \grave{C}^T \\ \grave{D}^T \end{bmatrix} \grave{Q} \begin{bmatrix} \grave{C} & \grave{D} \end{bmatrix} < 0.$$

The equivalence (d) $\Leftrightarrow$ (e) follows from the KYP Lemma [20,15].

C Proof of Theorem 5

Before proving the equivalence between the Theorem's two statements, some notation needs to be defined. To this end, let the controller $K(s) \in \boldsymbol{\mathcal{K}}_G^{TF}$ have a state-space realisation $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$, where $A_K \in \mathbb{R}^{s_K \times s_K}$ and $D_K \in \mathbb{R}^{p \times q}$. The order s_K of this controller is not yet known (i.e. s_K is a variable), as the set $\boldsymbol{\mathcal{K}}_G^{TF}$ contains all internally stabilising controllers for G of *any* order. Define

$$\begin{bmatrix} \hat{A} & \hat{B}_1 & \hat{B}_2 & \hat{B}_3 \\ \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} & \hat{D}_{13} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} & \hat{D}_{23} \\ \hat{C}_3 & \hat{D}_{31} & \hat{D}_{32} & \Phi_K^T \end{bmatrix} := \left[\begin{array}{cc:c:c:cc} \tilde{A} & 0 & \tilde{B}_1 & \tilde{B}_2 & 0 & \tilde{B}_3 \\ 0 & 0 & 0 & 0 & I_{s_K} & 0 \\ \hdashline \tilde{C}_1 & 0 & \tilde{D}_{11} & \tilde{D}_{12} & 0 & \tilde{D}_{13} \\ \hdashline \tilde{C}_2 & 0 & \tilde{D}_{21} & \tilde{D}_{22} & 0 & \tilde{D}_{23} \\ \hdashline 0 & I_{s_K} & 0 & 0 & A_K^T & C_K^T \\ \tilde{C}_3 & 0 & \tilde{D}_{31} & \tilde{D}_{32} & B_K^T & D_K^T \end{array}\right],$$

$$\text{and} \quad \begin{bmatrix} \tilde{A}_{cl} & \tilde{B}_{1cl} & \tilde{B}_{2\,cl} \\ \tilde{C}_{1cl} & \tilde{D}_{11cl} & \tilde{D}_{12\,cl} \\ \tilde{C}_{2\,cl} & \tilde{D}_{21cl} & \tilde{D}_{22\,cl} \end{bmatrix} := \begin{bmatrix} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{bmatrix} + \begin{bmatrix} \hat{B}_3 \\ \hat{D}_{13} \\ \hat{D}_{23} \end{bmatrix} \Phi_K \begin{bmatrix} \hat{C}_3 & \hat{D}_{31} & \hat{D}_{32} \end{bmatrix}.$$

Then $\mathcal{F}_l\left(\tilde{G}, K\right) = \left[\begin{array}{c|cc} \tilde{A}_{cl} & \tilde{B}_{1cl} & \tilde{B}_{2\,cl} \\ \hline \tilde{C}_{1cl} & \tilde{D}_{11cl} & \tilde{D}_{12\,cl} \\ \tilde{C}_{2\,cl} & \tilde{D}_{21cl} & \tilde{D}_{22\,cl} \end{array}\right]$ and $K \in \boldsymbol{\mathcal{K}}_G^{TF}$ if and only if $K \in \boldsymbol{\mathcal{K}}_{\tilde{G}}^{TF}$.

Statements (i) and (ii) in the Theorem will now be connected by a sequence of equivalent reformulations.

(a) $\min_{K \in \boldsymbol{\mathcal{K}}_G^{TF}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\left(G, K\right)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1.$

(b) $\exists\, K \in \boldsymbol{\mathcal{K}}^{TF}_{\tilde{G}}$ such that

$$\left[\mathcal{F}_l\Big(\tilde{G}(j\omega), K(j\omega)\Big)^T\right]^* \left[\mathcal{F}_l\Big(\tilde{G}(j\omega), K(j\omega)\Big)^T\right] < \begin{pmatrix} I_r & 0 \\ 0 & \bar{W}(j\omega)^*\bar{W}(j\omega) \end{pmatrix} \quad \forall \omega \in \mathbb{R}\cup\{\infty\}.$$

This easily follows by using the definition of $\tilde{G}(s)$.

(c) $\exists\, K \in \boldsymbol{\mathcal{K}}^{TF}_{\tilde{G}}$ such that

$$\begin{pmatrix} \mathcal{F}_l\Big(\tilde{G}(j\omega), K(j\omega)\Big)^T \\ \begin{bmatrix} I_r & 0 \\ 0 & T^o_{\breve{W}}(j\omega) \end{bmatrix} \end{pmatrix}^* \begin{pmatrix} I_{r+m} & 0 \\ 0 & \begin{bmatrix} -I_r & 0 \\ 0 & -\breve{W} \end{bmatrix} \end{pmatrix} \begin{pmatrix} \mathcal{F}_l\Big(\tilde{G}(j\omega), K(j\omega)\Big)^T \\ \begin{bmatrix} I_r & 0 \\ 0 & T^o_{\breve{W}}(j\omega) \end{bmatrix} \end{pmatrix} < 0$$

for all $\omega \in \mathbb{R}\cup\{\infty\}$. Observe that (b) $\Leftrightarrow$ (c) follows from simple algebraic manipulations after replacing $\bar{W}(j\omega)^*\bar{W}(j\omega)$ with $T^o_{\breve{W}}(j\omega)^*\breve{W}T^o_{\breve{W}}(j\omega)$.

(d) $\exists\, s_{_K} \in \mathbb{Z}_+$ and $\Phi_K \in \mathbb{R}^{(s_{_K}+p)\times(s_{_K}+q)}$ such that

$$\tilde{A}_{cl} \text{ is Hurwitz,}$$

$$\begin{bmatrix} \Big(j\omega I - \acute{A}\Big)^{-1}\acute{B} \\ I \end{bmatrix}^* \begin{bmatrix} \acute{C}^T \\ \acute{D}^T \end{bmatrix} \acute{Q} \begin{bmatrix} \acute{C} & \acute{D} \end{bmatrix} \begin{bmatrix} \Big(j\omega I - \acute{A}\Big)^{-1}\acute{B} \\ I \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R}\cup\{\infty\},$$

where $\acute{A}$, $\acute{B}$, $\acute{C}$, $\acute{D}$ and $\acute{Q}$ are defined by

$$\begin{bmatrix} \acute{A} & \acute{B} \\ \acute{C} & \acute{D} \end{bmatrix} := \left[\begin{array}{cc|cc} A_{\bar{W}} & 0 & 0 & B_{\bar{W}} \\ 0 & \tilde{A}^T_{cl} & \tilde{C}^T_{1cl} & \tilde{C}^T_{2\,cl} \\ \hline 0 & \tilde{B}^T_{1cl} & \tilde{D}^T_{11cl} & \tilde{D}^T_{21cl} \\ 0 & \tilde{B}^T_{2\,cl} & \tilde{D}^T_{12\,cl} & \tilde{D}^T_{22\,cl} \\ 0 & 0 & I_r & 0 \\ I_{s_W} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \end{array}\right],$$

$$\acute{Q} := \operatorname{diag}\left(I_r,\, I_m,\, -I_r,\, -\begin{bmatrix} 0 & \breve{W}_{12} \\ \breve{W}^T_{12} & \breve{W}_{22} \end{bmatrix}\right).$$

The equivalence (c) $\Leftrightarrow$ (d) follows by noting that $K(s)$ internally stabilises $\tilde{G}(s)$ if $\tilde{A}_{cl}$ is Hurwitz, and $\tilde{A}_{cl}$ is Hurwitz if there is a stabilisable and detectable state-space realisation for $K(s)$ which internally stabilises $\tilde{G}(s)$ [7, Lemma A.4.1].

(e) $\exists\, s_K \in \mathbb{Z}_+$, $\Phi_K \in \mathbb{R}^{(s_K+p)\times(s_K+q)}$ and $X = X^T \in \mathbb{R}^{(s_W+s_{\tilde{G}}+s_K)\times(s_W+s_{\tilde{G}}+s_K)}$ such that

$$X > 0,$$

$$\begin{bmatrix} X\acute{A} + \acute{A}^T X & X\acute{B} \\ \acute{B}^T X & 0 \end{bmatrix} + \begin{bmatrix} \acute{C}^T \\ \acute{D}^T \end{bmatrix} \acute{Q} \begin{bmatrix} \acute{C} & \acute{D} \end{bmatrix} < 0, \tag{C.1}$$

where $\acute{A}$, $\acute{B}$, $\acute{C}$, $\acute{D}$ and $\acute{Q}$ are defined as in (d) above. Since inequality (C.1) implicitly guarantees that $X\acute{A} + \acute{A}^T X < -\acute{C}^T\acute{Q}\,\acute{C} \leq 0$, a standard Lyapunov type argument gives $X > 0$ if and only if $\acute{A}$ is Hurwitz. Now $\acute{A}$ is Hurwitz if and only if $\tilde{A}_{cl}$ is Hurwitz, as $A_{\bar{W}}$ is already restricted to be Hurwitz in the set $\Xi_{(A_{\bar{W}},B_{\bar{W}})}$. Then (d) $\Leftrightarrow$ (e) follows from a straightforward application of the KYP Lemma [20,15].

(f) $\exists\, s_K \in \mathbb{Z}_+$, $\Phi_K \in \mathbb{R}^{(s_K+p)\times(s_K+q)}$ and $X = X^T \in \mathbb{R}^{(s_W+s_{\tilde{G}}+s_K)\times(s_W+s_{\tilde{G}}+s_K)}$ such that

$$X > 0,$$

$$\begin{pmatrix} X\begin{bmatrix} A_{\bar{W}} & 0 \\ 0 & \tilde{A}_{cl}^T \end{bmatrix} + \{\cdot\}^T & X\begin{bmatrix} 0 & B_{\bar{W}} \\ \tilde{C}_{1cl}^T & \tilde{C}_{2\,cl}^T \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ \tilde{B}_{1cl} & \tilde{B}_{2\,cl} \end{bmatrix} \\ * & \begin{bmatrix} -I_r & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11cl} & \tilde{D}_{12\,cl} \\ \tilde{D}_{21cl} & \tilde{D}_{22\,cl} \end{bmatrix} \\ * & * & \begin{bmatrix} -I_r & 0 \\ 0 & -I_m \end{bmatrix} \end{pmatrix}$$

$$< \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix} \breve{W} \left(\begin{bmatrix} I_{s_W} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right).$$

Note that (e) $\Leftrightarrow$ (f) follows by applying Schur Complement Lemma [9] around the (3,3)-block of the above inequality and then re-arranging to give inequality (C.1).

(g) $\exists\, s_{_K} \in \mathbb{Z}_+$, $\Phi_K \in \mathbb{R}^{(s_{_K}+p)\times(s_{_K}+q)}$ and $X = X^T \in \mathbb{R}^{(s_{_W}+s_{_{\tilde G}}+s_{_K})\times(s_{_W}+s_{_{\tilde G}}+s_{_K})}$ such that

$$X > 0,$$
$$F + U^T \Phi_K V + V^T \Phi_K^T U < 0,$$

where the real matrices F, U and V are defined as in Corollary 6. This equivalence follows by simply extracting Φ_K from the closed-loop matrices.

(h) $\exists\, s_{_K} \in \mathbb{Z}_+$ and $X = X^T \in \mathbb{R}^{(s_{_W}+s_{_{\tilde G}}+s_{_K})\times(s_{_W}+s_{_{\tilde G}}+s_{_K})}$ such that

$$X > 0,$$
$$\Psi_U^T F \Psi_U < 0 \quad \text{and} \quad \Psi_V^T F \Psi_V < 0,$$

where F is defined as in (g) and Ψ_U and Ψ_V are matrices with columns that form bases for the null spaces of U and V respectively. One possible choice of Ψ_U and Ψ_V is

$$\Psi_U := \begin{pmatrix} \begin{bmatrix} I_{s_{_W}} & 0 \\ 0 & \psi_1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \psi_2 \\ 0 & \psi_3 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I_r & 0 \\ 0 & I_m \end{bmatrix} \end{pmatrix}$$

and

$$\Psi_V := \begin{pmatrix} X^{-1} & 0 & 0 \\ 0 & I_{r+n} & 0 \\ 0 & 0 & I_{r+m} \end{pmatrix} \begin{pmatrix} \begin{bmatrix} I_{s_{_W}} & 0 \\ 0 & \psi_4 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I_r & 0 \\ 0 & I_n \end{bmatrix} \\ \begin{bmatrix} 0 & \psi_5 \\ 0 & \psi_6 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix}.$$

Here, the columns of $\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix}$ (resp. $\begin{bmatrix}\psi_4\\ \psi_5\\ \psi_6\end{bmatrix}$) form bases for the null space of $\begin{bmatrix}\tilde{B}_3^T & \tilde{D}_{13}^T & \tilde{D}_{23}^T\end{bmatrix}$ (resp. $\begin{bmatrix}\tilde{C}_3 & \tilde{D}_{31} & \tilde{D}_{32}\end{bmatrix}$). The equivalence (g) $\Leftrightarrow$ (h) follows from the Projection Lemma [5,2].

(i) $\exists\, s_K \in \mathbb{Z}_+$, $P = P^T \in \mathbb{R}^{(s_W+s_{\tilde{G}})\times(s_W+s_{\tilde{G}})}$ and $Q = Q^T \in \mathbb{R}^{(s_W+s_{\tilde{G}})\times(s_W+s_{\tilde{G}})}$ such that

$$P > 0, \quad Q > 0,$$

$$\begin{bmatrix}P & I\\ I & Q\end{bmatrix} \geq 0, \quad \operatorname{rank}\left(P - Q^{-1}\right) \leq s_K,$$

$$\Psi_P^T \cdot \begin{pmatrix} P\begin{bmatrix}A_{\bar{W}} & 0\\ 0 & \tilde{A}^T\end{bmatrix} + \{\cdot\}^T & P\begin{bmatrix}0 & B_{\bar{W}}\\ \tilde{C}_1^T & \tilde{C}_2^T\end{bmatrix} & \begin{bmatrix}0 & 0\\ \tilde{B}_1 & \tilde{B}_2\end{bmatrix}\\ * & \begin{bmatrix}-I_r & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}\tilde{D}_{11} & \tilde{D}_{12}\\ \tilde{D}_{21} & \tilde{D}_{22}\end{bmatrix}\\ * & * & \begin{bmatrix}-I_r & 0\\ 0 & -I_m\end{bmatrix}\end{pmatrix} \cdot \Psi_P$$

$$< \begin{pmatrix}\begin{bmatrix}I_{s_W} & 0\\ 0 & \psi_3^T\end{bmatrix}\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\end{pmatrix} \breve{W} \left(\begin{bmatrix}I_{s_W} & 0\\ 0 & \psi_3\end{bmatrix} \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right),$$

$$\Psi_Q^T \cdot \begin{pmatrix} \begin{bmatrix} A_{\bar{W}} & 0 \\ 0 & \tilde{A}^T \end{bmatrix} Q + \{\cdot\}^T & Q \begin{bmatrix} 0 & 0 \\ \tilde{B}_1 & \tilde{B}_2 \end{bmatrix} & \begin{bmatrix} 0 & B_{\bar{W}} \\ \tilde{C}_1^T & \tilde{C}_2^T \end{bmatrix} \\ * & \begin{bmatrix} -I_r & 0 \\ 0 & -I_m \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11}^T & \tilde{D}_{21}^T \\ \tilde{D}_{12}^T & \tilde{D}_{22}^T \end{bmatrix} \\ * & * & \begin{bmatrix} -I_r & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix} \cdot \Psi_Q$$

$$< \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & \psi_4^T \end{bmatrix} Q \begin{bmatrix} I_{s_W} & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \end{pmatrix} \breve{W} \left(\begin{bmatrix} I_{s_W} & 0 \\ 0 & 0 \end{bmatrix} Q \begin{bmatrix} I_{s_W} & 0 \\ 0 & \psi_4 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \right), \quad \text{(C.2)}$$

where Ψ_P and Ψ_Q are defined as in Part (ii) of the Theorem. Note that (h) $\Leftrightarrow$ (i) follows after some algebra by exploiting the all-zero rows/columns and through the application of the Decoupling Lemma [13, Lemma 6.2].

(j) $\exists\, s_K \in \mathbb{Z}_+$, $P = P^T \in \mathbb{R}^{(s_W+s_{\tilde{G}})\times(s_W+s_{\tilde{G}})}$, $R = R^T \in \mathbb{R}^{s_W \times s_W}$, $S \in \mathbb{R}^{s_W \times s_{\tilde{G}}}$ and $T = T^T \in \mathbb{R}^{s_{\tilde{G}} \times s_{\tilde{G}}}$ such that the LMI constraints given in Part (ii) of Theorem 5 hold together with

$$\operatorname{rank}\left(\begin{bmatrix} I_{s_W} & S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} P \begin{bmatrix} I_{s_W} & 0 \\ S^T & I_{s_{\tilde{G}}} \end{bmatrix} - \begin{bmatrix} R & 0 \\ 0 & T^{-1} \end{bmatrix} \right) \le s_K.$$

The equivalence (i) $\Leftrightarrow$ (j) follows by observing the following four facts:

- Any $Q = Q^T \in \mathbb{R}^{(s_W+s_{\tilde{G}})\times(s_W+s_{\tilde{G}})}$ satisfying $Q > 0$ can be decomposed as follows:

$$Q = \begin{bmatrix} I_{s_W} & 0 \\ S^T & I_{s_{\tilde{G}}} \end{bmatrix} \begin{bmatrix} R^{-1} & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} I_{s_W} & S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix},$$

  where $R = R^T \in \mathbb{R}^{s_W \times s_W}$, $S \in \mathbb{R}^{s_W \times s_{\tilde{G}}}$ and $T = T^T \in \mathbb{R}^{s_{\tilde{G}} \times s_{\tilde{G}}}$ are *completely independent* variables which fully parametrise Q. Furthermore, $Q > 0$ if and only if $R > 0$ and $T > 0$.

- Using such a decomposition for Q,

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad \begin{pmatrix} P & \begin{bmatrix} I_{s_W} & -S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} \\ \begin{bmatrix} I_{s_W} & 0 \\ -S^T & I_{s_{\tilde{G}}} \end{bmatrix} & \begin{bmatrix} R^{-1} & 0 \\ 0 & T \end{bmatrix} \end{pmatrix} \geq 0$$

$$\Leftrightarrow \quad \begin{pmatrix} P & \begin{bmatrix} R & -S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} \\ \begin{bmatrix} R & 0 \\ -S^T & I_{s_{\tilde{G}}} \end{bmatrix} & \begin{bmatrix} R & 0 \\ 0 & T \end{bmatrix} \end{pmatrix} \geq 0.$$

- Again, using the above decomposition for Q,

$$\begin{aligned} &\operatorname{rank}\left(P - Q^{-1}\right) \\ &= \operatorname{rank}\left(P - \begin{bmatrix} I_{s_W} & -S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} I_{s_W} & 0 \\ -S^T & I_{s_{\tilde{G}}} \end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix} I_{s_W} & S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} P \begin{bmatrix} I_{s_W} & 0 \\ S^T & I_{s_{\tilde{G}}} \end{bmatrix} - \begin{bmatrix} R & 0 \\ 0 & T^{-1} \end{bmatrix}\right). \end{aligned}$$

- Finally, the equivalence between inequality (C.2) and the last LMI constraint in Part (ii) of Theorem 5 is obtained through the application of the congruence transformation

$$\begin{pmatrix} \begin{bmatrix} R & -S\psi_4 \\ 0 & I \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} I_r & 0 \\ 0 & I_n \end{bmatrix} \end{pmatrix}$$

on inequality (C.2) and some algebra to rearrange the result.

(k) The proof is completed by noting that for any $s_K \geq (s_W + s_{\tilde{G}})$, the 'rank' constraint in (j) is redundant and the remaining conditions are exactly those stated in Part (ii) of the Theorem. The controller order s_K can be chosen as desired since the set $\boldsymbol{\mathcal{K}}_G^{TF}$ contains all internally stabilising controllers for G of *any* order (as stated at the beginning of the proof).

D Proof of Corollary 6

The additional 'rank' condition for the existence of internally stabilising controllers of order s_K directly follows from Step (j) in the proof of Theorem 5 given in Appendix C, whereas the LMI for the construction of such controllers follows from Step (g) of the same proof on noting that it is always possible to construct an $X = X^T \in \mathbb{R}^{(s_W+s_{\tilde{G}}+s_K)\times(s_W+s_{\tilde{G}}+s_K)}$ satisfying

$$X = \begin{bmatrix} P & \blacklozenge \\ \blacklozenge & \blacklozenge \end{bmatrix} > 0 \qquad \text{and} \qquad X^{-1} = \begin{bmatrix} Q & \blacklozenge \\ \blacklozenge & \blacklozenge \end{bmatrix},$$

where $\blacklozenge$ denotes "Don't Care" elements. This is because given

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} > 0,$$

it is easy to verify that

$$X^{-1} = \begin{bmatrix} (X_{11} - X_{12}X_{22}^{-1}X_{12}^T)^{-1} & \blacklozenge \\ \blacklozenge & \blacklozenge \end{bmatrix}.$$

Consequently, if $X_{11} = P$ and $(X_{11} - X_{12}X_{22}^{-1}X_{12}^T)^{-1} = Q$, then one possible way of constructing the matrix X using the given P and Q is by letting $X_{22} = I$ and computing X_{12} from the following factorisation $P - Q^{-1} = X_{12}X_{12}^T$.