Algebraic Characterisation of the H^∞ and H^2 Norms for Linear Continuous-Time Periodic Systems*

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Abstract

It is well-known that linear, periodically time-varying, continuous-time systems are formally equivalent to so-called lifted representations that are shift-invariant, but have spatially infinite-dimensional inputs and outputs. By shift invariance, corresponding frequency-domain representations can be constructed. Indeed, it makes sense to use the H^∞ norm of the associated frequency-domain symbol as a measure of system size. In fact, this is equal to the $L^2_{[0,\infty)}$ induced norm of the system. As an alternative measure of system size a generalisation of the H^2 norm, which characterises the impulsive response of the system, can also be defined. The purpose of this paper is to establish *finite-dimensional, algebraic* characterisations of these norms, for linear continuous-time periodic systems.

1 Introduction

The H^∞ and H^2 norms are widely used in control and signal processing as measures of performance and robustness [1]. In this paper finite-dimensional, *algebraic* characterisations of these norms are developed for linear, continuous-time, periodically time-varying, state-space, system models. Such models arise in the study and design of many systems, such as asymmetric rotating machinery [2], systems subject to periodic loading [3, 4], and systems of orbiting bodies [5, 6].

The time-lifting technique, introduced by [7, 8, 9] within the context of continuous-time systems, is central to the development of the main results of this paper. It is used here to convert the time-varying, continuous-time norm computation problems considered into *shift-invariant*, discrete-time problems. Standard discrete-time techniques can then be used to obtain the algebraic characterisations required. The idea of lifting systems in this way to obtain shift-invariant equivalent formulations of certain problems can actually be traced back some forty-five years to the work of Kranc [10], in which the context was purely discrete-time.

Algebraic characterisation of the H^∞ and H^2 norm for periodic systems is also considered in [11]. Their approach involves skewed truncation of a lifted frequency response operator and yields approximate characterisations that improve with order of truncation. The characterisations here, on the other hand, are exact.

2 Preliminaries

In this section some notation and elementary results are collected. The integers, reals and complex numbers are denoted by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. The non-negative integers, non-negative reals, the unit circle and the open unit disc in $\mathbb{C}$ are respectively denoted by $\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{T}$ and $\mathbb{D}$.

Throughout, several function spaces are used as signal spaces. Given $T \in (0,\infty]$, the sym-

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bol $L^2_{[0,T)}(\mathbb{R}^m)$ denotes the set of square-integrable functions $f : [0,T) \to \mathbb{R}^m$ with inner product

$$\langle x, y\rangle_{L^2_{[0,T)}(\mathbb{R}^m)} := \int_0^T \langle x(t), y(t)\rangle_{\mathbb{R}^m}\, dt.$$

The set of square-summable, infinite sequences in a Hilbert space E is denoted by $\ell^2(E)$ and the inner product on $\ell^2(E)$ is defined by

$$\langle x, y\rangle_{\ell^2(E)} := \textstyle\sum_{k\in\mathbb{Z}_+} \langle x_k, y_k\rangle_E.$$

The relationship between $L^2_{\mathbb{R}_+}(\mathbb{R}^m)$ and $\ell^2(L^2_{[0,h)}(\mathbb{R}^m))$ via the time-lifting isomorphism $\mathbf{W} : L^2_{\mathbb{R}_+}(\mathbb{R}^m) \to \ell^2(L^2_{[0,h)}(\mathbb{R}^m))$, defined by [7, 8, 9]

$$\mathbf{W} : u(t) \mapsto \hat{u}_k(\tau) := u(\tau + kh);\quad \tau \in [0,h)$$

for any $u \in L^2_{\mathbb{R}_+}(\mathbb{R}^m)$ and some real $h > 0$, is central to the development of the results that follow. Using this isomorphism [7, 8, 9], the state-space realisation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t); \\ y(t) &= C(t)x(t) + D(t)u(t), \end{aligned} \tag{1}$$

of a given h-periodic, continuous-time system $\mathbf{P} : \mathrm{dom}(\mathbf{P}) \subset L^2_{\mathbb{R}_+}(\mathbb{R}^m) \to L^2_{\mathbb{R}_+}(\mathbb{R}^p)$,[1] is *equivalent* to a *shift-invariant*, discrete-time, state-space realisation

$$\hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}\hat{u}_k;\quad \hat{y}_k = \hat{C}\hat{x}_k + \hat{D}\hat{u}_k, \tag{2}$$

where $\hat{x}_k = x(kh)$, $\hat{u}_k = \mathbf{W}u$, $\hat{y}_k = \mathbf{W}y$,

$$\begin{aligned} \hat{A} &: \mathbb{R}^n \to \mathbb{R}^n, \\ \hat{B} &: L^2_{[0,h)}(\mathbb{R}^m) \to \mathbb{R}^n, \\ \hat{C} &: \mathbb{R}^n \to L^2_{[0,h)}(\mathbb{R}^p), \\ \hat{D} &: L^2_{[0,h)}(\mathbb{R}^m) \to L^2_{[0,h)}(\mathbb{R}^p), \end{aligned}$$

$n = \dim(x(t))$, $m = \dim(u(t))$ and $p = \dim(y(t))$. The operators $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ can be evaluated by considering how $\mathbf{P}$ evolves over each $[kh,(k+1)h)$ interval of time, $k \in \mathbb{Z}$. This yields (cf. [7, 8]):

$$\hat{A} := \hat{x}_k \mapsto \Phi_A(h,0)\hat{x}_k; \tag{3}$$

$$\hat{B} := \hat{u}_k \mapsto \int_0^h \Phi_A(h,\eta)B(\eta)\hat{u}_k(\eta)\, d\eta; \tag{4}$$

$$\hat{C} := \hat{x}_k \mapsto C(\tau)\Phi_A(\tau,0)\hat{x}_k; \tag{5}$$

$$\hat{D} := \hat{u}_k \mapsto D(\tau)\hat{u}_k(\tau) + C(\tau)\int_0^\tau \Phi_A(h,\eta)B(\eta)\hat{u}_k(\eta)\, d\eta, \tag{6}$$

for $\tau \in [0,h)$, where $\Phi_A(\eta,\tau)$ is the unique *state transition matrix* satisfying

$$\frac{\partial}{\partial\eta}\Phi_A(\eta,\tau) = A(\eta)\Phi_A(\eta,\tau) \text{ and } \Phi_A(\tau,\tau) = I.$$

Note that although the operators $\hat{B}$ and $\hat{C}$ are infinite-dimensional, they have finite rank since the state dimension is still finite.

By shift-invariance of the discrete-time realisation (2), it follows (via the standard $\mathbf{Z}$-Transform isomorphism, $\mathbf{Z} : u_k \mapsto \sum_{k\in\mathbb{Z}} u_k\varphi^k$) that $\mathbf{P}$ can also be characterised as a *multiplication* operator with frequency domain symbol

$$P(\varphi) = \varphi\hat{C}(I - \varphi\hat{A})^{-1}\hat{B} + \hat{D}. \tag{7}$$

This fact is central to the development of the main results that follow.

Before going on, some elementary facts are gathered. These can be found (or easily derived from results) in any text on linear analysis (e.g. [12]). Given Hilbert spaces E and F, the set of bounded linear operators $\mathbf{X} : E \to F$ is denoted by $\mathcal{L}(E,F)$. The induced norm of $\mathbf{X} \in \mathcal{L}(E,F)$ is denoted by $\|\mathbf{X}\|_{E\to F}$ (the subscript may be dropped when convenient), and the adjoint $\mathbf{X}^* : F \to E$ is the unique bounded linear operator satisfying $\langle \mathbf{X}u, y\rangle_F = \langle u, \mathbf{X}^*y\rangle_E$. The spectrum of $\mathbf{X} \in \mathcal{L}(E) := \mathcal{L}(E,E)$ is denoted by $\mathrm{spec}(\mathbf{X})$, which is defined by $\mathrm{spec}(\mathbf{X}) := \{\lambda \in \mathbb{C} : \lambda I - \mathbf{X} \text{ is not invertible in } \mathcal{L}(E)\}$, and the spectral radius by $\mathrm{rad}(\mathbf{X}) := \sup_{\lambda\in\mathrm{spec}(\mathbf{X})} |\lambda|$.

Proposition 2.1 *Suppose that $Y \in \mathcal{L}(E,F)$ and $X \in \mathcal{L}(F,E)$. Then*

(i) The operator $I - YX$ is non-singular if and only if $I - XY$ is non-singular.

[1] i.e. $A(t+kh) = A(t)$, $B(t+kh) = B(t)$, $C(t+kh) = C(t)$ and $D(t+kh) = D(t)$ for all $k \in \mathbb{Z}_+$

(ii) If the norm $\|X\|_{E\to F} = 1$ then $I - X^*X$ is singular.

Proposition 2.2 *Suppose that $F(\varphi)$ is a continuous function on $\mathbb{D} \cup \mathbb{T}$ and $\|F(0)\| < 1$. Then $\|F(\varphi)\| < 1$ for each $\varphi \in \mathbb{D} \cup \mathbb{T}$ if and only if $I - F(\varphi)^*F(\varphi)$ is non-singular for each $\varphi \in \mathbb{D}\cup\mathbb{T}$.*

Proof: (only if): Since $\mathrm{rad}(F^*(\varphi)F(\varphi)) \leq \|F(\varphi)^*F(\varphi)\| = \|F(\varphi)\|^2 < 1$ for all $\varphi \in \mathbb{D} \cup \mathbb{T}$, it follows that $I - F(\varphi)^*F(\varphi)$ is non-singular for all $\varphi \in \mathbb{D} \cup \mathbb{T}$.

(if): Suppose that for some $\varphi \in \mathbb{D}\cup\mathbb{T}$, $\|F(\varphi)\| \geq 1$. By continuity of F and the fact that $\|F(0)\| < 1$, there exists a $\varphi_0 \in \mathbb{D}\cup\mathbb{T}$ such that $\|F(\varphi_0)\| = 1$. Hence, using Prop. 2.1, $I - F(\varphi_0)^*F(\varphi_0)$ must be singular. This is a contradiction. ∎

3 Characterisation of the H^∞ Norm

Recall from Section 2 that an h-periodic continuous-time system $\mathbf{P}$, with finite-dimensional state-space realisation (1), is equivalent (via the $\mathbf{W}$ and $\mathbf{Z}$ isomorphisms) to a multiplication operator with frequency domain symbol $P(\varphi)$, characterised in (7). Indeed, when $\mathbf{P}$ is "stable" (i.e. a bounded operator) its $L^2_{\mathbb{R}_+}$-induced norm is equal to the infinity norm of P:

$$\|P\|_\infty = \sup_{\varphi\in\mathbb{D}} \|\varphi\hat{C}(I - \varphi\hat{A})^{-1}\hat{B} + \hat{D}\|.$$

$\mathbf{P}$ "stable" corresponds to $\mathrm{spec}(\hat{A}) \subset \mathbb{D}$, and in this case $P(\varphi)$ is analytic on $\mathbb{D}$, so that the maximum modulus principle applies to give $\|P\|_\infty = \sup_{\varphi\in\mathbb{T}} \|P(\varphi)\|$. Note that at each frequency, the symbol $P(\varphi)$ is an infinite-dimensional operator. The subsequent result yields a finite-dimensional equivalent symbol which satisfies the same infinity norm condition as the symbol of interest (note that the norms may be different, but one will be less than 1 if and only if the other is less than 1).

Theorem 3.1 *Suppose that $\|\hat{D}\| < 1$, and let $\underline{B}$ and $\underline{C}$ be finite-dimensional matrices satisfying*

$$\begin{aligned} \underline{B}\underline{B}^* &= \hat{B}(I - \hat{D}^*\hat{D})^{-1}\hat{B}^*, \\ \underline{C}^*\underline{C} &= \hat{C}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C}. \end{aligned}$$

Define $\underline{A} := \hat{A} + \hat{B}(I - \hat{D}^*\hat{D})^{-1}\hat{D}^*\hat{C}$. Then the following are equivalent:

(i) $\mathrm{spec}(\hat{A}) \subset \mathbb{D}$ *and* $\|\varphi\hat{C}(I-\varphi\hat{A})^{-1}\hat{B}+\hat{D}\|_\infty < 1$;

(ii) $\mathrm{spec}(\underline{A}) \subset \mathbb{D}$ *and* $\|\varphi\underline{C}(I - \varphi\underline{A})^{-1}\underline{B}\|_\infty < 1$,

where $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are defined in equations (3–6).

Remark 3.2 *Given formulae for the* finite-dimensional *matrices $\underline{A}$, $\underline{B}$ and $\underline{C}$ (cf. Appendix A), an algebraic characterisation of the H^∞ norm of P can be obtained via Theorem 3.1, using standard discrete-time state-space techniques [1]. Testing of the condition $\|\hat{D}\| < 1$ can be carried out as described in [13].* ♣

Proof: The proof is similar to the development of the main result in [14].

First it is shown that the "stability" of $\underline{A}$ is related to that of $\hat{A}$. Suppose that $\mathrm{spec}(\hat{A}) \subset \mathbb{D}$ and that $\|\varphi\hat{C}(I - \varphi\hat{A})^{-1}\hat{B} + \hat{D}\|_\infty < 1$. Then using Prop. 2.1, note that

$$\begin{gathered} I - \lambda\underline{A} = I - \lambda\hat{A} - \lambda\hat{B}(I - \hat{D}^*\hat{D})^{-1}\hat{D}^*\hat{C} \\ \text{non-singular for } \lambda \in \mathbb{D} \\ \Updownarrow \\ I - (I - \lambda\hat{A})^{-1}\lambda\hat{B}(I - \hat{D}^*\hat{D})^{-1}\hat{D}^*\hat{C} \\ \text{non-singular for } \lambda \in \mathbb{D} \\ \Updownarrow \\ I - \hat{D}^*\hat{C}(I - \lambda\hat{A})^{-1}\lambda\hat{B}(I - \hat{D}^*\hat{D})^{-1} \\ \text{non-singular for } \lambda \in \mathbb{D} \\ \Updownarrow \\ I - \hat{D}^*\hat{D} - \hat{D}^*\hat{C}(I - \lambda\hat{A})^{-1}\lambda\hat{B} \\ \text{non-singular for } \lambda \in \mathbb{D} \\ \Updownarrow \\ I - \hat{D}^*(\hat{C}(I - \lambda\hat{A})^{-1}\lambda\hat{B} + \hat{D}) \\ \text{non-singular for } \lambda \in \mathbb{D}. \end{gathered}$$

Since $\|\hat{D}^*\| < 1$ and $\sup_{\lambda\in\mathbb{D}} \|\lambda\hat{C}(I - \lambda\hat{A})^{-1}\hat{B} + \hat{D}\| < 1$, it follows $I - \hat{D}^*(\hat{C}(I - \lambda\hat{A})^{-1}\lambda\hat{B} +$

$\hat{D})$ is non-singular for $\lambda \in \mathbb{D}$ and hence, that $\mathrm{spec}(\underline{A}) \subset \mathbb{D}$. Similarly, by essentially reversing the argument, if $\mathrm{spec}(\underline{A}) \subset \mathbb{D}$ and $\|\varphi\underline{C}(I - \varphi\underline{A})^{-1}\underline{B}\|_\infty < 1 \Leftrightarrow \|\varphi(I - \hat{D}\hat{D}^*)^{-\frac{1}{2}}\hat{C}(I - \varphi\underline{A})^{-1}\hat{B}(I - \hat{D}^*\hat{D})^{-\frac{1}{2}}\|_\infty < 1$, then $\mathrm{spec}(\hat{A}) \subset \mathbb{D}$.

Now consider condition *(i)*. The norm condition is equivalent to $\|\varphi\hat{C}(I - \varphi\hat{A})^{-1}\hat{B} + \hat{D}\| < 1$ for all $\varphi \in \mathbb{D} \cup \mathbb{T}$. Furthermore, repeated application of Props. 2.1 and 2.2 yields the following sequence of equivalent statements:[2]

$$\|\varphi\hat{C}(I - \varphi\hat{A})^{-1}\hat{B} + \hat{D}\| < 1$$

for all $\varphi \in \mathbb{D} \cup \mathbb{T} \Leftrightarrow$

$$I - (\hat{C}(I - \varphi\hat{A})^{-1}\varphi\hat{B} + \hat{D})^* \times (\hat{C}(I - \varphi\hat{A})^{-1}\varphi\hat{B} + \hat{D})$$

is non-singular for all $\varphi \in \mathbb{D} \cup \mathbb{T} \Leftrightarrow$

$$(I - \hat{D}^*\hat{D}) - \begin{pmatrix} (\varphi\hat{B})^* & \hat{D}^*\hat{C} \end{pmatrix} \times \begin{pmatrix} (I - \varphi\hat{A})^{-*}\hat{C}^*\hat{C}(I - \varphi\hat{A})^{-*} & (I - \varphi\hat{A})^{-*} \\ (I - \varphi\hat{A})^{-1} & 0 \end{pmatrix} \begin{pmatrix} \varphi\hat{B} \\ \hat{C}^*\hat{D} \end{pmatrix}$$

is non-singular for all $\varphi \in \mathbb{D} \cup \mathbb{T} \Leftrightarrow$

$$\begin{pmatrix} (I - \varphi\hat{A})^{-*}\hat{C}^*\hat{C}(I - \varphi\hat{A})^{-1} & (I - \varphi\hat{A})^{-*} \\ (I - \varphi\hat{A})^{-1} & 0 \end{pmatrix}^{-1} - \begin{pmatrix} \varphi\hat{B} \\ \hat{C}^*\hat{D} \end{pmatrix} (I - \hat{D}^*\hat{D})^{-1} \begin{pmatrix} (\varphi\hat{B})^* & \hat{D}^*\hat{C} \end{pmatrix}$$

is non-singular for all $\varphi \in \mathbb{D} \cup \mathbb{T} \Leftrightarrow$

$$\begin{pmatrix} 0 & (I - \varphi\underline{A}) \\ (I - \varphi\underline{A})^* & -\hat{C}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C} \end{pmatrix} - \begin{pmatrix} \varphi\hat{B} \\ 0 \end{pmatrix} (I - \hat{D}^*\hat{D})^{-1} \begin{pmatrix} (\varphi\hat{B})^* & 0 \end{pmatrix}$$

is non-singular for all $\varphi \in \mathbb{D} \cup \mathbb{T} \Leftrightarrow$

$$I - (I - \hat{D}^*\hat{D})^{-\frac{1}{2}} \begin{pmatrix} \varphi\hat{B} \\ 0 \end{pmatrix}^* \times \begin{pmatrix} 0 & (I - \varphi\underline{A}) \\ (I - \varphi\underline{A})^* & -\hat{C}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C} \end{pmatrix}^{-1} \times \begin{pmatrix} \varphi\hat{B} \\ 0 \end{pmatrix} (I - \hat{D}^*\hat{D})^{-\frac{1}{2}}$$

is non-singular for all $\varphi \in \mathbb{D} \cup \mathbb{T} \Leftrightarrow$

$$I - (I - \hat{D}^*\hat{D})^{-\frac{1}{2}} \begin{pmatrix} \varphi\hat{B} \\ 0 \end{pmatrix}^* \times \begin{pmatrix} (I - \varphi\underline{A})^{-*}\hat{C}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C}(I - \varphi\underline{A})^{-1} & (I - \varphi\underline{A})^{-*} \\ (I - \varphi\underline{A})^{-1} & 0 \end{pmatrix} \times \begin{pmatrix} \varphi\hat{B} \\ 0 \end{pmatrix} (I - \hat{D}^*\hat{D})^{-\frac{1}{2}}$$

is non-singular for all $\varphi \in \mathbb{D} \cup \mathbb{T} \Leftrightarrow$

$$I - \begin{pmatrix} \varphi\underline{B} \\ 0 \end{pmatrix}^* \times \begin{pmatrix} (I - \varphi\underline{A})^{-*}\underline{C}^*\underline{C}(I - \varphi\underline{A})^{-1} & (I - \varphi\underline{A})^{-*} \\ (I - \varphi\underline{A})^{-1} & 0 \end{pmatrix} \begin{pmatrix} \varphi\underline{B} \\ 0 \end{pmatrix}$$

is non-singular for all $\varphi \in \mathbb{D} \cup \mathbb{T} \Leftrightarrow$

$$\|\varphi\underline{C}(I - \varphi\underline{A})^{-1}\underline{B}\| < 1 \text{ for all } \varphi \in \mathbb{D} \cup \mathbb{T},$$

as required. ∎

[2] In view of the arguments given above, $(I - \varphi\underline{A})^{-1}$ exists for all $\varphi \in \mathbb{D} \cup \mathbb{T}$ provided $\mathrm{spec}(\hat{A}) \subset \mathbb{D}$ and that $\|\varphi\hat{C}(I - \varphi\hat{A})^{-1}\hat{B} + \hat{D}\|_\infty < 1$.

4 Characterisation of the H^2 Norm

In this section a useful characterisation of the H^2 norm is employed. This characterisation is often used in time-varying settings (cf. [15] for example). Let e_i denote the i-th canonical basis vector in $\mathbb{R}^m$, where m is the dimension of the input space, and $g_{i,\tau}(t)$ denotes the response of the h-periodic system $\mathbf{P}$ defined by the state-space equations (1), to the impulsive input $\delta(t - \tau)e_i$ for some $\tau \in [0, h)$. Making the standing assumptions that $D(t) \equiv 0$ and that $\mathbf{P}$ is stable (i.e. $\mathrm{spec}(\hat{A}) \subset \mathbb{D}$), the response $g_{i,\tau} \in L^2_{\mathbb{R}_+}(\mathbb{R}^p)$ and the H^2 norm is defined to be

$$\|\mathbf{P}\|_2^2 := \frac{1}{h} \int_0^h \sum_{i=1}^m \langle g_{i,\tau}, g_{i,\tau} \rangle_{L^2_{\mathbb{R}_+}(\mathbb{R}^p)} \, d\tau.$$

Note that as defined here, the H^2 norm is the average energy of the response of the system to impulses on each of the inputs, occurring some time within the interval $[0, h)$ (recall that h is the period of the time-varying behaviour of the system). An alternative would be to just consider the energy of the response to an impulse at 0, in which case the characterisation of the H^2 norm below would be completely algebraic.

Now define

$$\Delta_{\hat{D},\tau} := v \in \mathbb{R}^m \mapsto (\hat{D}\delta(t-\tau)v)(t) \in L^2_{[0,h)}(\mathbb{R}^p) \quad (8)$$

and

$$\Delta_{\hat{B},\tau} := v \in \mathbb{R}^m \mapsto (\hat{B}\delta(t-\tau)v) \in \mathbb{R}^n, \quad (9)$$

where $\hat{D}$ and $\hat{B}$ are defined in equations (6) and (4). Working in the time-lifted domain, it follows that

$$\begin{aligned}
\langle g_{i,\tau}, g_{i,\tau}\rangle_{L^2_{\mathbb{R}_+}(\mathbb{R}^p)} &= \langle \Delta_{\hat{D},\tau}e_i, \Delta_{\hat{D},\tau}e_i\rangle_{L^2_{[0,h)}(\mathbb{R}^p)} + \\
&\sum_{k=1}^{\infty} \langle \hat{C}\hat{A}^k\Delta_{\hat{B},\tau}e_i, \hat{C}\hat{A}^k\Delta_{\hat{B},\tau}e_i\rangle_{L^2_{[0,h)}(\mathbb{R}^p)}, \\
&= \langle e_i, (\Delta_{\hat{D},\tau})^*\Delta_{\hat{D},\tau}e_i\rangle_{\mathbb{R}^m} + \\
&\sum_{k=1}^{\infty} \langle e_i, (\Delta_{\hat{B},\tau})^*(\hat{A}^*)^k\hat{C}^*\hat{C}\hat{A}^k\Delta_{\hat{B},\tau}e_i\rangle_{\mathbb{R}^m}, \\
&= \langle e_i, (\Delta_{\hat{D},\tau})^*\Delta_{\hat{D},\tau}e_i\rangle_{\mathbb{R}^m} + \\
&\langle e_i, (\Delta_{\hat{B},\tau})^* L_o \Delta_{\hat{B},\tau}e_i\rangle_{\mathbb{R}^m},
\end{aligned}$$

where $L_o \geq 0$ is the solution to the Lyapunov equation $\hat{A}^*L_o\hat{A} - L_o + \hat{C}^*\hat{C} = 0$. Note that the last expression above involves only finite-dimensional matrices; $\hat{A}$, $\Delta_{\hat{B},\tau}$, $\hat{C}^*\hat{C}$ and $(\Delta_{\hat{D},\tau})^*\Delta_{\hat{D},\tau}$ are all matrices and explicit formulae are given in Appendix B. The following theorem is now immediate.

Theorem 4.1 *Given a stable, h-periodic system* $\mathbf{P}$ *with state-space realisation (1) and* $D(t) \equiv 0$,

$$\|\mathbf{P}\|_2^2 = \frac{1}{h}\int_0^h \mathrm{trace}\{\, (\Delta_{\hat{D},\tau})^*\Delta_{\hat{D},\tau} + (\Delta_{\hat{B},\tau})^* L_o \Delta_{\hat{B},\tau}\}\, d\tau,$$

where $\Delta_{\hat{D},\tau}$ *and* $\Delta_{\hat{B},\tau}$ *are defined in equations (8–9), and* $L_o \geq 0$ *is the observability gramian that satisfies*

$$\hat{A}^*L_o\hat{A} - L_o + \hat{C}^*\hat{C} = 0,$$

with $\hat{A}$ *and* $\hat{C}$ *as defined in equation (3) and (5) respectively.*

A State-Space Formulae for the H^∞ Case

In this section, explicit formulae are given for the following matrices, which appear in Section 3:

$$\begin{aligned}
\underline{A} &:= \hat{A} + \hat{B}(I - \hat{D}^*\hat{D})^{-1}\hat{D}^*\hat{C}; \\
\underline{BB^*} &:= \hat{B}(I - \hat{D}^*\hat{D})^{-1}\hat{B}^*; \\
\underline{C^*}\,\underline{C} &:= \hat{C}^*(I - \hat{D}\hat{D}^*)^{-1}\hat{C}.
\end{aligned}$$

The formulae can be obtained via a two-point boundary-value problem approach outlined in [16], yielding the following. Let

$$E(t) := \begin{pmatrix} E_{11}(t) & E_{12}(t) \\ E_{21}(t) & E_{22}(t) \end{pmatrix},$$

where the matrix valued functions $E_{ij}(t)$ are defined by:

$$\begin{aligned}
E_{22}(t) &= -E_{11}(t)^* := A(t) + B(I - D^*D)^{-1}D^*C(t); \\
E_{12}(t) &:= -C^*(I - DD^*)^{-1}C(t); \\
E_{21}(t) &:= B(I - D^*D)^{-1}B(t).
\end{aligned}$$

Now define

$$Q = \Phi_E(h, 0) := \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix},$$

where $\Phi_E(\cdot,\cdot)$ denotes the state transition matrix corresponding to E. Then,

$$\begin{aligned}
\underline{A} &= Q_{22} - Q_{21}Q_{11}^{-1}Q_{12}, \\
\underline{BB^*} &= Q_{21}Q_{11}^{-1}, \\
\underline{C^*}\,\underline{C} &= -Q_{11}^{-1}Q_{12}.
\end{aligned}$$

The matrix Q can be calculated explicitly by simply integrating the system $\dot{x}(t) = E(t)x(t)$; $x(0) = I$, forward in time up to $t = h$. If E were constant this would only involve computation of a matrix exponential.

B State-Space Formulae for the H^2 Case

Again, expressions for the finite-dimensional matrices $\hat{A}$, $\hat{C}^*\hat{C}$, $\Delta_{\hat{B},\tau}$ and $(\Delta_{\hat{D},\tau})^*\Delta_{\hat{D},\tau}$, which appear in Section 4, can be derived using the two-boundary value problem approach outlined in [16]. Recall that by assumption $D(t) \equiv 0$. Let

$$E(t) := \begin{pmatrix} E_{11}(t) & E_{12}(t) \\ 0 & E_{22}(t) \end{pmatrix},$$

where the matrix valued functions $E_{ij}(t)$ are defined by:

$$\begin{aligned} E_{22}(t) &= -E_{11}(t)^* := A(t); \\ E_{12}(t) &:= -C^* C(t). \end{aligned}$$

Now define

$$Q(\eta) = \Phi_E(h,\eta) := \begin{pmatrix} Q_{11}(\eta) & Q_{12}(\eta) \\ 0 & Q_{22}(\eta) \end{pmatrix},$$

where $\Phi_E(\cdot,\cdot)$ denotes the state transition matrix corresponding to E. Then,

$$\begin{aligned} \hat{A} &= Q_{22}(0), \\ \hat{C}^*\hat{C} &= -Q_{11}(0)^{-1}Q_{12}(0), \\ \Delta_{\hat{B},\tau} &= Q_{22}(\tau)B(\tau), \\ \Delta_{\hat{D},\tau} &= -Q_{11}(\tau)^{-1}Q_{12}(\tau)B(\tau). \end{aligned}$$

The matrix $Q(\eta)$ can be evaluated by integrating the system $\dot{x}(t) = E(t)x(t);\ x(\eta) = I$, forward in time up to $t = h$.

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