A note on robust stability analysis for feedback interconnections of time-varying linear systems

M. Saeed Akram[1] and Michael Cantoni[2]

***Abstract*— A generalization of Vinnicombe's ν-gap metric and corresponding robust feedback stability results are proposed for a class of linear time-varying (LTV) systems in "Robust stability analysis of time-varying linear systems," SIAM J. Control and Optimization, Vol. 51, pp. 353-379, 2013. An error in the analysis presented therein leads to an omission in the time-domain definition of a ν-gap metric for the class of continuous-time systems studied. This omission and the underlying error in the development are corrected herein. Specifically, the omission is a norm-coercivity constraint to be satisfied in conjunction with the family of Fredholm index conditions that generalize the winding-number condition in the frequency-domain definition for time-invariant systems.**

***Index Terms*— Gap metric; continuous-time systems.**

I. INTRODUCTION

The contribution of this paper is to correct an error in [1], where a linear time-varying generalization of Vinnicombe's ν-gap metric (see [2], [3]) is proposed. The error leads to an omission in the proposed definition of a generalized metric on a class of causal LTV systems that admit normalized coprime graph representations with compact Hankel operators for all partitions of time into past and future. Correcting the error, without substantially deviating from the analytical approach, leads to a norm-coercivity constraint to be satisfied in conjunction with the family of Fredholm index conditions used to define the metric in [1]. Together, these conditions generalize the determinant and winding number conditions in the standard frequency-domain definition of the ν-gap for time-invariant systems [2]. It is interesting to note that the additional condition at the centre of this corrigendum appears in the precursors [4], [5] of the paper [1], although the error underlying the issues corrected here is also present in these earlier reports.

The next section is used to gather preliminary definitions and results, including aspects of linear operator theory, a description of the class of systems for which an LTV generalization of the ν-gap is eventually defined, and related system theoretic results. The error in the development of the main robust feedback stability result of [1] is rectified in Section III. The corrected result is used to define an LTV generalization of the ν-gap metric and establish a corresponding ν-gap robust stability result in Section IV.

*This work was supported by the Australian Research Council (DP130104510) and the Higher Education Commission (HEC) Pakistan.

[1]M.S. Akram is with the Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan, mrsaeedakram@comsats.edu.pk

[2]M. Cantoni is with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, Australia, cantoni@unimelb.edu.au

II. PRELIMINARIES

A. Linear operators

A linear mapping $X : \mathrm{dom}(X) \subset \mathcal{H}_1 \to \mathcal{H}_2$ between a subspace $\mathrm{dom}(X)$ of a Hilbert space $\mathcal{H}_1$ and another $\mathcal{H}_2$ is such that $X(\alpha u + \beta v) = \alpha X(u) + \beta X(v) \in \mathcal{H}_2$ for all scalars α, β and $u, v \in \mathrm{dom}(X)$; for convenience, $X(u)$ is written Xu. The kernel of X is denoted by $\ker(X) := \{u \,|\, Xu = 0; u \in \mathrm{dom}(X)\}$, the image of the domain under X by $\mathrm{img}(X) := \{y \,|\, y = Xu; u \in \mathrm{dom}(X)\}$, the graph by $\mathrm{gr}(X) := \{[\begin{smallmatrix} u \\ y \end{smallmatrix}] \mid u \in \mathrm{dom}(X);\ y = Xu\}$ and the inverse graph by $\mathrm{gr}'(X) := \{[\begin{smallmatrix} y \\ u \end{smallmatrix}] \mid [\begin{smallmatrix} u \\ y \end{smallmatrix}] \in \mathrm{gr}(X)\}$. The mapping X is called a *bounded* operator if there exists a constant $c > 0$ such that $\|Xu\|_{\mathcal{H}_2} \le c\|u\|_{\mathcal{H}_1}$ for all $u \in \mathrm{dom}(X)$, where $\|\cdot\|_{\mathcal{H}} = (\langle \cdot,\cdot \rangle_{\mathcal{H}})^{1/2}$ is the norm induced by the inner-product $\langle \cdot,\cdot \rangle_{\mathcal{H}}$. Given a subspace $\mathcal{V} \subset \mathrm{dom}(X) \subset \mathcal{H}_1$, the closure is denoted by $\mathrm{cl}(\mathcal{V})$, the orthogonal complement by $\mathcal{V}^{\perp} := \{u \in \mathcal{H}_1 \mid \langle u, v \rangle_{\mathcal{H}_1} = 0\ \forall\ v \in \mathcal{V}\}$, the restriction of X to $\mathcal{V}$ by $X|_{\mathcal{V}} : \mathcal{V} \to \mathcal{H}_2$, and $X\mathcal{V}$ means $\mathrm{img}(X|_{\mathcal{V}})$.

The space of all bounded operators $X : \mathcal{H}_1 \to \mathcal{H}_2$ (n.b., $\mathrm{dom}(X) = \mathcal{H}_1$) is denoted by $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$, or $\mathscr{B}(\mathcal{H})$ when $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. If $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is bijective, then the inverse map $X^{-1} \in \mathscr{B}(\mathcal{H}_2, \mathcal{H}_1)$ (see, e.g., [6, Thm. 5.7]) and X is said to be *invertible*. For each $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$, the adjoint operator $X^* \in \mathscr{B}(\mathcal{H}_2, \mathcal{H}_1)$ uniquely satisfies (see, e.g., [6, Ch.11])

$$\langle Xu, y \rangle_{\mathcal{H}_2} = \langle u, X^* y \rangle_{\mathcal{H}_1} \text{ for all } u \in \mathcal{H}_1, y \in \mathcal{H}_2. \quad (1)$$

Note that $(X^*)^* = X$. The identity operator is denoted by $I := u \mapsto u$ and the zero operator by $O := u \mapsto 0$.

Lemma 1: ([7, Sec. 2.11]). For $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$, the following hold: (i) $\ker(X) = \mathrm{img}(X^*)^{\perp}$; (ii) $\ker(X^*) = \mathrm{img}(X)^{\perp}$; (iii) $\mathrm{cl}(\mathrm{img}(X)) = \ker(X^*)^{\perp}$; and (iv) $\mathrm{cl}(\mathrm{img}(X^*)) = \ker(X)^{\perp}$.

Given $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$, define the following gains:

$$\gamma(X) := \sup_{\|u\|_{\mathcal{H}_1}=1} \|Xu\|_{\mathcal{H}_2};\ \mu(X) := \inf_{\|u\|_{\mathcal{H}_1}=1} \|Xu\|_{\mathcal{H}_2}. \quad (2)$$

Lemma 2: ([8, Sec. 1.2-1.3]). For $X, Y \in \mathscr{B}(\mathcal{H})$, the following hold: (i) $\gamma(X^*) = \gamma(X)$; (ii) $\gamma(X+Y) \le \gamma(X) + \gamma(Y)$; (iii) $\gamma(XY) \le \gamma(X)\gamma(Y)$; and (iv) If $\gamma(I - X) < 1$, then X is invertible.

Lemma 3: ([8, Sec. 2.5]). For $X, Y \in \mathscr{B}(\mathcal{H})$, the following hold: (i) X is one-to-one with $\mathrm{img}(X) = \mathrm{cl}(\mathrm{img}(X))$ if and only if $\mu(X) > 0$; (ii) If X is invertible, then $\mu(X) = 1/\gamma(X^{-1})$; (iii) X is invertible if and only if $\mu(X) = \mu(X^*) > 0$; (iv) $|\mu(X) - \mu(Y)| \le \gamma(X - Y)$; and (v) $\mu(XY) \ge \mu(X)\mu(Y)$.

Lemma 4: If either (a) $\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right] \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2 \times \mathcal{H}_3)$ is an isometry (i.e., $X^*X + Y^*Y = I$), or (b) $\begin{bmatrix}X & Y\end{bmatrix} \in \mathscr{B}(\mathcal{H}_1 \times \mathcal{H}_2, \mathcal{H}_3)$ is a co-isometry (i.e., $XX^* + YY^* = I$) with X invertible, then $\gamma(Y)^2 = 1 - \mu(X)^2$.

Proof: If $\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right]$ is an isometry, then for any non-zero $u \in \mathcal{H}_1$ the use of (1) and bi-linearity of inner-products yields the equalities $1 = \langle \left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right] u, \left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right] u\rangle_{\mathcal{H}_2 \times \mathcal{H}_3} / \langle u,u\rangle_{\mathcal{H}_1} = \langle Yu, Yu\rangle_{\mathcal{H}_3} / \langle u,u\rangle_{\mathcal{H}_1} + \langle Xu, Xu\rangle_{\mathcal{H}_2} / \langle u,u\rangle_{\mathcal{H}_1}$. As such, part (a) implies $\gamma(Y)^2 = 1 - \mu(X)^2$ by the definitions (2).

On the other hand, if $\begin{bmatrix}X & Y\end{bmatrix}$ is a co-isometry, then $\left[\begin{smallmatrix}X^*\\Y^*\end{smallmatrix}\right]$ is an isometry, whereby $\gamma(Y^*)^2 = 1 - \mu(X^*)^2$, as shown above. With X is invertible, it follows that $\gamma(Y)^2 = 1 - \mu(X)^2$, since $\mu(X^*) = \mu(X)$ by Lemma 3(iii) and $\gamma(Y) = \gamma(Y^*)$ by Lemma 2(i). ■

Remark 5: Invertibility of X is missing from the part (b) condition in the corresponding result [11, Lem. 3], which is employed in [1], giving rise to the aforementioned error. In the subsequent re-development, there is no recourse to the part (b) condition, only the part (a) condition. ◇

It is said that $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is compact if for any bounded sequence $\{u_k\} \subset \mathcal{H}_1$, a subsequence of $\{Xu_k\}$ converges in $\mathcal{H}_2$. Finally, $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is called *Fredholm* if both dim $\ker(X)$ and dim $\ker(X^*)$ are finite, where dim denotes the dimension of a subspace. In this case, $\mathrm{img}(X)$ is closed [7, Thm. 15.2.1] and the Fredholm index is given by

$$\mathrm{ind}(X) := \dim \ker(X) - \dim \ker(X^*). \tag{3}$$

Note that $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is bijective if and only if X is Fredholm with $\mathrm{ind}(X) = \dim \ker(X) - \dim \ker(X^*) = 0$.

Lemma 6: ([7, Sec. 15.2]). The following hold:

(i) If $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is Fredholm, then X^* is Fredholm with $\mathrm{ind}(X^*) = -\mathrm{ind}(X)$;
(ii) If $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $Y \in \mathscr{B}(\mathcal{H}_2, \mathcal{H}_3)$ are Fredholm, then YX is Fredholm with $\mathrm{ind}(YX) = \mathrm{ind}(Y) + \mathrm{ind}(X)$;
(iii) If $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is Fredholm and $K \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is compact, then $X + K$ is Fredholm with $\mathrm{ind}(X+K) = \mathrm{ind}(X)$;
(iv) If $X \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is Fredholm and $Y \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ is such that $\gamma(Y) < \mu(X)$, then $X + Y$ is Fredholm with $\mathrm{ind}(X+Y) = \mathrm{ind}(X)$.

B. Signals and systems

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the natural, integer and real numbers, respectively. In this paper, systems are causal linear mappings between spaces of finite-energy signals with support (i.e., the subset of $\mathbb{R}$ on which the signal is non-zero) that is (non-uniformly) bounded below. Specifically, systems operate on signals in domains contained within the subspace $\mathcal{L}_{2+} := \bigcup_{\tau \in \mathbb{R}} P_\tau \mathcal{L}_2$, where $\mathcal{L}_2$ denotes the space of square integrable functions $f : \mathbb{R} \to \mathbb{R}$ (modulo those that are non-zero on sets of measure zero), which is a Hilbert space when endowed with the inner-product $\langle u, v\rangle = \int_{\mathbb{R}} u(t)v(t)\mathrm{d}t$ and induced norm denoted by $\|\cdot\|_2$ (i.e., $\mathcal{L}_2$ is complete w.r.t. this norm [13, Thm. 11.42]), and $P_\tau \in \mathscr{B}(\mathcal{L}_2)$ is defined by $(P_\tau u)(t) = u(t)$ for $t > \tau$ and $(P_\tau u)(t) = 0$ otherwise, for all $u \in \mathcal{L}_2, \tau \in \mathbb{R}$. Let $Q_\tau := I - P_\tau$. Note that P_τ and Q_τ are orthogonal projections on $\mathcal{L}_2$, and $\|u\|_2^2 = \|P_\tau u\|_2^2 + \|Q_\tau u\|_2^2 \ \forall \ \tau \in \mathbb{R}$. Moreover, $P_\tau P_\sigma = P_\sigma P_\tau = P_\sigma$ and $Q_\tau Q_\sigma = Q_\sigma Q_\tau = Q_\tau$ for $\tau < \sigma \in \mathbb{R}$. Now given any $h > 0$ and $u \in \mathcal{L}_2$, note that $\|u\|_2^2 = \lim_{N\to\infty} \sum_{k=-N}^{N} \|(Q_{(k+1)h} - Q_{kh})u\|_2^2$ [12, Thm. 18.13]. Hence, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sum_{k=-m}^{-n} \|(Q_{(k+1)h} - Q_{kh})u\|_2^2 + \sum_{k=n}^{m} \|(Q_{(k+1)h} - Q_{kh})u\|_2^2 < \varepsilon$ for all $m \geq n > N$. Moreover, for any $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$, $\|(Q_{(k+1)h} - Q_{kh})u\|_2^2 = \|P_\tau(Q_{(k+1)h} - Q_{kh})u\|_2^2 + \|Q_\tau(Q_{(k+1)h} - Q_{kh})u\|_2^2 = \|(Q_{(k+1)h} - Q_{kh})P_\tau u\|_2^2 + \|(Q_{(k+1)h} - Q_{kh})Q_\tau u\|_2^2$. Now fixing $n > N$, $\lim_{m\to\infty} \sum_{k=-m}^{-n} \|(Q_{(k+1)h} - Q_{kh})u\|_2^2 < \varepsilon$ by the monotone convergence theorem [13, Thm. 3.14]. Furthermore, with $\tau < -n$, $\|u - P_\tau u\|_2^2 = \|Q_\tau u\|_2^2 = \lim_{m\to\infty} \sum_{k=-m}^{-n} \|(Q_{(k+1)h} - Q_{kh})Q_\tau u\|_2^2 < \varepsilon$. Therefore, $P_\tau u \to u$ as $\tau \to -\infty$ for every $u \in \mathcal{L}_2$, and thus, $\mathcal{L}_2 = \mathrm{cl}(\mathcal{L}_{2+})$.

For given $X \in \mathscr{B}(\mathcal{L}_2)$ and $\tau \in \mathbb{R}$, define the Toeplitz-Wiener-Hopf operators

$$T_\tau(X) := P_\tau X|_{P_\tau \mathcal{L}_2} \text{ and } B_\tau(X) := Q_\tau X|_{Q_\tau \mathcal{L}_2},$$

and the Hankel operators

$$H_\tau(X) := P_\tau X|_{Q_\tau \mathcal{L}_2} \text{ and } J_\tau(X) := Q_\tau X|_{P_\tau \mathcal{L}_2},$$

which are all bounded. A bounded operator $X \in \mathscr{B}(\mathcal{L}_2)$ is said to be *causal* if $Q_\tau X = Q_\tau X Q_\tau$ (or equivalently $XP_\tau = P_\tau X P_\tau$) for all $\tau \in \mathbb{R}$, and *anticausal* if X^* is causal. More generally, a linear $M : \mathrm{dom}(M) \subset \mathcal{L}_2 \to \mathcal{L}_2$ is said to be causal if $\left[\begin{smallmatrix}0\\y_\tau\end{smallmatrix}\right] \in Q_\tau \mathrm{gr}(M)$ implies $y_\tau = 0$ for all $\tau \in \mathbb{R}$.

Lemma 7: ([1, Lem. 2.6]). The following hold for given $X \in \mathscr{B}(\mathcal{L}_2)$ and $Y \in \mathscr{B}(\mathcal{L}_2)$:

(i) If X is causal, then $T_\tau(X)$ is causal for $\tau \in \mathbb{R}$;
(ii) $(T_\tau(X))^* = T_\tau(X^*)$ and $(H_\tau(X))^* = J_\tau(X^*)$ for $\tau \in \mathbb{R}$.
(iii) If $T_\tau(X)$ is causal for all $\tau \in \mathbb{R}$, then X is causal;
(iv) The mixed Toeplitz-Wiener-Hopf and Hankel decomposition $T_\tau(YX) = T_\tau(Y)T_\tau(X) + H_\tau(Y)J_\tau(X)$ holds for $\tau \in \mathbb{R}$, whereby $T_\tau(YX) = T_\tau(Y)T_\tau(X)$ if X is causal, or Y is anti-causal, as this condition implies $J_\tau(X) = 0$, or $H_\tau(Y) = 0$, respectively;
(v) Given $\tau \in \mathbb{R}$, the operator $T_\tau(X)$ has bounded inverse $(T_\tau(X))^{-1} : P_\tau \mathcal{L}_2 \to P_\tau \mathcal{L}_2$ if and only if $\mu(T_\tau(X)) > 0$ and $T_\tau(X)$ is Fredholm with $\mathrm{ind}(T_\tau(X)) = 0$;
(vi) If X is causal, then $\mu(X) = \inf_{\tau \in \mathbb{R}} \mu(T_\tau(X))$;
(vii) If X is causal, then $\gamma(X) = \sup_{\tau \in \mathbb{R}} \gamma(T_\tau(X))$.

Definition 8: $\mathscr{C}_+$ is the class of all causal linear mappings $M : \mathrm{dom}(M) \subset \mathcal{L}_{2+} \to \mathcal{L}_{2+}$ for which there exist causal operators $V, U, \tilde{V}, \tilde{U}, X, Y, \tilde{X}, \tilde{Y} \in \mathscr{B}(\mathcal{L}_2)$ such that the following properties hold:

(A) $\begin{bmatrix}X & Y\\ -\tilde{U} & \tilde{V}\end{bmatrix}\begin{bmatrix}V & -\tilde{Y}\\ U & \tilde{X}\end{bmatrix} = \begin{bmatrix}V & -\tilde{Y}\\ U & \tilde{X}\end{bmatrix}\begin{bmatrix}X & Y\\ -\tilde{U} & \tilde{V}\end{bmatrix} = I$;
(B) $G^*G = I$ and $\tilde{G}\tilde{G}^* = I$, where
$$G := \begin{bmatrix}V\\U\end{bmatrix} \quad \text{and} \quad \tilde{G} := \begin{bmatrix}-\tilde{U} & \tilde{V}\end{bmatrix};$$
(C) $\mathrm{img}(G) = \ker(\tilde{G})$ and $\mathrm{gr}(M) \cap P_\tau \mathcal{L}_2 = \mathrm{img}(T_\tau(G)) = \ker(T_\tau(\tilde{G}))$ for all $\tau \in \mathbb{R}$;
(D) $H_\tau(G)$ and $H_\tau(\tilde{G})$ are compact for all $\tau \in \mathbb{R}$.

Remark 9: As shown in [1], the causal input-output maps generated by stabilizable and detectable LTV state-space models are elements of $\mathscr{C}_+$. See [14] for computationally tractable constructions of G and $\tilde{G}$ in the special case of time-periodic state-space models. ◇

Remark 10: Statement of the requirement $\mathrm{img}(G) = \ker(\tilde{G})$ is overlooked in [1], although it is proved for LTV state-space models therein. ◇

In view of property (A) in Definition 8, the causal operators $G \in \mathscr{B}(\mathcal{L}_2, \mathcal{L}_2 \times \mathcal{L}_2)$ and $\tilde{G} \in \mathscr{B}(\mathcal{L}_2 \times \mathcal{L}_2, \mathcal{L}_2)$ defined in property (B) have causal bounded left and right inverses $Z = \begin{bmatrix} X & Y \end{bmatrix}$ and $\tilde{Z} = \begin{bmatrix} -\tilde{Y}^* & \tilde{X}^* \end{bmatrix}^*$, respectively. As such, by virtue of property (C), G is called a *strong (or coprime)* right and $\tilde{G}$ a *strong (or coprime)* left representation of the graph of the system $M : \mathrm{dom}(M) \subset \mathcal{L}_{2+} \to \mathcal{L}_{2+}$. The normalization property (B) and compactness property (D) play a role in establishing a key robust stability result in Section III. Strong right and left representations for the *inverse* graph of M are given by $\left[\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right] G$ and $\tilde{G} \left[\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right]$, respectively.

C. Feedback stability

Consider the feedback interconnection of causal systems $M, \Delta \in \mathscr{C}_+$ defined by the following equations:

$$[M, \Delta] := \begin{cases} e_1 = & -\Delta e_2 + r_1, \\ e_2 = & -M e_1 + r_2, \end{cases}$$

where r_1 and r_2 are exogenous inputs and the signals e_1 and e_2 are the internal signals at the input to M and the input to Δ, respectively. For any $\tau \in \mathbb{R}$, consider the restricted operator

$$F_\tau := \begin{bmatrix} I & \Delta \\ M & I \end{bmatrix} \Big|_{(\mathrm{dom}(M) \times \mathrm{dom}(\Delta)) \cap \left[\begin{smallmatrix} P_\tau & O \\ O & P_\tau \end{smallmatrix}\right] (\mathcal{L}_2 \times \mathcal{L}_2)}.$$

By causality, it follows that $\mathrm{img}(F_\tau) \subset P_\tau \mathcal{L}_2$. The following notion of feedback stability is taken in line with the generalized notion introduced in [4], [5], [1]. See [9] and [10] for related work.

Definition 11: For $M, \Delta \in \mathscr{C}_+$, the feedback interconnection $[M, \Delta]$ is said to be *stable* whenever the following properties hold: (A) F_τ is injective with $\mathrm{img}(F_\tau) = P_\tau \mathcal{L}_2$ for all $\tau \in \mathbb{R}$; and (B) the inverse map $F_\tau^{-1} : P_\tau \mathcal{L}_2 \to (\mathrm{dom}(M) \times \mathrm{dom}(\Delta)) \cap \left[\begin{smallmatrix} P_\tau & O \\ O & P_\tau \end{smallmatrix}\right] (\mathcal{L}_2 \times \mathcal{L}_2) \subset P_\tau \mathcal{L}_2$ is such that $\sup_{\tau \in \mathbb{R}} \gamma(F_\tau^{-1}) < \infty$.

Remark 12: It is of note that if $[M, \Delta]$ is stable, then $F_\tau^{-1} \in \mathscr{B}(P_\tau \mathcal{L}_2, P_\tau \mathcal{L}_2)$ is causal for all $\tau \in \mathbb{R}$ [1, Thm. 3.5]. That is, causality is built into the notion of feedback stability specified in Definition 11.

III. A robust feedback stability result

In this section the graph representation properties that hold for systems in $\mathscr{C}_+$ are exploited to establish a robust stability result for feedback interconnections. While the analytical approach is not substantially different from [1], the development presented below is organized around highlighting and rectifying deficiencies in the formulation of a number results therein. The re-formulation of these results leads to the aforementioned correction in the generalized time-domain definition of Vinnicombe's ν-gap metric for LTV systems; see Section IV.

Henceforth, the two causal open-loop systems in $\mathscr{C}_+$ comprising a feedback interconnection are denoted by $M : \mathrm{dom}(M) \subset \mathcal{L}_{2+} \to \mathcal{L}_{2+}$ and $\Delta : \mathrm{dom}(\Delta) \subset \mathcal{L}_{2+} \to \mathcal{L}_{2+}$, respectively, sometimes with subscripts to distinguish different instances of a component. Normalized strong right and left graph representations of $M_k : \mathrm{dom}(M_k) \subset \mathcal{L}_{2+} \to \mathcal{L}_{2+}$ in $\mathscr{C}_+$ are denoted by G_k and $\tilde{G}_k$, respectively, in line with Definition 8 above; likewise, normalized strong right and left *inverse* graph representations of a given causal system $\Delta_k : \mathrm{dom}(\Delta_k) \subset \mathcal{L}_{2+} \to \mathcal{L}_{2+}$ in $\mathscr{C}_+$ are denoted by Γ_k and $\tilde{\Gamma}_k$, respectively. The respective causal left and right inverses are denoted by Z_k, $\tilde{Z}_k$, Ω_k and $\tilde{\Omega}_k$, so that

$$I = Z_k G_k = \tilde{G}_k \tilde{Z}_k = \Omega_k \Gamma_k = \tilde{\Gamma}_k \tilde{\Omega}_k. \tag{4}$$

A useful characterization of feedback stability in terms of graph representations is provided in the next result. While the result is correctly formulated as Theorem 3.7 in [1], the proof presented for parts (iv) and (v) makes use of Lemma 3.1 in [1], which is not correct for omission of the condition $\gamma(\tilde{G}\tilde{\Gamma}^*) < 1$ needed in addition to $\mu(\tilde{\Gamma}G) > 0$ to establish bounded invertibility of $\tilde{\Gamma}G$ and $\tilde{G}\Gamma$; the error relates to Remark 5. A different path is taken below.

Proposition 13: For $M, \Delta \in \mathscr{C}_+$, the following statements are equivalent:

(i) The feedback interconnection $[M, \Delta]$ is stable;
(ii) $\mu(\tilde{G}\Gamma) > 0$ and $T_\tau(\tilde{G}\Gamma)$ is Fredholm with $\mathrm{ind}(T_\tau(\tilde{G}\Gamma)) = 0$ for every $\tau \in \mathbb{R}$;
(iii) $\mu(\tilde{\Gamma}G) > 0$ and $T_\tau(\tilde{\Gamma}G)$ is Fredholm with $\mathrm{ind}(T_\tau(\tilde{\Gamma}G)) = 0$ for every $\tau \in \mathbb{R}$;
(iv) $\tilde{G}\Gamma$ is causally invertible in $\mathscr{B}(\mathcal{L}_2)$;
(v) $\tilde{\Gamma}G$ is causally invertible in $\mathscr{B}(\mathcal{L}_2)$.

Proof: The equivalence of (i) to the invertibility of $T_\tau(\tilde{G}\Gamma)$ in $\mathscr{B}(P_\tau \mathcal{L}_2)$ for each $\tau \in \mathbb{R}$, with $\sup_{\tau \in \mathbb{R}} \gamma((T_\tau(\tilde{G}\Gamma))^{-1}) < \infty$, follows by virtue of the formula

$$F_\tau^{-1} = \begin{bmatrix} -I & O \\ O & I \end{bmatrix} T_\tau(\Gamma)(T_\tau(\tilde{G}\Gamma))^{-1} T_\tau(\tilde{G}) + \begin{bmatrix} I & O \\ O & O \end{bmatrix}, \tag{5}$$

which is given correctly in [1, Lem. 3.6]. Following [1], application of Lemma 7(v-vi) yields the equivalence to statement (ii) above. Equivalence to part (iii) follows similarly.

The equivalence of (ii) and (iv) is now established, to correct the proof of this given in [1]. The equivalence of (iii) and (vi) holds similarly. First, it is shown that (ii) implies (iv). Since $\mu(\tilde{G}\Gamma) > 0$, it follows by Lemma 3 that $\mathrm{img}(\tilde{G}\Gamma) = \mathrm{cl}(\mathrm{img}(\tilde{G}\Gamma))$ and $\ker(\tilde{G}\Gamma) = \{0\}$. Moreover, Lemma 7(vi) yields $\mu(T_\tau(\tilde{G}\Gamma)) > 0$ for all $\tau \in \mathbb{R}$, which with $T_\tau(\tilde{G}\Gamma)$ Fredholm and $\mathrm{ind}(T_\tau(\tilde{G}\Gamma)) = 0$, implies $\mathrm{img}(T_\tau(\tilde{G}\Gamma)) = P_\tau \mathcal{L}_2$. Exploiting the causality of $\tilde{G}\Gamma$, it follows that

$$\mathrm{img}(\tilde{G}\Gamma) \supset \bigcup_{\tau \in \mathbb{R}} \mathrm{img}(T_\tau(\tilde{G}\Gamma)) = \bigcup_{\tau \in \mathbb{R}} P_\tau \mathcal{L}_2 = \mathcal{L}_{2+},$$

whereby $\mathcal{L}_2 \supset \mathrm{img}(\tilde{G}\Gamma) = \mathrm{cl}(\mathrm{img}(\tilde{G}\Gamma)) \supset \mathrm{cl}(\mathcal{L}_{2+}) = \mathcal{L}_2$. That is, $\mathrm{img}(\tilde{G}\Gamma) = \mathcal{L}_2$, and thus, $\tilde{G}\Gamma$ is boundedly invertible by the inverse-mapping theorem [6, Theorem 5.7]. Finally, it is shown that $(\tilde{G}\Gamma)^{-1}$ is causal. By the previously established implication (ii)⇒(i), and Definition 11, it follows that $F_\tau^{-1} \in \mathscr{B}(P_\tau\mathcal{L}_2)$ is causal for every $\tau \in \mathbb{R}$; see Remark 12. Rearrangement of (5) gives

$$(T_\tau(\tilde{G}\Gamma))^{-1} = T_\tau(\Omega)\Big(\begin{bmatrix}-I & O\\ O & I\end{bmatrix}F_\tau^{-1} + \begin{bmatrix}I & O\\ O & O\end{bmatrix}\Big)T_\tau(\tilde{Z}),$$

where Ω and $\tilde{Z}$ are causal left and right inverses of Γ and $\tilde{G}$, respectively, as in (4) so that $T_\tau(\Omega)T_\tau(\Gamma) = I$ and $T_\tau(\tilde{G})T_\tau(\tilde{Z})$ by the causality of graph representations and the left/right inverses and Lemma 15(iv). As such, $(T_\tau(\tilde{G}\Gamma))^{-1}$ is causal for all $\tau \in \mathbb{R}$. Applying Lemma 7(iv) to $(\tilde{G}\Gamma)^{-1}\tilde{G}\Gamma = I$, gives

$$I = T_\tau((\tilde{G}\Gamma)^{-1}\tilde{G}\Gamma) = T_\tau((\tilde{G}\Gamma)^{-1})T_\tau(\tilde{G}\Gamma),$$

which implies $T_\tau((\tilde{G}\Gamma)^{-1}) = (T_\tau(\tilde{G}\Gamma))^{-1}$ for all $\tau \in \mathbb{R}$ by uniqueness of the inverse. Using this and the causality of $(T_\tau(\tilde{G}\Gamma))^{-1}$, it follows that $T_\tau((\tilde{G}\Gamma)^{-1})$ is causal for all $\tau \in \mathbb{R}$, which implies $(\tilde{G}\Gamma)^{-1}$ is causal by Lemma 7(iii).

Now (ii) is proved by assuming (iv) holds. If the causal operator $\tilde{G}\Gamma$ has bounded causal inverse, then $\tilde{G}\Gamma(\tilde{G}\Gamma)^{-1} = I = (\tilde{G}\Gamma)^{-1}\tilde{G}\Gamma$ and Lemma 7(iv) gives

$$I = T_\tau(\tilde{G}\Gamma(\tilde{G}\Gamma)^{-1}) = T_\tau(\tilde{G}\Gamma)T_\tau((\tilde{G}\Gamma)^{-1}) \text{ and}$$
$$I = T_\tau((\tilde{G}\Gamma)^{-1}\tilde{G}\Gamma) = T_\tau((\tilde{G}\Gamma)^{-1})T_\tau(\tilde{G}\Gamma),$$

whereby $T_\tau(\tilde{G}\Gamma)$ is invertible and satisfies the required Fredholm index condition for $\tau \in \mathbb{R}$. Moreover, $\mu(\tilde{G}\Gamma) = 1/\gamma((\tilde{G}\Gamma)^{-1}) > 0$ by Lemma 3. ∎

The normalization property of right and left graph presentations in property (B) of Definition 8, and the compactness property (D), are not used to establish Proposition 13. These properties facilitate the establishment of a robust stability result for uncertain feedback interconnections. In the rest of this section, a consequence of property (D) is noted first, in order to elucidate a consequence of property (D), two additional theorems are established to summarize useful consequences of property (B), and finally, the main robust feedback stability result is presented.

Lemma 14: ([1, Lem. 3.2]). For $\Delta_1, \Delta_2 \in \mathscr{C}_+$, $\tau \in \mathbb{R}$ and $k,l \in \{1,2\}$, each of the Hankel operators $H_\tau(\tilde{\Gamma}_k\tilde{\Gamma}_l^*)$, $J_\tau(\tilde{\Gamma}_k\tilde{\Gamma}_l^*)$, $H_\tau(\Gamma_k^*\Gamma_l)$ and $J_\tau(\Gamma_k^*\Gamma_l)$ is compact.

The next lemma reformulates a subset of the equalities given in (3.3) and (3.4) of [1], by relating the validity of these to the invertibility of specific operators; see part (iv). This reformulation corrects an error related to the issue noted Remark 5. Part (v) does not appear in [1].

Theorem 15: For $\Delta_1, \Delta_2 \in \mathscr{C}_+$ and $k,l \in \{1,2\}$, the following hold:

(i) Γ_k, $\tilde{\Gamma}_k^*$, $\begin{bmatrix}\Gamma_l^*\\ \tilde{\Gamma}_l\end{bmatrix}$, $\begin{bmatrix}\Gamma_l^*\Gamma_k\\ \tilde{\Gamma}_l\Gamma_k\end{bmatrix}$ and $\begin{bmatrix}\Gamma_l^*\tilde{\Gamma}_k^*\\ \tilde{\Gamma}_l\tilde{\Gamma}_k^*\end{bmatrix}$ are isometries;
(ii) $\gamma(\tilde{\Gamma}_l\Gamma_k) = \sqrt{1-\mu(\Gamma_l^*\Gamma_k)^2}$ and $\gamma(\Gamma_l^*\Gamma_k) = \sqrt{1-\mu(\tilde{\Gamma}_l\Gamma_k)^2}$;
(iii) $\mu(\tilde{\Gamma}_l\tilde{\Gamma}_k^*) = \sqrt{1-\gamma(\tilde{\Gamma}_k\Gamma_l)^2} = \mu(\Gamma_k^*\Gamma_l)$;
(iv) Individually (and thus collectively), each of the operators $\Gamma_l^*\Gamma_k$, $\tilde{\Gamma}_l\tilde{\Gamma}_k^*$, $\Gamma_k^*\Gamma_l$ and $\tilde{\Gamma}_k\tilde{\Gamma}_l^*$ is invertible in $\mathscr{B}(\mathcal{L}_2)$ if and only if $\mu(\tilde{\Gamma}_k\tilde{\Gamma}_l^*) = \mu(\Gamma_k^*\Gamma_l) = \mu(\Gamma_l^*\Gamma_k) = \mu(\tilde{\Gamma}_l\tilde{\Gamma}_k^*) > 0$, which is equivalent to $\gamma(\tilde{\Gamma}_l\Gamma_k) = \gamma(\tilde{\Gamma}_k\Gamma_l) < 1$;
(v) $\gamma(\tilde{\Gamma}_l\Gamma_k) = \gamma(\tilde{\Gamma}_k\Gamma_l) < 1$ if and only if $\mu(\Gamma_l^*\Gamma_k) > 0$ and $\gamma(\tilde{\Gamma}_k\Gamma_l) < 1$.

Proof: (i) Using the Bezout identity specified in property (A) of Definition 8 and the normalization property in property (B), it follows that the inverse graph representations Γ_k and $\tilde{\Gamma}_k$ of Δ_k are such that

$$\Gamma_k^*\Gamma_k = I,\ \tilde{\Gamma}_k\tilde{\Gamma}_k^* = I, \text{ and } \begin{bmatrix}\Gamma_l^*\\ \tilde{\Gamma}_l\end{bmatrix}\begin{bmatrix}\Gamma_l & \tilde{\Gamma}_l^*\end{bmatrix} = \begin{bmatrix}I & O\\ O & I\end{bmatrix}. \quad (6)$$

Moreover,

$$\ker\Big(\begin{bmatrix}\tilde{\Gamma}_l\\ \Gamma_l^*\end{bmatrix}\Big) = \ker(\tilde{\Gamma}_l) \cap \operatorname{img}(\Gamma_l)^\perp$$
$$= \operatorname{img}(\Gamma_l) \cap \operatorname{img}(\Gamma_l)^\perp = \{0\},$$

where the first equality holds by Lemma 1(ii) and the second equality holds by the identity $\ker(\tilde{\Gamma}_l) = \operatorname{img}(\Gamma_l)$ in property (C) of Definition 8. In view of this and (6), $\begin{bmatrix}\Gamma_l & \tilde{\Gamma}_l^*\end{bmatrix}$ is also the right inverse of $\begin{bmatrix}\Gamma_l & \tilde{\Gamma}_l^*\end{bmatrix}^*$, whereby $\Gamma_l\Gamma_l^* + \tilde{\Gamma}_l^*\tilde{\Gamma}_l = I$ (the inverse of a bijective operator is unique). Now using (6), $(\Gamma_k^*\Gamma_l)(\Gamma_l^*\Gamma_k) + (\Gamma_k^*\tilde{\Gamma}_l^*)(\tilde{\Gamma}_l\Gamma_k) = \Gamma_k^*(\Gamma_l\Gamma_l^* + \tilde{\Gamma}_l^*\tilde{\Gamma}_l)\Gamma_k = I$, as claimed. Similarly, $\begin{bmatrix}(\Gamma_l^*\tilde{\Gamma}_k^*)^* & (\tilde{\Gamma}_l\tilde{\Gamma}_k^*)^*\end{bmatrix}^*$ is an isometry.

(ii) These equalities follow by direct application of Lemma 4 to the third isometry in part (i). Note that $\left[\begin{smallmatrix}O & I\\ I & O\end{smallmatrix}\right]\begin{bmatrix}(\Gamma_l^*\Gamma_k)^* & (\tilde{\Gamma}_l\Gamma_k)^*\end{bmatrix}^*$ is also an isometry.

(iii) By Lemma 4 and the last isometry in part (i),

$$\mu(\tilde{\Gamma}_l\tilde{\Gamma}_k^*) = \sqrt{1-\gamma(\Gamma_l^*\tilde{\Gamma}_k^*)^2} = \sqrt{1-\gamma(\tilde{\Gamma}_k\Gamma_l)^2},$$

where the last equality above holds by Lemma 2(i). Applying the first equality in part (ii), it follows that

$$\sqrt{1-\gamma(\tilde{\Gamma}_k\Gamma_l)^2} = \mu(\Gamma_k^*\Gamma_l),$$

to give the result claimed.

(iv) By Lemma 3(i), Lemma 3(iii) and the identity $(\Gamma_l^*\Gamma_k)^* = \Gamma_k^*\Gamma_l$, bounded invertibility of $\Gamma_l^*\Gamma_k$ is equivalent to $\mu(\Gamma_l^*\Gamma_k) = \mu(\Gamma_k^*\Gamma_l) > 0$, and thus, bounded invertibility of $\Gamma_k^*\Gamma_l$. Similarly, bounded invertibility of $\tilde{\Gamma}_l\tilde{\Gamma}_k^*$ is equivalent to $\mu(\tilde{\Gamma}_l\tilde{\Gamma}_k^*) = \mu(\tilde{\Gamma}_k\tilde{\Gamma}_l^*) > 0$, and thus, bounded invertibility of $\tilde{\Gamma}_k\tilde{\Gamma}_l^*$. Now using the identity $\mu(\Gamma_l^*\Gamma_k) = \mu(\tilde{\Gamma}_k\tilde{\Gamma}_l^*)$ stated in part (iii), it follows that the invertibility of any one of the four operator compositions above (e.g. $\Gamma_l^*\Gamma_k$) is equivalent to $\mu(\tilde{\Gamma}_k\tilde{\Gamma}_l^*) = \mu(\Gamma_k^*\Gamma_l) = \mu(\Gamma_l^*\Gamma_k) = \mu(\tilde{\Gamma}_l\tilde{\Gamma}_k^*) > 0$, and thus, equivalent to invertibility of each the three remaining compositions. This establishes the first equivalence claimed in (iv). The final equivalence follows directly by the equalities established in part (iii).

(v) If $\gamma(\tilde{\Gamma}_k\Gamma_l) = \gamma(\tilde{\Gamma}_l\Gamma_k) < 1$, then $\mu(\Gamma_l^*\Gamma_k) > 0$ by part (iii), which establishes necessity of the statement in (v). To establish sufficiency, the equivalences in part (iv) mean that it is enough to show that $\mu(\Gamma_l^*\Gamma_k) > 0$ and $\gamma(\tilde{\Gamma}_k\Gamma_l) < 1$ implies invertibility of $\Gamma_l^*\Gamma_k$ (or any one of the other three) in $\mathscr{B}(\mathcal{L}_2)$. To this end, first note that $\mu(\Gamma_l^*\Gamma_k) > 0$ implies $\gamma(\tilde{\Gamma}_l\Gamma_k) < 1$ by part (iii), and thus, that $\gamma(\Gamma_k^*\tilde{\Gamma}_l^*\tilde{\Gamma}_l\Gamma_k) \le \gamma((\tilde{\Gamma}_l\Gamma_k)^*)\gamma(\tilde{\Gamma}_l\Gamma_k) < 1$ by Lemma 2(i) and Lemma 2(iii). Similarly, note that $\gamma(\tilde{\Gamma}_k\Gamma_l) < 1$ implies $\gamma(\Gamma_l^*\tilde{\Gamma}_k^*\tilde{\Gamma}_k\Gamma_l) \le \gamma((\tilde{\Gamma}_k\Gamma_l)^*)\gamma(\tilde{\Gamma}_k\Gamma_l) < 1$.

As such, applying Lemma 2(iv), the graph representation identity (6), and the part (i) identity $I = \Gamma_l \Gamma_l^* + \tilde{\Gamma}_l^* \tilde{\Gamma}_l$, it follows that

$$(\Gamma_l^* \Gamma_k)^* (\Gamma_l^* \Gamma_k) = \Gamma_k^* (\Gamma_l \Gamma_l^*) \Gamma_k = \Gamma_k^* (I - \tilde{\Gamma}_l^* \tilde{\Gamma}_l) \Gamma_k = I - \Gamma_k^* \tilde{\Gamma}_l^* \tilde{\Gamma}_l \Gamma_k \tag{7}$$

and

$$(\Gamma_l^* \Gamma_k) \Gamma_k^* \Gamma_l = (\Gamma_k^* \Gamma_l)^* (\Gamma_k^* \Gamma_l) = I - \Gamma_l^* \tilde{\Gamma}_k^* \tilde{\Gamma}_k \Gamma_l \tag{8}$$

are invertible in $\mathscr{B}(\mathcal{L}_2)$. Invertibility of the latter implies $\text{img}(\Gamma_l^* \Gamma_k) = \mathcal{L}_2$, and invertibility of the former implies $\{0\} = \ker((\Gamma_l^* \Gamma_k)^* \Gamma_l^* \Gamma_k) = \ker(\Gamma_l^* \Gamma_k) \cup \{w \in \mathcal{L}_2 \mid \Gamma_l^* \Gamma_k w \in \ker((\Gamma_l^* \Gamma_k)^*)\} = \ker(\Gamma_l^* \Gamma_k)$, where the last equality holds since $\ker((\Gamma_l^* \Gamma_k)^*) = \text{img}(\Gamma_l^* \Gamma_k)^\perp$ by Lemma 4(ii). As such, $\Gamma_l^* \Gamma_k \in \mathscr{B}(\mathcal{L}_2)$ is bijective, and therefore, $(\Gamma_l^* \Gamma_k)^{-1} \in \mathscr{B}(\mathcal{L}_2)$ by the inverse mapping theorem [6, Thm. 5.7]. ∎

The following corrects Lemma 3.3 in [1], where the condition $\mu(\Gamma_1^* \Gamma_2) > 0$ is missing because of the aforementioned error in the development of a number of identities that are corrected in Theorem 15 above.

Theorem 16: For $\Delta_1, \Delta_2 \in \mathscr{C}_+$, if $\gamma(\tilde{\Gamma}_k \Gamma_l) < 1$ and $\mu(\Gamma_l^* \Gamma_k) > 0$, then the following hold for $\tau \in \mathbb{R}$ and $k, l \in \{1, 2\}$:

(i) $\tilde{\Gamma}_k \tilde{\Gamma}_l^*$ and $\Gamma_k^* \Gamma_l$ are invertible in $\mathscr{B}(\mathcal{L}_2)$;
(ii) $T_\tau(\Gamma_k^* \Gamma_l)$ and $T_\tau(\tilde{\Gamma}_k \tilde{\Gamma}_l^*)$ are Fredholm;
(iii) $\text{ind}(T_\tau(\tilde{\Gamma}_k \tilde{\Gamma}_l^*)) = -\text{ind}(T_\tau(\Gamma_k^* \Gamma_l))$.

Proof: (i) Given $\gamma(\tilde{\Gamma}_k \Gamma_l) < 1$ and $\mu(\Gamma_l^* \Gamma_k) > 0$, the statement is a direct consequence of Theorem 15(iv-v), which also yields $\gamma(\tilde{\Gamma}_k \Gamma_l) = \gamma(\tilde{\Gamma}_l \Gamma_k) < 1$, whereby

$$\gamma(T_\tau((\tilde{\Gamma}_l \Gamma_k)^* \tilde{\Gamma}_l \Gamma_k)) = \gamma(P_\tau (\tilde{\Gamma}_l \Gamma_k)^* \tilde{\Gamma}_l \Gamma_k P_\tau) \leq \gamma((\tilde{\Gamma}_l \Gamma_k)^* \tilde{\Gamma}_l \Gamma_k) < 1 \tag{9}$$

for every $\tau \in \mathbb{R}$, and similarly,

$$\gamma(T_\tau((\tilde{\Gamma}_k \Gamma_l)^* \tilde{\Gamma}_k \Gamma_l)) < 1. \tag{10}$$

(ii) It is shown that $T_\tau(\Gamma_k^* \Gamma_l)$ is Fredholm; that $T_\tau(\tilde{\Gamma}_k \tilde{\Gamma}_l^*)$ is Fredholm holds by a corresponding line of argument. First, using (7), note that $T_\tau((\Gamma_l^* \Gamma_k)^* \Gamma_l^* \Gamma_k) = I - T_\tau(\Gamma_k^* \tilde{\Gamma}_l^* \tilde{\Gamma}_l \Gamma_k)$. Applying Lemma 6(iv) and (9) then gives $T_\tau((\Gamma_l^* \Gamma_k)^* \Gamma_l^* \Gamma_k)$ is Fredholm with $\text{ind}(T_\tau((\Gamma_l^* \Gamma_k)^* \Gamma_l^* \Gamma_k)) = \text{ind}(I) = 0$, whereby $T_\tau(\Gamma_k^* \Gamma_l (\Gamma_k^* \Gamma_l)^*) = T_\tau((\Gamma_l^* \Gamma_k)^* \Gamma_l^* \Gamma_k)$ is Fredholm with $\text{ind}(T_\tau(\Gamma_k^* \Gamma_l (\Gamma_k^* \Gamma_l)^*)) = 0$. Similarly, (8) and (10) can be used to show that $T_\tau((\Gamma_k^* \Gamma_l)^* \Gamma_k^* \Gamma_l)$ is Fredholm with $\text{ind}(T_\tau((\Gamma_k^* \Gamma_l)^* \Gamma_k^* \Gamma_l)) = 0$. Moreover, Lemma 7(ii) and Lemma 7(iv) apply to yield the decompositions

$$(T_\tau(\Gamma_k^* \Gamma_l))^* T_\tau(\Gamma_k^* \Gamma_l) = T_\tau((\Gamma_k^* \Gamma_l)^* \Gamma_k^* \Gamma_l) - H_\tau(\Gamma_l^* \Gamma_k) J_\tau(\Gamma_k^* \Gamma_l)$$

and

$$T_\tau(\Gamma_k^* \Gamma_l)(T_\tau(\Gamma_k^* \Gamma_l))^* = T_\tau(\Gamma_k^* \Gamma_l (\Gamma_k^* \Gamma_l)^*) - H_\tau(\Gamma_k^* \Gamma_l) J_\tau(\Gamma_l^* \Gamma_k),$$

which with Lemma 6(iii), Lemma 14 and [6, Thm. 13.2] (i.e., the composition of a compact operator and a bounded one is compact), lead to the property that $(T_\tau(\Gamma_k^* \Gamma_l))^* T_\tau(\Gamma_k^* \Gamma_l)$ and $T_\tau(\Gamma_k^* \Gamma_l)(T_\tau(\Gamma_k^* \Gamma_l))^*$ are Fredholm. In particular dim $\ker((T_\tau(\Gamma_k^* \Gamma_l))^* T_\tau(\Gamma_k^* \Gamma_l))$ and dim $\ker(T_\tau(\Gamma_k^* \Gamma_l)(T_\tau(\Gamma_k^* \Gamma_l))^*)$ are finite by definition. Now $\ker((T_\tau(\Gamma_l^* \Gamma_k))^*) = \text{img}(T_\tau(\Gamma_l^* \Gamma_k))^\perp$ by Lemma 4(ii), and therefore, $\ker((T_\tau(\Gamma_k^* \Gamma_l))^* T_\tau(\Gamma_k^* \Gamma_l)) = \ker(T_\tau(\Gamma_k^* \Gamma_l))$. Similarly, $\ker(T_\tau(\Gamma_k^* \Gamma_l)(T_\tau(\Gamma_k^* \Gamma_l))^*) = \ker((T_\tau(\Gamma_k^* \Gamma_l))^*)$. As such, dim $\ker(T_\tau(\Gamma_k^* \Gamma_l))$ and dim $\ker((T_\tau(\Gamma_k^* \Gamma_l))^*)$ are finite, whereby $T_\tau(\Gamma_k^* \Gamma_l)$ is Fredholm, as claimed.

(iii) This statement follows as shown in the proof of [1, Lem. 3.3(ii)], which is reproduced below for completeness. First note that $T_\tau(\tilde{\Gamma}_k^*)$ has a bounded left inverse; e.g., $T_\tau(\tilde{\Omega}_k^*) \in \mathscr{L}(P_\tau \mathcal{L}_2, P_\tau \mathcal{L}_2 \times P_\tau \mathcal{L}_2)$ with $\tilde{\Omega}_k$ as defined in (4), since $\tilde{\Omega}_k$ is causal and therefore, $I = T_\tau(\tilde{\Gamma}_k \tilde{\Omega}_k) = T_\tau(\tilde{\Gamma}_k) T_\tau(\tilde{\Omega}_k)$ by Lemma 7(iv), so that $I = (T_\tau(\tilde{\Omega}_k))^* (T_\tau(\tilde{\Gamma}_k))^* = T_\tau(\tilde{\Omega}_k^*) T_\tau(\tilde{\Gamma}_k^*)$, where the last equality holds by Lemma 7(ii). Similarly $T_\tau(\Gamma_k)$ has a bounded left inverse; e.g., $T_\tau(\Omega_k)$ as defined in (4). It follows that $\mu(T_\tau(\Gamma_k)) \geq 1/\gamma(T_\tau(\Omega_k)) > 0$ and $\mu(T_\tau(\tilde{\Gamma}_k^*)) \geq 1/\gamma(T_\tau(\tilde{\Omega}_k^*)) > 0$, which with Lemma 3(i), gives $\text{img}(T_\tau(\Gamma_k)) = \text{cl}(\text{img}(T_\tau(\Gamma_k)))$ and $\text{img}(T_\tau(\tilde{\Gamma}_k^*)) = \text{cl}(\text{img}(T_\tau(\tilde{\Gamma}_k^*)))$. Indeed, applying the property

$$\text{img}(T_\tau(\Gamma_k)) = \ker(T_\tau(\tilde{\Gamma}_k)) \tag{11}$$

specified in part (C) of Definition 8, Lemma 7(ii), and Lemma 1, the following holds:

$$\ker(T_\tau(\Gamma_k^*)) = \text{img}(T_\tau(\Gamma_k))^\perp = \ker(T_\tau(\tilde{\Gamma}_k))^\perp = \text{cl}(\text{img}(T_\tau(\tilde{\Gamma}_k^*))) = \text{img}(T_\tau(\tilde{\Gamma}_k^*)). \tag{12}$$

Now, by Lemma 7(iv), $T_\tau(\tilde{\Gamma}_k \tilde{\Gamma}_l^*) = T_\tau(\tilde{\Gamma}_k) T_\tau(\tilde{\Gamma}_l^*) + H_\tau(\tilde{\Gamma}_k) J_\tau(\tilde{\Gamma}_l^*)$, which gives $T_\tau(\tilde{\Gamma}_k) T_\tau(\tilde{\Gamma}_l^*)$ is Fredholm with $\text{ind}(T_\tau(\tilde{\Gamma}_k) T_\tau(\tilde{\Gamma}_l^*)) = \text{ind}(T_\tau(\tilde{\Gamma}_k \tilde{\Gamma}_l^*))$ by Lemma 6(iii) and the property (D) in Definition 8, whereby $H_\tau(\tilde{\Gamma}_k)$ is compact, and thus, $H_\tau(\tilde{\Gamma}_k) J_\tau(\tilde{\Gamma}_l^*)$ is compact as the composition of a compact operator and a bounded one [6, Thm. 13.2]. Furthermore,

$$\begin{aligned}
\text{ind}(T_\tau(\tilde{\Gamma}_k \tilde{\Gamma}_l^*)) &= \text{ind}(T_\tau(\tilde{\Gamma}_k) T_\tau(\tilde{\Gamma}_l^*)) \\
&= \dim \ker(T_\tau(\tilde{\Gamma}_k) T_\tau(\tilde{\Gamma}_l^*)) - \dim \ker(T_\tau(\tilde{\Gamma}_l) T_\tau(\tilde{\Gamma}_k^*)) \\
&= \dim \ker(T_\tau(\tilde{\Gamma}_k)) \cap \text{img}(T_\tau(\tilde{\Gamma}_l^*)) - \dim \ker(T_\tau(\tilde{\Gamma}_l)) \cap \text{img}(T_\tau(\tilde{\Gamma}_k^*)) \\
&= \dim \text{img}(T_\tau(\Gamma_k)) \cap \ker(T_\tau(\Gamma_l^*))) - \dim(\text{img}(T_\tau(\Gamma_l)) \cap \ker(T_\tau(\Gamma_k^*))) \\
&= \dim \ker(T_\tau(\Gamma_l^*) T_\tau(\Gamma_k)) - \dim \ker(T_\tau(\Gamma_k^*) T_\tau(\Gamma_l)) \\
&= -\text{ind}(T_\tau(\Gamma_k^*) T_\tau(\Gamma_l)) = -\text{ind}(T_\tau(\Gamma_k^* \Gamma_l)),
\end{aligned}$$

where the second equality holds by definition and Lemma 7(ii), the third because $\ker(T_\tau(\tilde{\Gamma}_l) T_\tau(\tilde{\Gamma}_k^*)) = \ker(T_\tau(\tilde{\Gamma}_k^*)) \cup \{w \in \mathcal{L}_2 \,|\, T_\tau(\tilde{\Gamma}_k^*) w \in \ker(T_\tau(\tilde{\Gamma}_l))\}$, which implies dim $\ker(T_\tau(\tilde{\Gamma}_l) T_\tau(\tilde{\Gamma}_k^*)) = \dim \ker(T_\tau(\tilde{\Gamma}_l)) \cap \text{img}(T_\tau(\tilde{\Gamma}_k^*))$ since $\ker(T_\tau(\tilde{\Gamma}_k^*)) = \{0\}$, the fourth by (11) and (12), the fifth in a similar fashion to the third, the sixth by definition and Lemma 7(ii), and the last by Lemma 7(iv) as Γ_k is casual. ∎

It is now possible to formulate and prove the main robust stability result, which subsequently gives rise to the

LTV generalization of the ν-gap. Then next result corrects Theorem 4.1 in [1].

Theorem 17: Given $M,\Delta_1,\Delta_2 \in \mathscr{C}_+$, suppose $[M,\Delta_1]$ is stable and $\gamma(\tilde{\Gamma}_2\Gamma_1) < \mu(\tilde{\Gamma}_1 G)$. If $\mu(\Gamma_1^*\Gamma_2) > 0$ and $\mathrm{ind}(T_\tau(\Gamma_1^*\Gamma_2)) = -\mathrm{ind}(T_\tau(\tilde{\Gamma}_1\tilde{\Gamma}_2^*)) = 0$ for all $\tau \in \mathbb{R}$, then $[M,\Delta_2]$ is stable.

Proof: Note that $\gamma(\tilde{\Gamma}_2\Gamma_1) < \mu(\tilde{\Gamma}_1 G) \le 1$ as $\mu(\tilde{\Gamma}_1 G) \le \gamma(\tilde{\Gamma}_1 G) \le \gamma(\tilde{\Gamma}_1)\gamma(G) = 1$. Also note that $\gamma(\tilde{\Gamma}_2\Gamma_1) < \mu(\tilde{\Gamma}_1 G)$ implies

$$\gamma(\tilde{\Gamma}_2\Gamma_1)\sqrt{1-\mu(\tilde{\Gamma}_1 G)^2} < \mu(\tilde{\Gamma}_1 G)\sqrt{1-\gamma(\tilde{\Gamma}_2\Gamma_1)^2}, \qquad (13)$$

which holds (trivially) when $\mu(\tilde{\Gamma}_1 G) = 1$, and by the strictly increasing nature of the mapping

$$x \mapsto x/\sqrt{1-x^2}, \qquad \text{for all } x \in [0,1),$$

otherwise. Now, if $\mu(\Gamma_1^*\Gamma_2) > 0$ in addition to $\gamma(\tilde{\Gamma}_2\Gamma_1) < 1$, then $\gamma(\tilde{\Gamma}_1\Gamma_2) = \gamma(\tilde{\Gamma}_2\Gamma_1) < 1$ and $\mu(\Gamma_1^*\Gamma_2) = \mu(\tilde{\Gamma}_2\tilde{\Gamma}_1^*) = \mu(\tilde{\Gamma}_1\tilde{\Gamma}_2^*) = \mu(\Gamma_2^*\Gamma_1) > 0$ by Theorem 15(iv-v), which with Lemma 2, Lemma 3(v), Theorem 15(iii), and (13), yields the inequalities

$$\begin{aligned}\gamma(\tilde{\Gamma}_2\Gamma_1\Gamma_1^* G) &\le \gamma(\tilde{\Gamma}_2\Gamma_1)\gamma(\Gamma_1^* G)\\ &= \gamma(\tilde{\Gamma}_2\Gamma_1)\sqrt{1-\mu(\tilde{\Gamma}_1 G)^2}\\ &< \mu(\tilde{\Gamma}_1 G)\sqrt{1-\gamma(\tilde{\Gamma}_2\Gamma_1)^2}\\ &= \mu(\tilde{\Gamma}_1 G)\mu(\tilde{\Gamma}_1\tilde{\Gamma}_2^*)\\ &= \mu(\tilde{\Gamma}_1 G)\mu(\tilde{\Gamma}_2\tilde{\Gamma}_1^*)\\ &\le \mu(\tilde{\Gamma}_2\tilde{\Gamma}_1^*\tilde{\Gamma}_1 G). \qquad (14)\end{aligned}$$

Moreover, $\tilde{\Gamma}_2\tilde{\Gamma}_1^*$ is boundedly invertible by Theorem 15(iv) and $T_\tau(\tilde{\Gamma}_2\tilde{\Gamma}_1^*)$ is Fredholm with $\mathrm{ind}(T_\tau(\tilde{\Gamma}_2\tilde{\Gamma}_1^*)) = -\mathrm{ind}(\Gamma_2^*\Gamma_1)$ by Theorem 16(ii-iii). Finally, note that $\tilde{\Gamma}_1 G$ is boundedly invertible by Proposition 13(v) as $[M,\Delta_1]$ is stable. Therefore, $\tilde{\Gamma}_2\tilde{\Gamma}_1^*\tilde{\Gamma}_1 G$ is boundedly invertible.

The rest of the proof follows the argument presented in the last part of the proof of [1, Thm. 4.1]. ■

IV. An LTV ν-gap metric

The robust stability result in Theorem 17 motivates the following generalized definition of a ν-gap metric for systems in the class $\mathscr{C}_+$; the metric property follows by arguments similar to those in [15], via modifications similar to the corresponding of [1] documented in Section III above. The following corrects Definition 4.2 in [1] by adding the condition $\mu(\Gamma_1^*\Gamma_2) > 0$ to the Fredholm index conditions to define the case in which the ν-gap is equal to $\gamma(\tilde{\Gamma}_2\Gamma_1)$. Consistency of the generalization with the time-invariant theory still follows as shown in [1, Sec. 4.1]. A corresponding ν-gap robust stability result is also stated below.

Definition 18: Let $\delta_\nu : \mathscr{C}_+ \times \mathscr{C}_+ \to [0,1]$ be defined by

$$\delta_\nu(\Delta_1,\Delta_2) := \begin{cases}\gamma(\tilde{\Gamma}_2\Gamma_1) & \text{if } \mu(\Gamma_1^*\Gamma_2) > 0 \text{ and } T_\tau(\Gamma_1^*\Gamma_2) \\ & \text{is Fredholm with} \\ & \mathrm{ind}(T_\tau(\Gamma_1^*\Gamma_2)) = 0\ \forall \tau \in \mathbb{R}, \\ 1 & \text{otherwise.}\end{cases}$$

where Γ_k and $\tilde{\Gamma}_k$ denote normalized right and left graph representations for Δ_k $(k = 1,2)$, as in Section III.

Corollary 19: ([1, Cor. 4.3]). Given $M,\Delta \in \mathscr{C}_+$, let

$$\beta(M,\Delta_k) := \begin{cases}\mu(\tilde{\Gamma}_k G) & \text{if } [M,\Delta_k] \text{ is stable,}\\ 0 & \text{otherwise.}\end{cases}$$

Note that $0 \le \beta(M,\Delta_k) \le 1$ for $k \in \{1,2\}$. Then

$$\arcsin\ \beta(M,\Delta_2) \ge \arcsin\ \beta(M,\Delta_1) - \arcsin\ \delta_\nu(\Delta_1,\Delta_2)$$

In particular, if $[M,\Delta_1]$ is stable and $\delta_\nu(\Delta_1,\Delta_2) < \beta(M,\Delta_1)$, then $[M,\Delta_2]$ is stable.

Proof: The result follows from Theorem 17 as shown in the proof of [1, Corol. 4.3], along the lines of the time-variant proof given in [2, Thm. 4.2]. ■

V. Conclusion

Aspects of the time-domain development of an LTV ν-gap metric in the paper [1] are rectified in the preceding sections. The corrections made ultimately lead to the addition of a condition to the family of Fredholm index conditions that generalize the determinant and winding number condition in the frequency-domain of the ν-gap for linear time-invariant systems.

References

[1] M. Cantoni, U. T. Jönsson, S.Z. Khong, Robust stability analysis for feedback interconnections of time-varying linear systems, SIAM J. Control Optim. 51 (2013), 353-379.

[2] G. Vinnicombe, Frequency domain uncertainty and the graph topology, IEEE Trans. Autom. Contr. 38 (1993), 1371-1383.

[3] G. Vinnicombe, Uncertainty and Feedback: $\mathscr{H}^\infty$ Loopshaping and the ν-Gap Metric, Imperial College Press, London, 2001.

[4] U. T. Jönsson and M. Cantoni, Robust stability analysis for feedback interconnections of unstable time-varying systems, Technical Report TRITA-MAT-2010-OS 04, Department of Mathematics, Royal Institute of Technology (KTH), Sweden, 2010.

[5] U. T. Jönsson and M. Cantoni, Robust stability analysis for feedback interconnections of unstable time-varying systems, in Proceedings of the American Control Conf., 2011, 3734-3741.

[6] B. Bollobas, Linear Analysis: An Introductory Course, Cambridge Math. Textbooks, Cambridge University Press, Cambridge, UK, 1990.

[7] I. Gohberg, S. Goldberg, M. A. Kaashoek, Basic Classes of Linear Operators, Birkhauser Verlag, 2003.

[8] A. Feintuch, Robust Control Theory in Hilbert Spaces, Applied Mathematics Sciences 130, Springer, 1998.

[9] W. Bian, M. French, and H. K. Pillai, An intrinsic behavioral approach to the gap metric, SIAM J. Control Optim., 47 (2008), 1939-1960.

[10] H. L. Trentelman and S. V. Gottimukkala, Distance between behaviors and rational representations, SIAM J. Control Optim., 51 (2013), 4211-4241.

[11] M. Cantoni, A characterisation of the gap metric for approximation problems, in Proceedings of the 45th IEEE Conf. on Decision and Control, 2006, 5359-5364.

[12] H.L. Royden and P.M. Fitzpatrick, Real Analysis, Pearson Inc., Fourth Edition, 2010.

[13] W. Rudin, Principles of Mathematical Analysis, McGraw-Hill, Third Edition, 1976.

[14] S. Vuglar, M. Cantoni, Computing the distance between time-periodic dynamical systems, Proceedings of 22nd International Symposium on Mathematical Theory of Networks and Systems, 2016.

[15] S.Z. Khong, M. Cantoni, Gap metrics for time-varying linear systems in a continuous-time setting, Systems & Control Letters, 70 (2014), 118-126.