NON-UNIFORM DFT FILTER BANKS DESIGN WITH SEMI-DEFINITE PROGRAMMING

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ABSTRACT

Uniform DFT filter banks are well studied but they are not always suitable for applications with non-uniform time frequency representations such as speech and image processing. Using an allpass tranformation, non-uniform spacings between frequency subbands can be obtained. However, this introduces additional aliasing due to non-linear phase distortions. This paper considers the design of non-uniform DFT filter banks with specified group delay by minimising aliasing whilst constraining magnitude and phase distortions to a prescribed tolerance.

1. INTRODUCTION

In multirate signal processing, the signal of interest is often decomposed into frequency subbands which can be processed in parallel at a lower sampling rate [6]. This provides a significant reduction in computational complexity [6] and ease of implementation [1]. Uniform filter banks (i.e. where the subbands are evenly spaced) have been well studied [6]. However, in many applications a non-uniform time frequency representation is more appropriate, for example, in the psychoacoustic model of human hearing, the cochlear decomposes speech signals into non-uniform subbands [9]. Consequently, non-uniform filter banks provide a natural means for processing such signals.

One of the main problem in filter bank design is the aliasing of the subbands due to non-perfect frequency selectivity of the subband filters. This problem is further compounded in non-uniform filter banks due to the non-linear phase characteristics of the filters obtained from the allpass transformation [8]. In [3], the sum of the aliasing and distortion was minimised. This formulation does not allow the designer to control neither the distortion nor the aliasing. Although weighting these two criteria provides some control on the solution, it is not clear how the weights should be selected. Moreover, there is no guarantee that any specified bounds on the aliasing or distortion are met.

This paper proposes a method for designing the analysis and synthesis filters of non-uniform DFT filter banks to meet specifications on the prototype response whilst achieving near perfect reconstruction. The resulting problem is a Quadratically Constrained Quadratic Program (QCQP) which can be efficiently solved by standard Semi-Definite Programming (SDP) techniques [7].

2. BACKGROUND

A non-uniform DFT filter bank can be obtained by replacing the delay elements in the uniform DFT filter bank with an allpass function $Q(z)$ [4]. The resulting non-uniform filter bank in polyphase form is shown in figure 1. The phase characteristics of $Q(z)$ determine the non-uniform spacing between the subbands.

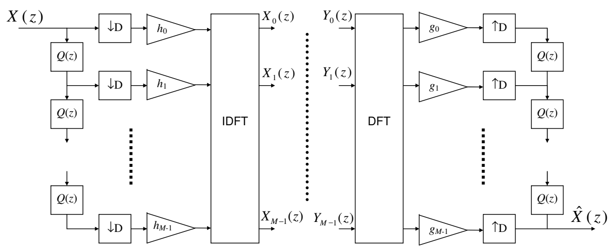

Fig. 1. Polyphase form of DFT filter banks

In the analysis filter bank of figure 1,

$$X_m(z) = \frac{1}{D}\sum_{d=0}^{D-1} X(z^{\frac{1}{D}} W_D^d)\mathbf{h}^T \phi_{m,d}(z) \qquad (1)$$

where

$$\begin{aligned} W_M &= e^{-j2\pi/M}, \\ \mathbf{h} &= [h_0, ..., h_{M-1}]^T, \\ \phi_{m,d}(z) &= [\phi_{m,d}^0(z), ..., \phi_{m,d}^{M-1}(z)]^T, \text{ and} \\ \phi_{m,d}^l(z) &= (W_M^{-m} Q(z^{\frac{1}{D}} W_D^d))^l. \end{aligned}$$

Note that the transfer function for the mth analysis filter is $H_m(z) = \mathbf{h}^T \phi_{m,0}(z^D)$.

A representative measure of the mean aliasing in the analysis bank is the inband aliasing

$$\beta(\mathbf{h}) = \sum_{d=0, d\neq D/2}^{D-1} \int_{-\pi}^{\pi} |\mathbf{h}^T \phi_{M/2,d}(e^{j\omega})|^2 d\omega. \quad (2)$$

The subband $m = M/2$ has the lowest signal to aliasing ratio and the aliasing term $d = D/2$ is omitted as it contains the desired signal content for this subband [3]. The performance criteria for the analysis filter banks are the inband aliasing (2) and the analysis distortion

$$\alpha(\mathbf{h}) = \int_{-\pi}^{\pi} |\mathbf{h}^T \phi_{0,0}(e^{j\omega D}) - \tilde{H}(e^{j\omega})|^2 d\omega \quad (3)$$

i.e. the deviation of the prototype filter H_0 from its desired response $\tilde{H}$, where $\tilde{H}$ is a lowpass filter with cutoff frequency ω_p and group delay τ_h:

$$\tilde{H}(e^{j\omega}) = e^{-j\omega\tau_h} \begin{cases} 1, \omega \in [-\omega_p, \omega_p] \\ 0, \omega \in \text{otherwise} \end{cases}$$

Ideally, we want both the inband aliasing $\beta(\mathbf{h})$ and the analysis distortion $\alpha(\mathbf{h})$ to be zero. However, this is not the case in practice and the design objective is to keep $\beta(\mathbf{h})$ and $\alpha(\mathbf{h})$ down to some acceptable level. In [3], the authors minimised the sum of $\beta(\mathbf{h})$ and $\alpha(\mathbf{h})$. The advantage of this approach lies in its simplicity, however, it is not possible to control the inband aliasing and the analysis distortion individually. Minimising a weighted sum of the two components can provide some control, but it is not clear how the weights should be chosen. A more sensible approach, presented in Section 3, is to minimise $\beta(\mathbf{h})$ and constrain $\alpha(\mathbf{h})$ to a prescribed tolerance.

For the synthesis filter bank, consider the reconstructed signal

$$\hat{X}(z) = \sum_{m=0}^{M-1} Y_m(z^D) \mathbf{g}^T \varphi_m(z) \quad (4)$$

where

$$\begin{aligned} \mathbf{g} &= [g_{0,...,}g_{M-1}]^T, \\ \varphi_m(z) &= [\varphi_m^0(z), ..., \varphi_m^{M-1}(z)]^T, \text{ and} \\ \varphi_m^l(z) &= (W_M^m Q(z)^{-1})^l Q(z)^{M-1}. \end{aligned}$$

Note that the transfer function mth of the synthesis filter is $G_m(z) = \mathbf{g}^T \varphi_m(z)$. The reconstruction accuracy of the filter bank can be evaluated by measuring the distortion to a known input X when there is no subband processing, i.e. $X_m = Y_m$. In this case,

$$\hat{X}(z) = \sum_{m=0}^{M-1} \sum_{d=0}^{D-1} X(zW_D^d) \mathbf{h}^T \mathbf{\Phi}_{m,d}(z) \mathbf{g} \quad (5)$$

where $\mathbf{\Phi}_{m,d}(z) = \frac{1}{D} \phi_{m,d}(z^D) \varphi_m^T(z)$. The overall response T of the filter bank is defined as [3], i.e.

$$T(z) = \mathbf{h}^T \mathbf{\Psi}(z) \mathbf{g} \quad (6)$$

where $\mathbf{\Psi}(z) = \sum_{m=0}^{M-1} \sum_{d=0}^{D-1} \mathbf{\Phi}_{m,d}(z)$.

In the design of the synthesis filters, we consider the overall performance of the filter bank for fixed analysis filter coefficients $\mathbf{h}$. Analogous to the analysis filter bank, the performance criteria are the residual aliasing

$$\delta(\mathbf{g}) = \sum_{m=0}^{M-1} \sum_{d=1}^{D-1} \int_{-\pi}^{\pi} |\mathbf{h}^T \mathbf{\Phi}_{m,d}(e^{j\omega}) \mathbf{g}|^2 d\omega, \quad (7)$$

and the overall distortion

$$\gamma(\mathbf{g}) = \int_{-\pi}^{\pi} |\mathbf{h}^T \mathbf{\Psi}(e^{j\omega}) \mathbf{g} - \tilde{T}(e^{j\omega})|^2 d\omega \quad (8)$$

where $\tilde{T}$ is the desired overall response taken here to be a unit gain with constant group delay τ_p:

$$\tilde{T}(e^{j\omega}) = e^{-j\omega\tau_T}, \omega \in [-\pi, \pi]$$

Ideally, both the residual aliasing $\delta(\mathbf{g})$ and the overall distortion $\gamma(\mathbf{g})$ should be zero. The design objective is to keep $\delta(\mathbf{g})$ and $\gamma(\mathbf{g})$ low. Again, similar to the design of the analysis filters, minimising the sum of $\delta(\mathbf{g})$ and $\gamma(\mathbf{g})$ as in [3] is not satisfactory. A more practical approach is to minimise $\delta(\mathbf{g})$ and constrain $\gamma(\mathbf{g})$ to a specified level.

3. PROBLEM FORMULATION AND SOLUTION

This section formulates the design of the analysis and synthesis filters according to the criteria discussed in Section 2.

For the design of the analysis filter bank, we seek the vector $\mathbf{h}$ which minimises the inband aliasing (2) while keeping the analysis distortion (3) below a prescribed value. This problem is a Quadratically Constrained Quadratic Program (QCQP):

$$\begin{aligned} \min \beta(\mathbf{h}) &= \mathbf{h}^T \mathbf{A}_0 \mathbf{h} & (9) \\ \text{s.t. } \alpha(\mathbf{h}) &= \mathbf{h}^T \mathbf{A}_1 \mathbf{h} - \mathbf{h}^T \mathbf{c}_1 + ||\tilde{H}||^2 \le K_h & (10) \end{aligned}$$

where

$$\begin{aligned} \mathbf{A}_0 &= \sum_{d=0, d\neq D/2}^{D-1} \int_{-\pi}^{\pi} \phi_{M/2,d}(e^{j\omega}) \phi_{M/2,d}^H(e^{j\omega}) d\omega \\ \mathbf{A}_1 &= \int_{-\pi}^{\pi} \phi_{0,0}(e^{j\omega D}) \phi_{0,0}^H(e^{j\omega D}) d\omega & (11) \\ \mathbf{c}_1 &= 2\int_{-\pi}^{\pi} Re\{\tilde{H}^*(e^{j\omega}) \phi_{0,0}(e^{j\omega D})\} d\omega & (12) \\ ||\tilde{H}||^2 &= \int_{-\pi}^{\pi} |\tilde{H}(e^{j\omega})|^2 d\omega \end{aligned}$$

Note that α is convex and hence, if $\mathbf{A}_0$ is positive definite the solution to problem (9-10) is unique if one exists. For $\mathbf{A}_0$ positive semidefinite, there exist non-trivial filter coefficients which achieve zero inband aliasing. Assuming $\mathbf{A}_1$ positive definite, $\mathbf{h}=0.5\mathbf{A}_1^{-1}\mathbf{c}_1$ achieves the minimum allowable analysis distortion of $||\tilde{H}||^2 - 0.25\mathbf{c}_1^T\mathbf{A}_1^{-1}\mathbf{c}_1$ and thus any K_h below this value would render problem (9-10) infeasible.

Similarly, for a given analysis filter bank parameterised by **h,** the design of the synthesis filters seeks a vector **g** which minimises the residual aliasing (7) while keeping the overall distortion (8) within a prescribed bound. The resulting problem is the QCQP:

$$\begin{aligned} \min \delta(\mathbf{g}) &= \mathbf{g}^T\mathbf{B}_0\mathbf{g} && (13) \\ \text{s.t. } \gamma(\mathbf{g}) &= \mathbf{g}^T\mathbf{B}_1\mathbf{g} - \mathbf{g}^T\mathbf{d}_1 + ||\tilde{T}||^2 \leq K_g && (14) \end{aligned}$$

where

$$\begin{aligned} \mathbf{B}_0 &= \sum_{m=0}^{M-1}\sum_{d=1}^{D-1}\int_{-\pi}^{\pi} \mathbf{\Phi}_{m,d}^H(e^{j\omega})\mathbf{h}^*\mathbf{h}^T\mathbf{\Phi}_{m,d}(e^{j\omega})d\omega \\ \mathbf{B}_1 &= \int_{-\pi}^{\pi} \mathbf{\Psi}^H(e^{j\omega})\mathbf{h}^*\mathbf{h}^T\mathbf{\Psi}(e^{j\omega})d\omega \\ \mathbf{d}_1 &= 2\int_{-\pi}^{\pi} Re\{\tilde{T}(e^{j\omega})\mathbf{\Psi}^T(e^{j\omega})\mathbf{h}\}d\omega \\ ||\tilde{T}||^2 &= \int_{-\pi}^{\pi} |\tilde{T}(e^{j\omega})|^2 d\omega \end{aligned}$$

Again, assuming $\mathbf{B}_0$ positive definite, the solution to problem (13-14) is unique if one exists. If $\mathbf{B}_1$ is positive definite, any K_g below the minimum achievable overall distortion, $||\tilde{T}||^2 - 0.25\mathbf{d}_1^T\mathbf{B}_1^{-1}\mathbf{d}_1^T$, means that problem (13-14) is infeasible.

The roles of the costs and constraints can be reversed for applications where the specifications are given in terms of the aliasing rather than the distortion.

QCQP's can be expressed as equivalent semi-definite programs (SDP) for which solutions can be efficiently computed [5], [7]. For a more comprehensive survey of SDP's, the reader is referred to [7] and references therein. The analysis filter design problem (9-10) can be written as the SDP:

$$\begin{aligned} &\min t \\ &\text{s.t.} \begin{bmatrix} \mathbf{I} & \mathbf{A}_0^{1/2}\mathbf{h} \\ (\mathbf{A}_0^{1/2}\mathbf{h})^T & t \end{bmatrix} \geq 0 \\ &\begin{bmatrix} \mathbf{I} & \mathbf{A}_1^{1/2}\mathbf{h} \\ (\mathbf{A}_1^{1/2}\mathbf{h})^T & \mathbf{c}_1^T\mathbf{h} - (||\tilde{H}||^2 - K_h) \end{bmatrix} \geq 0 \end{aligned}$$

and the synthesis filter design problem (13-14) as:

$$\begin{aligned} &\min t \\ &\text{s.t.} \begin{bmatrix} \mathbf{I} & \mathbf{B}_0^{1/2}\mathbf{g} \\ (\mathbf{B}_0^{1/2}\mathbf{g})^T & t \end{bmatrix} \geq 0 \\ &\begin{bmatrix} \mathbf{I} & \mathbf{B}_1^{1/2}\mathbf{g} \\ (\mathbf{B}_1^{1/2}\mathbf{g})^T & \mathbf{d}_1^T\mathbf{g} - (||\tilde{T}||^2 - K_g) \end{bmatrix} \geq 0 \end{aligned}$$

The contribution to the analysis distortion (3) outside of the interval $[-\omega_p, \omega_p]$ is effectively an aliasing component which has already been accounted for in the definition of β (2). Hence, (11) and (12) can be taken as

$$\begin{aligned} \mathbf{A}_1 &= \int_{-\omega_p}^{\omega_p} \phi_{0,0}(e^{j\omega D})\phi_{0,0}^H(e^{j\omega D})d\omega \\ \mathbf{c}_1 &= 2\int_{-\omega_p}^{\omega_p} Re\{e^{j\omega\tau_h}\phi_{0,0}(e^{j\omega D})\}d\omega \end{aligned}$$

without adversely affecting the solution.

4. DESIGN EXAMPLE

Consider a non-uniform filter bank with $M = 16$ channels and non-critical decimation $D = 2$. For the allpass transformation, the first order Laguerre allpass filter with a uniformity coefficient of $\mu = 0.4$ is used. Remark: if $\mu = 0$, this reduces to the uniform DFT case. The desired analysis filter has a gain of 1 in the passband $[-\omega_p, \omega_p]$ with $\omega_p = \pi/M$ and group delay $\tau_h = (M-1)/2$. The desired overall response has again of 1 and group delay $\tau_T = M$.

A design based on [3], i.e. minimising the sum of the aliasing and distortion, yields an analysis distortion of $\alpha = 8.46 \times 10^{-3}$ and an overall distortion of $\gamma = 5.94 \times 10^{-1}$ which is unacceptable for speech applications. We need to push both the analysis distortion and the overall distortion further down. Suppose that we want the analysis and overall distortion to be less than $K_h = 5.1$ and $K_g = 9.7$ respectively. Since we know the bound for a feasible solution we can set the constraint larger than that and trade off the inband and residual aliasing respectively. The design task is to determine the analysis and synthesis filter with minimal aliasing while satisfying these requirements. This can be achieved using the proposed QCQP approach. Solutions to the relevant SDP's are computed numerically using the software package [5]. The minimum inband and residual aliasing are respectively 1.95×10^{-11} and 4.50×10^{-11}.

Figures 2 and 4 show the magnitude responses of the analysis filters based on [3] and QCQP respectively. The approach in [3] yields sidelobes at -25db while QCQP yields sidelobles at -47db. This is quite a significant improvement. Figures 3 and 5 show the overall magnitude response ripples obtained from [3] and QCQP respectively. The QCQP design yields an overall response ripple 10^{-3} times smaller than the design based on [3].

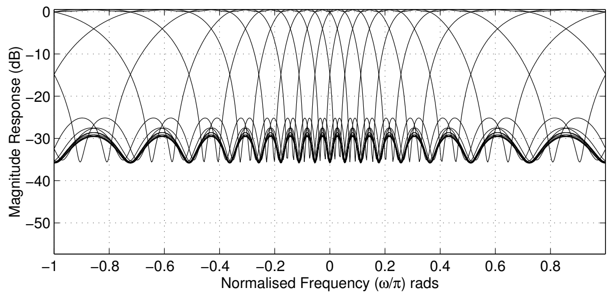

Fig. 2. Analysis filter responses of design based on [3]

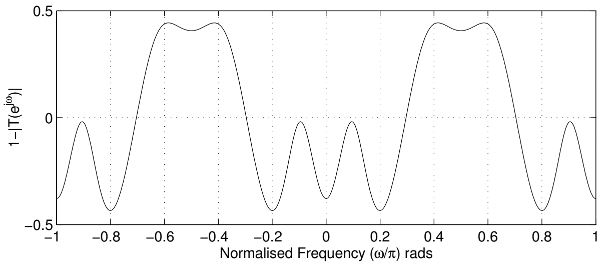

Fig. 3. Overall response ripple of design based on [3]

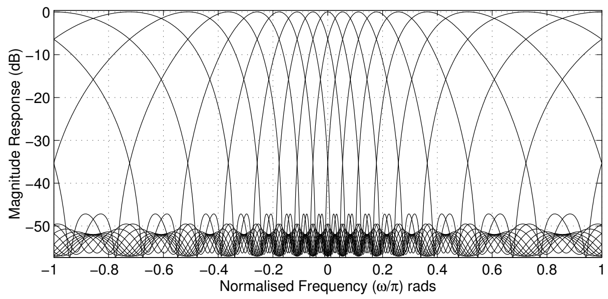

Fig. 4. Analysis filter responses of QCQP design

5. CONCLUSION

An efficient approach for the design of non-uniform DFT analysis and synthesis filter banks to meet specifications on the distortion whilst achieving near perfect reconstruction has been proposed. The design solution amounts to solving two Quadratically Constrained Quadratic Programs. It has been demonstrated that the proposed approach provides an efficient and versatile tool for the design of non-uniform filter banks. Numerical comparison with the approach proposed in [3] shows a dramatic improvement. Extension of

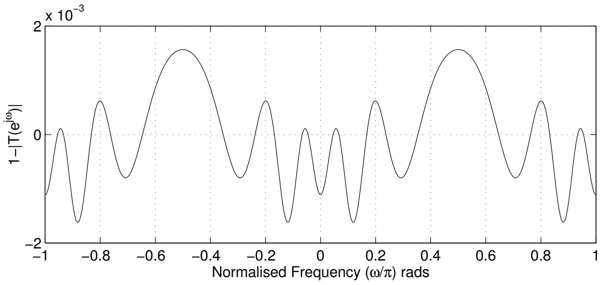

Fig. 5. Overall response ripple of QCQP design

our approach to generalised DFT filter banks such as [2] is under investigation.

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