Lecture Notes for
436-351 Thermofluids 2
Unit 1: Potential Flow

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Preliminaries

Main Text

- Anderson J. D. **Fundamentals of Aerodynamics**, *McGraw-Hill*

Suggested Reading

- Vallentine H. R. **Applied hydrodynamics**
- Lamb **Hydrodynamics**
- Streeter **Fluid dynamics**
- Milne & Thomson **Theoretical Aerodynamics**

Assessment (Unit 1 only)

70 % End of semester examination
20 % Assignment
10 % Prac

Contents

Chapter 1

Introduction

We wish to predict fluid motion, that is the flow patterns and associated forces they create (eg. lift and drag). In many cases this is a difficult task and several different approaches may be required.

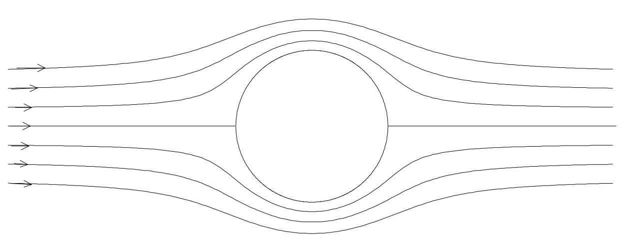
Analytical — Simplify equations

Computational — Model the physics that is not understood

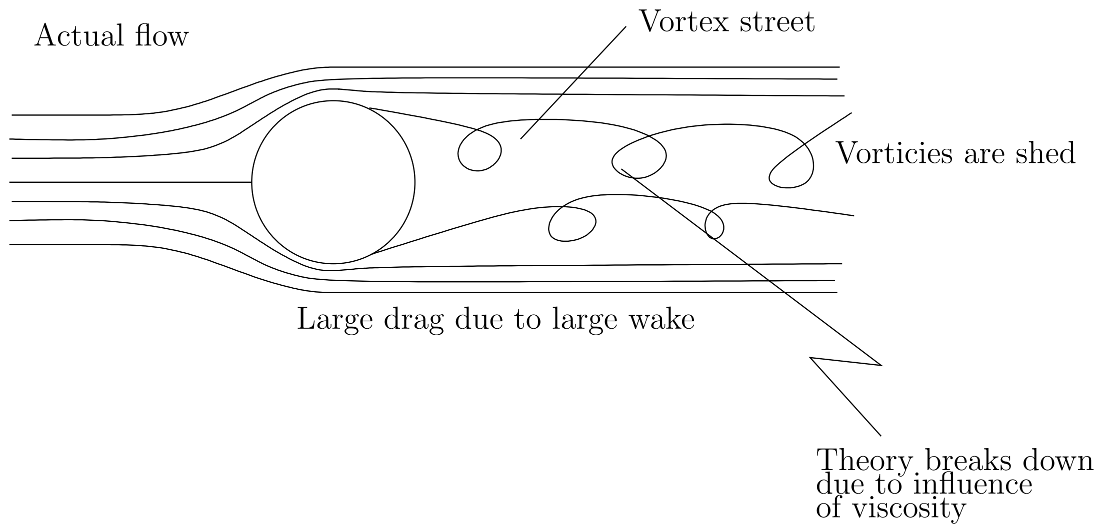
Experimental — Test scale models

Here we will consider the analytical method known as classical hydrodynamics. This involves the study of ideal fluids, by ideal we mean incompressible and frictionless (inviscid), ie. $\rho =$constant and $\mu = 0$.

For example consider flow around a cylinder,
Classical model

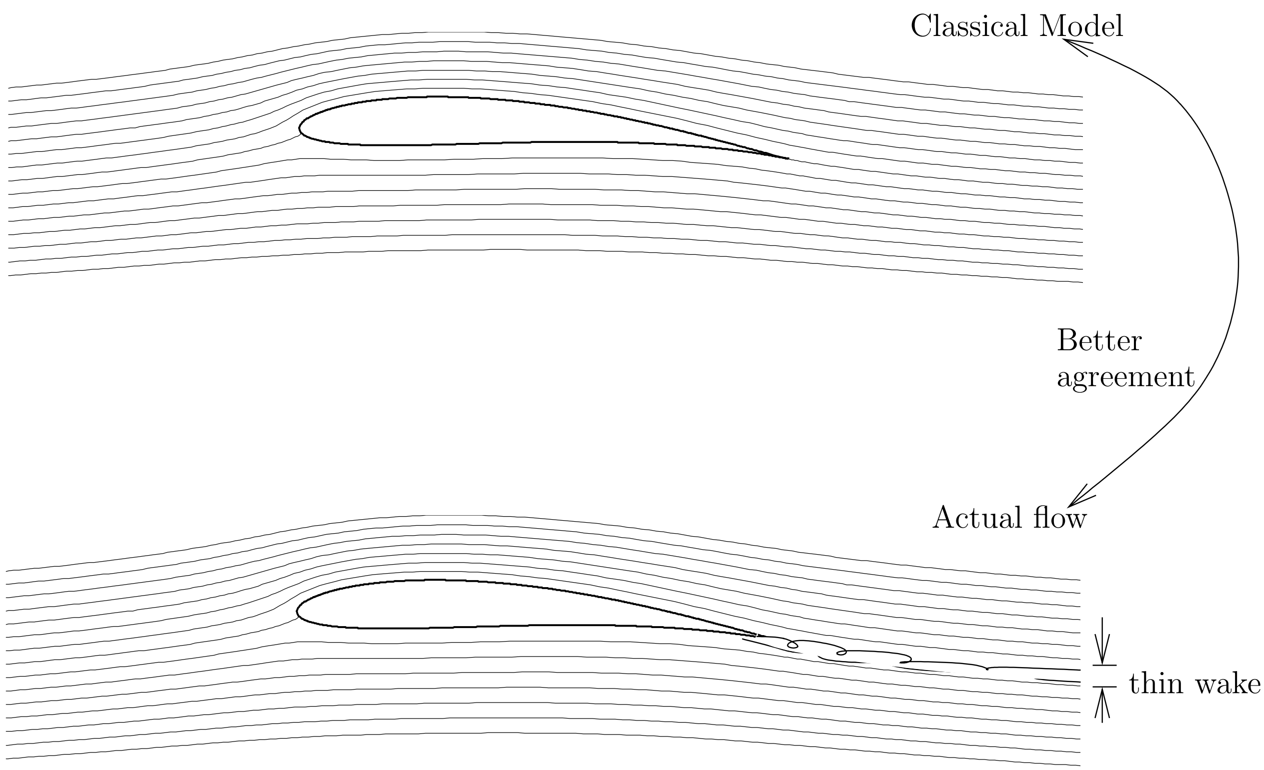


Drag$= 0$



The discrepency between the above is known as d'Alembert's paradox. Engineers initially largely ignored the classical approach. However in many fluid flows friction is only important in regions such as boundary layers and wakes. Outside of these regions the fluid may be considered frictionless.

A more useful application would be a streamlined body.



Chapter 2

Some Preliminary Concepts

2.1 Concept of steady and unsteady flow

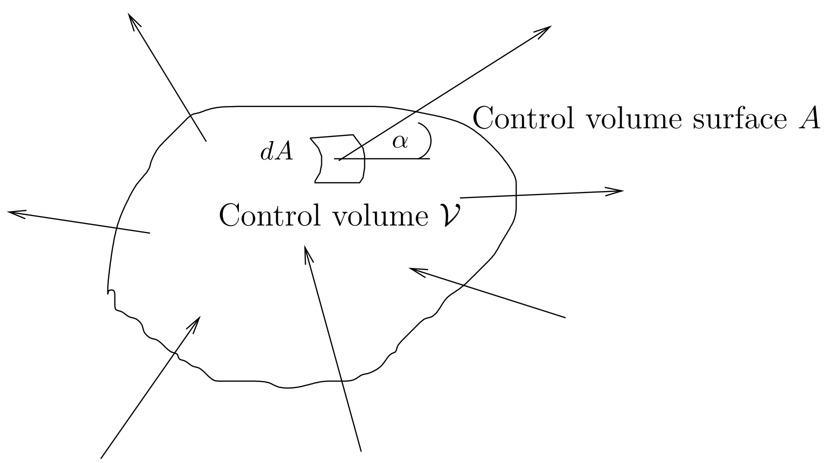
In general the velocity field consist of three velocity components,

$$\underset{\sim}{\mathbf{V}} = u\underset{\sim}{\mathbf{i}} + v\underset{\sim}{\mathbf{j}} + w\underset{\sim}{\mathbf{k}} \tag{2.1}$$

is a function of space and time i.e.

$$\underset{\sim}{\mathbf{V}} = \underset{\sim}{\mathbf{V}}(x, y, z, t). \tag{2.2}$$

If the velocity components are a function of space alone and are not a function of time we have steady flow, ie $\underset{\sim}{\mathbf{V}} = \underset{\sim}{\mathbf{V}}(x, y, z)$. Consider <u>continuity</u>



From last year,

$$\begin{aligned} \int_A \rho V \cos\alpha \cdot dA &= 0 \quad \text{(steady flow)} \\ &= -\frac{d}{dt}\int_{\mathcal{V}} \rho d\mathcal{V} \quad \text{(unsteady flow)} \end{aligned}$$

The above is in integral form, we can also write it in differential form,

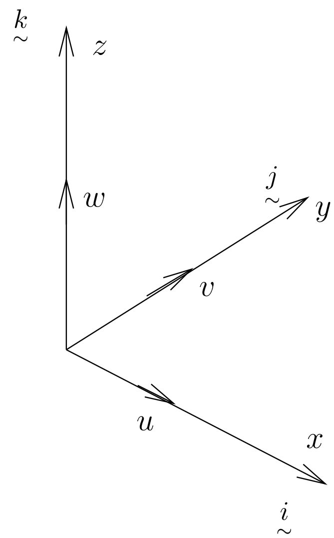
$$\begin{aligned}\nabla \cdot \rho \underset{\sim}{\mathbf{V}} &= 0 \quad \text{(steady flow)} \\ &= -\frac{\partial \rho}{\partial t} \quad \text{(unsteady flow)}\end{aligned}$$

Since we limit ourselves to incompressible flow (ie. ρ =constant)

$$\left.\begin{aligned}\nabla \cdot \underset{\sim}{\mathbf{V}} = 0 \\ \therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,.\end{aligned}\right\} \quad \text{steady or unsteady}$$

Note we are using the Cartesian coordinate system where

NB:



$$\underset{\sim}{\mathbf{V}} = \underset{\sim}{\mathbf{i}}u + \underset{\sim}{\mathbf{j}}v + \underset{\sim}{\mathbf{k}}w$$

2.2 Pathlines and Streamlines

In Fluid mechanics, it is important to visualise the flow field. Many fundamental concepts of Fluid mechanics can be understood by sketching how the flow looks like. In order to "visualise" the flow field, it is critical that one comprehend the concept of **streamlines** and **pathlines**.

- A **streamline** is defined as a curve whose tangent at any point is in the direction of the velocity vector at that point. It is the snapshot of the flowfield at any instant in time. For unsteady flows the streamline pattern is different at different times.

- A **pathline** is the line traced out by fluid particle as it moves through the flow field. It represent the path of a massless fluid particle moving in a flow field.

For steady flow, pathlines and streamlines coincide. They do not coincide for unsteady flows.

<u>Exercise 2.1:</u> Show that for a three-dimensional (sometimes written as [3]) flow field, the mathematical equation for stream line can be written as

$$\begin{aligned} wdy - vdz &= 0 \\ udz - wdx &= 0 \\ vdx - udy &= 0 \end{aligned} \tag{2.3}$$

u is the velocity in the x direction and v is the velocity in the y direction and w is the velocity in the z direction. For two-dimensional (sometimes written as [2]) flows, only the third relatioship is important

$$vdx - udy = 0 \tag{2.4}$$

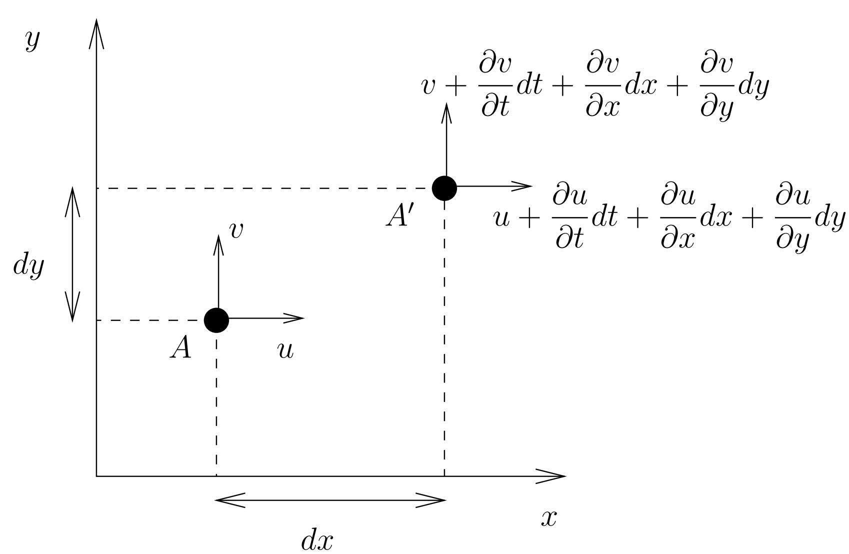
<u>Exercise 2.2:</u> Find the equations for streamlines and pathlines for the flow field given by the following expressions

(a) $\underset{\sim}{\mathbf{V}} = x\underset{\sim}{\mathbf{i}} - y\underset{\sim}{\mathbf{j}}$

(b) $\underset{\sim}{\mathbf{V}} = x\underset{\sim}{\mathbf{i}} + yt\underset{\sim}{\mathbf{j}}$

2.3 Concept of total derivative (substantial or Lagrangian derivative)

Differentiation following the motion of the fluid.



Particle goes from A to A' in time dt so acceleration in the $x-$direction is,

$$\begin{aligned}
a_x = \frac{Du}{Dt} &= \frac{\text{Change in } u \text{ velocity}}{dt} \\
&= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial t}\right) \\
&= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}
\end{aligned}$$

Note we use $\frac{D}{Dt}$ to denote the total derivative, ie.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

similarly in $y-$direction

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

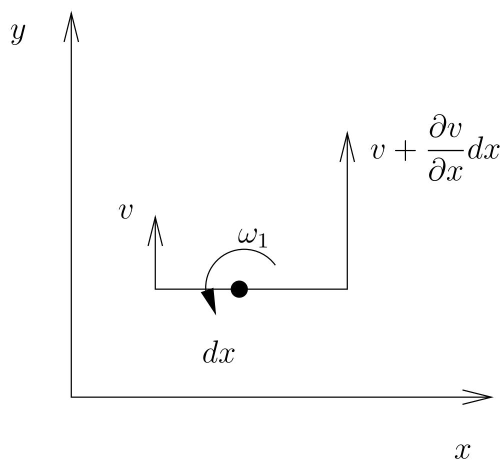
When we consider steady flow all derivatives of velocity with respect to time are zero

$$a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \tag{2.5}$$

$$a_y = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \tag{2.6}$$

2.4 Vorticity and Angular Velocity

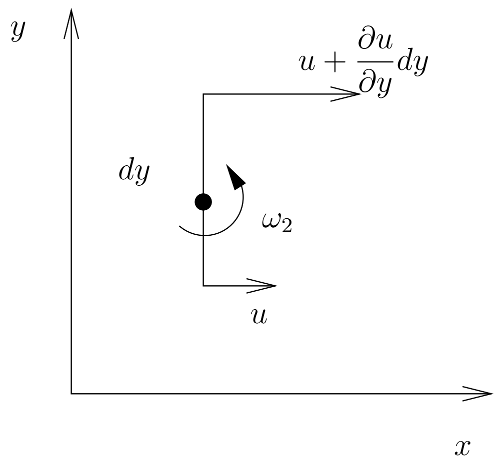
Consider fluid element dx



its angular velocity is

$$\begin{aligned}
\omega_1 &= \frac{v + \frac{\partial v}{\partial x}dx - v}{dx} \\
&= \frac{\partial v}{\partial x}\ .
\end{aligned}$$

Similarly consider fluid element dy



its angular velocity is

$$\begin{aligned}\omega_2 &= \frac{u - u - \frac{\partial u}{\partial y} dy}{dy} \\ &= -\frac{\partial u}{\partial y} \, .\end{aligned}$$

Hence

$$\boxed{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega_1 + \omega_2}$$

and the above is called vorticity or rotation and is denoted by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \, . \tag{2.7}$$

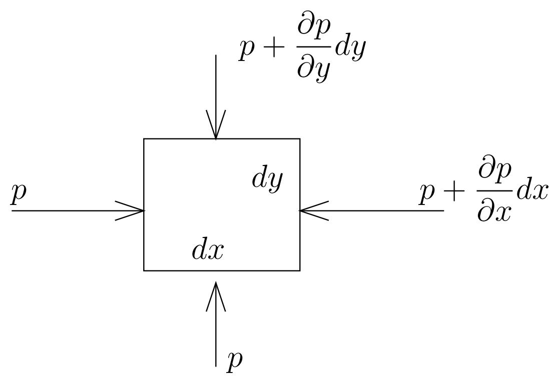
It is defined to be the sum of the angular velocities of two mutually perpendicular fluid lines.

Chapter 3

Momentum equations: Euler's equations of motion

Forces on a particle

Consider [2] flow, frictionless fluid and ignore body forces (gravity).



Resultant external force in the $x-$direction,

$$\begin{aligned} F_x =& p dy - \left(p + \frac{\partial p}{\partial x} dx \right) dy \\ =& - \frac{\partial p}{\partial x} dx dy \ , \end{aligned}$$

similarly

$$F_y = - \frac{\partial p}{\partial y} dx dy \ .$$

Now Newton's equation of motion says

$$\begin{aligned} m a_x =& F_x \\ m a_y =& F_y \end{aligned}$$

and the mass of the element is $m = \rho dxdy$. Using Eqs. (2.5) and (2.5) the following two equations are obtained

$$\rho dxdy\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}dxdy$$
$$\rho dxdy\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}dxdy$$

$$\boxed{\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}}$$

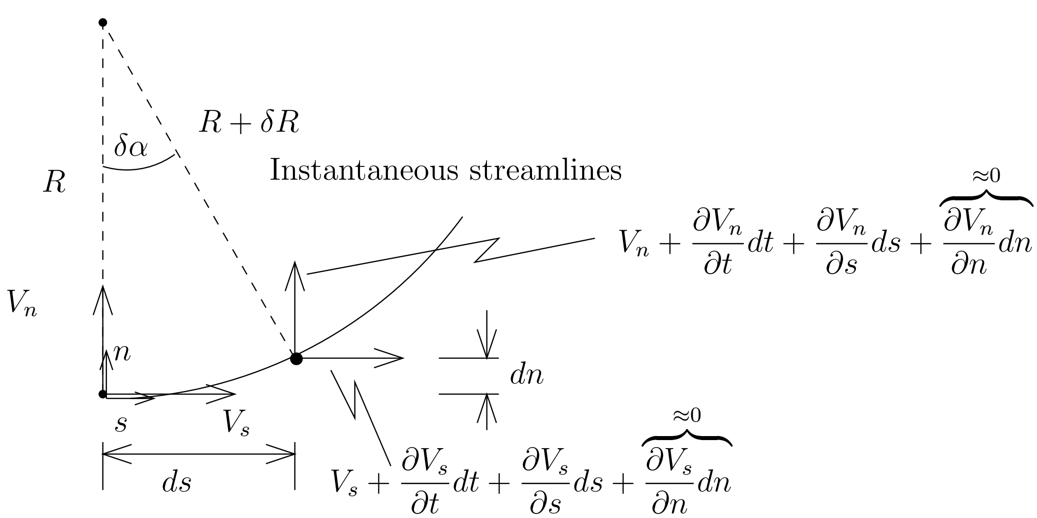
These are the Euler equations of motion in [2] steady flow (Cartesian coordinates).

If we had considered unsteady flow $\partial u/\partial t \neq 0$ and $\partial v/\partial t \neq 0$ then

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}}$$

These are the Euler equations of motion in [2] unsteady flow (Cartesian coordinates).

The above equations can be derived in other coordinate systems eg. streamline curve linear co-ord.



Note: $dn = ds\tan(d\alpha) \approx dsd\alpha \approx 0$

$$\frac{\partial V_n}{\partial s} \approx \frac{V_s}{R}, \qquad V_n = 0$$

$$\boxed{\begin{aligned}\frac{\partial V_s}{\partial t} + V_s\frac{\partial V_s}{\partial s} &= -\frac{1}{\rho}\frac{\partial p}{\partial s}\\ \frac{\partial V_n}{\partial t} + \frac{V_s^2}{R} &= -\frac{1}{\rho}\frac{\partial p}{\partial n}\end{aligned}}$$

These are the Euler equations of motion in [2] unsteady flow (streamline curve linear coordinates).

3.1 Pressure Forces: Bernoulli's Equation

In many engineering applications, it is important to calculate the pressure forces at various points in the fluid. For inviscid flows, Bernoulli's equation is usually used to calculate pressure forces. Bernoulli's equation is given by

$$p + \frac{1}{2}\rho V^2 = \text{constant} \tag{3.1}$$

For an inviscid fluid, Eq. (3.1) is valid along a streamline.

Proof:

Consider the steady x momentum equation in [2]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Multiply the above equation by dx gives

$$u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

Using Eq. (2.4) on the second term on the LHS of the above equation gives

$$\begin{aligned}u\frac{\partial u}{\partial x}dx + u\frac{\partial u}{\partial y}dy &= -\frac{1}{\rho}\frac{\partial p}{\partial x}dx\\ u\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) &= -\frac{1}{\rho}\frac{\partial p}{\partial x}dx\\ udu &= -\frac{1}{\rho}\frac{\partial p}{\partial x}dx\\ \frac{1}{2}du^2 &= -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \end{aligned} \tag{3.2}$$

Repeating similar steps for the y momentum equation gives

$$\frac{1}{2}dv^2 = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \tag{3.3}$$

Adding Eqs. (3.2) and (3.3) gives

$$\frac{1}{2}d\left(u^2+v^2\right) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy\right)$$

$$\frac{1}{2}d\left(V^2\right) = -\frac{1}{\rho}dp$$
$$dp = -\rho V dV \tag{3.4}$$

where $V^2 = u^2 + v^2$. If we assume that ρ is a constant, we can integrate the above equation along a streamline to obtain

$$\int_{p_1}^{p_2} dp = -\rho \int_{V_1}^{V_2} V dV$$
$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 \tag{3.5}$$

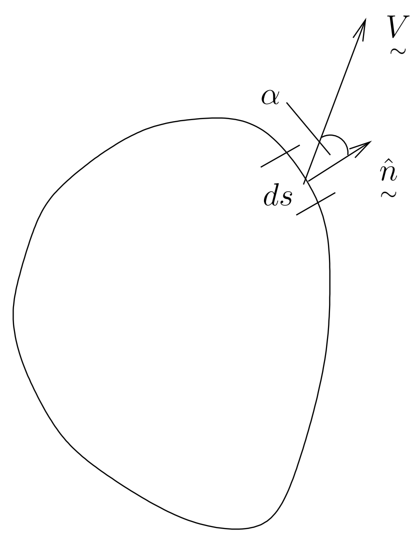
Exercise 3.1: The analysis above show that Bernoulli's equation (Eq. (3.1)) is valid only along a streamline. However, if the flow is inviscid **and** irrotational, it can be shown that Eq. (3.1) is valid anywhere in the flow field. Prove that the previous statement is true.

Chapter 4

Velocity Potential and Stream function

4.1 Concept of a stream function ψ

The stream function is related to the rate at which fluid volume is streaming *across* and elementary arc, ds.
For **Incompressible [2] flow.**



From continuity we have

$$\oint \underbrace{V \cos \alpha ds}_{d\psi} = 0 \qquad \text{or} \qquad \oint \underset{\sim}{V} \cdot \underset{\sim}{\hat{n}} ds = 0$$

where $V = |\underset{\sim}{\mathbf{V}}|$

$$\text{ie.} \qquad d\psi = V \cos(\alpha) ds = \underset{\sim}{\mathbf{V}} \cdot \underset{\sim}{\hat{n}} ds$$

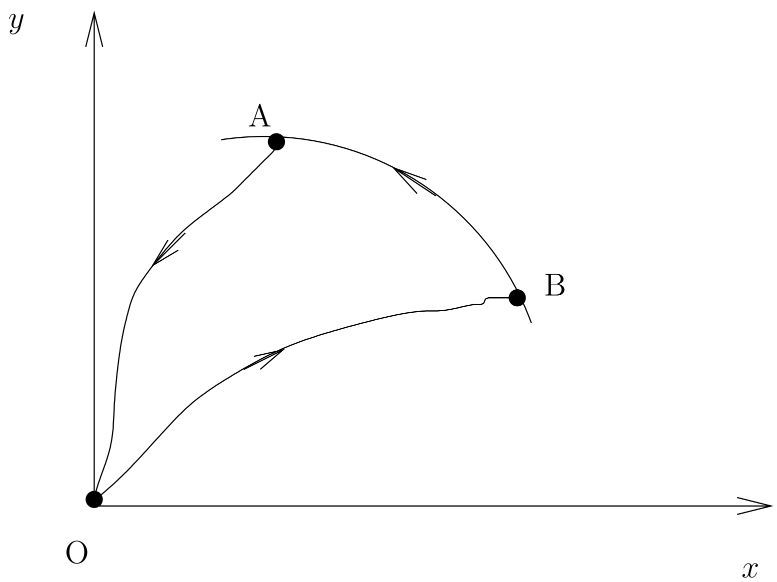
This means

$$\oint d\psi = 0$$

where ψ a scalar point function and $d\psi$ is an exact differential ie.

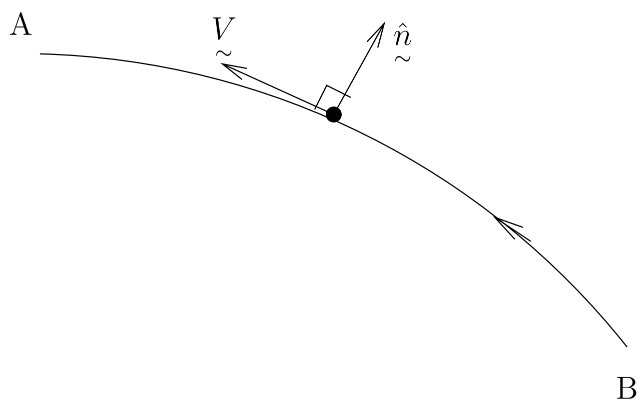
$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

4.1.1 How does ψ behave alone a streamline



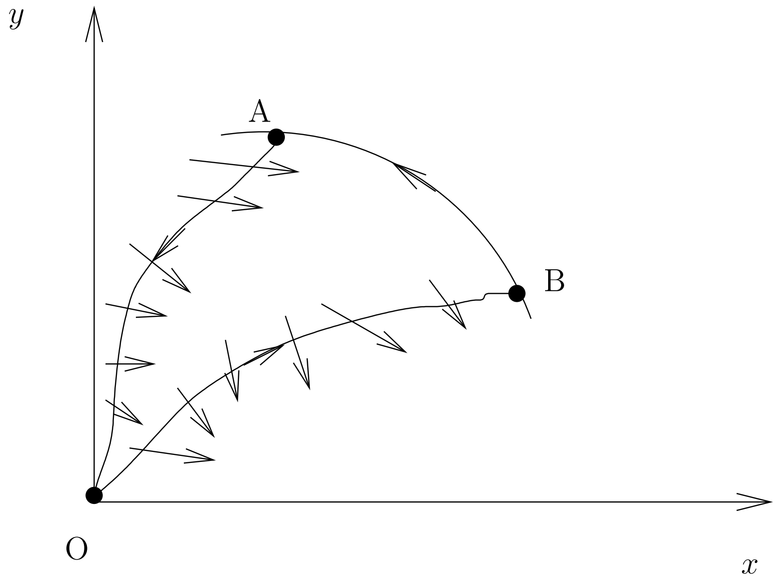
$$\oint_{OBAO} d\psi = \int_O^B d\psi + \int_B^A d\psi + \int_A^O d\psi = 0$$
$$= (\psi_B - \psi_O) + (\psi_A - \psi_B) + (\psi_O - \psi_A)$$

Along the streamline $\alpha = 90^o \Rightarrow \cos\alpha = 0$



$$\therefore d\psi = V \cos\alpha ds = 0 \qquad \text{along a streamline}$$
$$\text{Hence} \qquad \boxed{\psi_A = \psi_B}$$

4.1.2 What is the physical meaning of ψ



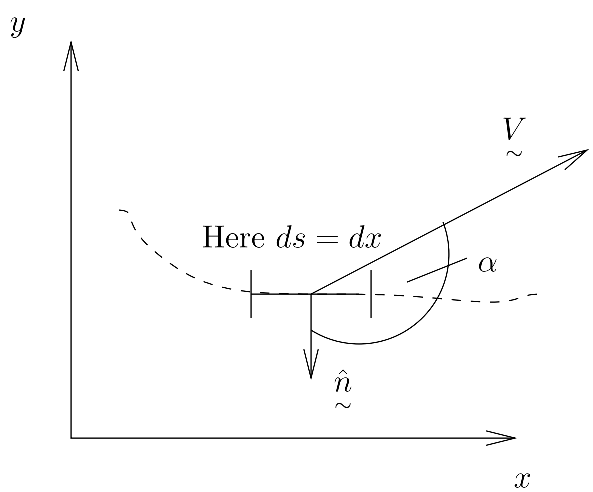
$$\psi_A - \psi_O = \text{ volume flux crossing OA}$$
$$\psi_B - \psi_O = \text{ volume flux crossing OA}$$

If AB is a streamline the above fluxes must be equal. This means that the difference in ψ between two points = the volume flux across *any* line joining the two points. Therefore a streamline is like a fence, across which flow cannot occur. Also the volume flux across a path between two streamlines is independant of the path.

4.1.3 What is the relationship between u, v and ψ

$$\text{Have} \qquad \frac{d\psi}{ds} = V\cos\alpha = \underset{\sim}{\mathbf{V}} \cdot \underset{\sim}{\hat{\mathbf{n}}}$$

Say we move a small amount in the x-dir

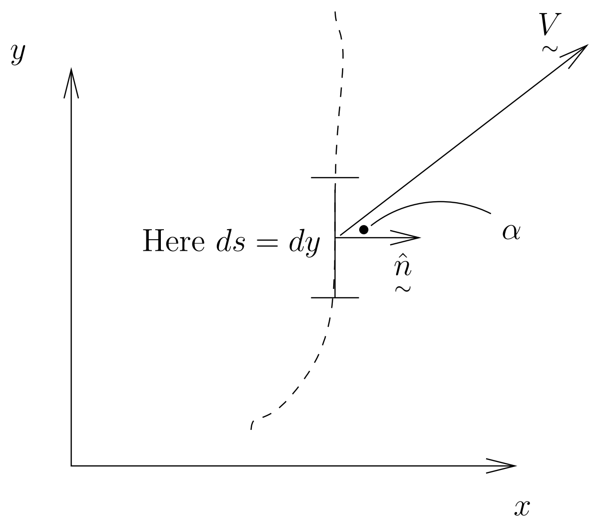


then $ds = dx$ and

$$\underset{\sim}{\mathbf{V}} \cdot \underset{\sim}{\hat{\mathbf{n}}} = -v$$

$$\therefore \boxed{\frac{\partial \psi}{\partial x} = -v}$$

Say we move a small amount in the y-dir

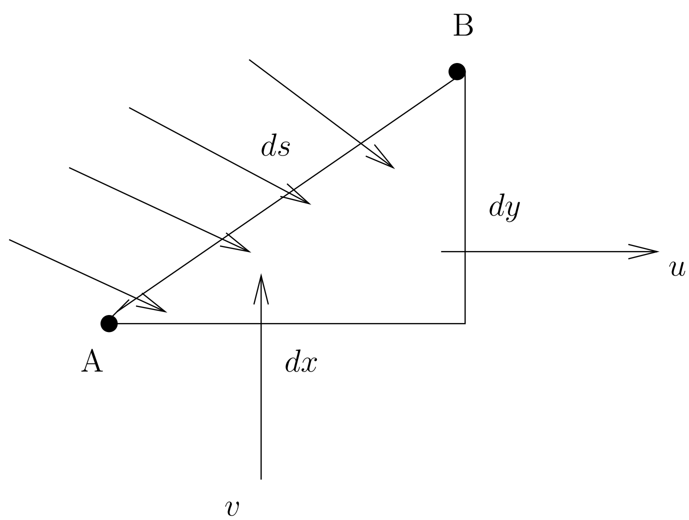


then $ds = dy$ and

$$\underset{\sim}{\mathbf{V}} \cdot \underset{\sim}{\hat{\mathbf{n}}} = u$$

$$\therefore \boxed{\frac{\partial \psi}{\partial y} = u}$$

Alternative derivation;



Let $d\psi =$ flux crossing AB

$$\underbrace{d\psi}_{\text{flux in across AB}} = \underbrace{udy}_{\text{flux out side}} - \underbrace{vdx}_{\text{flux in bottom}}$$

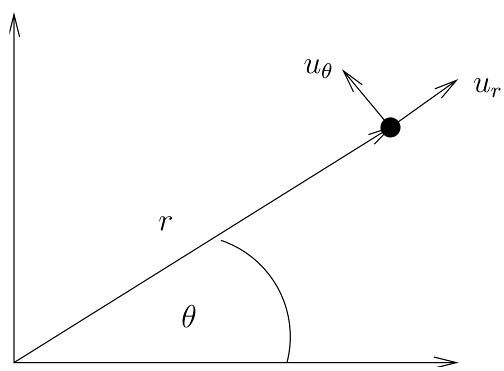
because $d\psi$ is an exact differential

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy\ .$$

Equating coefficient of dx and dy gives

$$\frac{\partial \psi}{\partial x} = -v \qquad \frac{\partial \psi}{\partial y} = u$$

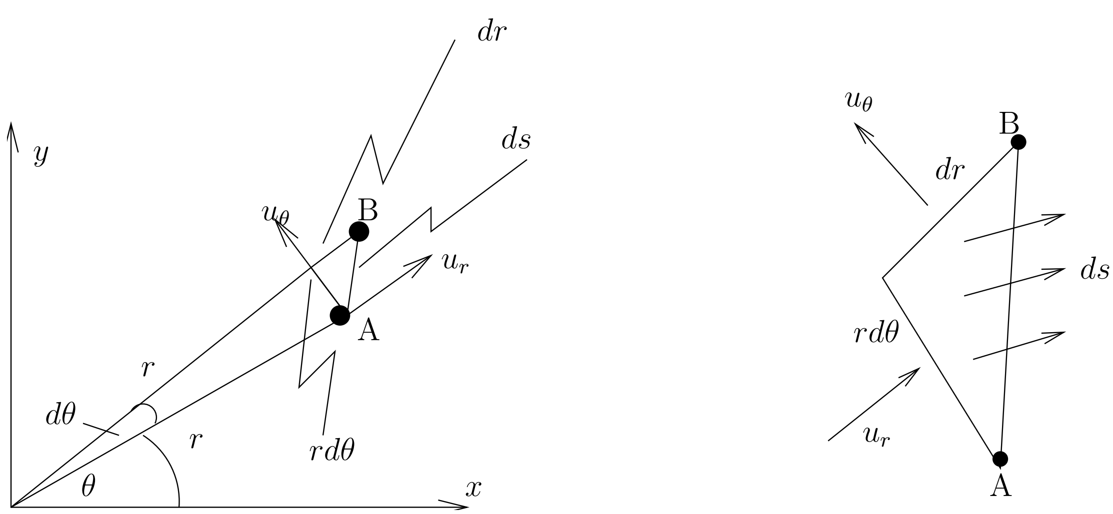
In polar coordinates



Convention

- u_r = radial component
- u_θ = tangential component

Note u_r and u_θ correspond with u and v when $\theta = 0$



$$d\psi = \text{ flux across AB}$$

$$d\psi = u_r r d\theta - u_\theta dr$$

$$\text{but} \quad d\psi = \frac{\partial \psi}{\partial r}dr + \frac{\partial \psi}{\partial \theta}d\theta$$

Equate coefficients of dr and $d\theta$

$$\boxed{\frac{\partial \psi}{\partial r} = -u_\theta}$$

$$\boxed{\frac{1}{r}\frac{\partial \psi}{\partial \theta} = u_r}$$

If we had considered compressible flow

$$\frac{\partial \psi}{\partial x} = -\frac{\rho}{\rho_0} v \qquad \frac{\partial \psi}{\partial r} = -\frac{\rho}{\rho_0} u_\theta$$
$$\frac{\partial \psi}{\partial y} = \frac{\rho}{\rho_0} u \qquad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\rho}{\rho_0} u_r$$

where ρ_0 is some reference density arbitraily chosen at some point in the flow. For incompressible flow $\rho/\rho_0 = 1$

Exercise 4.1: For the flow defined by the stream function $\psi = V_\infty y$:

(a) Plot the streamlines.

(b) Find the x and y components of velocity at any point.

(c) Find the volume flow rate per unit width flowing between the streamlines $y = 1$ and $y = 2$.

Exercise 4.2: An inviscid flow is bounded by a wavy wall at $y = H$ and a plane wall at $y = 0$. The stream function is

$$\psi = A\left(e^{-ky} - e^{ky}\right)\sin(kx) + By^2 \tag{4.1}$$

(a) Obtain an expression for the velocity field.

(b) Is the flow rotational or irrotational ?

(c) Find the pressure distribution on the plane wall surface, given that $p = 0$ at $[0, 0]$.

Exercise 4.3: The flow around a corner can be defined with the streamfunction, $\Psi = kxy$

(a) Find the value of k if you are given that the volume flow rate of a line drawn between (0,0) and (1,1) is $2m^3/s$.

(b) Is the flow field irrotational ?

(c) Given that the pressure at (0,0) is p_0, what is the pressure distribution along the two walls.

(d) Pretend that the streamline going through the point (2,3) is a wall. Find the pressure distribution along this wall.

4.2 General equation for ψ

It has been shown that along a streamline ψ is constant. Therefore if we can determine the stream function we are then able to plot (or sketch) the streamlines for a given flow.

Use Eulers equation of motion to determine a general equation for ψ. Assuming steady flow we have;

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{4.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}\ . \tag{4.3}$$

we want to eliminate pressure, so differentiating (4.2) wrt y and (4.3) wrt x

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y} = \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} \tag{4.4}$$

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial y\partial x} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x\partial y} \tag{4.5}$$

Subtract (4.4) from (4.5) and assume

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y} = \frac{1}{\rho}\frac{\partial^2 p}{\partial y\partial x} \qquad \text{ie. } p \text{ is a regular function}$$

$$0 = u\frac{\partial}{\partial x}\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} + v\frac{\partial}{\partial y}\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} + \left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\}\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\}$$

but from continuity

$$\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} = 0$$

hence

$$u\frac{\partial}{\partial x}\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} + v\frac{\partial}{\partial y}\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} = 0$$

$$\text{OR} \qquad \frac{D}{Dt}\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} = 0 \tag{4.6}$$

The term $\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\}$ is the vorticity which was defined in section 2.4

Note in streamline coordinates

$$\zeta = \frac{V_s}{R} - \frac{\partial V_s}{\partial n}$$

Therefore (4.6) says that

$$\boxed{\frac{D\zeta}{Dt} = 0} \tag{4.7}$$

This means if we follow a fluid element its vorticity (ζ) does not change. Since this is steady flow following a fluid element $\Rightarrow$ travelling on a streamline. Hence streamlines are lines of constant ζ as well as ψ.

Since we have $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$ substituting into (2.7)

$$\zeta = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

$$\begin{aligned} \therefore \zeta &= -(\nabla \cdot \nabla)\psi & \qquad \nabla^2 &= \text{Laplacian operator} \\ &= -\nabla^2 \psi & &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned}$$

$$\boxed{\nabla^2 \psi = -\zeta}$$

Equation (4.7) becomes

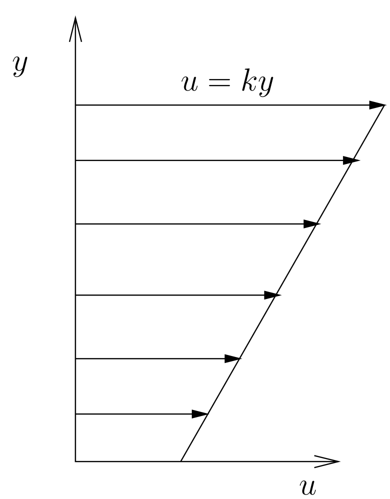
$$\boxed{\frac{D}{Dt}\left\{\nabla^2 \psi\right\} = 0} \qquad \textbf{Helmholtz's equation}$$

This is effectively the Euler equation and continuity expressed in terms of the stream function (ψ) for the case of incompressible [2] flow.

Exercise 4.4: Repeat the steps above and show that Helmholtz's equation is valid for inviscid, incompressible AND *unsteady* flows.

Two special cases

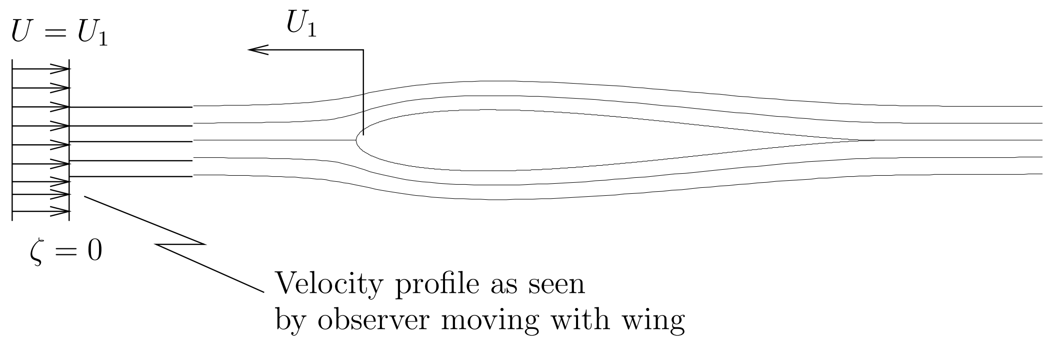
1. Uniform vorticity upstream



Here every streamline has the same vorticity and since ζ remains constant along a streamline then ζ is constant everywhere.

$$\Rightarrow \boxed{\nabla^2 \psi = -\zeta = \text{ const.}} \qquad \textbf{Poisson's equation} \qquad (4.8)$$

2. Zero upstream vorticity
   Occurs quite a lot in practice ie



Now ζ = constant along streamlines, therefore $\zeta = 0$ everywhere. With $\zeta = 0$ everywhere we have irrotational flow.

$$\Rightarrow \boxed{\nabla^2\psi = 0} \quad \textbf{Laplace equation} \tag{4.9}$$

This leads to what is know as potential flow and we can say if the flow is irrotational ($\zeta = 0$) the stream function (ψ) will satisfy the Laplace equation. The beauty of the Laplace equation is that it is **LINEAR**. This means if we have a series of simple flow solutions eg. ψ_1, ψ_2, ψ_3 then the solution to more complex flows can be obtained by superposition of the simple flows eg.

$$\underbrace{\psi}_{\text{complicated flow}} = \underbrace{\psi_1 + \psi_2 + \psi_3 + \cdots}_{\text{simple flows}}$$

Note: the Laplace operator in polar coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \qquad \underbrace{+\left(\frac{\partial^2}{\partial z^2}\right)}_{\text{In the case of polar cylindrical}}$$

Some more about vorticity

Vorticity is really a [3] vector $\underset{\sim}{\mathbf{\Omega}}$

$$\underset{\sim}{\mathbf{\Omega}} = \underset{\sim}{\mathbf{i}}\xi + \underset{\sim}{\mathbf{j}}\eta + \underset{\sim}{\mathbf{k}}\zeta$$

it can be evaluated by taking the curl of the velocity vector ie.

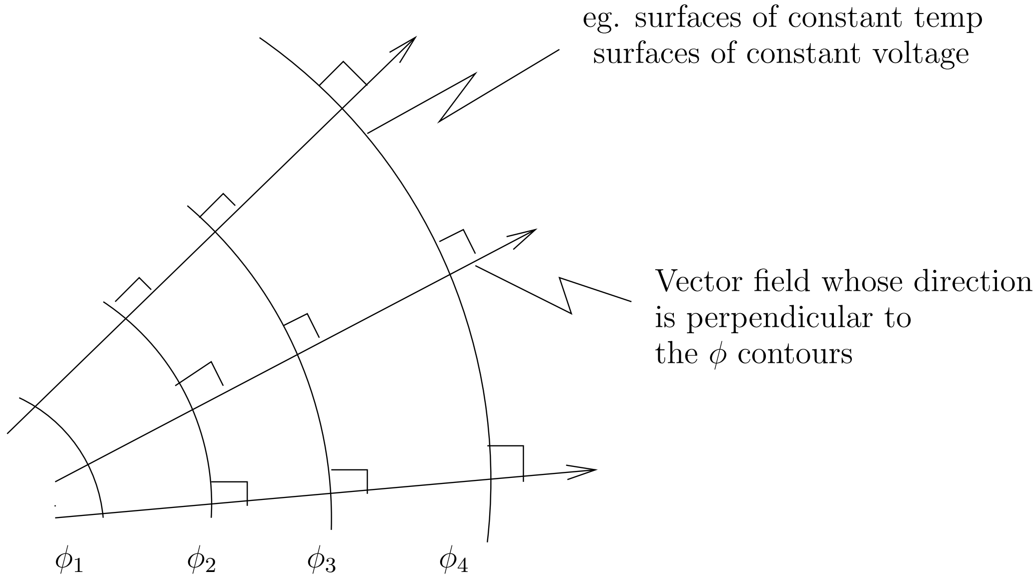
$$\begin{aligned}
\underset{\sim}{\mathbf{\Omega}} &= \text{curl}\,(\underset{\sim}{\mathbf{V}}) \\
&= \nabla \times \underset{\sim}{V} \\
&= \begin{vmatrix} \underset{\sim}{\mathbf{i}} & \underset{\sim}{\mathbf{j}} & \underset{\sim}{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\
&= \underset{\sim}{\mathbf{i}}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \underset{\sim}{\mathbf{j}}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \underset{\sim}{\mathbf{k}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)
\end{aligned}$$

hence for [2] flow

$$\underset{\sim}{\mathbf{\Omega}} = \underset{\sim}{\mathbf{k}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \qquad |\underset{\sim}{\mathbf{\Omega}}| = \zeta$$

4.3 Concept of a scalar point function

If we have a scalar point function, ie. $\phi = \phi(x, y, z)$ then surfaces of constant ϕ will form plates.



Often we can denote a vector field by

$$\begin{aligned}\underset{\sim}{\mathbf{V}} &= \nabla\phi \\ &= \text{grad}\,\phi\end{aligned}$$

where

$$\nabla = \underset{\sim}{\mathbf{i}}\frac{\partial}{\partial x} + \underset{\sim}{\mathbf{j}}\frac{\partial}{\partial y} + \underset{\sim}{\mathbf{k}}\frac{\partial}{\partial z}\ .$$

There are many vector fields in nature that can be defined this way;

Vector field	Scalar function
Current flux	Voltage potential
Heat flux	Temperature
Gravitational force	Potential energy

The scalar function is called the *potential* for the vector field and if a vector field possesses a potential it is called a *conservative* field.

Often in fluid flow (but not always) the velocity field is a vector point function $\underset{\sim}{\mathbf{V}}(x, y, z)$ which possesses potential scalar function $\phi(x, y, z)$,

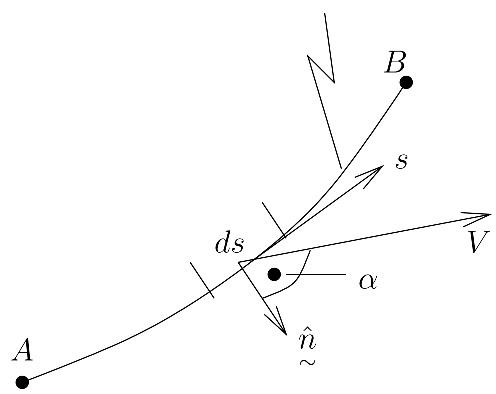
$$\underset{\sim}{\mathbf{V}}(x, y, z) = \nabla\phi(x, y, z)\ .$$

When this happens such a flow is called *potential flow* and ϕ is referred to as the velocity potential.

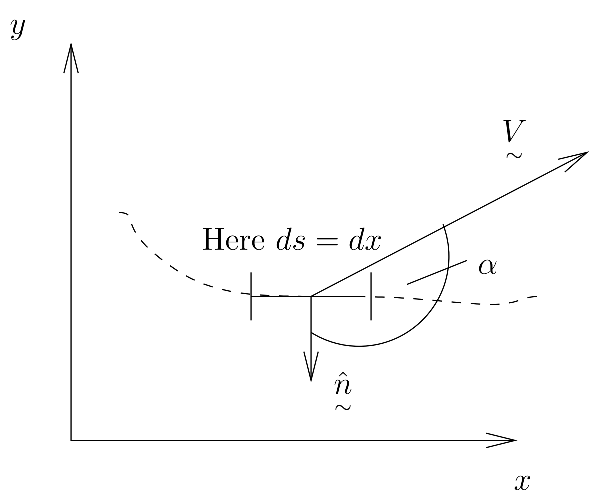
4.4 Concept of velocity potential ϕ

The velocity potential is analogous to the stream function. The stream function is related to the rate of flow *across* an small arc, ds, but the velocity potential, ϕ is related to the rate of flow *along* ds.

$$\text{Let } d\phi = V\,ds \sin\alpha$$
$$\text{then } \phi_A - \phi_B = \int_A^B V \sin\alpha ds$$
$$\text{or } \frac{d\phi}{ds} = V \sin\alpha$$



Say we move a small amount in the x-dir

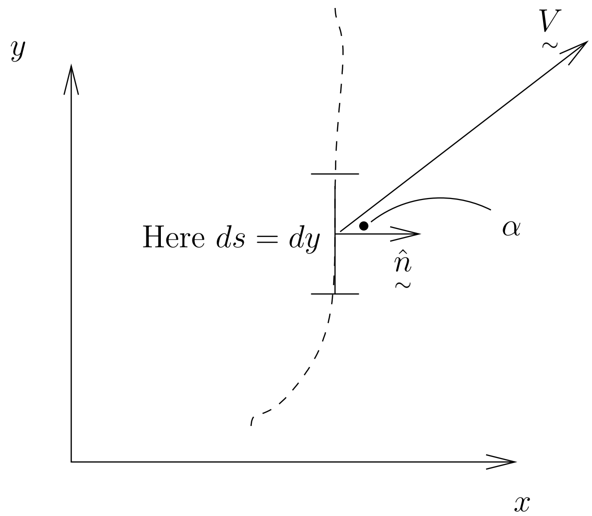


then $ds = dx$ and

$$V \sin\alpha = u$$

$$\therefore \boxed{\frac{\partial\phi}{\partial x} = u}$$

Say we move a small amount in the y-dir



then $ds = dy$ and

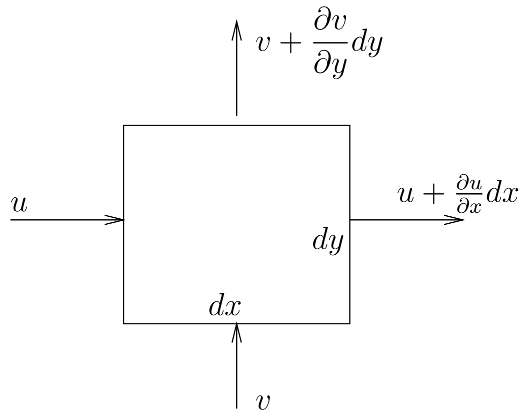
$$V \sin\alpha = v$$

$$\therefore \boxed{\frac{\partial \phi}{\partial y} = v}$$

$$\begin{aligned}
\text{Now } \underset{\sim}{\mathbf{V}} &= \underset{\sim}{\mathbf{i}} u + \underset{\sim}{\mathbf{j}} v \\
&= \underset{\sim}{\mathbf{i}} \frac{\partial \phi}{\partial x} + \underset{\sim}{\mathbf{j}} \frac{\partial \phi}{\partial y} \\
\therefore \underset{\sim}{\mathbf{V}} &= \nabla \phi \quad (= \text{grad}\, \phi)
\end{aligned}$$

$\boxed{\text{This means that } \phi \text{ is the potential function of velocity}}$

To find the governing equation for ϕ use volume flux technique.



From volume flux balance (ie. what goes in must come out)

$$u dy \cdot 1 + v dx \cdot 1 = \left(u + \frac{\partial u}{\partial x} dx\right) dy + \left(v + \frac{\partial v}{\partial y} dy\right) dx$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \quad \leftarrow \quad \text{continuity equation}$$

But

$$u = \frac{\partial \phi}{\partial x} \ \rightarrow \ \frac{\partial u}{\partial x} = \frac{\partial^2 \phi}{\partial x^2}$$

and

$$v = \frac{\partial \phi}{\partial y} \ \rightarrow \ \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial y^2}$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\Rightarrow \boxed{\nabla^2 \phi = 0}$$

Therefore the velocity potential function ϕ like the stream function ψ follows the Laplace equation (harmonic functions). Hence we can use superposition of solutions, ie. to obtain a complex flow just add simple flows together.

Chapter 5

Some Simple Solutions

5.1 Some simple solutions

We will find the solution (ie. stream function ψ) for some simple flows.

5.1.1 Parallel flow

For the parallel flow with uniform velocity U_∞ shown in Figure 5.1, we have

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x}$$

In this case $u = U_\infty$, $v = 0$

$$\therefore \frac{\partial \psi}{\partial x} = 0\,, \qquad \frac{\partial \psi}{\partial y} = U_\infty$$

This gives two partial differential equations which can be solved by integration

$$\frac{\partial \psi}{\partial x} = 0$$
$$\psi = f_1(y)$$

and

$$\frac{\partial \psi}{\partial y} = U_\infty$$
$$\psi = U_\infty y + f_2(x)$$

where $f_1(y)$ and $f_2(x)$ are functions of integration. These equations are compatible only if $f_2(x) = k$ where k is an arbitrary constant. For convenience the value of ψ is normally set to zero when $y = 0 \Rightarrow K = 0$. Hence

$$\boxed{\psi = U_\infty y} \qquad \text{parallel flow (left to right)}$$

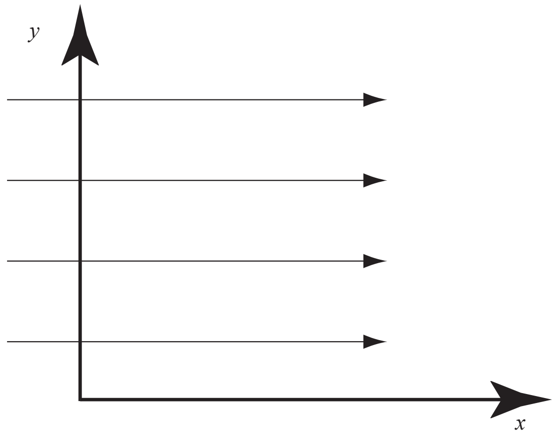
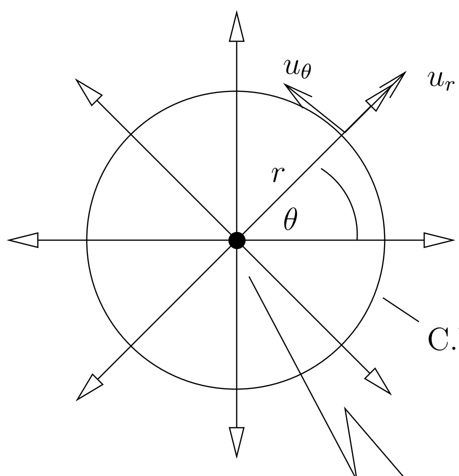


Figure 5.1: Parallel flow from left to right

5.1.2 Source flow

In source flow we have Q m^{3}s^{-1} emerging from a point and flowing in the radial direction ie.



The volume flow rate through the control volume surface with unit depth is given by

$$2\pi r \cdot 1 \cdot u_r = Q \qquad \text{(vol. flux)}$$

hence

$$u_r = \frac{Q}{2\pi r}\ , \qquad u_\theta = 0 \text{ By definition of a source}$$

. We have

$$\frac{\partial \psi}{\partial r} = -u_\theta\ , \qquad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = u_r$$

hence in this case

$$\frac{\partial \psi}{\partial r} = 0 \,, \qquad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi r}$$

$$\begin{aligned} \frac{\partial \psi}{\partial r} &= 0 \\ \Rightarrow \psi &= f_1(\theta) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{r}\frac{\partial \psi}{\partial \theta} &= \frac{Q}{2\pi r} \\ \Rightarrow \psi &= \frac{Q}{2\pi}\theta + f_2(r) \,. \end{aligned}$$

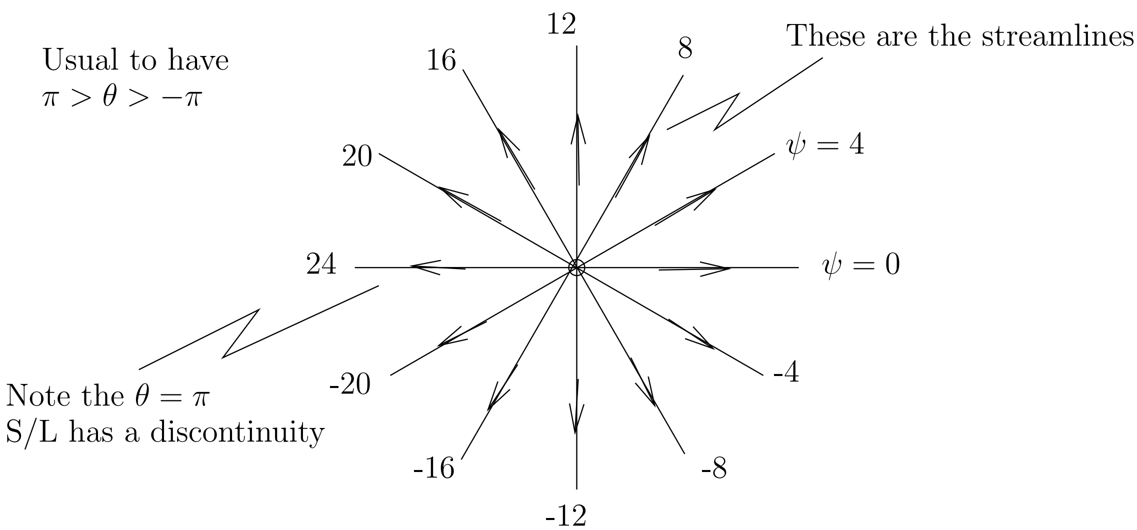
These two equations are compatible only if $f_2(r) = k$ usually $k = 0$ when $\theta = 0$.

$$\boxed{\psi = \frac{Q}{2\pi}\theta} \qquad \text{source in polar coordinates}$$

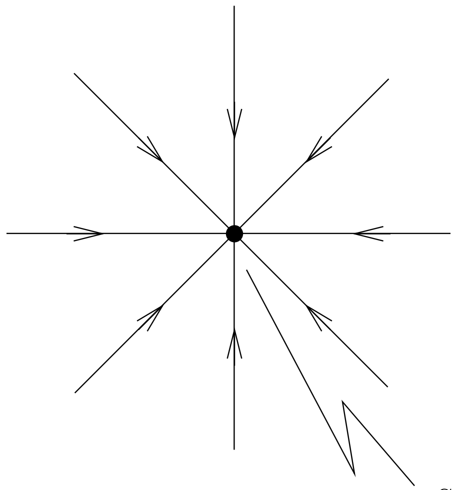
In cartesian

$$\psi = \frac{Q}{2\pi} \arctan\left(\frac{y}{x}\right)$$

<u>Example:</u> $Q = 48$ units gives the following ψ values when plotted at $\pi/6$ intervals



5.1.3 Sink flow



Show that for a sink

$$\boxed{\psi = -\frac{Q}{2\pi}\theta}$$

Sink
Strength$= Q$ $\mathrm{m^3s^{-1}}$

Exercise 5.1: Follow the steps outlined above and see if you can derive the stream function for typical flows shown in Figure 5.2.

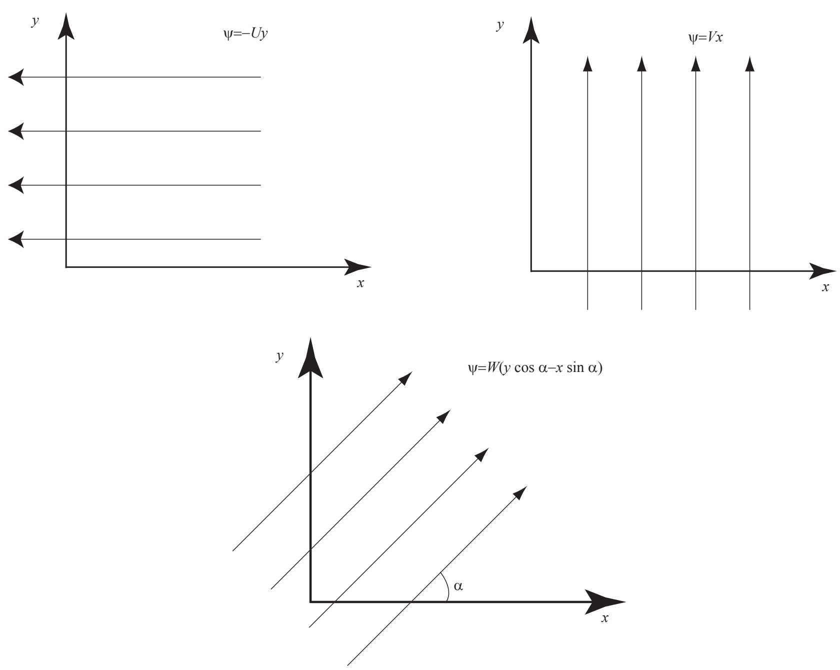
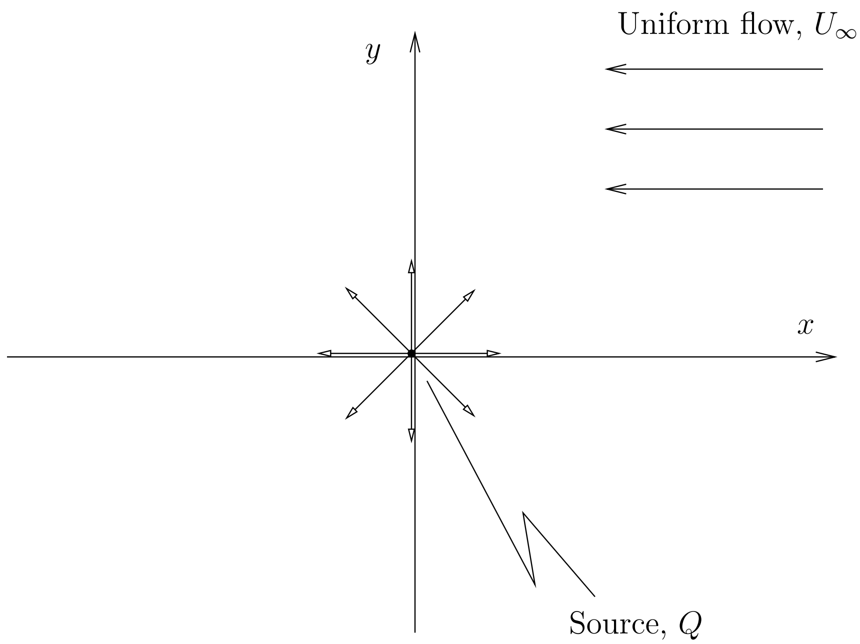


Figure 5.2: Examples of some typical stream function

5.2 More complex flow solutions

Source combines with uniform stream



$$\begin{aligned}
\text{For uniform flow} \qquad \psi_1 &= -U_\infty y \\
\text{For source} \qquad \psi_2 &= \frac{Q\theta}{2\pi} = \frac{Q}{2\pi}\arctan\left(\frac{y}{x}\right)
\end{aligned}$$

Since the Laplace equation is linear we can add these solutions to get the solution for the new flow

$$\begin{aligned}
\psi &= \psi_1 + \psi_2 \\
\therefore \psi &= -U_\infty y + \frac{Q}{2\pi}\arctan\left(\frac{y}{x}\right)
\end{aligned}$$

We want to sketch this flow, ie. plot lines of constant ψ. To do this we first find the **stagnation points**, which are points where $u = v = 0$. The velocity components in the new flow are;

$$\begin{aligned}
u &= \frac{\partial \psi}{\partial y} \\
&= -U_\infty + \frac{Q}{2\pi}\frac{1}{\left(1+\frac{y^2}{x^2}\right)}\frac{1}{x} \\
&= -U_\infty + \frac{Q}{2\pi}\frac{x}{(x^2+y^2)} \qquad \text{(A)}
\end{aligned}$$

and

$$\begin{aligned} v &= -\frac{\partial \psi}{\partial x} \\ &= -\frac{Q}{2\pi}\frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{-y}{x^2}\right) \\ &= \frac{Q}{2\pi}\frac{y}{(x^2+y^2)} \end{aligned} \tag{B}$$

Lets assume the stagnation point(s) occurs at x_0, y_0.
From (B)

$$0 = \frac{Q}{2\pi}\frac{y}{(x^2+y^2)}$$

$$\therefore y_0 = 0 \qquad \text{ie. stagnation point occurs on the } x\text{-axis}$$

and from (A)

$$U_\infty = \frac{Q}{2\pi}\frac{x}{(x^2+y^2)}$$

$$\therefore x_0 = \frac{Q}{2\pi U_\infty}$$

So there exists one stagnation point at $(\frac{Q}{2\pi U_\infty}, 0)$.

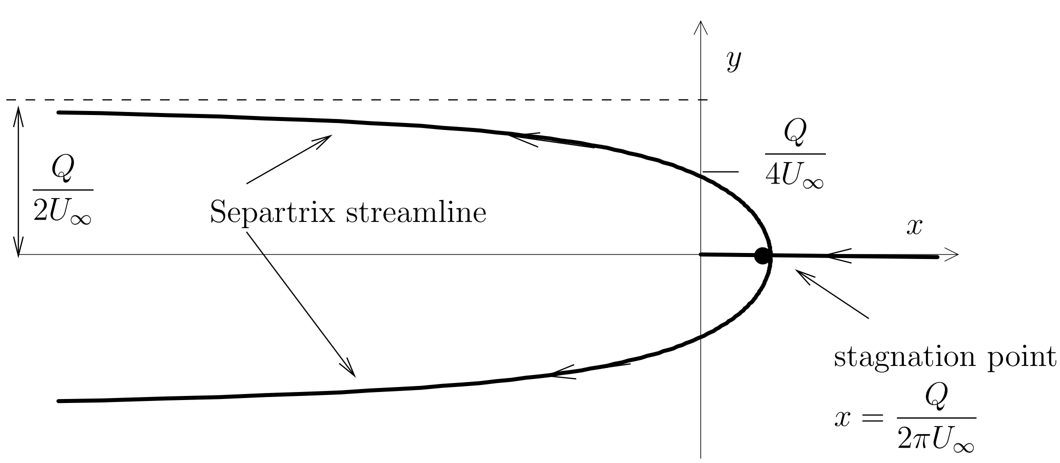
Streamlines that pass through the stagnation points are called sepratrix streamlines. The value of ψ on the sepratrix must be constant $= \psi|_{x_0,y_0}$ and in this case $\psi|_{x_0,y_0} = 0$. Lets plot this streamline

$$\psi = 0 = -U_\infty y + \frac{Q}{2\pi}\arctan\left(\frac{y}{x}\right)$$

The solution has two branches

$$y = 0$$

$$\text{and} \qquad \mathrm{x} = \mathrm{y}\cot\left(\frac{2\pi \mathrm{U}_\infty \mathrm{y}}{\mathrm{Q}}\right)$$



Locate x intercepts

$$x = \lim_{y \to 0} \frac{y}{\tan\left(\frac{2\pi y U_\infty}{Q}\right)}$$

Use L'Hopital's rule

$$x = \lim_{y \to 0} \frac{1}{\frac{2\pi U_\infty}{Q}\sec^2\left(\frac{2\pi y U_\infty}{Q}\right)}$$

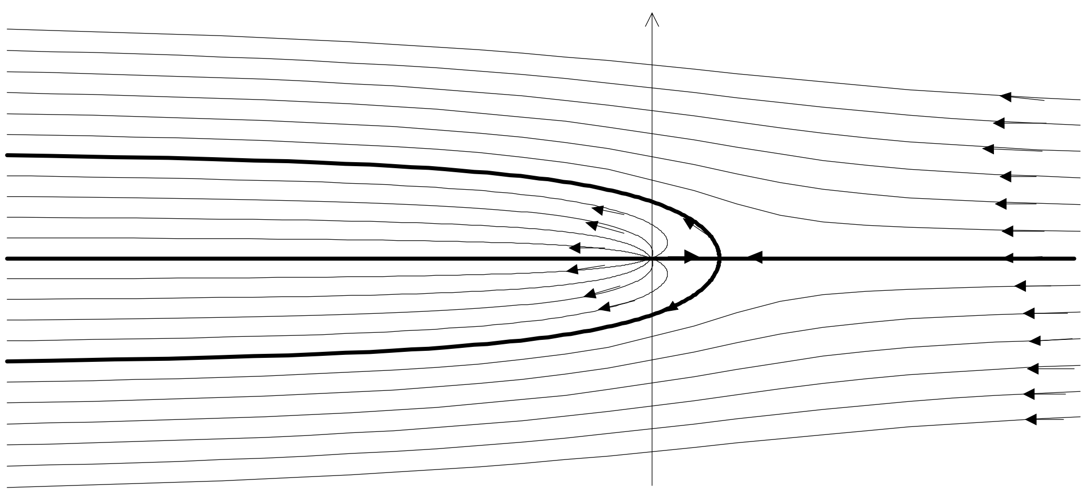
$$x = \frac{Q}{2\pi U_\infty} \qquad \text{ie. the stagnation point, as expected}$$

Locate y intercepts ($x = 0$)

$$-U_\infty y + \frac{Q}{2\pi}\frac{\pi}{2} = 0$$

$$\therefore y = \frac{Q}{4U_\infty}$$

Also for $x \to \infty$, $y \to \frac{Q}{2U_\infty}$.
We can now sketch the overall flow pattern



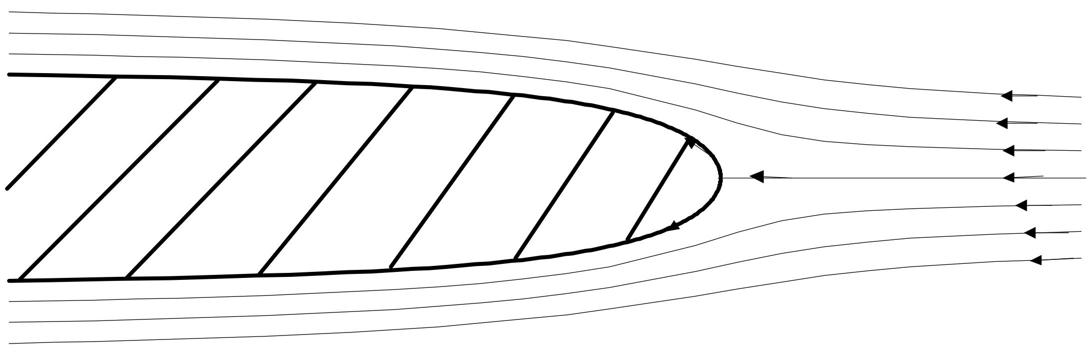
To aid sketching;

- Find stagnation points and note at a stagnation point 2 streamlines come in and two streamlines come out
- sketch sepratrix streamline
- consider flow close to origin (ie. source dominates) and in far field (ie. uniform flow dominates)
- streamlines cannot cross each other
- adjacent streamlines must flow in the same direction.

Now any streamline can be replaced by a solid boundary, eg. we can replace the $\psi = 0$ streamline with a solid boundary. Hence we have solved the flow field about a body whose shape is

$$x = y \cot\left(\frac{2\pi U_\infty y}{Q}\right)$$

and such a shape is called a Half-Rankine body (or semi-infinite body) ie.



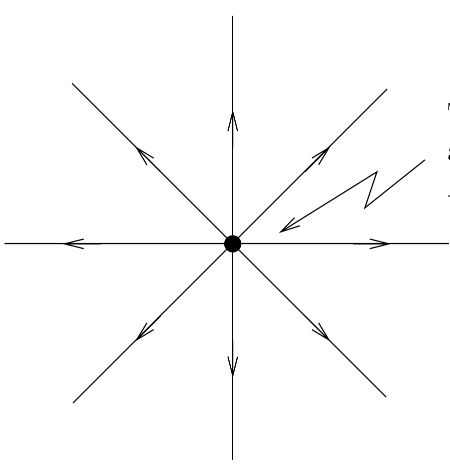
Exercise 5.2: Repeat the derivation outlined in Section 5.2 with the free stream velocity going from left to right. Assume that $Q/(2\pi U_\infty) = 1$. In addition, plot the pressure coefficient, C_p, along the centerline of the body. The solution to this exercise is shown in Figure 5.3.

5.3 Singularities

There are in general two types of singularities

1. Irregular singularity eg. Source/Sink, Vortex $u = v = \pm\infty$
2. Regular singularity (or saddle), $u = v = 0$ stagnation point

Irregular singularity



This is called an irregular singularity
and we cannot Taylor Series expand about this point.
Also a discontinuity exists

Regular singularity

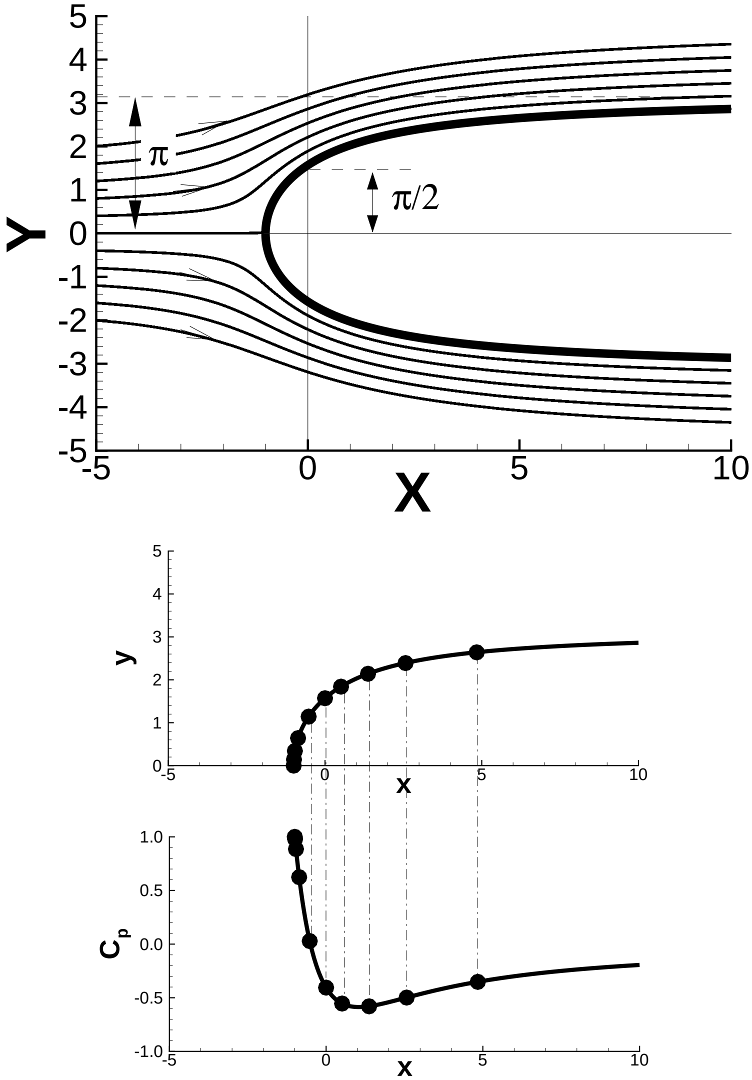
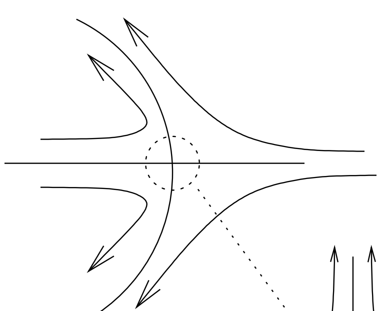
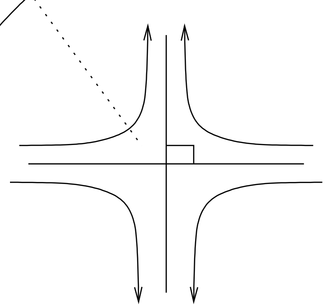


Figure 5.3: Solution to Exercise 5.2



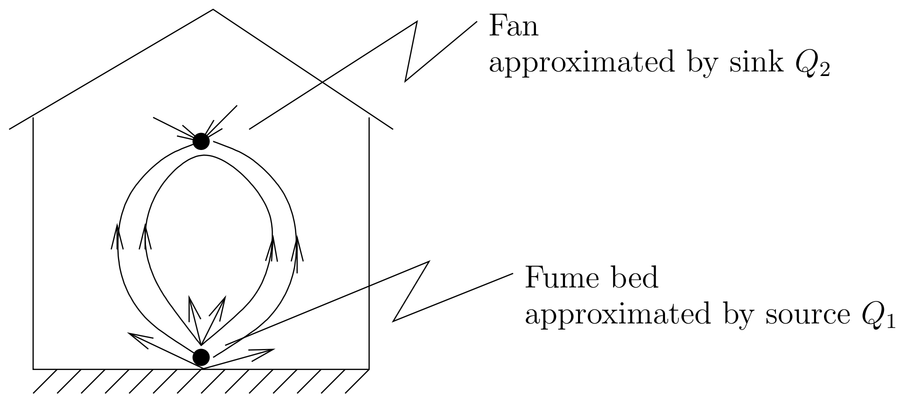
This is a regular sigularity and we can Taylor Series expand about this point Also all functions are continuous as we go through this point. This point is also known as a saddle.



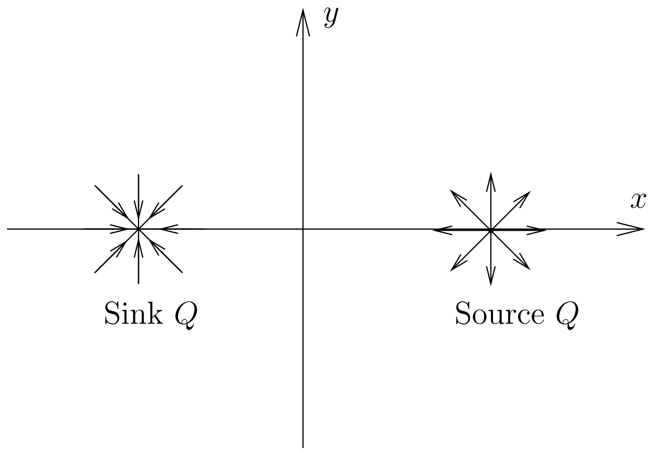
This is what we see if we approach the singularity, ie. rectangular hyperbolae. Note; 90^o only if vorticity is zero.

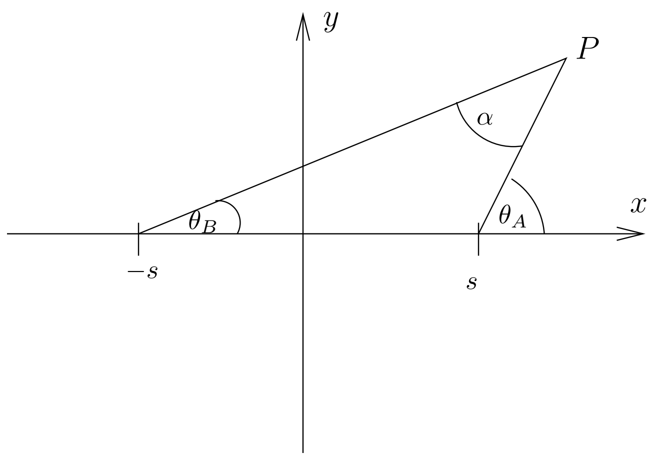
Note: Sources and sinks are called irregular singularities since they cannot occur in practice (ie. $u = v = \infty$). However they can be used to approximate certain practical situations.

Example



5.4 Source and Sink





Have

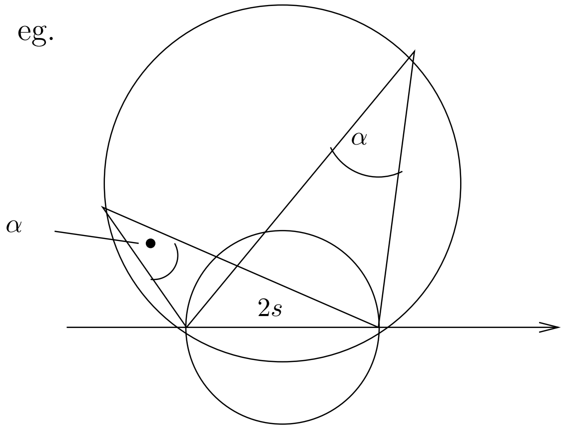
$$\psi_A = \frac{Q}{2\pi}\theta_A$$
$$\psi_B = \frac{-Q}{2\pi}\theta_B$$

hence

$$\begin{aligned}\psi &= \psi_A + \psi_B \\ &= \frac{Q}{2\pi}(\theta_A - \theta_B) \\ &= \frac{Q}{2\pi}\,\alpha\end{aligned}$$

Plot the streamlines, ie. lines of $\psi =$ constant $\Rightarrow \alpha =$ constant. It can be show that the locus of $\alpha =$ constant corresponds to circles all intersecting the $x-$axis at $-s$ and $+s$.

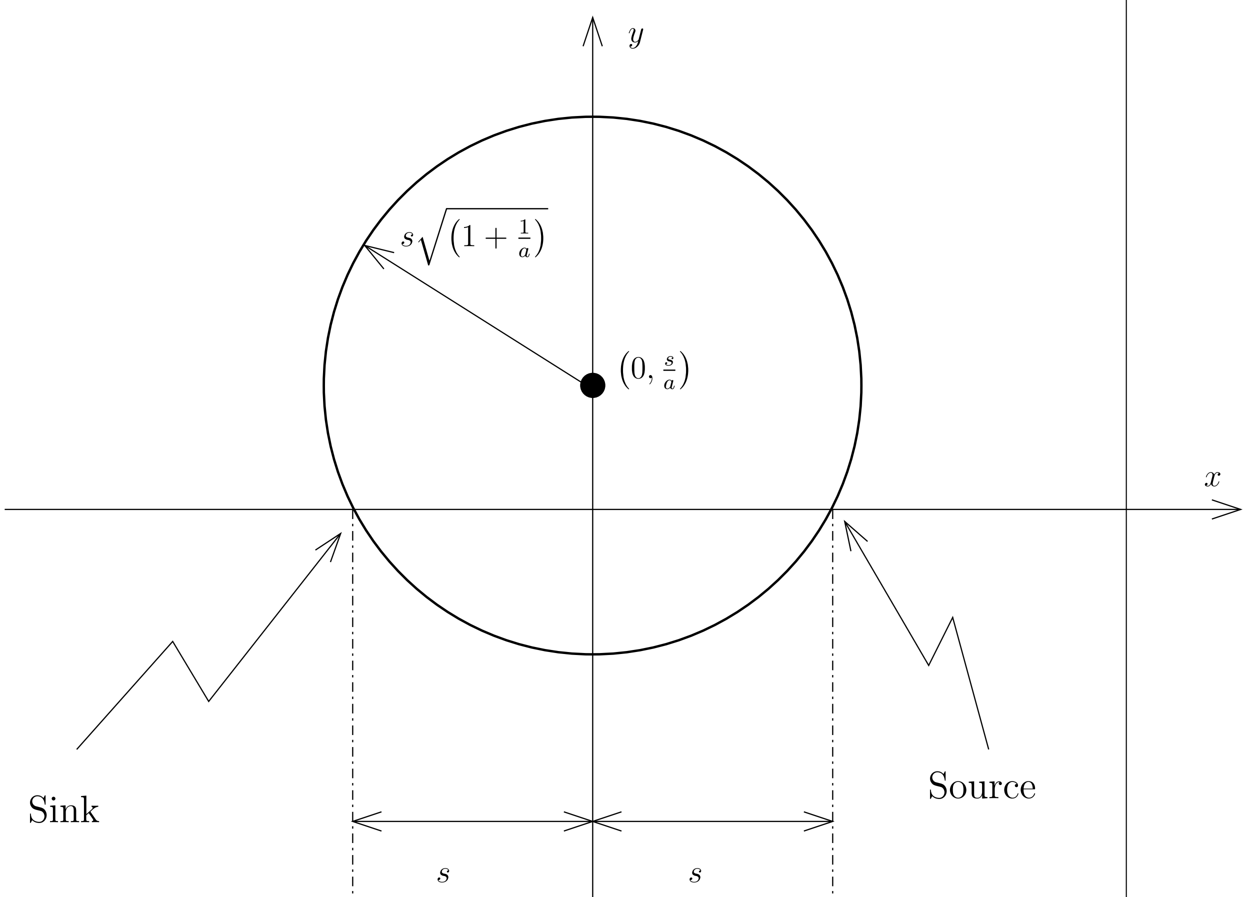
eg.



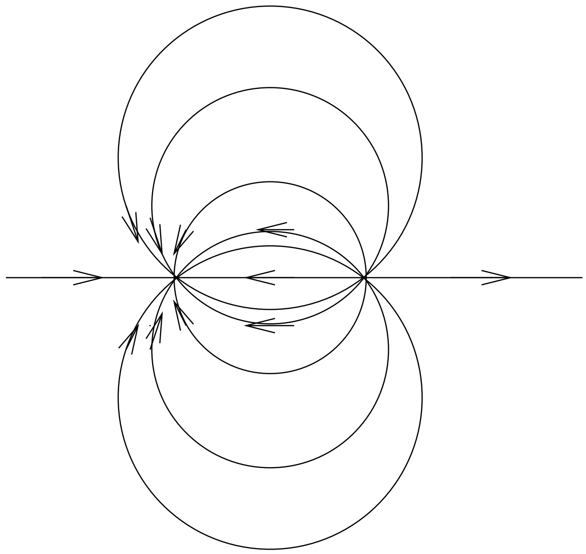
Exercise 5.3: Show that lines of $\psi =$ const. (ie. $\alpha =$ const.) gives the family of equations

$$x^2 + \left(y - \frac{s}{a}\right)^2 = s^2\left(1 + \frac{1}{a^2}\right)$$

where $a = \tan\alpha = \tan(2\pi\Psi/Q)$



Hence the streamline pattern is;



5.4.1 Superimpose flow right to left

Now let us add a flow from *right to left*, then the new flow stream function is

$$\psi = -U_\infty y + \frac{Q}{2\pi}(\theta_A - \theta_B)\ .$$

Know,

$$\begin{aligned}\tan(\theta_A - \theta_B) =& \frac{\tan\theta_A - \tan\theta_B}{1+\tan\theta_A\tan\theta_B}\\ \therefore \theta_A - \theta_B =& \arctan\left\{\frac{\frac{y}{x-s} - \frac{y}{x+s}}{1+\frac{y^2}{x^2-s^2}}\right\}\end{aligned}$$

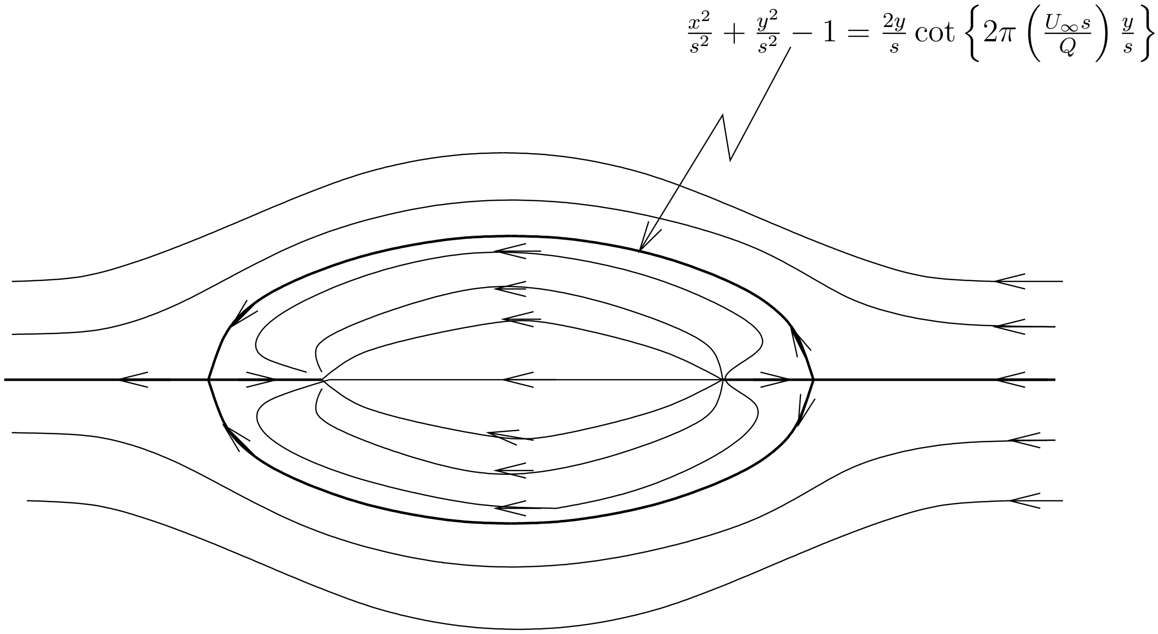
therefore the stream function in cartesian coordinates is

$$\psi = -U_\infty y + \frac{Q}{2\pi}\arctan\left\{\frac{\frac{y}{x-s} - \frac{y}{x+s}}{1+\frac{y^2}{x^2-s^2}}\right\}\ .$$

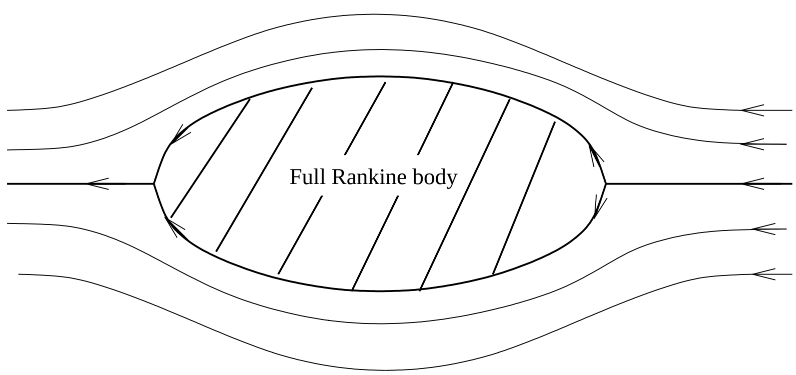
It can be shown that the stagnation points lie on the $\psi = 0$ streamline and this is called the separatrix streamline, lets sketch it. The solution for $\psi = 0$ has two branches;

$$\begin{aligned} y &= 0\\ \text{and} \qquad x^2 + y^2 - s^2 &= 2ys\cot\left(\frac{2\pi U_\infty y}{Q}\right)\end{aligned}$$

$$\text{ie.} \qquad \boxed{\frac{x^2}{s^2} + \frac{y^2}{s^2} - 1 = \frac{2y}{s}\cot\left\{2\pi\left(\frac{U_\infty s}{Q}\right)\frac{y}{s}\right\}} \qquad \psi = 0 \Rightarrow \text{Oval shape} \tag{5.1}$$



Like before the $\psi = 0$ streamline can be replaced by a solid body. Hence we have infact solved the problem of flow past a body whose shape is given by equation (5.1).



This is called a full Rankine body. The shape of the body depends on the non-dimensional parameter $\frac{U_\infty s}{Q}$ while the scale (size) depends on the length scale s.

Exercise 5.4: Show that the stagnation points for the full Rankine body occur at $x = \pm s\sqrt{Q/(U_\infty \pi s) + 1}$.

5.4.2 Superimpose flow left to right

The flow pattern changes dramatically if the free stream flow is in the opposite direction. Consider the case when the flow is from *left to right*

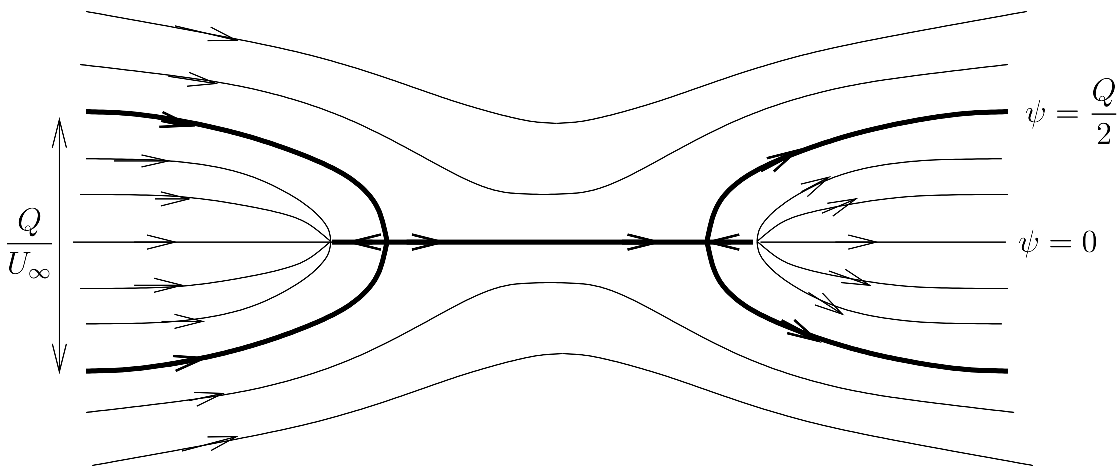
$$\psi = U_\infty y + \frac{Q}{2\pi} \arctan \left\{ \frac{\frac{y}{x-s} - \frac{y}{x+s}}{1 + \frac{y^2}{x^2 - s^2}} \right\} .$$

Find the stagnation points ie.

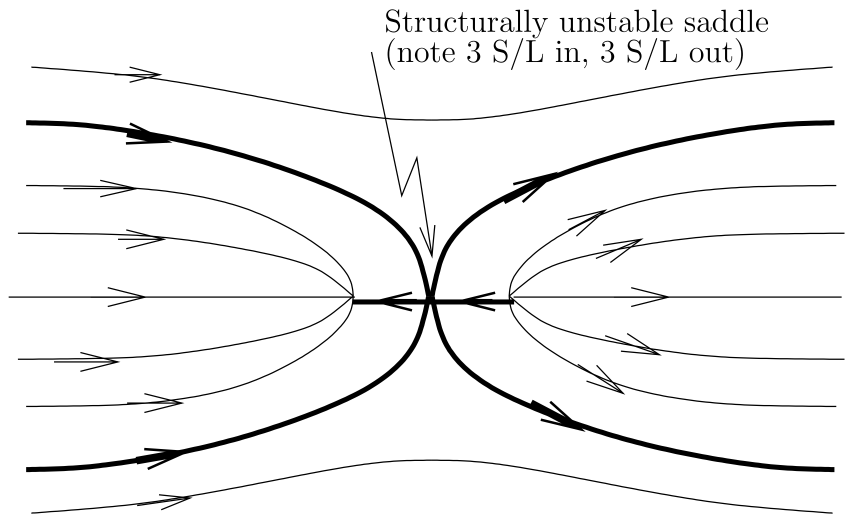
$$u = 0 \qquad \text{and} \qquad v = 0$$
$$\rightarrow \frac{\partial \psi}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial \psi}{\partial x} = 0$$

It turns out (exercise show this) that the location of the stagnation points depends on the strength of the parameter $\frac{U_\infty s}{Q}$, there are three cases;

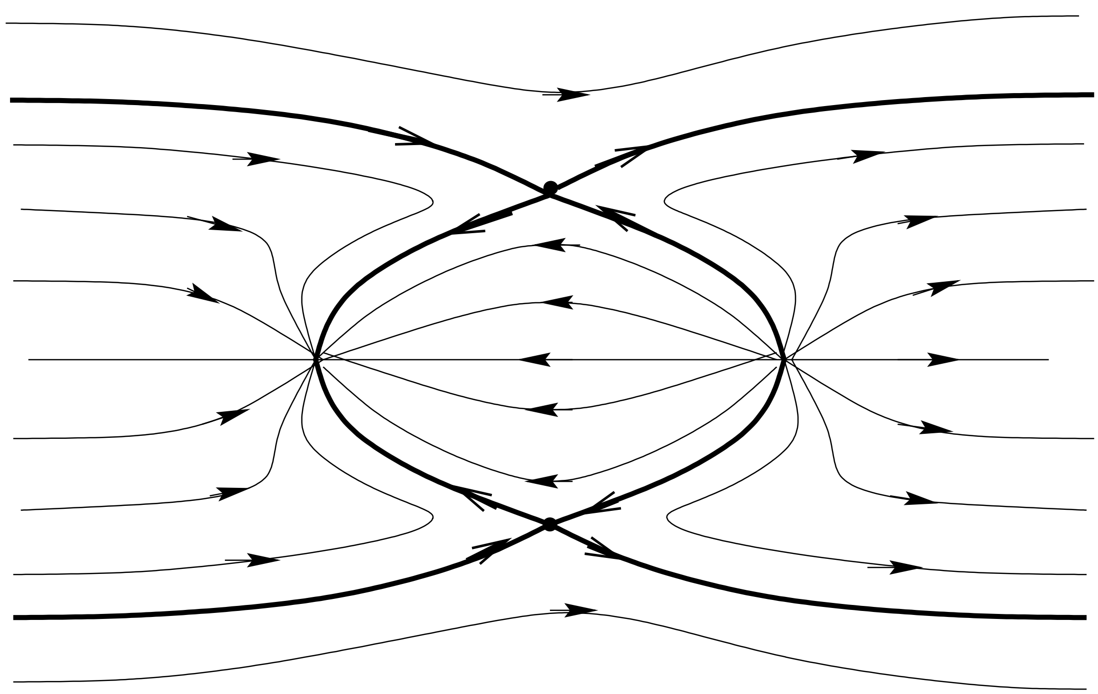
1. $\frac{U_\infty s}{Q} > 1/\pi$, then the stagnation points are on the $x-$axis at $x = \pm s\sqrt{1 - Q/(U_\infty \pi s)}$.



2. $\frac{U_\infty s}{Q} = 1/\pi$, then the stagnation points are at the origin (repeated root) this is a degenerate case (unstable saddle)



3. $\frac{U_\infty s}{Q} < 1/\pi$, then the stagnation points are on the $y-$axis at $y = \pm s\sqrt{Q/(U_\infty \pi s) - 1}$.



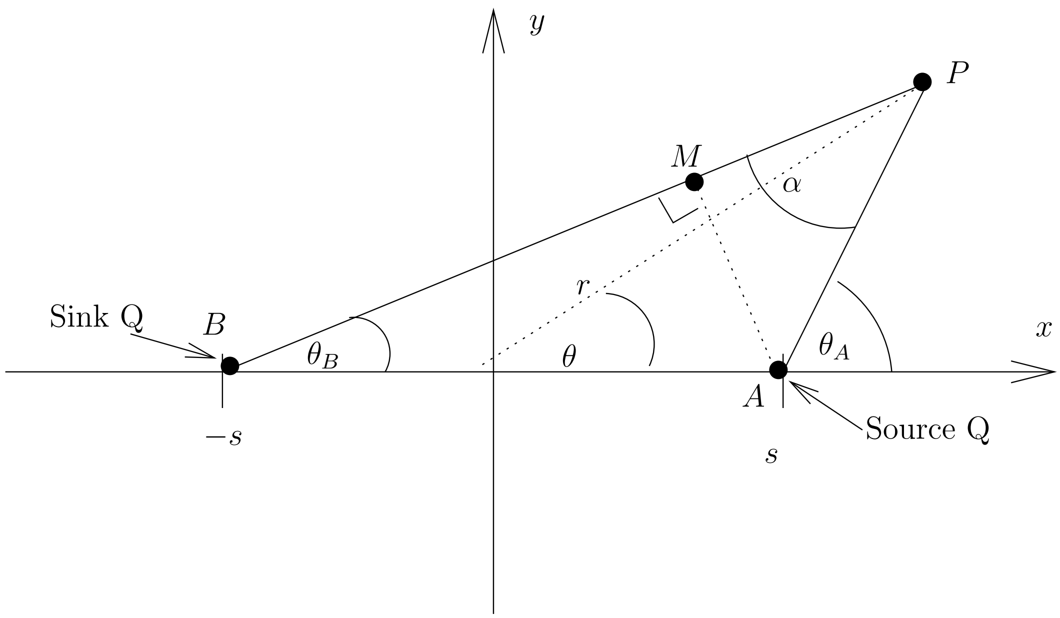
Note as the parameter $\frac{U_\infty s}{Q}$ is varied the saddles (stagnation points) move, merge and split. When this happens it is called a bifurcation.

5.5 Flow past a circular cylinder

This is a special case of the Rankine body where the spacing between the source and sink goes to zero.

5.5.1 The doublet

When a source and sink of equal strength are superimposed upon one and other we get a doublet. First consider source/sink pair spaced $2s$ apart;



then let the source and the sink move together (ie. $s \to 0$), such that the product Qs remains constant (K). Then as $s \to 0$

$$\alpha = \frac{\overline{AM}}{r} = \frac{2s\sin\theta}{r}$$

We know (from last lecture)

$$\psi = \frac{Q}{2\pi}\alpha$$
$$\therefore \psi = \frac{Q}{2\pi}\frac{2s\sin\theta}{r}$$

Hence

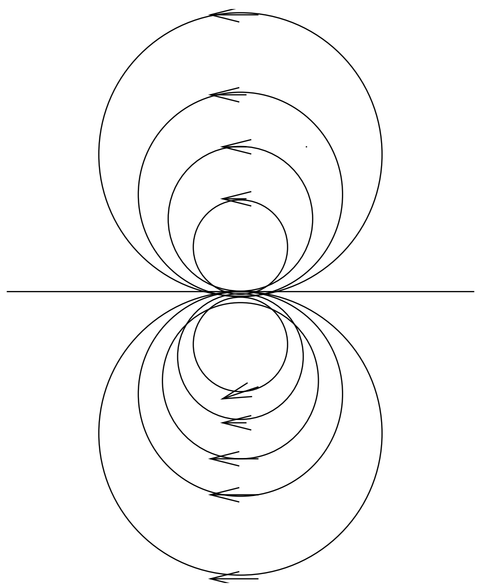
$$\boxed{\psi = \frac{Qs\sin\theta}{\pi r} = \frac{K\sin\theta}{\pi r}} \qquad \text{Doublet .}$$

Note as $s \to 0$, $Q \to \infty$ so that $Qs = K$ remains a constant. In Cartesian coordinates

$$\sin\theta = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}$$
$$r = (x^2+y^2)^{\frac{1}{2}}$$

$$\therefore \boxed{\psi = \frac{Ky}{\pi(x^2+y^2)}}$$

Streamlines are a family of circles, whose centres lie on the $y-$axis and which pass through the origin.



5.5.2 Doublet with uniform flow

Now we are going to add parallel flow from right to left to the doublet. Hence the stream function for this flow is

$$\psi = -U_\infty y + \frac{Ky}{\pi(x^2+y^2)}$$

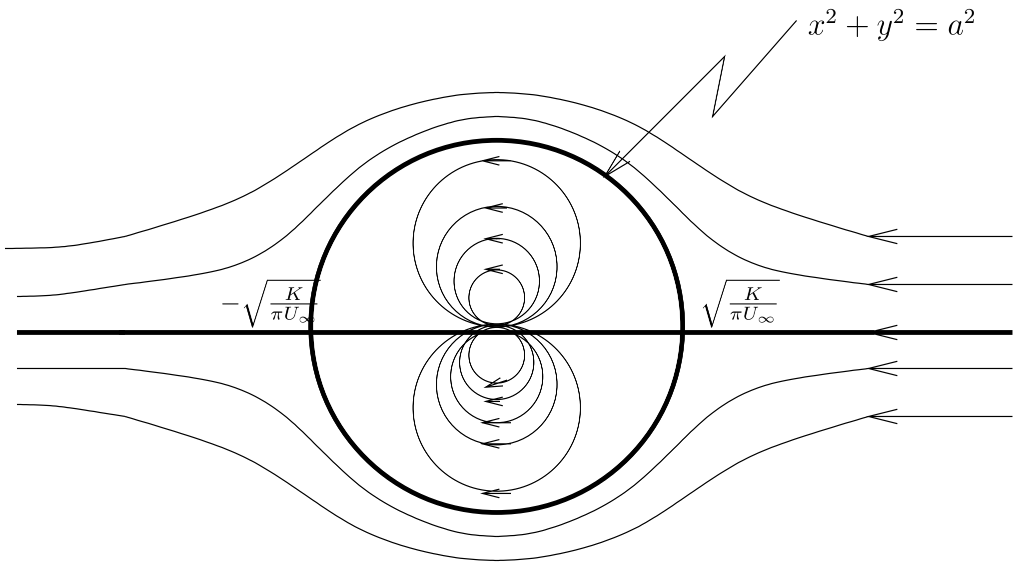
Find stagnation points ie. $u = 0$ and $v = 0$; exercise show stagnation points at

$$x = \pm\sqrt{\frac{K}{\pi U_\infty}}, \quad y = 0 \ .$$

Consider the sepratrix streamline, it passes through the stagnation point and corresponds to the $\psi = 0$ streamline. The solution for $\psi = 0$ has two branches

$$y = 0$$

$$\text{and} \quad x^2 + y^2 = \frac{K}{\pi U_\infty} = a^2 \quad \leftarrow \text{equation to a circle with radius } a = \sqrt{\frac{K}{\pi U_\infty}}$$



Again we can replace the $\psi = 0$ streamline with a solid body and hence we have solved for the flow past a circular cylinder.

Exercise 5.5: Find an expression for the velocity on the surface of the cylinder ($\Psi = 0$ streamline). Use this expression to find the pressure distribution and hence the lift and drag forces on the cylinder.

Exercise 5.6:
Two half cylinders of outer radius a are joined together in a uniform potential flow, as shown in figure 5.4. A hole is to be drilled at an angle θ such that there will be no nett force between the half cylinders at the joints. Determine the angle θ assuming the internal pressure P_{int} to be equal to the static pressure on the external surface of the cylinder at the point where the hold is drilled.

Hint: Remember from potential flow theory that the predicted pressure on the surface of a cylinder is given by

$$P = P_\infty + \frac{1}{2}\rho U_\infty^2 - 2\rho U_\infty^2 \sin^2\theta \tag{5.1}$$

Exercise 5.7: Integrate Eq. (5.1) and show that the lift and drag on a circular cylinder as predicted by potential flow theory is zero.

5.6 Circulation

We wish to solve flow about bodies that produce lift. This can be achieved by introducing circulation around the body. Circulation is the line integral of velocity around a closed loop. Suppose we are in a flow field where the velocity at one of the points is V.

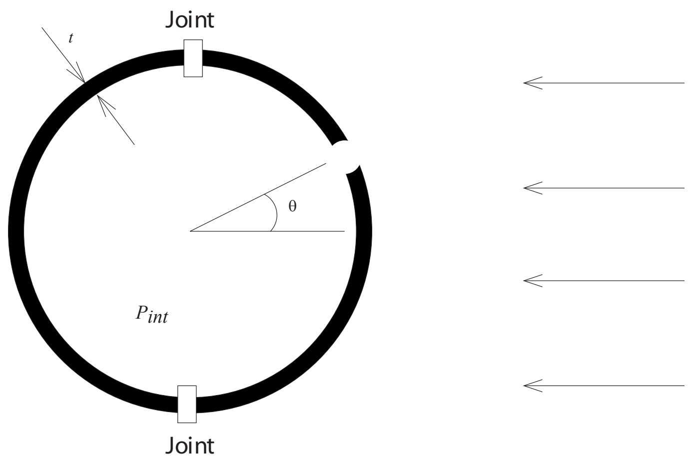
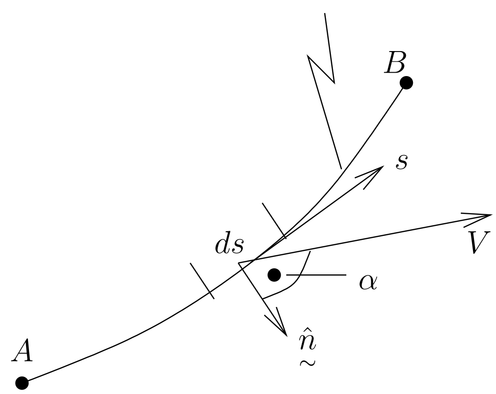


Figure 5.4: Half-cylinder configuration described in Exercise 5.6



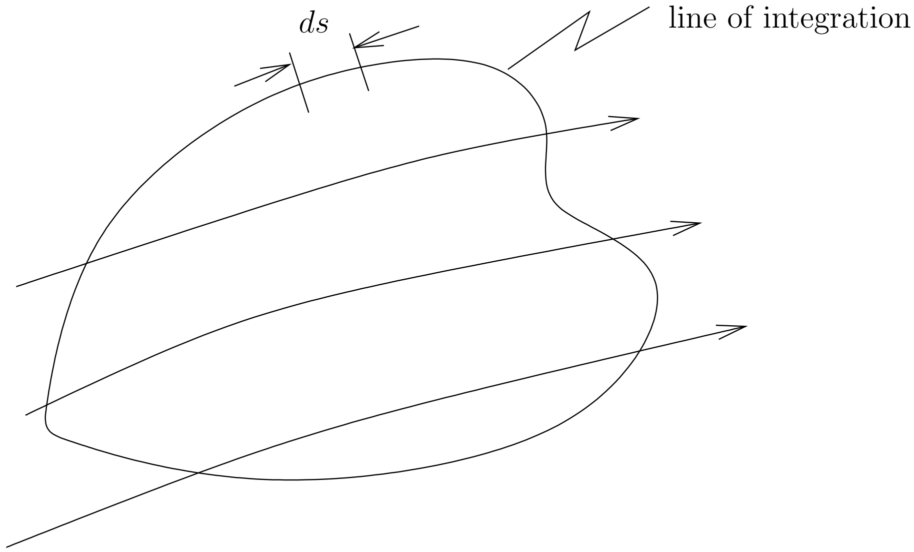
Line integral of velocity from A to B is equal to the component of velocity along the line from A to B, and we will denote this integral by L_{AB} ie

$$L_{AB} = \int_A^B V \sin \alpha ds \ .$$

This expression is analogous do work done if we replaced V with force F

$$\text{work done } = \int_A^B F \sin \alpha ds \ .$$

When the curve is closed it gives the circulation

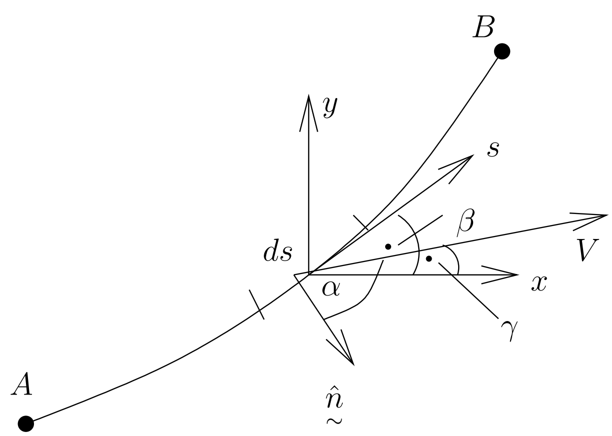


$$\Gamma = \oint V \sin\alpha ds$$

The above expression can also be expressed in terms of the velocity components u and v ie.

$$\Gamma = \oint V \sin\alpha ds = \oint (udx + vdy)$$

Proof:

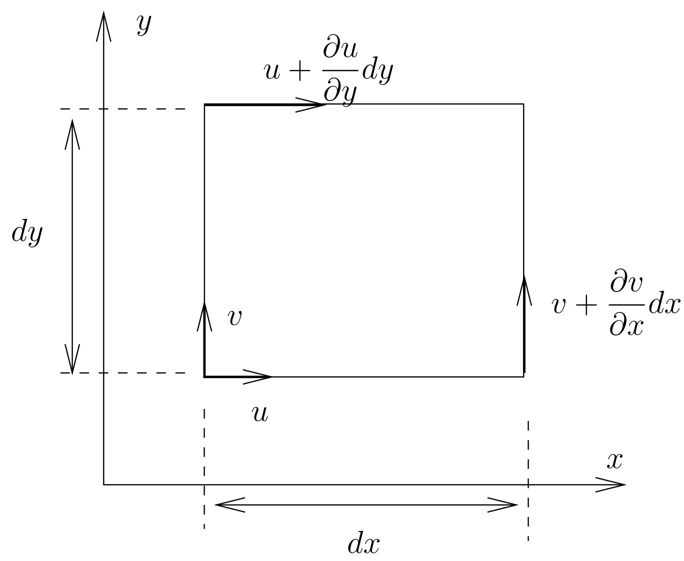


$$\begin{aligned} udx + vdy =& \underbrace{V\cos\gamma}_{u}\ \underbrace{ds\cos\beta}_{dx} + \underbrace{V\sin\gamma}_{v}\ \underbrace{ds\sin\beta}_{dy} \\ =& V ds(\cos\gamma\cos\beta + \sin\gamma\sin\beta) \\ =& V ds(\cos(\beta-\gamma)) \\ =& V ds\sin\alpha \end{aligned}$$

Actually circulation is closely related to vorticity, infact

$$\Gamma = \int_A \zeta dA$$

That is circulation is the area integral of vorticity. Consider an infinitesimal fluid element and evaluate the line integral around this element.



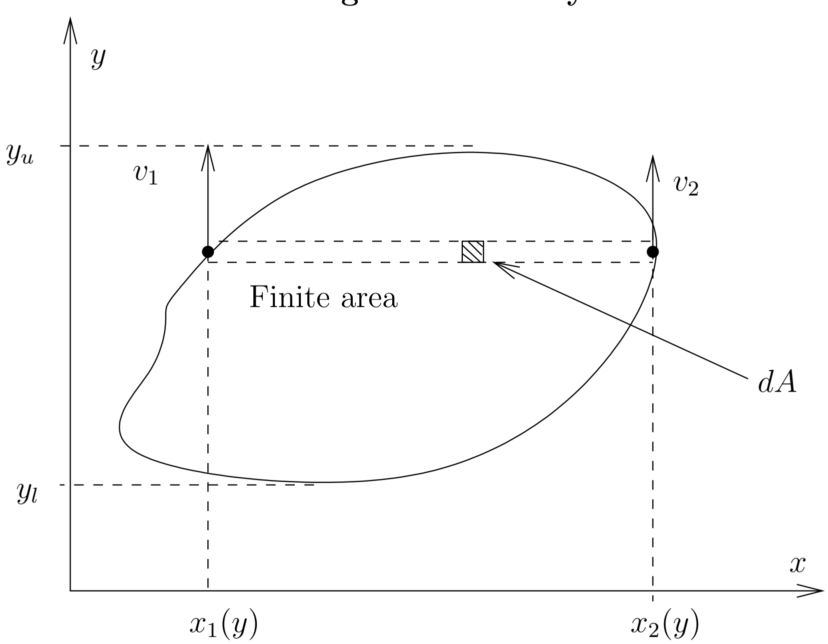
$$d\Gamma = udx + \left(v + \frac{\partial v}{\partial x}dx\right)dy + \left(u + \frac{\partial u}{\partial y}dy\right)(-dx) + v(-dy)$$

$$\therefore d\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$

$$\therefore \boxed{d\Gamma = \zeta dxdy} \Rightarrow \quad \text{circulation} = \text{vorticity} \times \text{area}$$

Hence we can say vorticity ζ = circulation around an element per unit area.

What is the area integral of vorticity over a finite area ?



Integrate over the area,

$$\iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$

$$= \iint_A \left(\frac{\partial v}{\partial x}\right)dxdy - \iint_A \left(\frac{\partial u}{\partial y}\right)dxdy \ .$$

Consider first term,

$$\begin{aligned}\iint_A \left(\frac{\partial v}{\partial x}\right) dxdy &= \int_{y_l}^{y_u} \left(\int_{x_1}^{x_2} \frac{\partial v}{\partial x} dx\right) dy \\ &= \int_{y_l}^{y_u} (v_2 - v_1) dy \\ &= \int_{y_l}^{y_u} v_2 dy + \int_{y_u}^{y_l} v_1 dy \\ &= \oint v dy \ .\end{aligned}$$

Similarly it can be shown that,

$$\iint_A \left(\frac{\partial u}{\partial y}\right) dxdy = -\oint u dx$$

hence

$$\boxed{\iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy = \oint (u dx + v dy)}$$

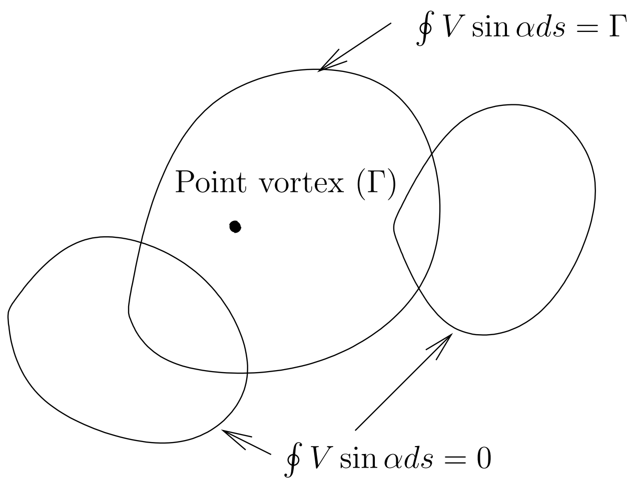
This expression implies the area integral of vorticity = line integral of velocity on a closed circuit around the area.

5.7 The point vortex

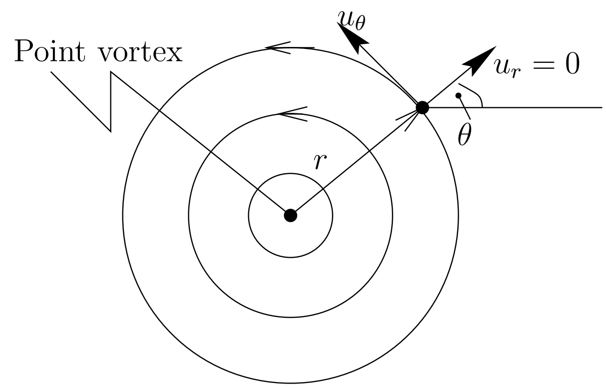
We wish to introduce circulation into potential flow problems but this requires we introduce vorticity since

$$\Gamma = \int_A \zeta dA \ .$$

However we want a irrotational flow field so put all the vorticity at a single point called a point vortex (this will be a singularity in the flow field). For a point vortex $A \to 0$ while $\zeta \to \infty$ such that Γ remains finite, ie. vorticity is concentrated at a point.



Derivation of the stream function for a point vortex with circulation (or strength) Γ.



Circulation = line integral around a closed circuit ie.

$$\Gamma = \oint V \sin \alpha ds$$

Now for the point vortex $V = u_\theta$ and $\alpha = \pi/2$ ie. $u_r = 0$

$$\Gamma = \int_0^{2\pi} u_\theta r d\theta$$
$$\therefore \Gamma = 2\pi r u_\theta$$

In polar coordinates we have

$$\frac{\partial \psi}{\partial r} = -u_\theta \quad \text{and} \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = u_r$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$
$$\Rightarrow \psi = f_1(r)$$

and

$$-\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$
$$\Rightarrow \psi = -\frac{\Gamma}{2\pi}\ln(r) + f_2(\theta)$$

Above are compatible if $f_2(\theta) = \text{const}$, choose the constant such that $\psi = 0$ at $r = b$ where b is some arbitrary value

$$\boxed{\psi = -\frac{\Gamma}{2\pi}\ln\left(\frac{r}{b}\right)} \quad \text{stream function for a potential vortex}$$

Check whether the point vortex satisfies the Laplace equation ie.

$$\nabla^2 \psi = 0$$

In polar coordinates

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial\psi^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial\psi^2}{\partial r^2}$$

For the point vortex we have

$$\frac{\partial\psi}{\partial r} = -\frac{\Gamma}{2\pi r}\,, \quad \frac{\partial^2\psi}{\partial r^2} = \frac{\Gamma}{2\pi r^2} \quad \text{and} \quad \frac{\partial\psi^2}{\partial\theta^2} = 0$$

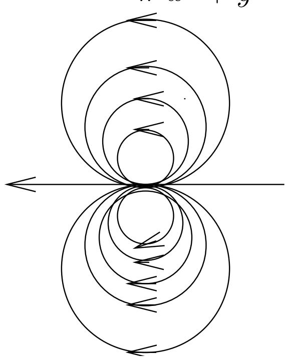
Hence,

$$\begin{aligned}\nabla^2\psi =& \frac{1}{r}\left(-\frac{\Gamma}{2\pi r}\right) + \frac{\Gamma}{2\pi r^2}\\ =& 0\end{aligned}$$

5.8 Flow past a circular cylinder with circulation

It has been shown that flow around a cylinder can be generated from a doublet in a uniform flow

$$\psi = \frac{K}{\pi}\frac{y}{x^2+y^2} \qquad\qquad \psi = -U_\infty y$$



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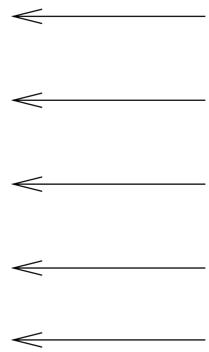
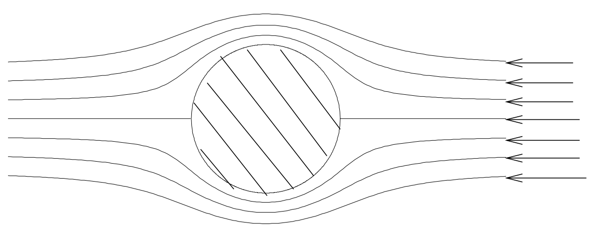


$$\begin{aligned}\psi_1 =& -U_\infty y + \frac{K}{\pi}\frac{y}{x^2+y^2}\\ =& -U_\infty \sin\theta\left(r - \frac{K}{\pi U_\infty}\frac{1}{r}\right)\end{aligned}$$

and the radius (a) of the cylinder generated by the above is

$$a = \sqrt{\frac{K}{\pi U_\infty}}$$



Because the flow pattern is symmetrical there is no lift generated.

In order to obtain lift we have to add circulation to the flow, this can be achieved by introduction of a point vortex. To achieve positive lift for the above configuration we require positive circulation (ie. anti-clockwise). The stream function for a vortex placed at the origin is

$$\psi_2 = \frac{-\Gamma}{2\pi}\ln\left(\frac{r}{b}\right) .$$

Hence the combined flow is

$$\begin{aligned}\psi =& \psi_1 + \psi_2\\ =& -U_\infty \sin\theta\left(r - \frac{a^2}{r}\right) - \frac{\Gamma}{2\pi}\ln\left(\frac{r}{b}\right)\end{aligned}$$

$$
\begin{aligned}
u_r =& \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\frac{1}{r}U_\infty \cos\theta\left(r - \frac{a^2}{r}\right) \\
=& -U_\infty \cos\theta \left(1 - \frac{a^2}{r^2}\right) \qquad (5.1)
\end{aligned}
$$

$$
u_\theta = -\frac{\partial \psi}{\partial r} = U_\infty \sin\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi}\frac{1}{r} \qquad (5.2)
$$

Now we want to find the stagnation points ie. $u_r = u_\theta = 0$. From (5.1)

$$
-U_\infty \cos\theta\left(1 - \frac{a^2}{r^2}\right) = 0
$$

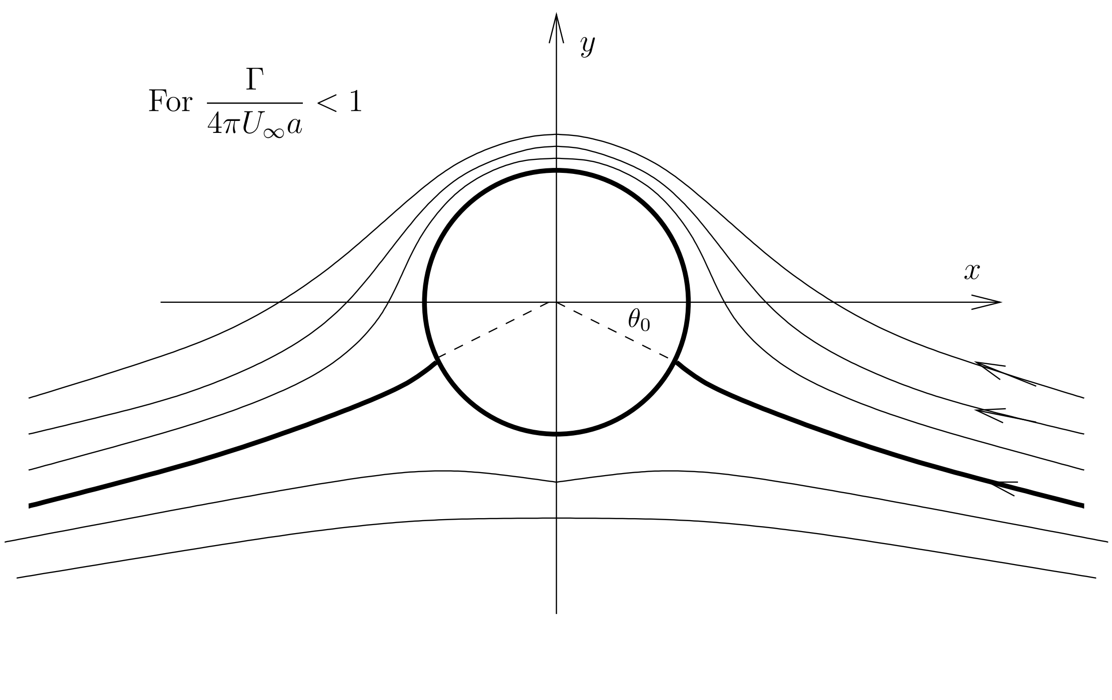
$$
\text{solutions are} \quad r = a \quad \text{or} \quad \theta = \frac{\pi}{2}
$$

Check if $u_\theta = 0$ has solutions for $r = a$, from (5.2);

$$
U_\infty \sin\theta\left(1 + \frac{a^2}{a^2}\right) + \frac{\Gamma}{2\pi a} = 0
$$

$$
\therefore \ \sin\theta = \frac{-\Gamma}{4U_\infty \pi a} \quad \text{solutions exist for } \frac{\Gamma}{4\pi U_\infty a} < 1
$$

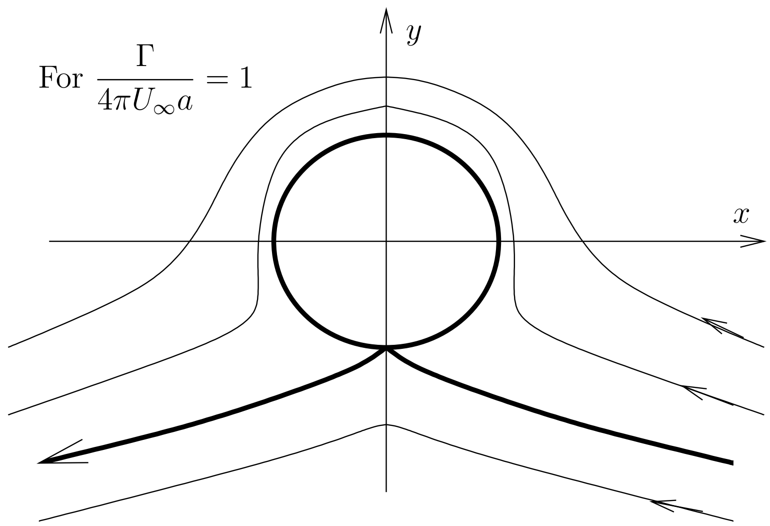
This means when the non-dimensional parameter $\frac{\Gamma}{4\pi U_\infty a} < 1$ there exists two stagnation points located on the surface of the cylinder ($r = a$) and at $\theta_0 = \sin^{-1}(\frac{-\Gamma}{4U_\infty \pi a})$, ie.



For $\frac{\Gamma}{4\pi U_\infty a} = 1$ these two stagnation points merge and are both located at

$$\theta = \sin^{-1}\left(\frac{-4U_\infty \pi a}{4U_\infty \pi a}\right) = -\frac{\pi}{2}, \qquad r = a$$

ie. one stagnation point

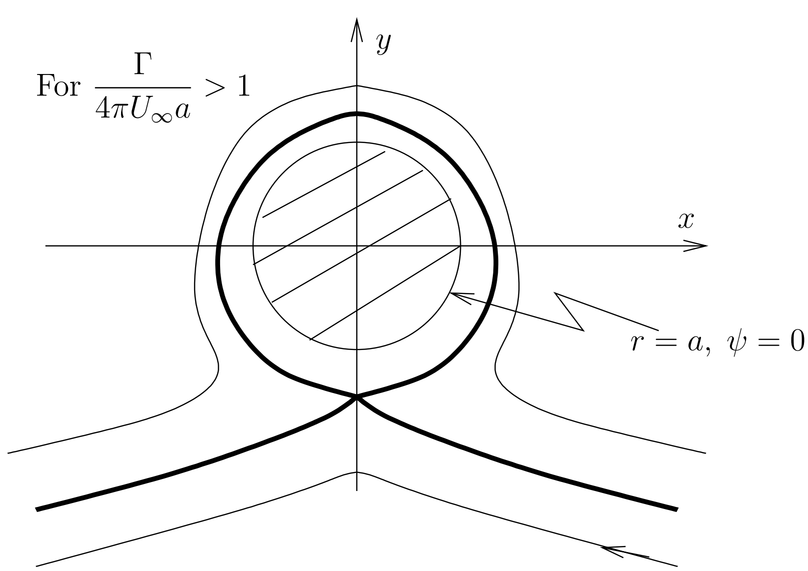


Now check if $v' = 0$ has solutions for $\theta = \frac{\pi}{2}$, from (5.2);

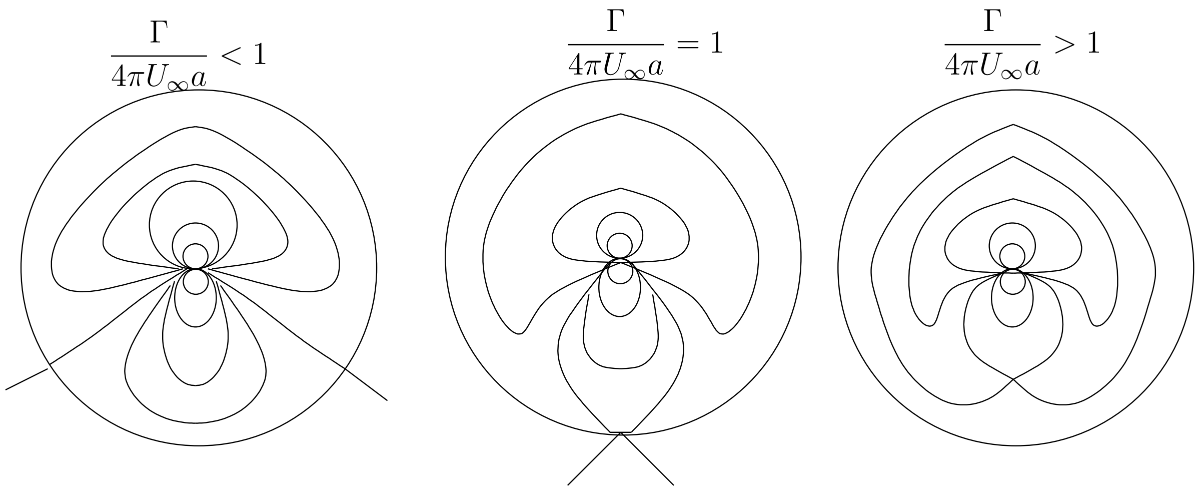
$$U_\infty\left(1 + \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi r} = 0$$

$$\therefore\ r^2 + \frac{\Gamma}{2\pi U_\infty} r + a^2 = 0 \Rightarrow \quad \text{2 real solutions exist for } \frac{\Gamma}{4\pi U_\infty a} > 1$$

This means when the non-dimensional parameter $\frac{\Gamma}{4\pi U_\infty a} > 1$ there exists two stagnation points located on the $y-$axis (one at $|r| < a$ and one at $|r| > a$) ie.

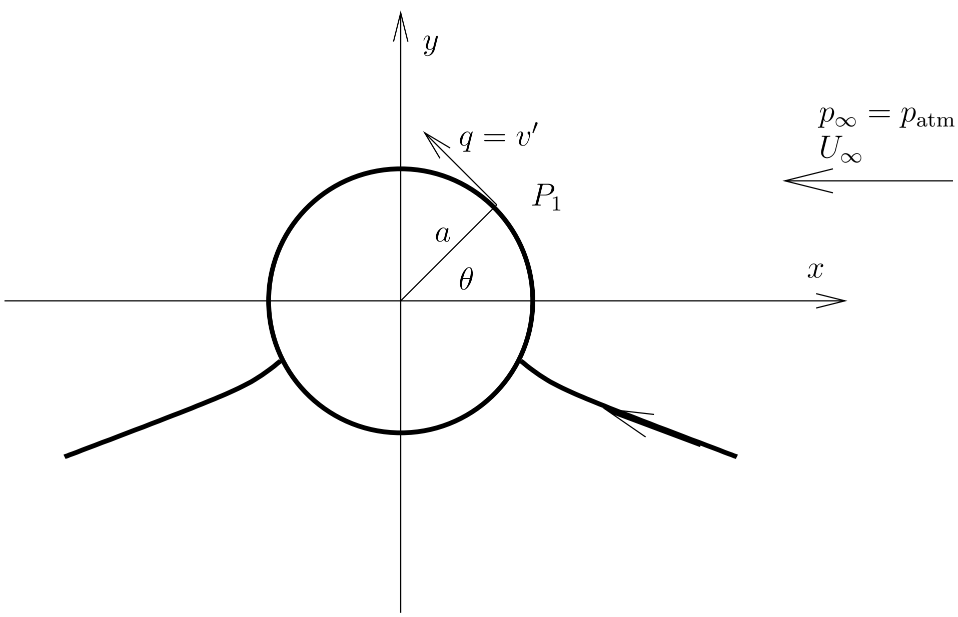


Note for all of the above cases the shape of the $\psi = 0$ streamline is preserved as a circle of radius$= a$. If we looked at the streamlines inside the circle we would see;



5.8.1 Pressure distribution

In order to determine the lift generated we need to know the pressure distribution around the cylinder. Let p be the static pressure at some point $P_1(a, \theta)$ and q be the resultant velocity ($q = \sqrt{u_r^2 + u_\theta^2}$).



Applying Bernoulli along the streamline $\psi = 0$ (ie. $r = a$) also note $q = u_\theta$ on the surface.

$$p + \frac{1}{2}\rho v'^2 = p_\infty + \frac{1}{2}\rho U_\infty^2 = p_t \quad = \text{total pressure}$$

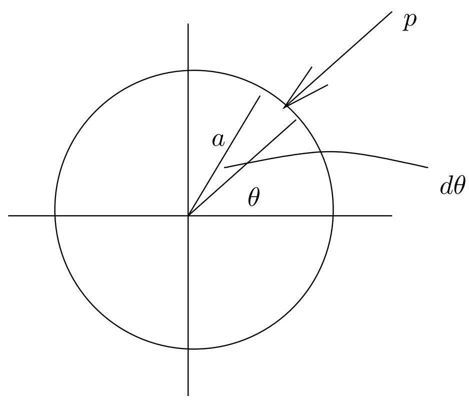
$$\therefore p = p_t - \frac{1}{2}\rho v'^2$$

Now we know the velocity distribution on the surface is;

$$u_\theta = 2U_\infty \sin\theta + \frac{\Gamma}{2\pi a}$$

$$\begin{aligned} p =& p_t - \frac{1}{2}\rho\left(2U_\infty \sin\theta + \frac{\Gamma}{2\pi a}\right)^2 \\ =& p_t - \frac{1}{2}\rho\left(4U_\infty^2 \sin^2\theta + \frac{2\Gamma U_\infty \sin\theta}{\pi a} + \frac{\Gamma^2}{4\pi^2 a^2}\right) \end{aligned}$$

Lift = normal force perpendicular to the free stream direction



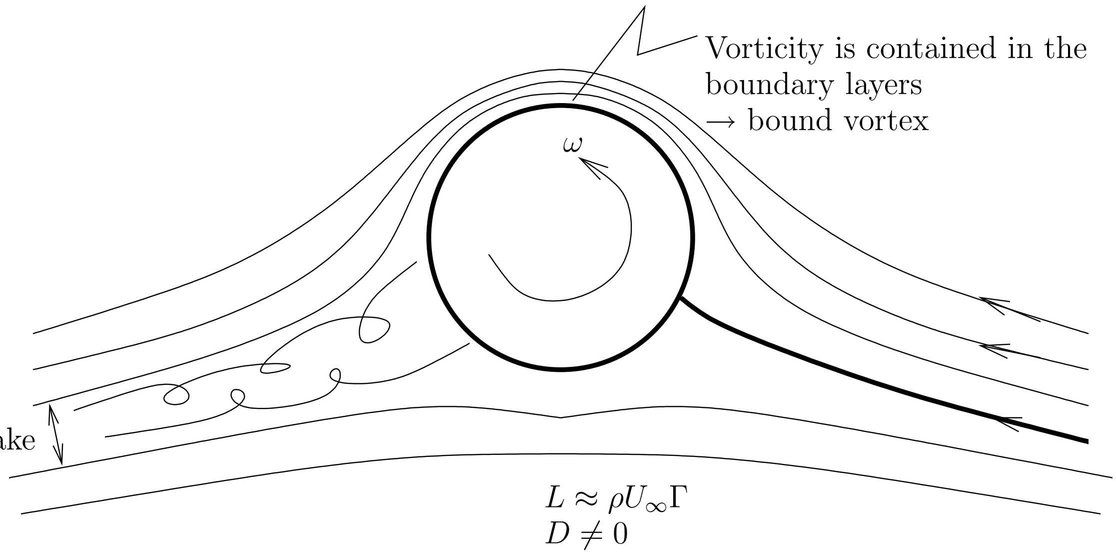
$$\begin{aligned} L =& -\int_0^{2\pi} pad\theta \cdot 1 \cdot \sin\theta \qquad \text{(lift per unit length)} \\ =& \int_0^{2\pi} \left[-p_t a \sin\theta + \frac{1}{2}\rho a\left(4U_\infty^2 \sin^3\theta + \frac{2\Gamma U_\infty \sin^2\theta}{\pi a} + \frac{\Gamma^2}{4\pi^2 a^2}\sin\theta\right)\right] d\theta \\ =& \int_0^{2\pi} \left[-p_t a \sin\theta + \frac{1}{2}\rho a\left(U_\infty^2 (3\sin\theta - \sin 3\theta) + \frac{\Gamma U_\infty}{\pi a}(1-\cos 2\theta) + \frac{\Gamma^2}{4\pi^2 a^2}\sin\theta\right)\right] d\theta \\ =& \frac{1}{2}\rho a\left(\frac{\Gamma U_\infty}{\pi a}\right) 2\pi \\ \therefore L =& \rho U_\infty \Gamma \end{aligned}$$

If the cylinder had a length of l

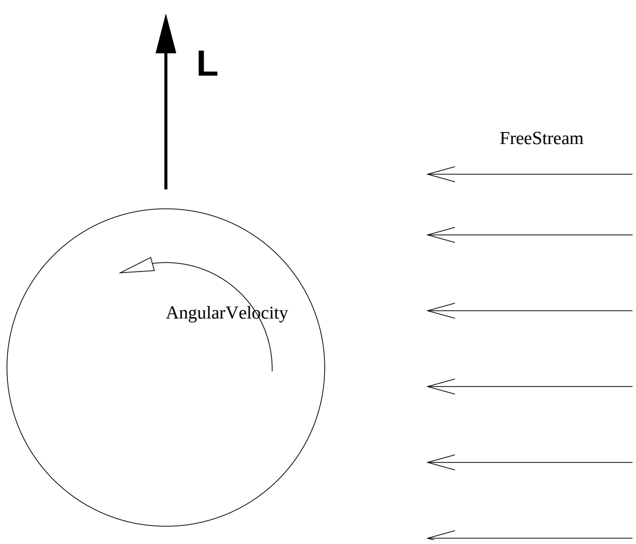
$$\Rightarrow \boxed{\text{total lift} = \rho U_\infty \Gamma l}$$

5.8.2 Magnus effect

The cylinder with circulation flow can be approximately achieved by spinning a cylinder in cross flow. The lift that results is called the Magnus effect.



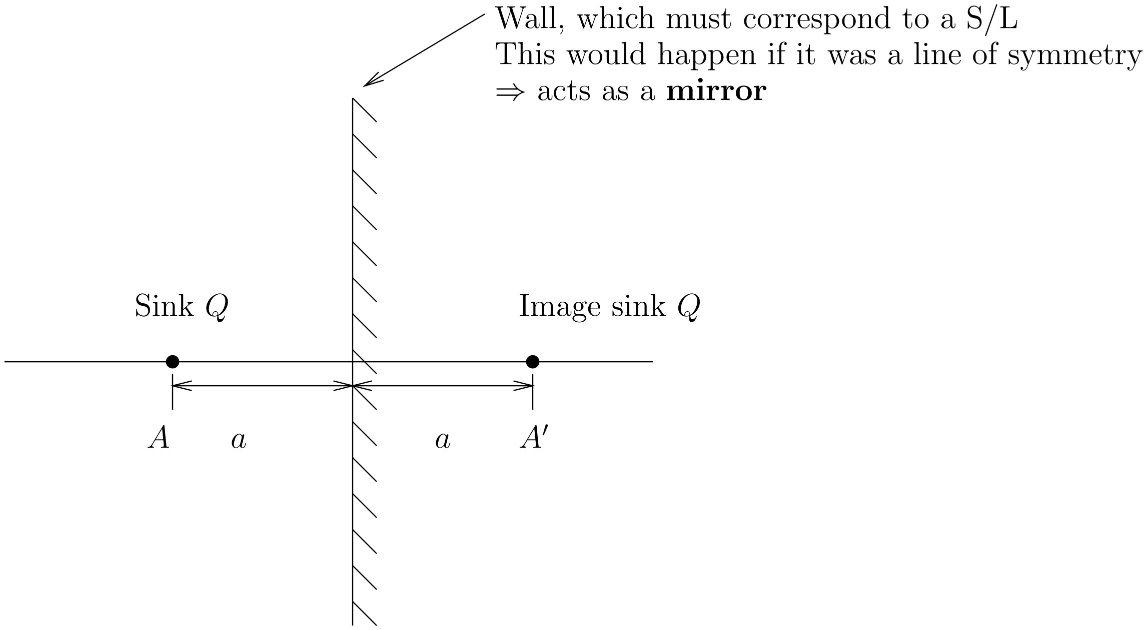
> **<u>Exercise 5.8:</u>**
> A cylinder of diameter 2.5 cm rotate as indicated at 3600 rpm in standard air which is flowing over the cylinder at 30 ms^{-1}. Estimate the lift per unit length of the cylinder.



5.9 Method of images

Often we want to study flow patterns in the vicinity of a solid plane boundary. To get the correct flow requires that the boundary corresponds to a streamline. This can be achieved by treating the boundary as a mirror and placing images of the flow structures behind the mirror.

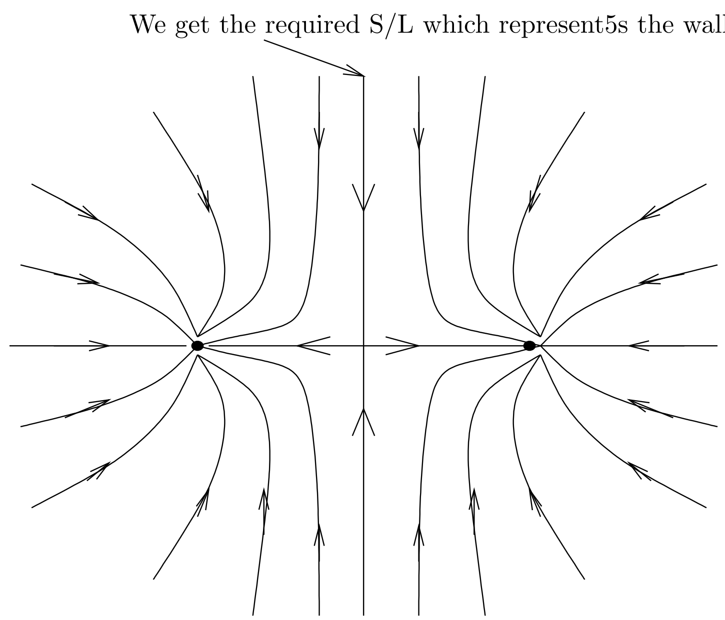
For example say we have a sink located near a plane wall;



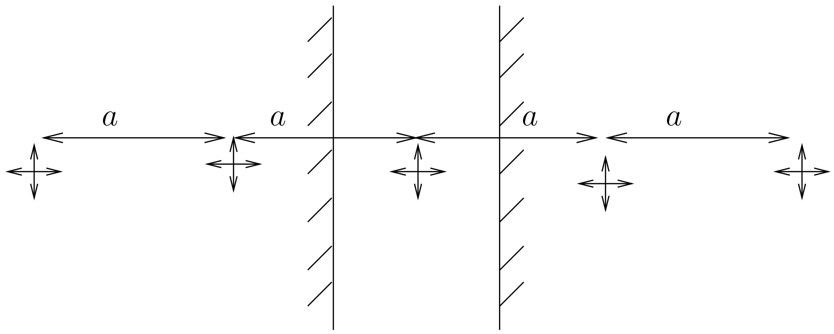
For the above the stream function would be

$$\begin{aligned}\psi =&\psi_A + \psi_{A'}\\ =&-\frac{Q}{2\pi}\theta_A - \frac{Q}{2\pi}\theta_{A'}\\ =&-\frac{Q}{2\pi}\left(\arctan\left\{\frac{y}{x+a}\right\} + \arctan\left\{\frac{y}{x-a}\right\}\right)\end{aligned}$$

which gives;



Another example is a source between parallel planes, in this case we get an infinite series of images (ie. like looking into a mirror when there is another mirror behind you)

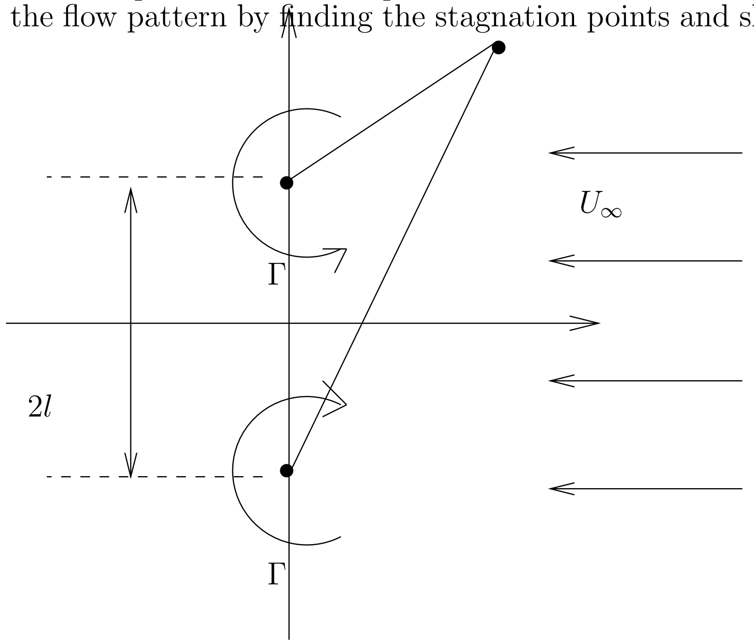


and the stream function is a series

$$\psi = \sum_{i=1}^{\infty} \left(\frac{Q}{2\pi} \theta_i \right)$$

5.10 Vortex pair

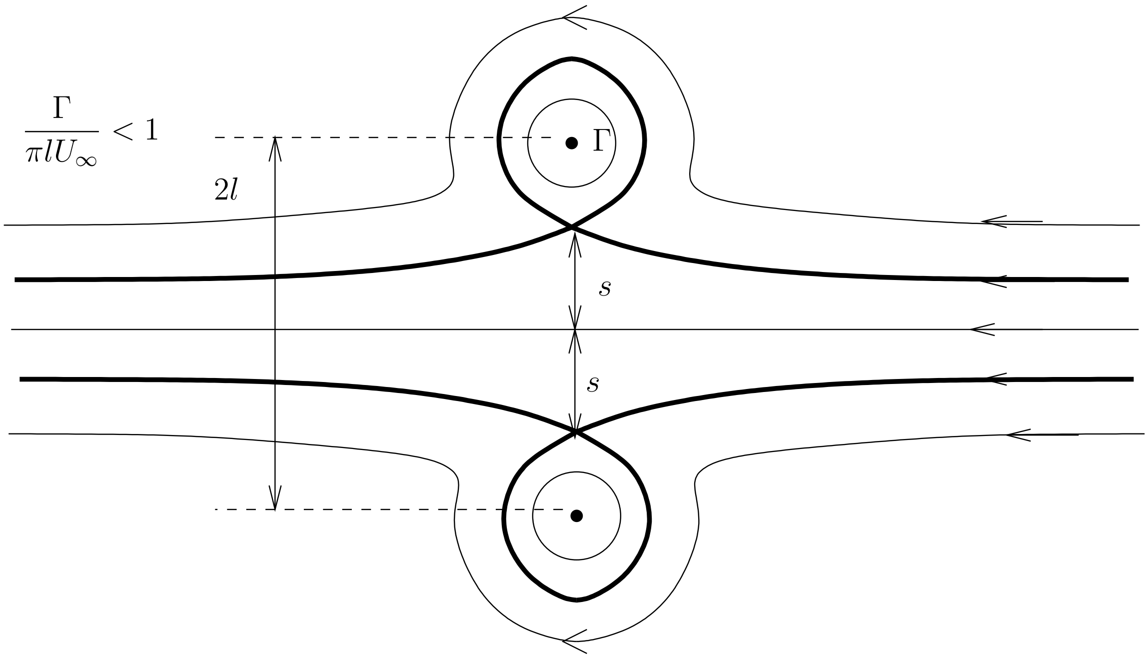
Imagine we have a vortex pair held fixed in space with a uniform flow superimposed. We can analysis the flow pattern by finding the stagnation points and sketching the flow.



$$\begin{aligned}
\psi =& \frac{-\Gamma}{2\pi}\ln r_A + \frac{\Gamma}{2\pi}\ln r_B - U_\infty y \\
=& \frac{-\Gamma}{2\pi}\ln\left(\frac{r_A}{r_B}\right) - U_\infty y \\
=& \frac{-\Gamma}{2\pi}\ln\left(\frac{(x^2+(y-l)^2)^{\frac{1}{2}}}{(x^2+(y+l)^2)^{\frac{1}{2}}}\right) - U_\infty y \\
=& \frac{-\Gamma}{4\pi}\ln\left(\frac{x^2+(y-l)^2}{x^2+(y+l)^2}\right) - U_\infty y
\end{aligned}$$

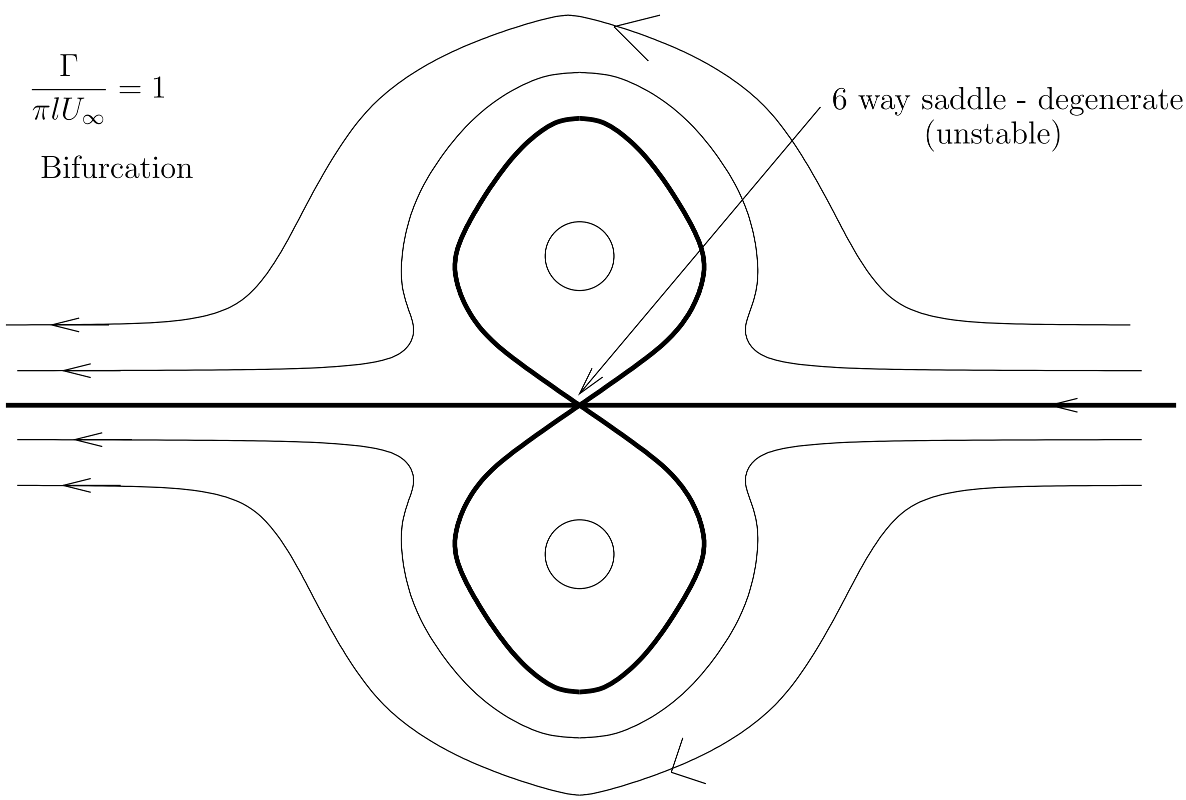
$$\left.\begin{aligned}
u = \frac{\partial\psi}{\partial y} = 0 \\
v = -\frac{\partial\psi}{\partial x} = 0
\end{aligned}\right\} \quad \text{for stagnation points}$$

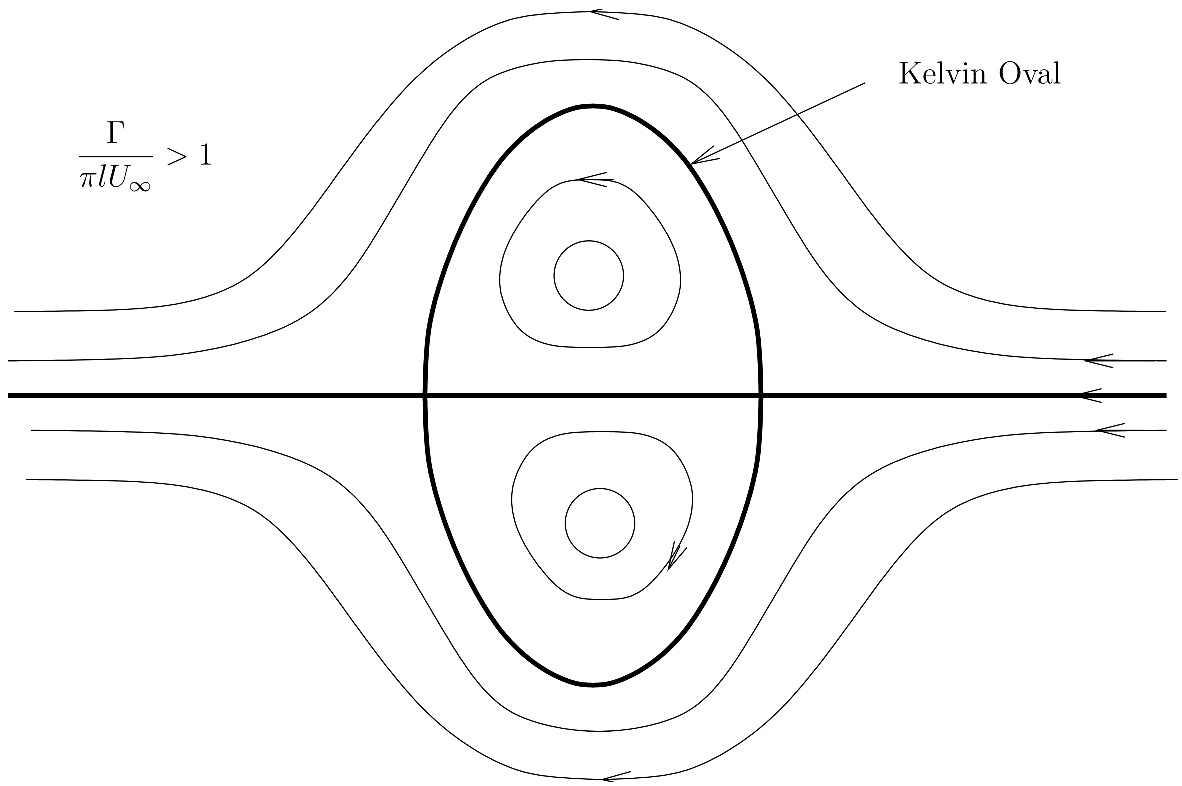
It turns out that we get different flow patterns depending on the strength of the non-dimensional parameter $\frac{\Gamma}{lU_\infty}$



Using dimensional analysis it can be shown that

$$\frac{s}{l} = f\left(\frac{\Gamma}{lU_\infty}\right)$$

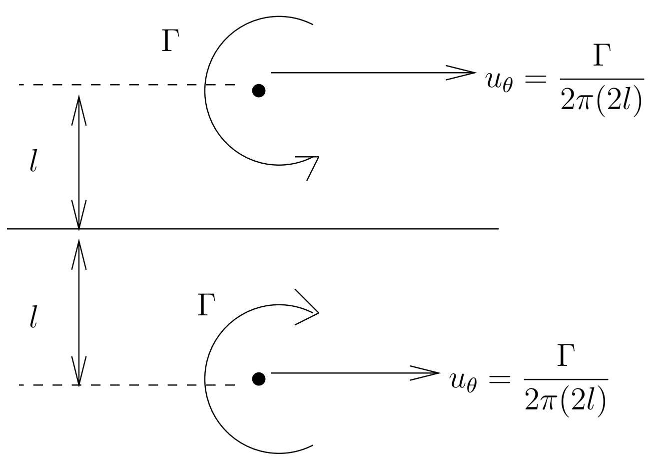




To find the shape of the Kelvin Oval consider $\psi = 0$ streamline, show this gives

$$x = \left[\frac{(y-l)^2 \exp\left(\frac{4\pi U_\infty y}{\Gamma}\right) - (y+l)^2}{1 - \exp\left(\frac{4\pi U_\infty y}{\Gamma}\right)} \right]^{\frac{1}{2}}$$

In real physical situations we cannot have vortex pairs fixed in space hence the pattern is unsteady. This is because the vortices induce each other along with a velocity.



Therefore each vortex moves with a velocity

$$u = \frac{\Gamma}{2\pi(2l)}$$

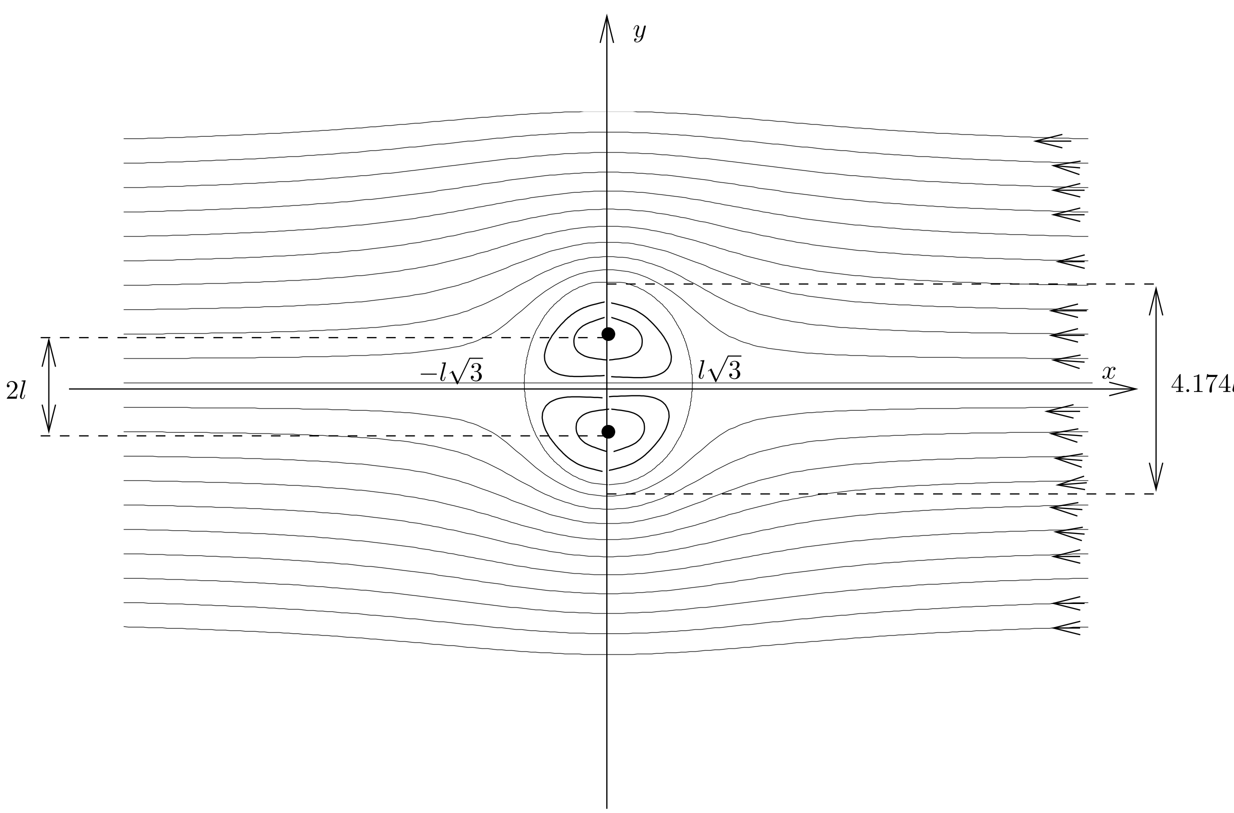
In order to achieve steady flow we must analyse the pattern in a frame of reference moving with the vortex pair, this implies we see a uniform flow of

$$U = \frac{\Gamma}{4\pi l} \quad \text{right to left}$$

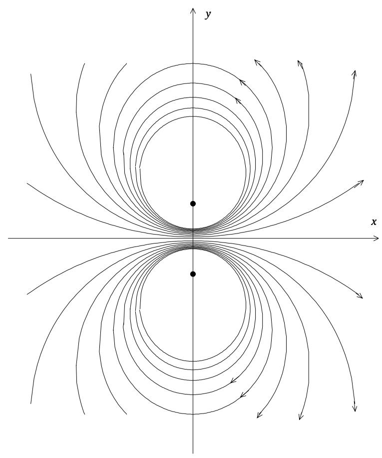
The strength of the non-dimensional parameter

$$\frac{\Gamma}{\pi U_\infty l} = 4$$

and hence the shape of the Kelvin Oval is fixed and the streamline pattern looks like

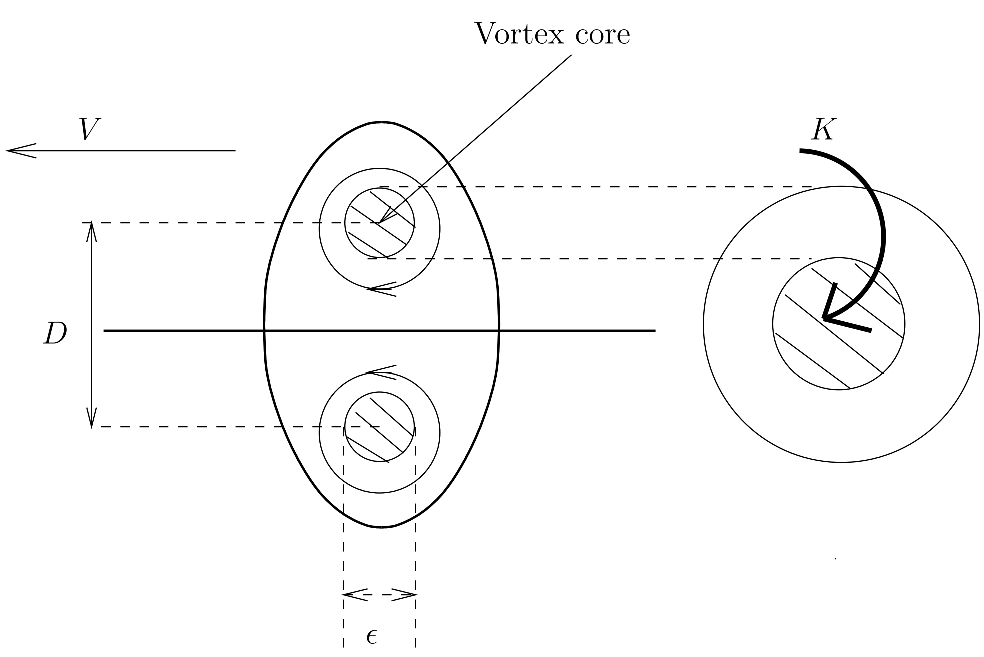


For a stationary observer the pattern is unsteady. However the instantaneous streamline pattern looks like



A smoke ring is an axisymetric version of this. As before we can have a range of flow patterns. However the maths is more complicated, owing to the vorticity being distributed over a finite area → the vortex core. The velocity of propagation depends on the size of the vortex core ϵ

$$V = \frac{\Gamma}{\pi D}\left(\ln\frac{4D}{\epsilon} - \frac{1}{4}\right)$$



5.11 Velocity Field in terms of velocity potential function ϕ

Point vortex

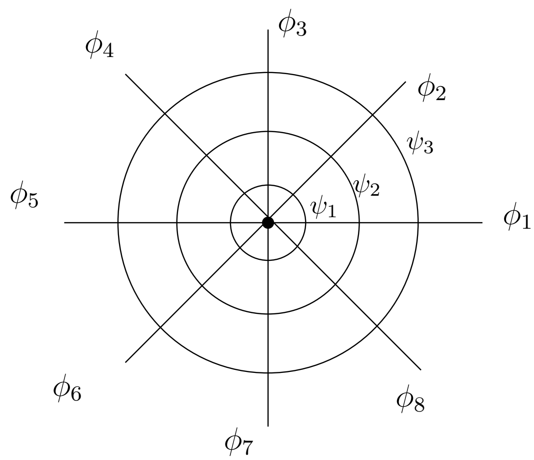
Have,

$$\begin{aligned} u_r &= 0 \ = \frac{\partial \phi}{\partial r} \\ u_\theta &= \frac{\Gamma}{2\pi r} \ = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \end{aligned}$$

integrating gives

$$\phi = \frac{\Gamma}{2\pi}\theta + c$$

Sketching lines of constant ϕ and ψ gives

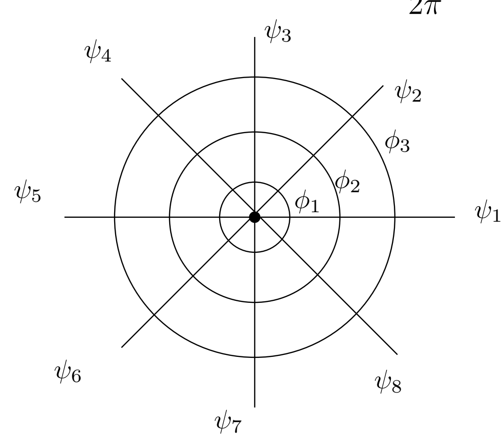


The stream function ψ and the velocity potential ϕ are orthogonal to each other. $\Rightarrow$ Conjugate harmonic functions.

Source

As an exercise show that the velocity potential for a source is given by $\phi = \dfrac{Q}{2\pi}\ln r$

and lines of constant ϕ and ψ look like $\rightarrow$



The table below shows the velocity potential and stream function of some simple cases.

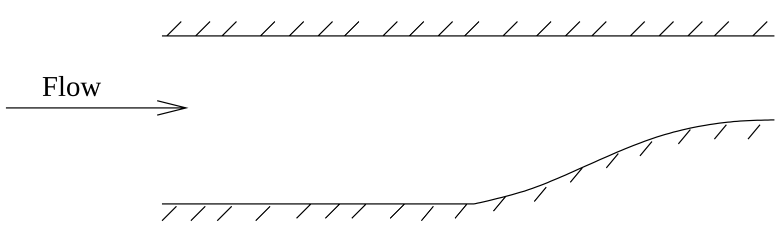
Flow	Velocity potential, ϕ	Streamfunction, Ψ
Uniform flow	$U_\infty y$	$U_\infty x$
Source	$\frac{Q}{2\pi}\ln\sqrt{x^2+y^2} = \frac{Q}{2\pi}\ln(r)$	$\frac{Q}{2\pi}\arctan(y/x) = \frac{Q\theta}{2\pi}$
Potential vortex (anticlockwise circulation)	$\frac{\Gamma}{2\pi}\arctan(y/x) = \frac{\Gamma\theta}{2\pi}$	$-\frac{\Gamma}{2\pi}\ln\sqrt{x^2+y^2} = -\frac{\Gamma}{2\pi}\ln(r)$
Doublet (anticlocwise top, clockwise bottom)	$\frac{K}{\pi}\frac{x}{x^2+y^2} = \frac{K}{\pi}\frac{\cos\theta}{r}$	$\frac{K}{\pi}\frac{y}{x^2+y^2} = \frac{K}{\pi}\frac{\sin\theta}{r}$

5.12 Electrical analogy

The flow of electrical current in a two dimensional conductor is analogous to irrotational flow and follows the Laplace equation

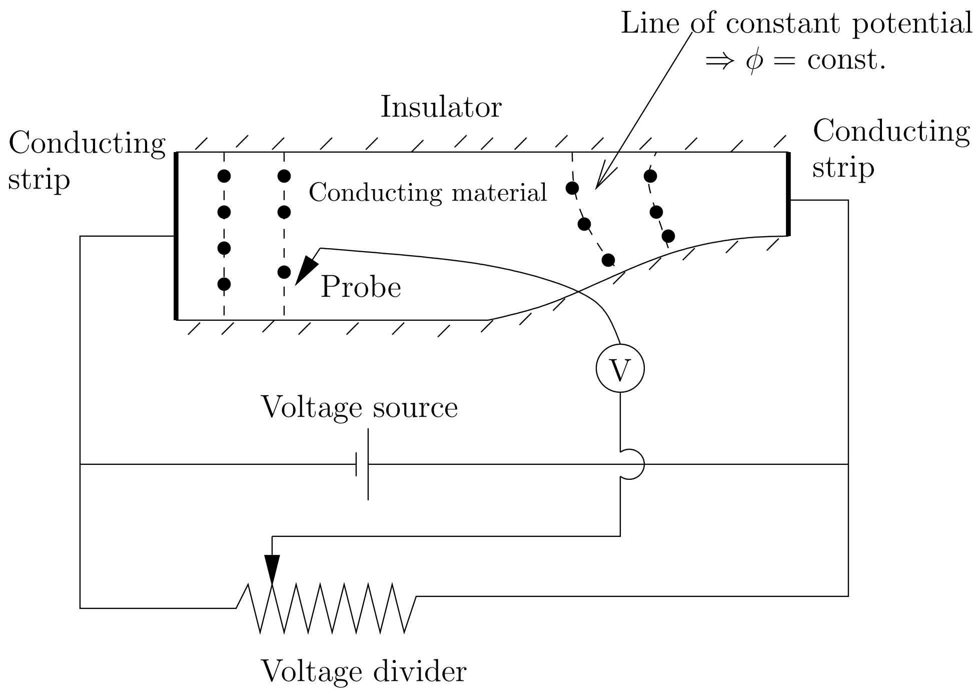
$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

where V is the electrical potential and is the counterpart of the velocity potential ϕ. Therefore we can use the electrical analogy to obtain the flow pattern through a conduit.



Method

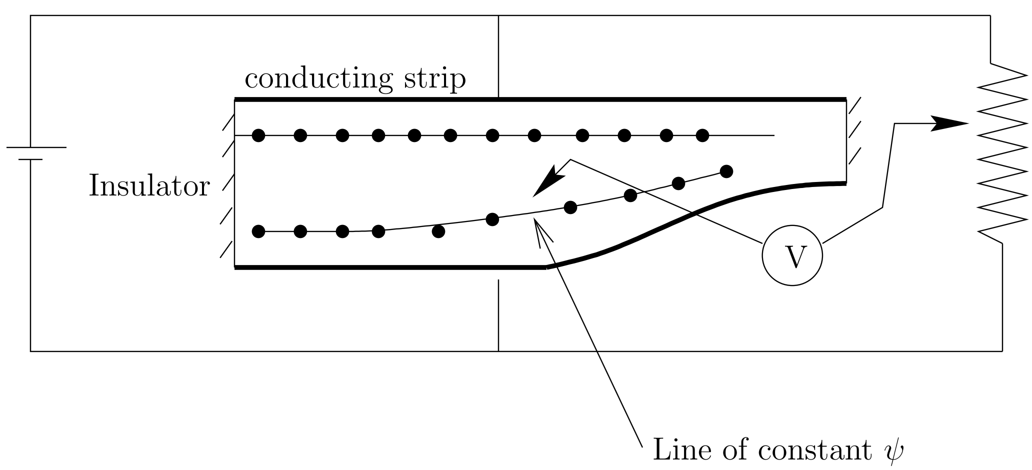
1. Cut a piece of conducting material into the shape of the conduit.



2. Establish a voltage drop along the conductor between the flow entrance and exit boundaries.

3. Use a potentiometer or voltmeter probe to locate line of constant potential.

To locate the lines of constant ψ we swap the conducting strips with the insulators and repeat the above.



Chapter 6

The complex potential function

6.1 Introduction

In order to extend the range of patterns we can analyse it is useful to define the complex potential function

$$w = \underbrace{\phi}_{\text{Potential function}} + i\underbrace{\psi}_{\text{Stream function}}$$

Applies only to flows which have both a stream function $\psi \Rightarrow$ **2 dimensional** and a velocity potential function $\phi \Rightarrow$ **irrotational**.
z is the complex variable

$$\begin{aligned} z =& x + iy \\ =& re^{i\theta} , \end{aligned}$$

it can be interpreted as a position vector. The complex potential function can then be expressed as a function of the complex variable z, ie put

$$w = f(z)$$

where f is an *analytic function* ($\Rightarrow$ finite number of singularities).

We need to prove an analytic function given by $w = f(z) = \phi + i\psi$ gives the solution to 2 dimensional irrotational flow (for example is $w = cz^2$ a valid solution ?)

Consider

$$w = A + iB = f(z)$$

where $z = x + iy$. Differentiate with respect to x

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x} = \frac{dw}{dz}$$

differentiate with respect to y

$$\frac{\partial w}{\partial y} = \frac{dw}{dz}\frac{\partial z}{\partial y} = i\frac{dw}{dz} .$$

Hence

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{1}{i}\frac{\partial w}{\partial y} \tag{6.1}$$

Also

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial A}{\partial x} + i\frac{\partial B}{\partial x} \quad \text{and} \\ \frac{\partial w}{\partial y} &= \frac{\partial A}{\partial y} + i\frac{\partial B}{\partial y}\end{aligned}$$

Hence from (6.1)

$$\begin{aligned}\frac{\partial A}{\partial x} + i\frac{\partial B}{\partial x} &= \frac{1}{i}\left(\frac{\partial A}{\partial y} + i\frac{\partial B}{\partial y}\right) \\ \therefore \frac{\partial A}{\partial x} + i\frac{\partial B}{\partial x} &= \frac{\partial B}{\partial y} - i\frac{\partial A}{\partial y}\end{aligned}$$

Equating real and imaginary parts

$$\left.\begin{aligned}\frac{\partial A}{\partial x} &= \frac{\partial B}{\partial y} \\ \frac{\partial B}{\partial x} &= \frac{-\partial A}{\partial y}\end{aligned}\right\} \quad \text{Cauchy-Riemann equations}$$

Hence,

$$\frac{\partial^2 A}{\partial x^2} = \frac{\partial^2 B}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 A}{\partial y^2} = \frac{-\partial^2 B}{\partial x \partial y}$$

therefore

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = 0\ .$$

Similarly show

$$\nabla^2 B = 0\ .$$

Hence we can choose $A = \phi$ and $B = \psi$ and

$$w = \phi + i\psi$$

Example: complex potential function for a point vortex
From earlier lectures we have derived that

$$\begin{aligned}\psi &= \frac{-\Gamma}{2\pi}\ln(r) \\ \phi &= \frac{\Gamma}{2\pi}\theta\end{aligned}$$

$$\begin{aligned}\Rightarrow w &= \frac{\Gamma}{2\pi}\theta - i\frac{\Gamma}{2\pi}\ln(r) \\ &= \frac{\Gamma}{2\pi}(\theta - i\ln(r))\end{aligned}$$

Complex potential function	Flow pattern
$w = U_\infty z$	Uniform Flow
$w = \frac{Q}{2\pi}\ln(z)$	Source
$w = -\frac{i\Gamma}{2\pi}\ln(z)$	Potential vortex (anticlockwise circulation)
$w = \frac{K}{\pi z} = \frac{\mu}{z}$	Doublet
$w = U_\infty\left(z + \frac{a^2}{z}\right)$	Flow past a cylinder of radius a
$w = U_\infty\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln(z)$	Flow past a cylinder of radius a with circulation Γ

Table 6.1: Examples of complex potential functions.

Now

$$\begin{aligned}\ln(z) =& \ln(re^{i\theta})\\ =& \ln(r) + i\theta\\ \therefore -i\ln(z) =& \theta - i\ln(r)\end{aligned}$$

Hence

$$\boxed{w = \frac{-i\Gamma}{2\pi}\ln(z)} \quad \text{complex potential for point vortex}$$

Exercise 6.1:
Follow the steps outlined in the above example and show that some complex potential functions of some of the flows you have seen before are as given in the Table 6.1.

6.2 Velocity components from w

From earlier lectures

$$\begin{aligned}\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} =& u\\ \frac{\partial\phi}{\partial y} = \frac{-\partial\psi}{\partial x} =& v\end{aligned}$$

Now

$$\begin{aligned}\frac{dw}{dz} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial z} \\ &= \left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right) \cdot 1 \\ &= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \\ \therefore \frac{dw}{dz} &= u - iv\end{aligned}$$

To find stagnation points we then solve

$$\boxed{\frac{dw}{dz} = 0} \quad \text{for } z$$

6.3 Example - Stagnation point flow

$$\begin{aligned}w &= cz^2 \\ &= c(x+iy)^2 \\ w &= c(x^2-y^2) + i2cxy\end{aligned}$$

But $w = \phi + i\psi$, equating real and imaginary parts

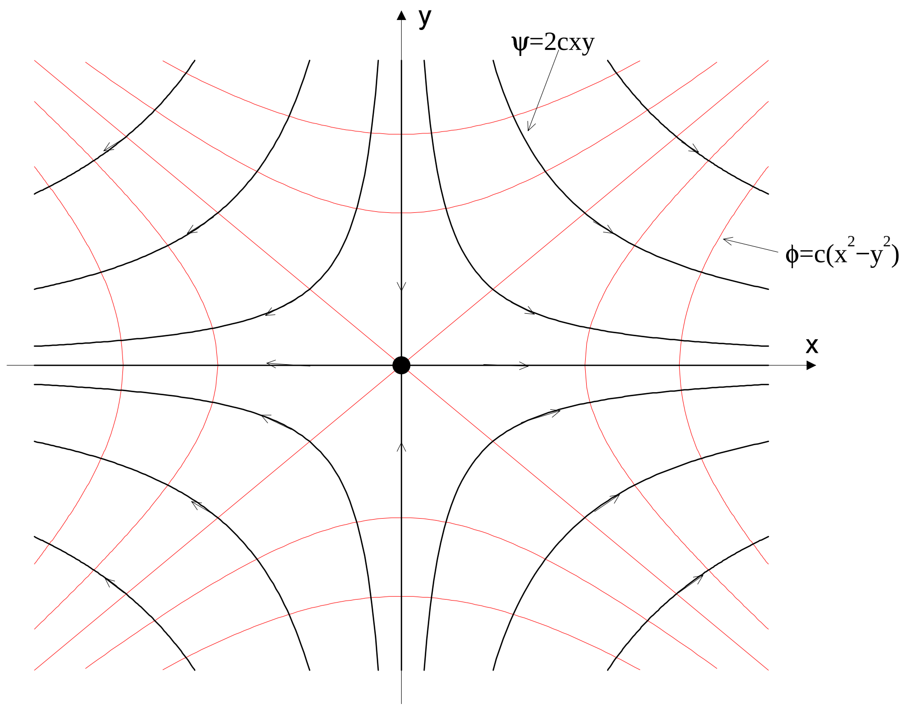
$$\left.\begin{aligned}\phi = c(x^2-y^2) \\ \psi = 2cxy\end{aligned}\right\} \quad \text{both satisfy Laplace equation}$$

Velocity

$$\begin{aligned}\frac{dw}{dz} &= u - iv \\ &= 2cz \quad \text{note stagnation point at } z = 0 \\ &= 2c(x+iy)\end{aligned}$$

equate real and imaginary parts

$$\begin{aligned}\Rightarrow u &= 2cx \\ v &= 2cy\end{aligned}$$



NB: lines of constant ψ and ϕ intersect at right angles.

Exercise 6.2:
A very long processing vat in a factor is giving off poisonous fumes at a rate of Q_f cubic units per unit length of vat. This vat is located at $x = 0$ and $y = 0$. At a height h directly above the vat, a long exhaust duct with uniform distributed openings along its length exist. This duct is sucking Q cubic units/unit length. The source of fumes from the vat can be regarded as a point source in two-dimensional flow and the exhaust duct can be regarded as a point sink.

(a) Write down the *complex potential function* for this problem. Remember that you HAVE TO use a sink image and a source image (the method of images) in order to correctly model the problem.

(b) From your answer in part (a), derive expressions for the u and v components of the velocity field.

6.4 Example: flow over a circular cylinder

From previous lectures, it was shown that the combination of a doublet with uniform flow gives a flow pattern that is similar to that of a uniform flow past a circular cylinder (see Fig. (6.1)). In this example, the flow over a circular cylinder will be analyse using the complex potential function w introduced in the previous lecture. For this flow, the complex potential function is given by the sum of the complex potential of uniform flow plus the complex potential of a doublet

$$
\begin{aligned}
w = & \quad w_{\text{uniform flow}} + & w_{\text{doublet}} \\
= & \quad U_\infty z + & \frac{\mu}{z} \\
= & \quad U_\infty \left[z + \frac{\left(\frac{\mu}{U_\infty}\right)}{z} \right]
\end{aligned}
$$

Hence

$$
w = U \left(z + \frac{a^2}{z} \right) \tag{6.2}
$$

where

$$
a = \sqrt{\frac{\mu}{U_\infty}}
$$

is there radius of the cylinder. Note that for large values of z,

$$
lim_{z \to \infty} w = U_\infty z = w_{\text{uniform flow}}.
$$

This mean that the flow pattern is approaches uniform flow at large distances from the origin. The stream function and velocity potential for this flow in cartesian coordinates can be obtained by substituting $z = x + iy$ into Eq. (6.2), so

$$
\begin{aligned}
w =& U_\infty \left(x + iy + \frac{a^2}{x + iy} \right) \\
=& U_\infty \left(x + iy + \frac{a^2 (x - iy)}{x^2 + y^2} \right)
\end{aligned}
$$

separate real and imaginary parts to get

$$
\begin{aligned}
w =& U_\infty x \left(1 + \frac{a^2}{x^2 + y^2} \right) & + i U_\infty y \left(1 - \frac{a^2}{x^2 + y^2} \right) \\
=& \phi & + i \psi
\end{aligned}
$$

Equating the real and imaginary parts to the the velocity potential and stream function for a flow past a cylinder going from left to right.

$$
\phi = U_\infty x \left(1 + \frac{a^2}{x^2 + y^2} \right)
$$

$$
\psi = U_\infty y \left(1 - \frac{a^2}{x^2 + y^2} \right)
$$

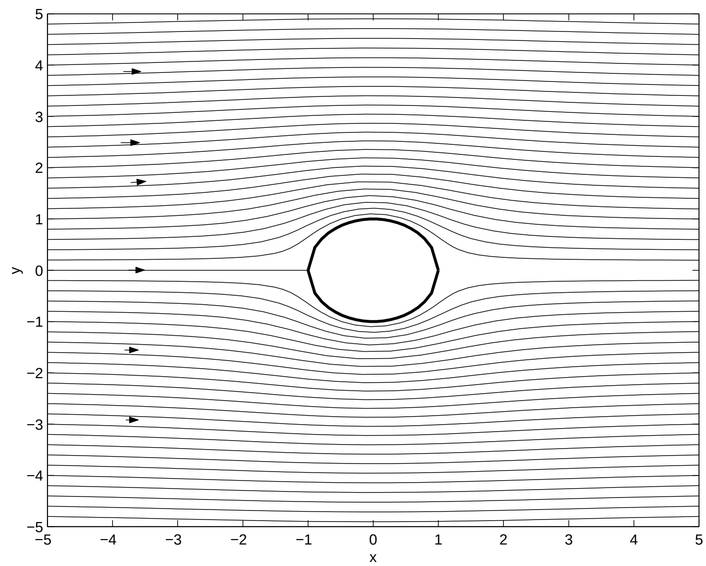


Figure 6.1: Flow over a circular cylinder obtained from the complex potential function $w = z + a^2/z$

To obtain the velocity field, calculate dw/dz. From Eq. (6.2),

$$\begin{aligned}\frac{dw}{dz} &= U_\infty\left(1 - \frac{a^2}{z^2}\right)\\ &= U_\infty\left(1 - \frac{a^2}{(x+iy)^2}\right)\\ &= U_\infty\left(1 - \frac{a^2}{(x^2 - y^2 + i2xy)}\right)\end{aligned}$$

Exercise 6.3:

Show that the above expression simplifies to

$$\begin{aligned}\frac{dw}{dz} =& U_\infty\left[1 - \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}\right] && +i\left[\frac{2U_\infty a^2 xy}{(x^2+y^2)^2}\right]\\ =& u && +i(-v)\end{aligned}$$

Hence

$$u = U_\infty\left[1 - \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}\right]$$

and

$$v = \left[\frac{-2U_\infty a^2 xy}{(x^2+y^2)^2}\right]$$

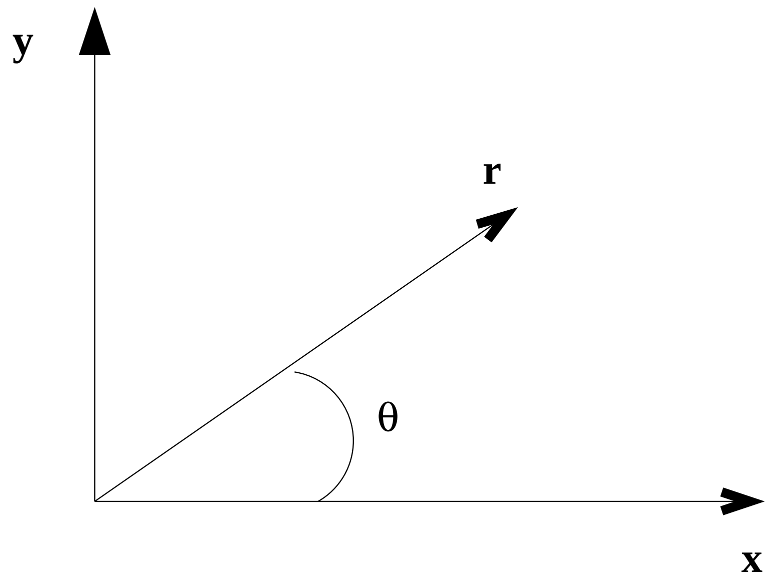


Figure 6.2: Cartesian and polar coordinate system

Sometimes, it is more convenient to work in polar coordnates (see Fig. 6.2). Let $z = re^{i\theta}$. Substitute this into Eq. (6.2) to obtain

$$
\begin{aligned}
w =& U_\infty \left(re^{i\theta} + \frac{a^2}{r} e^{-i\theta} \right) \\
=& U_\infty \left(r(\cos(\theta + i \sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta) \right)
\end{aligned}
$$

Grouping real and imaginary parts will give

$$
w = \left[\left(r + \frac{a^2}{r} \right) \cos\theta + i \left(r - \frac{a^2}{r} \right) \sin\theta \right]
$$

Hence, the velocity potential and the stream function are given by

$$
\phi = \left(r + \frac{a^2}{r} \right) \cos\theta
$$

$$
\psi = \left(r - \frac{a^2}{r} \right) \sin\theta
$$

To obtain the velocity field,

$$\begin{aligned}
\frac{dw}{dz} &= U_\infty\left(1-\frac{a^2}{z^2}\right) \\
&= U_\infty\left(1-\frac{a^2}{r^2}e^{-i2\theta}\right) \qquad \text{(use } z = re^{i\theta}\text{)} \\
&= U_\infty\left(1-\frac{a^2}{r^2}(\cos(2\theta)-i\sin(2\theta))\right) \\
&= u - iv
\end{aligned}$$

Equating real and imaginary parts will give

$$u = U_\infty\left(1-\frac{a^2}{r^2}\cos(2\theta)\right) \tag{6.3}$$

$$v = -U_\infty\left(\frac{a^2}{r^2}\sin(2\theta)\right) \tag{6.4}$$

Note that u and v are the Cartesian velocity components and **NOT** the radial and tangential velocity components.

Exercise 6.4:
Prove that

$$u = U_\infty\left(1-\frac{a^2}{r^2}\cos(2\theta)\right) = U_\infty\left[1-\frac{a^2(x^2-y^2)}{(x^2+y^2)^2}\right]$$

and

$$v = -U_\infty\left(\frac{a^2}{r^2}\sin(2\theta)\right) = \left[\frac{-2U_\infty a^2 xy}{(x^2+y^2)^2}\right]$$

From Eqs. (6.3) and (6.4) the speed, V, of the fluid at any point is given by

$$\begin{aligned}
V^2 &= u^2 + v^2 \\
&= U_\infty^2\left(1-\frac{a^2}{r^2}\cos(2\theta)\right)^2 + U^2\left(\frac{a^2}{r^2}\sin(2\theta)\right)^2 \\
&= U_\infty^2\left(1-2\frac{a^2}{r^2}\cos^2(2\theta)+\frac{a^4}{r^4}(\cos^2(2\theta)+\sin^2(2\theta))\right) \\
&= U_\infty^2\left(1-2\frac{a^2}{r^2}\cos(2\theta)+\frac{a^4}{r^4}\right) \\
&= U_\infty^2\left(1-2\frac{a^2}{r^2}\cos(2\theta)+\frac{a^4}{r^4}\right)
\end{aligned}$$

On the surface of the cylinder, $r = a$, so

$$\begin{aligned}
V^2 &= U_\infty^2\left(2-2\cos(2\theta)\right) \\
&= 2U_\infty^2\left(1-\cos(2\theta)\right) \\
&= 2U_\infty^2\left(1-\cos(2\theta)\right) \\
&= 4U_\infty^2\sin^2(\theta) \qquad \text{(remember that } \cos(2\theta) = \cos^2\theta - \sin^2\theta\text{)}
\end{aligned}$$

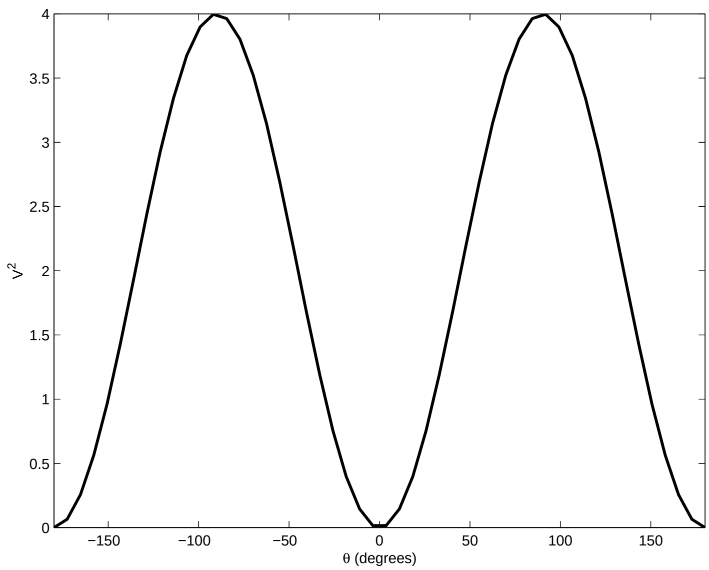


Figure 6.3: V^2 distribution of flow over a circular cylinder

V^2 distribution on the surface of the cylinder is shown in Fig. 6.3. The velocity of the fluid is zero at $\theta = 0^o$ and $\theta = 180^o$. Maximum velocity occur on the sides of the cylinder at $\theta = 90^o$ and $\theta = -90^o$.

Pressure distribution on the surface of the cylinder can be found by using Benoulli's equation. Thus, if the flow is steady, and the pressure at a great distance is p_∞,

$$\begin{aligned} p_\infty + \frac{1}{2}\rho U_\infty^2 &= p_{cylinder} + \frac{1}{2}\rho V^2 \\ &= p_{cylinder} + \frac{1}{2}\rho 4U_\infty^2 \sin^2(\theta) \end{aligned}$$

therefore

$$p_{cylinder} = p_\infty + \frac{1}{2}\rho U_\infty^2 (1 - 4\sin^2\theta)$$

and

$$C_p = \frac{p_{cylinder} - p_\infty}{\left(\frac{1}{2}\rho U_\infty^2\right)} = 1 - 4\sin^2\theta$$

A plot of C_p vs θ is shown in Fig. 6.4. The value of C_p is 1 at the front stagnation point ($\theta = 0$). As the side of the cylinder, $\theta = \pi/2$ and the value of C_p drops to -3. C_p then increases to 1 at the rear stagnation point of the cylinder ($\theta = \pi$).

Exercise 6.5:

Determine the points on the cylinder where $p_{cylinder} = p_\infty$

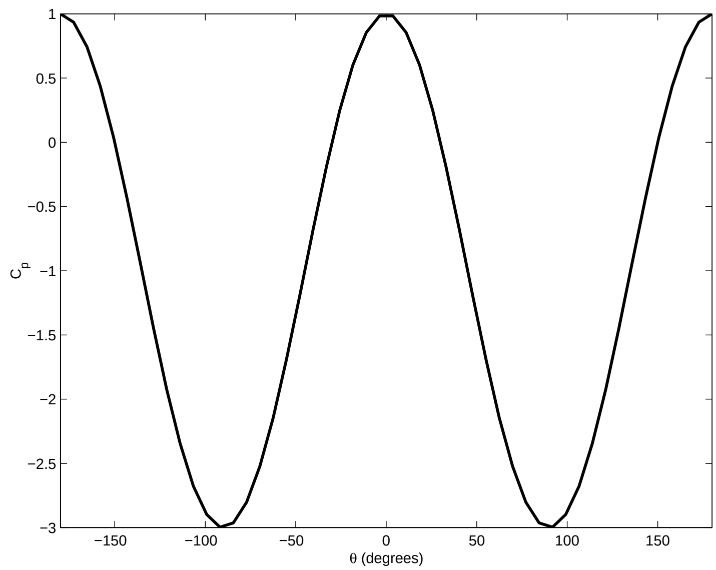


Figure 6.4: C_p distribution of flow over a circular cylinder

Exercise 6.6:

- Show, from first principles, that the radial and tangential velocity components of the flow is related to the complex potential function, w by

$$e^{i\theta}\frac{dw}{dz} = u_r - iu_\theta \tag{6.5}$$

- The complex potential function, w, of the flow over a circular cylinder can be expressed a combination of free stream velocity U_∞ and doublet with strength, μ.

$$w = U_\infty z + \frac{\mu}{z} \tag{6.6}$$

  - Differentiate Eq. (6.6) and use Eq. (6.5) to find the expression for u_r and u_θ on the surface of the cylinder expressed in cylindrical coordinates.
  - Find the pressure coefficient, C_p, on the surface of the cylinder.

Exercise 6.7:

A mathematical model of the flow in a factory with an exhaust duct (of strength Q) and a fume bed (of strength Q_f) with cross flow (U_∞) is given by the complex potential function

$$w = U_\infty z + \frac{Q_f}{\pi} \ln z - \frac{Q}{2\pi} \left[\ln(z - ih) + \ln(z + ih) \right] \tag{6.7}$$

where h is the distance between the exahust duct and the fume bed.

(a) Use the Root Locus analysis (c.f. refer to your Control Theory lecture notes) to locate the stagnation points in the flow field.

(b) Indicate how the location of the stagnation point changes for different values of Q_f/Q and Q/U_∞.

(c) Sketch the flow pattern for various values of Q_f/Q and Q/U_∞.

Chapter 7

Conformal Transformations

A large amount of airfoil theory has been developed by distorting flow around a cylinder to flow around an airfoil. The essential feature of the distortion is that the potential flow being distorted ends up also as potential flow.

The most common Conformal transformation is the Jowkowski transformation which is given by

$$f(z) = z + \frac{c^2}{z}$$

To see how this transformation changes flow pattern in the z (or $x - y$) plane, substitute $z = x + iy$ into the expression above to get

$$\begin{aligned}
\zeta = \xi + i\eta &= z + \frac{c^2}{z} \\
&= x + iy + \frac{c^2}{x + iy} \\
&= (x + iy)\frac{(x + iy)(x - iy)}{(x + iy)(x - iy)} + \frac{c^2(x - iy)}{(x + iy)(x - iy)} \\
&= \frac{(x + iy)(x^2 + y^2) + c^2(x - iy)}{(x^2 + y^2)} \\
&= x\left(1 + \frac{c^2}{x^2 + y^2}\right) + iy\left(1 - \frac{c^2}{x^2 + y^2}\right)
\end{aligned}$$

This means that

$$\xi = x\left(1 + \frac{c^2}{x^2 + y^2}\right)$$

and

$$\eta = y\left(1 - \frac{c^2}{x^2 + y^2}\right)$$

For a circle of radius r in the z plane, x and y are related by

$$x^2 + y^2 = r^2,$$

hence,

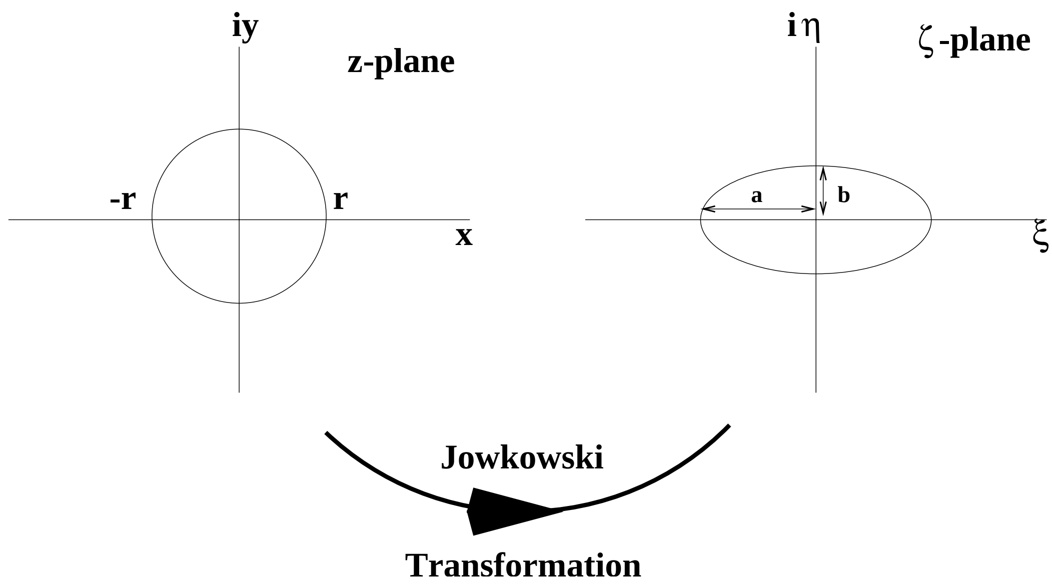


Figure 7.1: Jowkowski Transformation, $f(z) = z + c^2/z$, applied to a circle on the z-plane of radius r. In this figure, $a = (r + c^2/r)$ and $b = (r - c^2/r)$.

$$\xi = x\left(1 + \frac{c^2}{r^2}\right)$$

and

$$\eta = y\left(1 - \frac{c^2}{r^2}\right)$$

So in the z plane,

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1,$$

and in the ζ plane

$$\frac{\xi^2}{\left(r + \frac{c^2}{r}\right)^2} + \frac{\eta^2}{\left(r - \frac{c^2}{r}\right)^2} = 1$$

The circle of radius r in the z-plane is seen to transform into an ellipse with semi-axes $a = (r + c^2/r)$ and $b = (r - c^2/r)$ in the ζ-plane (see Fig. (7.1)), provided $c < r$. In the special case where $r = c$, $a = 2c$ and $b = 0$. This means that if the circle in the z-plane that we wish to transform has a radius c, it will be transformed to an infinitely thin plate of length $4r$ in the ζ-plane.

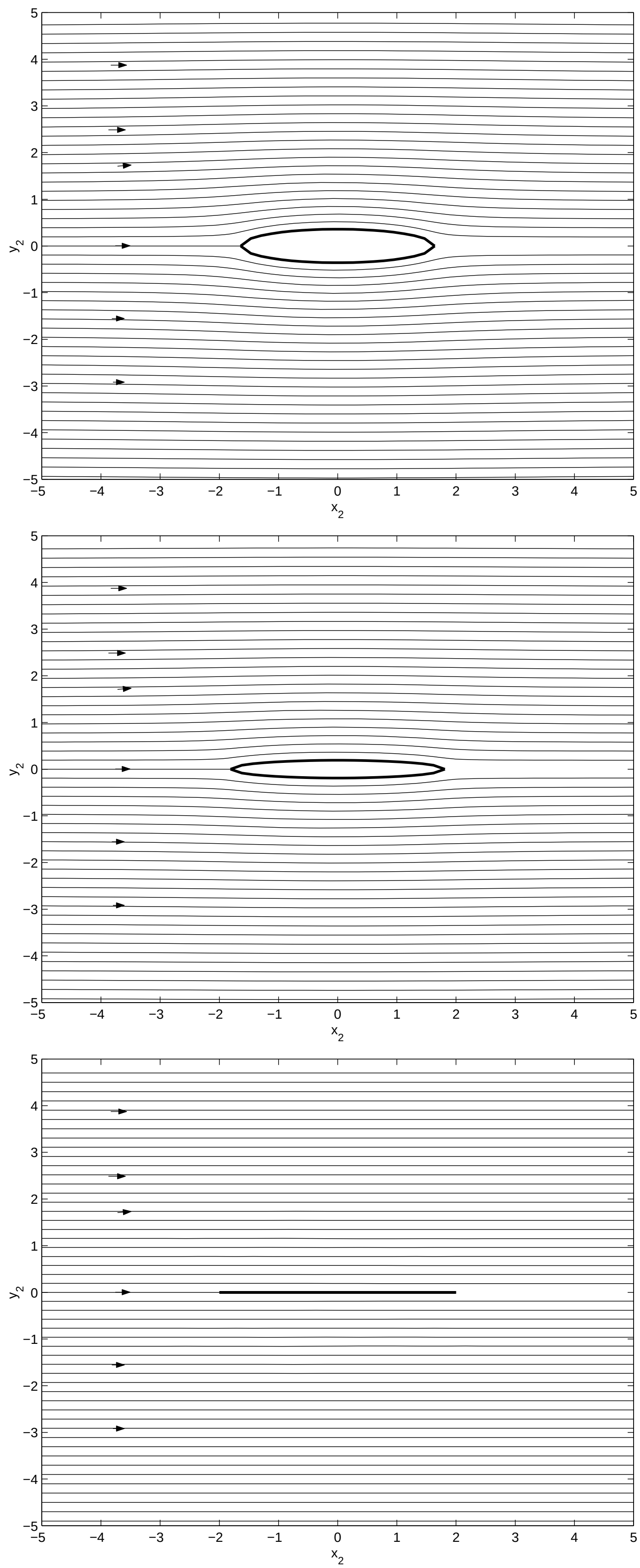


Figure 7.2: Flow over an ellipse obtained by applying the Jowkowski transformation on flow over a circular cylinder. The top figure was calculated with c=0.8, middle figure with $c = 0.9$ and the bottom figure with $c = 1.0$.

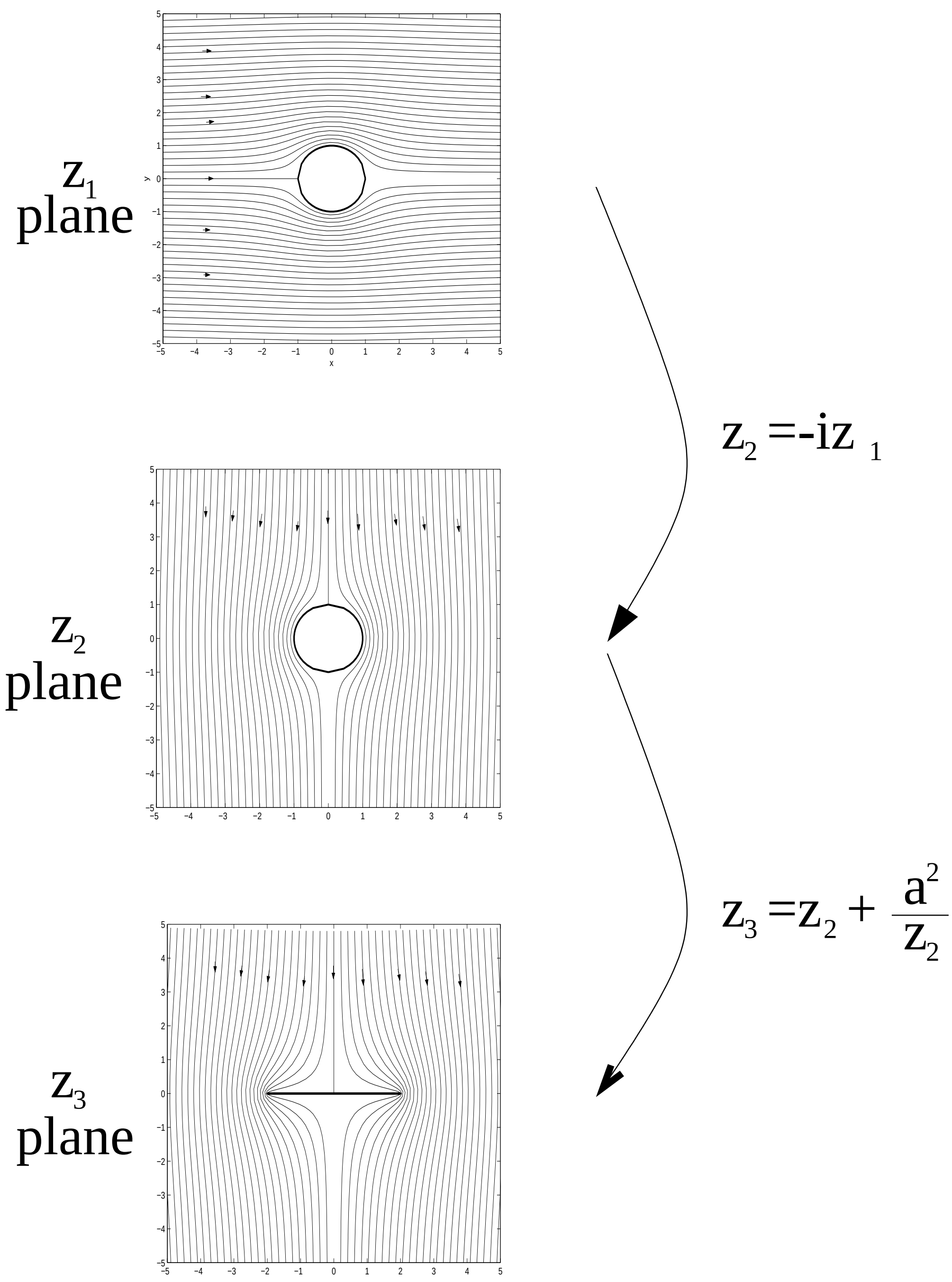


Figure 7.3: Figure showing the various conformal transformation used to obtain the flow over a flat plate.

Exercise 7.1: Show that the following transformation

$$z_2 = -iz_1 \tag{7.1}$$

rotates a flow 90^o in the clockwise direction.

The result above could be used to analyse the flow over a flat plate. If the flow in the z_1 plane is rotated by -90^o by the transformation $z_2 = -iz_1$, the flow in the z_2-plane will still be a flow over a circular cylinder but with the main flow direction going from top to bottom (see Fig. 7.3). If the $z_3 = z_2 + a^2/z_2$ is now applied to the flow in the z_2-plane, the flow pattern perpendicular to the flat plate is observed. The successive transformation leading the flow over a flat plate is

$$w = z_1 + \frac{a^2}{z_1}$$

$$z_2 = -iz_1$$

$$z_3 = z_2 + \frac{a^2}{z_2}$$

7.1 Conformal Transformation of velocities

In the z-plane, the components of velocity, u and v, are given by the expression

$$\frac{dw}{dz} = u - iv. \tag{7.1}$$

In the ζ-plane, the components of velocity, $\hat{u}$ and $\hat{v}$, are given by

$$\frac{dw}{d\zeta} = \hat{u} - i\hat{v}. \tag{7.2}$$

Using Eq. (7.1), Eq. (7.2) can be rewritten as

$$\begin{aligned} \hat{u} - i\hat{v} &= \frac{dw}{d\zeta} && (7.3) \\ &= \frac{dw}{dz}\frac{dz}{d\zeta} && (7.4) \\ &= (u - iv)\left(\frac{dz}{d\zeta}\right) && (7.5) \end{aligned}$$

In general, $dz/d\zeta$ is a complex quantity. We will just let

$$\frac{dz}{d\zeta} = A + iB. \tag{7.6}$$

From Eq. (7.5), we can determine the velocity in the ζ-plane knowing the velocity in the z-plane. To determine that, we substitute Eq. (7.6) into Eq. (7.5) to obtain

$$\hat{u} - i\hat{v} = (u - iv)(A + iB). \tag{7.7}$$

Take the complex conjugate of Eq. (7.7) we get

$$\hat{u} + i\hat{v} = (u + iv)(A - iB). \tag{7.8}$$

Multiplying Eqs. (7.7) and (7.8) gives

$$\hat{u}^2 + \hat{v}^2 = \left(u^2 + v^2\right)\left(A^2 + B^2\right). \tag{7.9}$$

Hence

$$\hat{q} = q\left|\frac{dz}{d\zeta}\right| \tag{7.10}$$

where

$$\hat{q} = \sqrt{\hat{u}^2 + \hat{v}^2}, \tag{7.11}$$

$$q = \sqrt{u^2 + v^2} \tag{7.12}$$

and

$$\left|\frac{dz}{d\zeta}\right| = \sqrt{A^2 + B^2} \tag{7.13}$$

Equation (7.10) shows that the velocity in the ζ-plane can be obtained from the fluid velocity in the z-plane by multiplying the fluid velocity in the z-plane by $|dz/d\zeta|$.

7.1.1 Example-Flow over a Flat Plate

From the previous lecture, it has been found that the flow over a flat plate can be obtained from the following sequence of transformation

$$z_2 = -iz_1 = -iz \tag{7.14}$$

$$\zeta = z_3 = z_2 + \frac{a^2}{z_2} \tag{7.15}$$

From Eqs. (7.14) and (7.15), we obtain

$$\begin{aligned}
\frac{d\zeta}{dz} &= -i - i\frac{a^2}{z^2} \\
&= -i(1 + \frac{a^2}{z^2}) \\
\left|\frac{d\zeta}{dz}\right| &= \left|1 + \frac{a^2}{z^2}\right| \\
&= \left|1 + \frac{a^2}{r^2 e^{i2\theta}}\right| \\
&= \left|1 + \frac{a^2}{r^2} e^{-i2\theta}\right| \\
&= \left|1 + \frac{a^2}{r^2}(\cos(2\theta) - i\sin(2\theta))\right|
\end{aligned}$$

on the surface of the cylinder, $r = a$

$$\begin{aligned}
\left|\frac{d\zeta}{dz}\right| &= |1+\cos(2\theta) - i\sin(2\theta)| \\
&= \sqrt{[1+\cos(2\theta)]^2 + \sin^2(2\theta)} \\
&= \sqrt{4\cos^2(\theta)} \\
&= 2\cos(\theta)
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\frac{dz}{d\zeta}\right| &= \frac{1}{\left|\frac{d\zeta}{dz}\right|} \\
&= \frac{1}{2\cos(\theta)}
\end{aligned}$$

Using Eq. (7.10) and remembering from the previous lecture that the velocity on the surface of the cylinder is $V = 2U_\infty \sin(\theta)$, we get

$$\begin{aligned}
\hat{q} &= q\left|\frac{dz}{d\zeta}\right| \\
&= 2U_\infty \sin(\theta)\frac{1}{2\cos(\theta)} \\
&= U_\infty \tan(\theta)
\end{aligned}$$

We now need to express $\tan(\theta)$ in terms of the variables in the ζ-plane, ξ and η. From Eqs. (7.14) and (7.15), we obtain

$$\begin{aligned}
\zeta &= -iz + \frac{a^2}{-iz} \\
&= -iz + \frac{ia^2}{z} \\
&= -ire^{i\theta} + \frac{ia^2}{re^{i\theta}} \\
&= -ire^{i\theta} + \frac{ia^2}{r}e^{-i\theta}
\end{aligned}$$

On the surface of the cylinder $r = a$, so

$$\begin{aligned}
\zeta &= -iae^{i\theta} + iae^{-i\theta} \\
&= ia(e^{-i\theta} - e^{i\theta}) \\
&= ia(\cos(\theta) - i\sin(\theta) - \cos(\theta) - i\sin(\theta)) \\
&= ia(-2i\sin(\theta)) \\
\xi + i\eta &= 2a\sin(\theta)
\end{aligned}$$

Equating real and imaginary parts gives

$$\xi = 2a\sin(\theta) \tag{7.16}$$

Using Pythagoras theorem gives

$$\tan(\theta) = \frac{\xi}{\sqrt{4a^2 - \xi^2}} \tag{7.17}$$

We know previously that

$$\begin{aligned} \hat{q} &= U_\infty \tan(\theta) \\ &= U_\infty \frac{\xi}{\sqrt{4a^2 - \xi^2}} \end{aligned}$$

To obtain the pressure distribution on the plate, use Bernoulli's theorem

$$\begin{aligned} p_\infty + \frac{1}{2}\rho U^2 &= p_{plate} + \frac{1}{2}\rho \hat{q}^2 \\ &= p_{plate} + \frac{1}{2}\rho U_\infty^2 \frac{\xi^2}{4a^2 - \xi^2} \end{aligned}$$

Hence,

$$C_p = \frac{p_{plate} - p_\infty}{\frac{1}{2}\rho U^2} = \left(1 - \frac{\xi^2}{4a^2 - \xi^2}\right)$$

Exercise 7.2:
Show that $p_{plate} = p_\infty$ at $\xi = \sqrt{2}a$

Exercise 7.3: The complex potential function for flow past a circular cylinder (with flow downwards i.e. in the negative y-direction is given by

$$w = U_\infty \left(iz + \frac{a^2}{iz}\right) = U_\infty \left(iz - \frac{ia^2}{z}\right)$$

Show that the velocity on the surface of the cylinder is given by

$$V = 2U_\infty \cos(\theta)$$

Use the Jowkowski transformation to show that the velocity on a the flow past a horizontal flat plate is given by

$$\hat{V} = \frac{U_\infty \xi}{\sqrt{4a^2 - \xi^2}}$$

7.2 Flow Over An Airfoil

We have shown that the Jowkowski transformation

$$\zeta = z + \frac{c^2}{z}$$

transforms a circle of radius a¡c into an ellipse. If we make $c = a$, then we find that the Jowkowski transformation changes the circle into a flat plate. The circle of radius c in the z-plane is the Jowkowski transforming circle.

The effects of moving a circle of radius a in the z-plane closer and closer to the Jowkowski transformation circle is shown in Fig. 7.4. It can be seen that when the circle of radius a touches the Jowkowski transformation circle, that point transforms to a very sharp trailing edge of an airfoil shaped body.

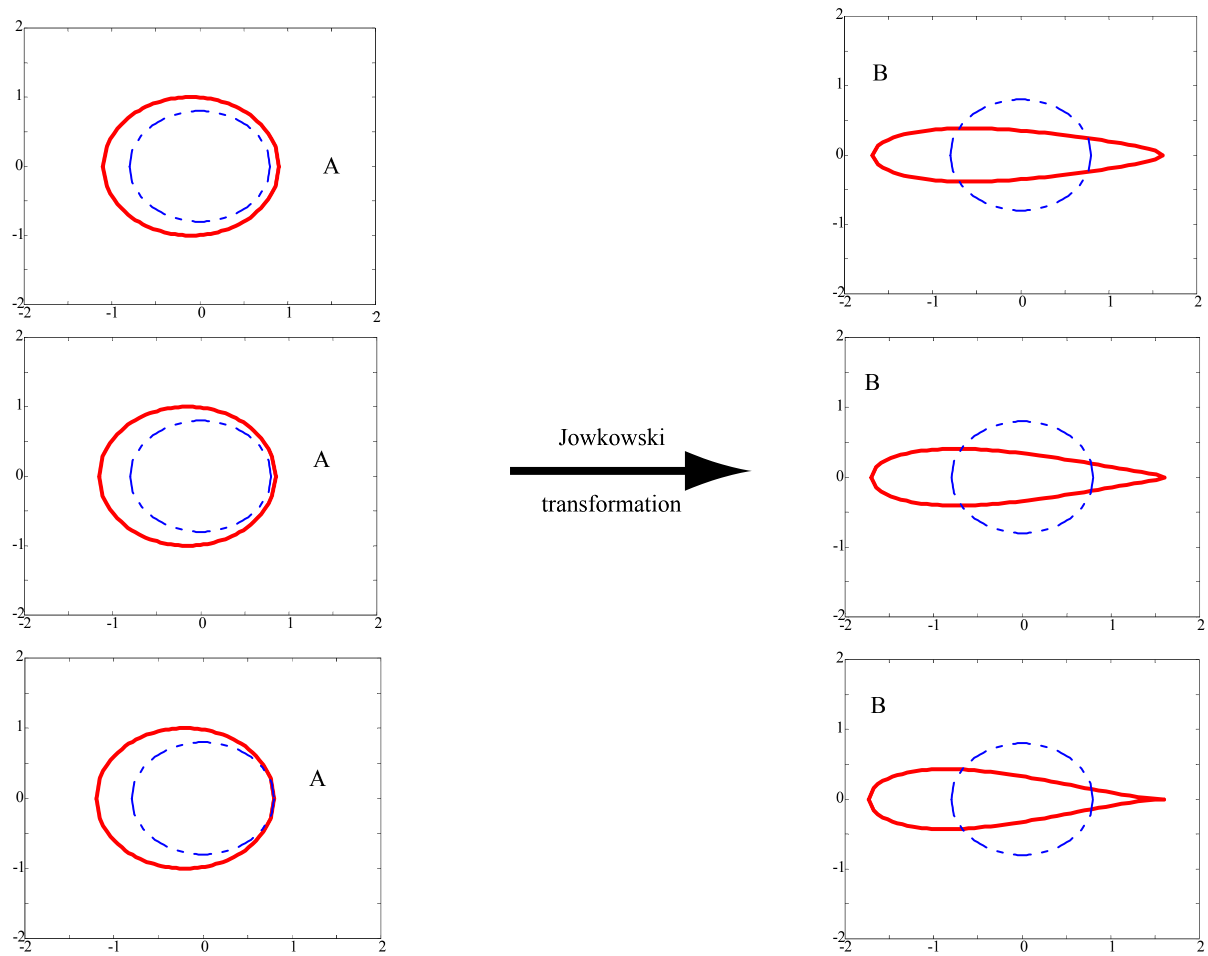


Figure 7.4: Figure showing the effects of moving a circle in the z-plane closer and closer to the Jowkowski transformation circle.