Frequency-Weighted Robust Fault Reconstruction Using a Sliding Mode Observer

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Abstract: A new method for the design of robust fault reconstruction filters is introduced in this paper. The method is based on shaping the map from disturbance inputs to the fault estimation error through frequency weighting functions and formulating the problem as a robust observer design where the input-output error map is minimized by solving a set of matrix inequalities. A numerical example consisting of ten states, two known inputs, four measured outputs, one disturbance and three faults is considered. Through this example, it is shown that the frequency shaping renders far superior performance than existing conventional methods when properly incorporated in the observer design problem.
1 Introduction

Fault detection and isolation (FDI) has been the subject of extensive research for some time now, but especially since early 1990s [1-4]. The interest in this line of research stems from its practical application to a variety of industries such as aerospace [5, 6], energy systems [7, 8], industrial applications [9], [10], and process control [11], to name a few. The main function of an FDI scheme is to detect a fault when it happens, which may then be acted on in a variety of ways, such as sending alarm signals, taking protection measures, or reconfiguring a running control scheme.

The most commonly used schemes are observer based [4]. Two approaches have been successfully extensively studied: the residual generation approach [2] and the direct approach [12]. In the residual generation approach, a signal is generated and flagged when an abnormal (faulty) condition takes place in the system. In the direct approach the fault is directly reconstructed using unknown input observer theory and its location and characteristics are determined online. A shortcoming of all of these approaches is that changes in the system operating condition and or variations in the system parameters result in estimation error and may lead to inaccurate detection of faults.

To overcome this problem, sliding mode observers have been proposed, for their ability to account for model inaccuracy. This particular class of observer based techniques was first reported by a group of researchers [13], [14], [15]. However, in these approaches the sliding motion is used to detect the presence of fault only and not its location or characteristics.
An improvement on this approach is reported in [16] [17] where the nonlinear term that maintains the sliding motion is used to reconstruct the fault and therefore provide full information about it. A major restriction of this approach was its inability to deal with simultaneous faults, such faults that occur simultaneously in actuators and sensors, for example. This restriction was subsequently relaxed by [18] where a sliding mode observer scheme capable of simultaneously reconstructing both sensor and actuator faults was proposed. However, in those works, there was no consideration for robustness of the fault reconstruction (from any nonlinearities or uncertainties). In [19] this work was extended by introducing a design method for the observer using Linear Matrix Inequalities (LMIs) such that the $L_2$ gain from the nonlinearities/uncertainties to the fault reconstruction is minimized.

In this paper, we propose a further improvement to [20] by introducing frequency shaping features in the design of sliding mode observers. The frequency characteristics of the input disturbances are first determined. Then, these disturbances are assumed to be the output of a filter (with the previously mentioned frequency characteristics). The filter dynamics are then augmented with the original system, and a robust sliding mode observer [19] is designed for the augmented system. This is shown to be quite effective in enhancing the design feasibility and thus producing far superior results in fault detection and identification. Furthermore, the proposed frequency-shaping approach is capable of detecting and reconstructing multiple simultaneous faults on line and in real time.
To illustrate the salient features of the new approach, we consider a 10th order system and inject different combinations of simultaneous faults in the state and output measurements. We will show that for this particular system, the frequency shaping produces exact fault reconstruction while the alternatively designed observer without frequency shaping fails to do so.

2 Problem Statement

In this paper we consider a system (plant) of the following description:

\[
\dot{x}_p = A_p x_p + B_p u + M_p f_x + Q_p \xi_p \tag{1}
\]

\[
y_p = C_p x_p + D_p u + N_p f_y \tag{2}
\]

where \( x_p \in \mathbb{R}^{n_p} \) is the state vector, \( u_p \in \mathbb{R}^m \) is the input vector, \( y_p \in \mathbb{R}^p \) is the output vector, \( f_x \in \mathbb{R}^q \) is the actuator faults vector, \( \xi_p \in \mathbb{R}^k \) is an unknown process disturbance vector, and \( f_y \in \mathbb{R}^r \) is the sensor faults vector. The matrices \( A_p, B_p, M_p, Q_p, C_p, D_p, N_p \) are appropriately dimensioned matrices associated with the standard state-space model of the plant. The term \( Q_p \xi_p \) may be considered to include both process disturbances and also plant uncertainties [4]. We assume (as in [17]) that matrices \( C_p, N_p, M_p \) all have full rank, and \( n_p > p \geq r + q \). The problem addressed in this paper is to estimate the actuator and sensor fault signals in the presence of the unknown disturbances using a sliding mode observer [21] of the form:
\[
\dot{x} = A\dot{x} + Bu - G_f e_y + G_n \nu \\
\hat{y} = C\dot{x} + Du
\]

(3) 

(4)

where \( \dot{x} \) is the state vector of the observer, \( \hat{y} \) is the output vector of the observer, \( e_y = \hat{y} - y \) is the output estimation error, \( \nu \) is a nonlinear switching term defined by

\[
\nu = -\rho \frac{e_y}{\|e_y\|}, \quad e_y \neq 0 \text{ where } \rho \text{ is a positive scalar, and } G_f \text{ and } G_n \text{ are gain matrices of the observer to be designed.}
\]

Conventionally this problem has been fully addressed and solved in [17] using an LMI based optimization approach. Defining the operator 

\[
T : \xi \mapsto e_f,
\]

the problem is conventionally formulated as follows: \( \min \|T\|_{\mathcal{H}_\infty} \) where \( e_f \) is the fault estimation error.

In this paper we extend existing results by introducing a frequency shaping filter into the design procedure thus allowing the design to be optimized over specific frequency ranges. The new problem formulation is, therefore, \( \min \|T\Omega\|_{\mathcal{H}_\infty} \) where \( \Omega \) is a frequency shaping filter matrix with state-space data \((A_\Omega, B_\Omega, C_\Omega, D_\Omega)\). We show that this extension results in a new LMI based problem, which when solved yields a significantly superior estimation outcome. This will be verified through an extensive set of comparison studies using a 10th order faulty system, where the faults under various scenarios are estimated by both this new approach and the conventional method [17].
3 Development of Fault Detection Method

3.1 State-Space Model

In the following the plant output described by equation (2) is first partitioned into fault-free and fault-dependent parts. The fault dependent part is then passed through a dynamic filter [18], which is then augmented with the plant state equation and the frequency shaping filter to produce a standard overall state-space model. Let \( T_r \in \mathbb{R}^{p \times p} \) be an orthogonal matrix such that:

\[
T_r N_p = \begin{bmatrix} 0 & p-r \\ \frac{-1}{N_2} & r \end{bmatrix} ; \quad T_r y_p = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 x_p + D_1 u \\ C_2 x_p + D_2 u + N_2 f_y \end{bmatrix} \frac{p-r}{r} \tag{5}
\]

where \( N_2 \) is invertible and \( C_1, C_2, D_1 \) and \( D_2 \) are appropriate partitions of \( C_p \) and \( D_p \).

Now introduce the following filter for the fault-dependent component of the output, \( y_2 \):

\[
\dot{z}_1 = A_f (z_1 - y_2) = A_f z_1 - A_f C_2 x_p - A_f D_2 u - A_f N_2 f_y
\]

where \( A_f \) is chosen as a stable matrix. Let a frequency weighted filter be defined as

\[
\dot{z}_2 = A_{\Omega} z_2 + B_{\Omega} \xi \text{ and its output be defined as } \xi_p = C_{\Omega} z_2 + D_{\Omega} \xi, \text{ where } \xi \in \mathbb{R}^k, \quad z_2 \in \mathbb{R}^h, \quad h \geq k. \text{ The matrices } A_{\Omega}, B_{\Omega}, C_{\Omega}, D_{\Omega} \text{ are appropriately dimensioned matrices that represent the state-space model of the filter. Substituting these definitions into (1) yields:}
\[
\dot{x}_p = A_p x_p + B_p u + M_p f_x + Q_p C_\Omega z_2 + Q_p D_\Omega \xi 
\]  \hspace{1cm} (7)

Augment equations (7), with the filter equations to obtain:

\[
\begin{bmatrix}
\dot{x}_p \\
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} =
\begin{bmatrix}
A_p & 0 & Q_p C_\Omega \\
-A_j C_2 & A_f & 0 \\
0 & 0 & A_\Omega
\end{bmatrix}
\begin{bmatrix}
x_p \\
z_1 \\
z_2
\end{bmatrix} +
\begin{bmatrix}
B_p \\
0 \\
0
\end{bmatrix} u +
\begin{bmatrix}
M_p & 0 \\
0 & -A_j N_2 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
f_x \\
f_y \\
0
\end{bmatrix} +
\begin{bmatrix}
Q_p D_\Omega \\
0 \\
B_\Omega
\end{bmatrix} \xi
\]  \hspace{1cm} (8)

\[
p-r\begin{bmatrix}
y_1 \\
z_1
\end{bmatrix} =
\begin{bmatrix}
C_1 & 0 & 0 \\
0 & I_r & 0 \\
0 & 0 & h
\end{bmatrix}
\begin{bmatrix}
x_p \\
z_1 \\
z_2
\end{bmatrix} +
\begin{bmatrix}
D_1 \\
0 \\
0
\end{bmatrix} u
\]  \hspace{1cm} (9)

The system of equations (8) - (9) can now be written in the standard state space form:

\[
\dot{x} = Ax + Bu + Mf + Q_\xi 
\]  \hspace{1cm} (10)

\[
y = Cx + Du 
\]  \hspace{1cm} (11)

A sliding mode observer in the form of (3) - (4) can now be designed for this system to reconstruct the fault signals \(f\), [21].

### 3.2 Robust Fault Reconstruction Using a Sliding Mode Observer

**Theorem 1:** If the following conditions are satisfied (i) \(\text{rank}(CM)=q+r\); and (ii) the zeros of \((A,M,C)\) (if any) are stable, then for the state-space model described by equations (10)
and (11) there exists a change of coordinates such that the triple \((A,M,C)\) can be re-written as:

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ M_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & T \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \quad M_o = \begin{bmatrix} 0 \\ M_o \end{bmatrix}
\]

(12)

where \(A_{11} \in \mathbb{R}^{(n-p) \times (n-p)}\). The matrix \(T \in \mathbb{R}^{p \times p}\) is orthogonal, \(Q_i \in \mathbb{R}^{(n-p) \times k}\), \(M_2 \in \mathbb{R}^{p \times q}\), and \(M_o \in \mathbb{R}^{r \times q}\) is invertible. Any unobservable modes of \((A_{11}, A_{21})\) are the invariant zeros of \((A,M,C)\) and are stable. Full proof is given in [21]. A sliding mode observer for this system of the form described by equations (3) and (4) has been proposed in [21], where, in the coordinates of (12), the gain matrix \(G_o\) is assumed to have the following structure

\[
G_o = \begin{bmatrix} -LT & T \end{bmatrix} P_o^{-1} \quad \text{where } P_o \in \mathbb{R}^{p \times p} \quad \text{is a symmetric positive definite matrix, and } L^o \in \mathbb{R}^{(n-p) \times (p-q)}.
\]

If we define the state estimation error as \(e = \hat{x} - x\), then from equations (10), (11), (3), and (4) the following error system is obtained:

\[
e = (A - G_jC)e + G_o \nu - Mf - Q \xi
\]

(13)

It has been proven in [19] that the error vector \(e\) is norm bounded, and a sliding motion will take place in finite time on the surface \(S_e = \{ e: Ce = 0 \}\) if the following conditions hold:

**Condition (i):** there exists a matrix \(P\) with the structure

\[
P = \begin{bmatrix} P_{11} & P_{12}L \\ L^TP_{11} & T^TP_{21} + L^TP_{11}L \end{bmatrix} > 0;
\]

\(P_{11} \in \mathbb{R}^{(n-p) \times (n-p)}\), that satisfies \(P(A - G_jC) + (A - G_jC)^T P < 0\).
Condition (ii): if the scalar $\rho$ in the switching function $\nu$ satisfies the following inequality

$$\rho > \frac{2 \| PT_{A_{21}} \| \times \| PQ \| \times \| \xi \|}{\mu} + (\| P_o T Q_2 \| \times \| \xi \|) + (\| P_o T M_2 \| \times \| f \|)$$

where $\mu = -\lambda_{\max} \left( P (A - G_i C) + (A - G_i C)^T P \right)$.

3.2.1 Observer Design

Introduce a further change of coordinates $T_L = \begin{bmatrix} I_{n-p} & L \\ 0 & T \end{bmatrix}$ on the parameters in (12) to obtain.

$$A = \begin{bmatrix} A_{11} + L A_{21} & -(A_{11} + L A_{21}) L T^T + (A_{12} + L A_{22}) T^T \\ T A_{21} & -T A_{21} L T^T + T A_{22} T^T \end{bmatrix}, M = \begin{bmatrix} 0 \\ T M_2 \end{bmatrix}$$  \hspace{1cm} (14)

$$C = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, Q = \begin{bmatrix} Q_1 + L Q_2 \\ T Q_2 \end{bmatrix}, G_u = \begin{bmatrix} 0 \\ P_o^{-1} \end{bmatrix}, G_i = \begin{bmatrix} G_{i,1} \\ G_{i,2} \end{bmatrix}$$  \hspace{1cm} (15)

The structure of $C$ in (15) implies that the bottom $p$ elements of the error $e$ are the output estimation error $e_y$. Partition the error system (13) in the new coordinate system as follows:

$$\dot{e}_i = (A_{11} + L A_{21}) e_i + \left( -\left( A_{11} + L A_{21} \right) L T^T + \left( A_{12} + L A_{22} \right) T^T - G_{i,1} \right) e_y - (Q_1 + L Q_2) \xi$$  \hspace{1cm} (16)

$$\dot{e}_y = T A_{21} e_i + \left( -T A_{21} L T^T + T A_{22} T^T - G_{i,2} \right) e_y - T Q_2 \xi - T M_2 f + P_o^{-1} \nu$$  \hspace{1cm} (17)

Assume that a sliding motion has occurred, and hence $e_y = \dot{e}_y = 0$. Then equations (16) and (17) reduce to:

$$\dot{e}_i = (A_{11} + L A_{21}) e_i - (Q_1 + L Q_2) \xi$$  \hspace{1cm} (18)
\[ 0 = TA_2 e_i - TQ_2 \hat{\xi} - TM_2 f + P_o^{-1} v_{eq} \]  \hspace{1cm} (19)

where \( v_{eq} \), which is a version of \( v \) required to achieve and maintain sliding motion. The signal \( v_{eq} \) is computable online by replacing \( v \) with \( v_\delta \) where

\[ v_\delta = -\rho \frac{e_y}{\| e_y \| + \delta} \]  \hspace{1cm} (20)

and \( \delta \) is a small positive scalar that governs the degree of accuracy of \( v_{eq} \) [21]. Furthermore, by replacing \( v \) with \( v_\delta \), there will be no singularity when \( e_y = 0 \), which is feasible in practice. This results in \( e_y \) being driven towards \( S_e \) but it will not slide perfectly on \( S_e \) and will be bounded inside a small boundary layer around \( S_e \). The term \( \delta \) in (20) needs to be chosen small enough so that the boundary layer is negligible (so that the fault reconstruction analysis in (18) - (19) is applicable, because it is based on the assumption that \( e_y = e_x = 0 \)), but not too small that it causes difficulties in terms of the numerical integration methods necessary to solve the differential equations. Now, we define an estimate for the fault signals as:

\[ \hat{f} = W T^T P_o^{-1} v_{eq} \]  \hspace{1cm} (21)

where \( W = \begin{bmatrix} W_1 & M_1^{-1} \end{bmatrix} \) with \( W_1 \in \mathbb{R}^{q \times (p - q)} \) being a design matrix [19].
Pre-multiply both sides of equation (19) by $WT^T$ and define $e_f = \hat{f} - f$ as the fault estimation error to get the following fault error equation:

$$e_f = -WA_\xi e_\xi + WQ_\xi \xi$$

(22)

Equations (18) and (22) show a state-space system from the uncertainty vector $\xi$ to the fault estimation error $e_f$. In the ideal case where there is zero uncertainty, the vector $e_f$ will be zero and the fault estimate $\hat{f}$ will be an exact replication of the fault $f$. The problem that needs to be solved now is to minimize the effect of the uncertainty on the fault estimation error, by appropriately choosing matrices $W_1$, $L^e$ and $G_f$. This problem can be formulated and solved using LMIs in the form of the Bounded Real Lemma [19].

The observer design is such that the $L_2$ gain from the uncertainty $\xi$ to the fault estimation error $e_f$ is minimized, i.e $\|T\Omega\|_{\infty} < \gamma$. The problem posed is therefore equivalent to the following: Minimize $\gamma$ with respect to the variables $P$, $W_1$, $E_2$ subject to the following matrix inequalities:

$$
\begin{bmatrix}
    P_{11}A_{11} + A_{11}^TP_{11} + P_{12}A_{21} + A_{21}^TP_{12} & -P_{11}Q_{1} - P_{12}Q_{2} & -A_{21}^TW^T \\
    -Q_{21}P_{11} - Q_{22}P_{12} & -\gamma I_k & Q_2^TW^T \\
    -WA_{21} & WQ_2 & -\gamma I_q
\end{bmatrix} < 0
$$

(23)

$$
\begin{bmatrix}
    PA + A^TP - \gamma C^T(D_dD^T_k)^\dagger C & -PB_d & E^T \\
    -B_d^TP & -\gamma I_{pvk} & H^T \\
    E & H & -\gamma I_q
\end{bmatrix} < 0
$$

(24)
\[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0, \quad P_{12} = [P_{121} \quad 0] \]  

(25)

where \( P_{11} = P_{11}^T, P_{22} = P_{22}^T \) with \( P_{11} \in \mathbb{R}^{(n-p) \times (n-p)}, P_{12} \in \mathbb{R}^{(n-p) \times p}, P_{121} \in \mathbb{R}^{(n-p) \times (p-q)}, P_{22} \in \mathbb{R}^{p \times p} \).

The other matrices are defined as: \( B_d = \begin{bmatrix} 0 & Q \end{bmatrix}; \ D_d = \begin{bmatrix} D_1 & 0 \end{bmatrix}; \ H = \begin{bmatrix} 0 & WQ_2 \end{bmatrix}; \) and

\[ E = \begin{bmatrix} -W & A_{21} \quad E_2 \end{bmatrix}. \]

**Solution:** Provided that conditions 1 and 2 are satisfied, the LMI solver will return the values of \( P, W_1 \) and \( \gamma \), which are then used to calculate \( L = P_{11}^{-1} P_{12} \);

\[ G_l = \gamma_o P^{-1} C^T (D_2 D_d^T)^{-1}; \quad P_o = T(P_2 - P_{12}^T P_{11}^{-1} P_{11}^T) T^T; \quad G_s = \begin{bmatrix} -LT^T \\ T^T \end{bmatrix} P_o^{-1}. \]

The parameters \( \gamma_o \in \mathbb{R}, D_1 \in \mathbb{R}^{p \times p} > 0 \) are user-defined design parameters which can be used as a means of tuning the gains. It is clear that when \( \gamma_o \) increases, the value of \( \gamma \) decreases, which results in \( G_l \) having a larger gain. Decreasing the gain of \( D_l \) has the same effect.

The final step is to perform the inverse of the first coordinate transform on \( G_l \) and \( G_n \) to get back to the original coordinate system. When the observer has been designed, let the transfer function from \( \xi \) to \( e_f \) be \( G(s) \). Therefore the transfer function from \( \hat{\xi}_p \) to \( e_f \) is \( \Omega^{-1}(s) \Omega(s) \) where \( \Omega(s) \) is the transfer function of the filter characteristics of \( \hat{\xi}_p \). In the case of the unweighted system, \( \Omega(s) \) is simply an identity matrix, which is the same as in [19]. However, for the frequency weighted case \( \Omega(s) \) is given. Thus, if the frequency content of the disturbance \( \hat{\xi}_p \) is known, then \( \Omega(s) \) can be chosen such that it has high
singular values at those particular frequencies. This will cause the transfer function from $\xi_{p}$ to $e_{f} \Omega^{-1}(s)G(s)$, to have low singular values at those frequencies, which is the main contribution of this paper.

It is worth mentioning that the objective of the design method proposed in this paper is to introduce additional flexibility into the design procedure. This is achieved by placing emphasis over a desired frequency range by adding frequency weights to ‘shape’ the solution. Therefore the conventional performance optimization problem has been greatly enhanced. The effect of this can be seen in (8) where a set of states associated with the frequency weighting filter is introduced. This is reflected in inequalities (23) - (25) and as a result the feasible domain of solutions to the affine matrix inequalities (23) - (25) is now a function of the frequency-weighting filter introduced. It should be borne in mind that according to Bode’s integral relationship [22] any performance gain over one frequency will generally be obtained at the expense of some performance deterioration over some other (complementary) frequency range or ranges. Therefore what is possible is that one may trade-off performance improvements over one set of frequency ranges (of interest) with possible performance deteriorations over a complementary set of frequency ranges (which are not of interest). This has been demonstrated by the example given in Section 4.


4 Design Examples

A $10^{th}$-order system with two known control inputs, one unknown external disturbance input has been generated to test the performance of the new approach presented in this paper. To illustrate the fault detection capabilities, we introduced two faults in the state equations, representing actuator failures, and one fault in the output measurements, representing sensor failure. The parameters for this test system in the notation of (1) - (2), which have been randomly generated by MATLAB, are listed below:

$$A_p = \begin{bmatrix}
-0.4925 & 0.7207 & -4.2222 & -1.3667 & -0.4711 & -1.5172 & 0.0549 & -0.4535 & 0.2361 \\
-0.1393 & -0.7289 & 3.9207 & 2.1717 & -0.3720 & -1.2001 & 0.0027 & -0.3274 & 0.1370 \\
4.3162 & -3.8987 & -0.9100 & 0.6429 & 1.3650 & -0.7686 & 0.2174 & -2.0595 & 0.0549 \\
1.1683 & -2.1948 & -0.5360 & -1.0590 & 0.5107 & -0.9407 & -0.3720 & 0.1370 & -0.3299 \\
1.1554 & 1.1440 & -1.2947 & 0.3915 & -0.1005 & -1.0590 & 0.5107 & -0.9407 & -0.3720 \\
0.7180 & 0.5586 & -0.1005 & 0.3915 & -0.1005 & -1.0590 & 0.5107 & -0.9407 & -0.3720 \\
1.3843 & -0.3244 & -0.7976 & 0.3805 & 1.7062 & -1.0943 & -0.2226 & 0.3156 & -2.2151 \\
0.0586 & 0.6315 & 1.0881 & -0.0872 & 0.0146 & 0.1598 & -1.2001 & -0.0027 & -1.4341 \\
0.5428 & -0.8011 & -0.9911 & 0.6768 & 0.7645 & -0.9814 & 0.2826 & 0.0826 & -0.7771 \\
0.1278 & -0.9436 & 2.0618 & 0.2103 & 0.1476 & 0.5753 & 2.2345 & -1.2172 & 1.0126 \\
\end{bmatrix}$$

$$B_p = \begin{bmatrix}
0.0579 & 0.0153 \\
0.3529 & 0.7468 \\
0.8132 & 0.4451 \\
0.0099 & 0.9318 \\
0.1389 & 0.4660 \\
0.2028 & 0.4186 \\
0.1987 & 0.8462 \\
0.6038 & 0.5252 \\
0.2722 & 0.2026 \\
0.1988 & 0.6721 \\
\end{bmatrix} ; \quad M_p = \begin{bmatrix}
0.9501 & 0.6154 \\
0.2311 & 0.7919 \\
0.6068 & 0.9218 \\
0.4860 & 0.7382 \\
0.8913 & 0.1763 \\
0.7621 & 0.4057 \\
0.4565 & 0.9355 \\
0.0185 & 0.9169 \\
0.8214 & 0.4103 \\
0.4447 & 0.8936 \\
\end{bmatrix} ; \quad Q_p = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} ; \quad N_p = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\end{bmatrix} ; \quad P = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\end{bmatrix}$$
\[
C_p = \begin{bmatrix}
1.4352 & -0.0312 & 0.8678 & 1.5256 & -0.2132 & 0 & 1.2579 & -0.9199 & -0.2263 & -0.0164 \\
0 & -1.0030 & 2.0718 & 2.1432 & 0 & 0 & 0.5941 & -1.4481 & 0 & 0.5941 \\
0.7431 & -1.0381 & -0.5944 & -0.7460 & 0 & 0.9790 & 0 & 0 & 0.6340 & -0.5223 \\
-0.1214 & 0.6286 & 0 & -1.5315 & 0.8222 & 0.4926 & -1.4178 & 0.0973 & 0.0390 & 0 \\
\end{bmatrix}
\]

In order to be able to design a frequency-shaping observer, the frequency characteristics of the unknown input disturbance is required to be known. While this is normally the case in practice, for this particular Matlab generated example, we arbitrarily select the frequency range of the external disturbance vector to be 0.005 to 0.5 rad./s. Furthermore, the disturbance \( \xi \) and faults \( f \) are assumed to be norm bounded by \( \| \xi \| < 1.5 \) and \( \| f \| < 3 \).

### 4.1 Sliding Mode Observer Design

In the following we design two sliding modes observers for the system described above. The design of the first is a straightforward application of existing theory [18] (without frequency weighting), while the design of the second is based on the approach of this paper, where frequency shaping is used. Later in section 5, detailed comparison of the two performances is provided.

#### 4.1.1 Observer design without frequency weighting

Combine \( x_p \) and \( z_1 \) to obtain:
\[
\begin{bmatrix}
\dot{x}_p \\
\dot{z}_1 \\
\end{bmatrix} =
\begin{bmatrix}
A_p & 0 \\
-A_fC_2 & A_f \\
\end{bmatrix}
\begin{bmatrix}
x_p \\
z_1 \\
\end{bmatrix} +
\begin{bmatrix}
B_p \\
-A_fD_2 \\
\end{bmatrix} u +
\begin{bmatrix}
M_p & 0 \\
0 & -A_fN_2 \\
\end{bmatrix}
\begin{bmatrix}
f_x \\
f_y \\
\end{bmatrix} +
\begin{bmatrix}
Q_p \\
0 \\
\end{bmatrix} \xi_p \\
\tag{26}
\end{align}
\]

\[
\begin{bmatrix}
y_1 \\
z_1 \\
\end{bmatrix} =
\begin{bmatrix}
C_1 & 0 \\
0 & I_r \\
\end{bmatrix}
\begin{bmatrix}
x_p \\
z_1 \\
\end{bmatrix} +
\begin{bmatrix}
D_1 \\
0 \\
\end{bmatrix} u
\tag{27}
\]

For the observer design, the design parameters were chosen as \( \gamma_o=100 \) and \( D_1=I_4 \), and \( A_f=-I \). The LMI toolbox returned the following parameters in the original coordinates.

\[
G_j = 10^4 \times
\begin{bmatrix}
2.9295 & -1.4116 & -4.5999 & -0.1229 \\
5.2783 & -2.5494 & -8.3009 & -0.2247 \\
1.5334 & -0.7416 & -2.4083 & -0.0662 \\
1.1882 & -0.5747 & -1.8661 & -0.0513 \\
0.8042 & -0.3843 & -1.2569 & -0.0319 \\
\end{bmatrix}
\]

\[
G_n =
\begin{bmatrix}
292.9464 & -141.1603 & -459.9889 & -12.2899 \\
527.8317 & -254.9427 & -830.0950 & -22.4707 \\
153.3355 & -74.1558 & -240.8253 & -6.6157 \\
118.8185 & -57.4675 & -186.6053 & -5.1304 \\
80.4192 & -38.4264 & -125.6947 & -3.1889 \\
\end{bmatrix}
\]

\[
G_j = 10^4 \times
\begin{bmatrix}
4.8865 & -2.3555 & -7.6789 & -0.2052 \\
3.0989 & -1.4977 & -4.8718 & -0.1327 \\
2.0905 & -1.0134 & -3.2899 & -0.0914 \\
3.2615 & -1.5713 & -5.1227 & -0.1365 \\
3.4758 & -1.6793 & -5.4644 & -0.1484 \\
-0.0012 & 0.0096 & 0.0085 & 0.0065 \\
\end{bmatrix}
\]

\[
W_{T^T} =
\begin{bmatrix}
1.1424 & -0.2896 & 0.6558 & 0 \\
0.4612 & -0.8031 & 0.3224 & 0 \\
-0.4010 & 0.2703 & -0.1325 & 1.0000 \\
\end{bmatrix}
\]

It is obvious that the value of $\rho$ can be calculated from condition (ii), as by assumption $\|f\|$ and $\|r\|$ are known. For the example presented in section 4.2.1, $\rho$ was chosen to be 200 (since the right-hand side of condition (ii) turned out to be 170, and this serves as a lower bound on $\rho$).
4.1.2 Observer design with frequency weighting

The spectral density of the frequency-shaping filter is shown in Figure 1.

![Figure 1: Frequency response of shaping filter](image)

In the design of an observer, the frequency shaping filter of figure 1 is used to shape the path between the external disturbance and the observer output error. For the system of equations (8),(9), the same design parameters were used as in the previous section. As a result the LMI toolbox has returned the following observer gains
For this case, in order to guarantee sliding motion on $\mathcal{S}$, a value of $\rho > 141$ must be chosen to satisfy condition (ii). Hence $\rho$ was picked to be 150. By way of comparison, the sigma plots for the frequency-weighted and frequency-unweighted cases are shown in Figure 2. Inspection of this figure reveals that a significant improvement in the attenuation of the error map from the disturbance input, $\xi_p$, to the fault estimation error, $e_f$, is obtained over the low frequency range of interest (0.003 – 0.3 rad./s) to the study.
5 Simulation Studies

In this section we report on the performance of the two observer-based fault detection schemes designed in section 4. The performance of each of the two schemes is tested through Simulink™. Three case studies are considered as outlined below. In the remainder of this paper we will refer to the frequency weighted design as “our observer”, and to the frequency unweighted design as “the alternative design”. In each of the three case studies, the system is subjected to the following conditions.

1. The control inputs are represented by step signals of magnitude 1, acting at 1 and 5 seconds on channels 1 and 2 respectively.

2. The disturbance is represented by a sine wave of amplitude 1, with a frequency of 0.03 radians per second applied at time $t = 10$ (s)
These control inputs and disturbances are shown in Figure 4.

Figure 3: The inputs and disturbance

In the following studies, the initial condition of the states is arbitrarily set to

\[ x_o = \begin{bmatrix} 0.0950 & 0.0231 & 0.0607 & 0.0486 & 0.0891 & 0.0762 & 0.0456 & 0.0019 & 0.0821 & 0.0445 \end{bmatrix} \]

and the observer initial states are set to zero value.

5.1 Case Study #1

This initial study tests the performance of the designed sliding mode observer. The responses of the output variables are shown in Figure 4. The figures demonstrate the convergence property of the observer, as the observer outputs converge to the true outputs of the plant after around 0.15 seconds. And once the convergence takes place, the observer emulates exactly the behaviour of the plant, as expected.
The responses of the fault detection filter are shown in Figure 6, for the following two scenarios: (i) when the fault detection observer is designed without frequency-weighting using existing theory [19]; and (ii) and when the fault detection observer is designed with frequency-weighting, using the novel approach proposed in this paper. The aim of this study is to compare the two cases and highlight the advantages that are apparent when frequency weighting is incorporated into the observer design procedure. This is evident in the figures where the frequency-weighted filter does not respond to any of the inputs or disturbances, while the traditional observer drifts away after approximately 10 (s), indicating the presence of some sort of fault when in fact there is none.
5.2 Case Study #2

In this study, we assume that the plant is in equilibrium operating under a normal (fault free) condition. We also assume that the observer was switched on for long enough for it to track the states of the plant. The same scenario as described in Case Study #1 is used, however, the following faults are now introduced. At time $t = 6$ (s), a fault is introduced into the first of the two fault input channels, then at time $t = 12$ (s), a further fault is introduced into the second of the two fault input channels. Finally, at time $t = 18$ (s), a sensor fault is introduced into the output fault channel. The purpose of this study is to (i) detect the occurrence of the faults; (ii) identify the seriousness of the fault by reconstructing it in its entirety; and (iii) compare the fault detection performances of the
proposed frequency weighted observer against an existing unweighted observer. Figure 6 shows the responses of the traditional and the frequency weighted observer based fault detection filters. The three figures show the reconstructions of the fault signals, the first two being the actuator faults, $f_x$, and the third being the sensor fault, $f_y$. From these figures it can be concluded that while the alternative observer fails to reconstruct the faults, the proposed frequency weighted observer is able to reconstruct the three simultaneous faults in their entirety in real-time. It can also be concluded that the proposed fault detection filter is insensitive to external disturbances, and therefore would only react when a fault or a combination of faults occur.

![Figure 6: Responses for case study #2.](image)
5.3 Case Study #3

In this study, we repeat the same scenario of case study 2, but with the following alterations. A fault in actuator 1 occurs at time $t = 8$ (s) and is maintained thereafter. A fault in actuator 2 occurs at time $t = 15$ (s), and is then cleared after 1 second. However the sensor channel is assumed to be fault-free for the entire period. The simulation results are shown in Figure 7. The first two figures show the reconstructions of the actuator faults by the two observers. The third figure shows the reconstruction of the sensor signal, which is fault free. Here again the figures further demonstrate the fact that while the alternative observer fails to accurately identify any of the faults, our frequency shaping approach provides exact detection and identification of all the faults, as well as provides information about their magnitude, all in real time. Note that when the fault on actuator 2 is cleared at $t = 16$ (s), the fault filter detects it by switching back to no-fault mode. It can be observed that while the frequency-weighted observer provides excellent fault reconstructions on all three fault channels, the fault reconstructions provided by the alternative observer tend to drift away from the true values which, after a period of time, will provide false fault signals.
6 Conclusions

A new fault detection scheme has been introduced in this paper. The scheme has been shown to be able to provide exact information about any fault or a combination of faults when and where they occur. The main advantages of the proposed approach are: (i) robustness can be incorporated in the design through inclusion of uncertainties or parameter variation in the model of the external input term; (ii) a previously ineffective design method can be made effective through the introduction of frequency shaping pre-filters, as demonstrated in the design example; and (iii) only one observer design is
required to detect any number of faults, provided that the output measurement contains enough information about the state of the system.

The new scheme has been tested on a 10th order system with a total of three inputs and four outputs. The system is assumed to have two possible actuators faults and one sensor fault. In addition, the system is also assumed to have an unknown disturbance. Two sliding mode fault detection filters are designed, with and without our frequency shaping approach. The performance of the two schemes is then tested through simulation using a number of different fault scenarios. Simulation results have shown that in all of the studied cases the alternative design failed to accurately reconstruct any of the faults, while our fault detection scheme was not only able to accurately detect the occurrence of faults, but also determine their exact locations.

References


