Reduced-order linear functional observer for linear systems

M.Aldeen and H.Trinh

Abstract: A new reduced-order linear functional observer is introduced in this paper. The order of the observer is proportional to the ratio of the number of output measurements to the number of inputs. It is shown that the observer asymptotically converges to any number of linear functionals when some minor conditions are met. A simple observer construction procedure, which is easy to implement on MATLAB, is provided. Numerical examples are considered to illustrate the properties of the observer.

List of Symbols

$x(t)$	n-dimensional state vector
$u(t)$	m-dimensional control input vector
$y(t)$	r-dimensional output vector
R^n	real vector space of dimension n
A	(nxn) real constant matrix
B	(nxm) real constant matrix
C	(rxn) real constant matrix
F	(mxn) state feedback real constant matrix
E	(pxp) stable real matrix
G	(pxr) real constant matrix
T	(pxn) real constant matrix
K	(mxp) real constant matrix
W	(mxr) real constant matrix
z	p-dimensional state vector
e	p-dimensional error vector; $e = z - Tx$
t_i	i-th column of T
f_i	i-th column of F
$[.]^T$	transpose of matrix [.]
$a \in b$	a belongs to b.

1 Introduction

It is often the case that, due to limited output measurement, full state feedback control laws can not, in general, be realised. However, in actual implementation a state feedback control law does not necessarily require the availability of the complete state vector, x. Rather, the implementation requires the feedback signals $Fx(t)$, which are linear functions of x, to be generated.

The possibility of constructing an observer of a lower dimension than that of a full state observer which might be employed to generate a single linear state functional was first explored by Bass and Gura [1] and Luenberger [2, 3]. Alternative design procedures for observing a scalar linear function of the state of a multiple-output system are presented in Moore [4], Murdoch [5] and Roman *et al* [6].

Fortmann and Williamson [7] were the first to reconstruct vector linear functions of the state for multiple-output systems. Later, minimal partial realisation theory and decision methods were used by Roman and Bullock [8], and Moore and Ledwich [4], respectively, to solve general vector state function problem for multiple-output systems.

Murdoch [9] proposed a simple sequential design procedure of a degenerate observer capable of generating multiple linear state functionals. The method does not require the reduction of the system to a number of single-output subsystems before a linear functional observer can be designed, as proposed in Fortmann and Williamson [7]. Another method based on the reduction of a state observer for a system in Luenberger companion form, where the state functionals are treated as additional outputs, was proposed by Fairman and Gupta [10]. By contrast, the approach of Sirisena [11] does not require transformation of the system into Luenberger companion form.

An interesting algorithm for the design of multifunctional observers was reported by Tsui [12]. The algorithm is based on finding coordinate transformation which relates the states of the system to those of the observer. In this process a lower Hessenberg form of the observable pair is required. The same author [13], showed that the order of the functional observer can be reduced to the sum of descending ordered observability indices of the system minus the number of the required functionals. This guarantees a lower order functional observer than the author's previously reported result, [13].

O'Reilly [14], provides an excellent read on observer theory. In Chapter 3, the author provides analysis of the theory of single- and multi-functional observers and their realisation.

In this paper, a new observer, which is capable of asymptotically estimating any vector state functionals, is introduced. The order of the observer is dependent on the ratio of the number of independent output measurements to the number of independent inputs. Thus for high-order systems with considerably more outputs than inputs, the order of the proposed linear functional observer could be as low as that proposed by Tsui [13] and Murdoch [9]. The observer construction procedure is very simple in that it requires the solution of a set of consistent linear algebraic equations. The procedure involves no linear transformation

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and can be easily carried out by constructing a short MATLAB.m function.

The new linear functional observer possesses three distinctive advantages. The first is that dynamics of the observer can be freely chosen. The second is that the order of the observer is low and comparable to that of Murdoch [9] and Tsui [13]. This provides considerable benefits, especially when used on large-interconnected systems such as power systems where the number of outputs can be much higher than the number of input. As will be shown in Section 5, a 15-order system with five outputs and two inputs requires only a fourth-order linear functional observer (the same requirement as in Murdoch [9] and Tsui [13]), while it requires a 15th-order full-order observer or a 10th-order reduced-order observer. The third advantage is the simplicity of the algorithm, where the design is accomplished by solving a set of consistent linear algebraic equations.

2 Statement of the problem

Consider a linear time-invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^r$ are the state, input and output vectors, respectively. Matrices $A \in R^{n \times n}$, $B \in R^{n \times m}$ and $C \in R^{r \times n}$ are constant. Without loss of generality, the pair (A, B) is assumed to be completely controllable and the pair (A, C) is assumed to be completely observable. It is further assumed that matrix C is in the canonical form

$$C = [C_1 \ 0] \tag{1c}$$

where $C_1 \in R^{r \times r}$ is of full rank. If C is not in the canonical form (1c), then the following transformation guarantees it

$$C = C_{old}\begin{bmatrix} C_{old} \\ Q \end{bmatrix}; \tag{1d}$$

where Q is the null space of C_{old}; ie.

$$Q = null(C_{old}). \tag{1e}$$

As eqn. 1 is completely controllable, a linear state feedback controller can be easily designed as

$$u(t) = Fx(t), \tag{2}$$

where $F \in R^{m \times n}$, to achieve specified performance objectives. The design can be carried out by using any existing control design approach, for instance, linear quadratic regulator or eigenvalues\eigenvectors assignment method.

The problem considered in this paper is that of designing a p-dimensional

$$\left(p \geq \frac{m(n-r)}{r}\right)$$

dynamical system to generate any required vector state function(s) of x. This is a particularly useful result for systems with more outputs than inputs, as will be demonstrated in example 1, Section 5.

Note that such an observer can be used to generate either the entire state vector x or any subset of it by letting F in eqn. 2 to be either I_n or to comprise those rows of I_n that correspond to the state variables to be estimated, respectively. This is demonstrated in example 2, Section 5.

3 Main result

Decompose, arbitrarily, the feedback gain matrix F as follows

$$F = KT + WC \tag{3}$$

where $K \in R^{m \times p}$, $T \in R^{p \times n}$, $W \in R^{m \times r}$ and $m < p \leq n$. Using Eqn. 3, the feedback control law, Eqn. 2, becomes

$$u(t) = Fx(t) = KTx(t) + Wy(t) = Kz(t) + Wy(t) \tag{4}$$

where

$$z(t) = Tx(t) \in R^p. \tag{5}$$

Consider the following system

$$\dot{z}(t) = Ez(t) + TBu(t) + Gy(t) \tag{6}$$

where $E \in R^{p \times p}$ and $G \in R^{p \times r}$.

Theorem 1: Eqn. 6 can be constructed to asymptotically generate linear functions of Eqn. 1, Fx, provided that the following conditions hold

$$\left\{\begin{array}{l} (i)\ F = KT + WC \\ (ii)\ E \text{ is stable} \\ (iii)\ GC - TA + ET = 0 \\ (iv)\ p \geq \dfrac{m(n-r)}{r} \end{array}\right\}.$$

Proof of Theorem 1: Condition (i) of the theorem is implied by Eqn. 3. Now define e as the error between the state of Eqn. 6, z, and its estimate, Tx, i.e. as

$$e(t) = z(t) - Tx(t). \tag{7}$$

Taking a derivative of Eqn. 7, gives

$$\begin{aligned} \dot{e}(t) &= \dot{z}(t) - T\dot{x}(t) \\ &= Ez(t) + TBu(t) + Gy(t) - TAx(t) - TBu(t) \\ &= Ez(t) + GCx(t) - TAx(t) \\ &= Ez(t) - ETx(t) + GCx(t) - TAx(t) + ETx(t). \end{aligned} \tag{8}$$

If

$$GC - TA + ET = 0, \tag{9}$$

then eqn. 8 becomes

$$\dot{e}(t) = Ee(t). \tag{10}$$

Hence the dynamics of e is governed by matrix E. It can, therefore, be concluded that Eqn. 6 can act as a linear functional observer for Eqn. 1, provided that E is stable and $GC - TA + ET = 0$. Thus conditions (ii) and (iii) of the Theorem are proven.

To prove condition (iv) of the Theorem, eqn. 3 is partitioned into two parts as follows

$$WC_1 = (F - KT)\begin{bmatrix} I_r \\ 0 \end{bmatrix} \tag{11a}$$

and

$$(F - KT)\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = 0. \tag{11b}$$

Let matrices $K \in R^{m\times p}$, $F \in R^{m\times n}$, and $T \in R^{p\times n}$ be expressed as follows:

$$K = \begin{bmatrix} k_{1,1} & k_{1,2} & \cdot & k_{1,p} \\ k_{2,1} & k_{2,2} & \cdot & k_{2,p} \\ \cdot & \cdot & \cdot & \cdot \\ k_{m,1} & k_{m,2} & \cdot & k_{m,p} \end{bmatrix}; \quad (12a)$$

$$F = [F_1 F_2] = \begin{bmatrix} f_{1,1} & f_{1,2} & \cdot & f_{1,n} \\ f_{2,1} & f_{2,2} & \cdot & f_{2,n} \\ \cdot & \cdot & \cdot & \cdot \\ f_{m,1} & f_{m,2} & \cdot & f_{m,n} \end{bmatrix}; \text{ and} \quad (12b)$$

$$T = [T_1 T_2] = \begin{bmatrix} t_{1,1} & t_{1,2} & \cdot & t_{1,n} \\ t_{2,1} & t_{2,2} & \cdot & t_{2,n} \\ \cdot & \cdot & \cdot & \cdot \\ t_{p,1} & t_{p,2} & \cdot & t_{p,n} \end{bmatrix} \quad (12c)$$

where $F_1 \in R^{m\times r}$; $F_2 \in R^{m\times(n-r)}$; $T_1 \in R^{p\times r}$ and $T_2 \in R^{p\times(n-r)}$. Incorporating eqns. 12 into 11b and rearranging gives

$$F_2 = KT_2 \quad (13a)$$

or in a matrix-vector form

$$\Omega t = f \quad (13b)$$

where:

$$\Omega = \begin{bmatrix} 0_{n-r\cdot r} & k_{11}I_{n-r} & 0_{n-r\cdot r} & k_{12}I_{n-r} & \cdot & k_{1p}I_{n-r} \\ 0_{n-r\cdot r} & k_{21}I_{n-r} & 0_{n-r\cdot r} & k_{22}I_{n-r} & \cdot & k_{2p}I_{n-r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0_{n-r\cdot r} & k_{m1}I_{n-r} & 0_{n-r\cdot r} & k_{m2}I_{n-r} & \cdot & k_{mp}I_{n-r} \end{bmatrix} \quad (13c)$$

$$t = [t_{1,1} \;\; t_{1,2} \;\; \cdot \;\; t_{1,n} \;\; t_{2,1} \;\; \cdot \;\; t_{2,n} \;\; \cdot \;\; t_{p,n-1} \;\; t_{p,n}]^T; \quad (13d)$$

$$f = [f_{1,r+1} \;\; f_{1,r+2} \;\; \cdot \;\; f_{1,n} \;\; f_{2,r+1} \;\; \cdot \;\; f_{2,r+n} \;\; \cdot \;\; f_{m,n-1} \;\; f_{m,n}]^T. \quad (13e)$$

A solution for eqn. 13a exists if K is chosen to have full row rank and

$$p \geq m. \quad (14)$$

eqn. 13a represents $m(n-r)$ equations in $p(n-r)$ unknowns. As $p \geq m$, $(p-m)$ elements of t_i; $i = r+1, r+2, \ldots, n$ can be chosen arbitrarily and the rest is uniquely determined (note that the elements of T_1 can be arbitrarily assigned). Although solving eqn. 13a is straightforward and satisfies condition (i) of the *Theorem*, there is no guarantee that the solution will satisfy condition (iii) simultaneously. Therefore the elements of T must be found to satisfy both conditions. This can be achieved by considering condition (iii)

$$GC - TA + ET = 0. \quad (15)$$

Since matrix C is assumed to take the form (1c), eqn. 15 can be partitioned into two parts:

$$GC_1 = (TA - ET)\begin{bmatrix} I_r \\ 0 \end{bmatrix} \quad (16a)$$

and

$$(TA - ET)\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = 0. \quad (16b)$$

or

$$\left\{[T_1 T_2]\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - E[T_1 T_2]\right\}\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = 0 \quad (16c)$$

or

$$\left\{[T_1 T_2]\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} - ET_2\right\} = 0 \quad (16d)$$

where $A_{11} \in R^{r\times r}$; $A_{12} \in R^{r\times(n-r)}$; $A_{21} \in R^{(n-r)\times r}$ and $A_{22} \in R^{(n-r)\times(n-r)}$. As commented on earlier $E \in R^{p\times p}$ can be chosen to be any matrix with a set of stable eigenvalues – a straightforward choice of E is a diagonal matrix. ie. $E = diag(\lambda_i)$; $i = 1, 2, \ldots, p$. eqn. 16dcan then be written as:

$$\Psi t = 0 \quad (17a)$$

where

$$\Psi = \begin{bmatrix} A_{12}^T & A_{22}^T - \lambda_1 I_{n-r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{12}^T & A_{22}^T - \lambda_2 I_{n-r} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & A_{12}^T & A_{22}^T - \lambda_p I_{n-r} \end{bmatrix}. \quad (17b)$$

The set of eqn. 17a is consistent, there are $p(n-r)$ equations in np unknowns, and therefore an infinite number of solutions to t can be found. However, as is the case with the set of eqns. 13b, a solution to Eqn. 17a may not be a solution to Eqn. 13b. However, if the set of eqn. 17a is augmented with the set of eqns. 13b, a set of $\{m(n-r) + p(n-r)\}$ equations in pn unknowns is obtained as

$$\prod t = f \quad (18a)$$

where

$$\prod = \begin{bmatrix} \Psi \\ \Omega \end{bmatrix}. \quad (18b)$$

Eqn. 18a contains a set of $\{m(n-r) + p(n-r)\}$ linear algebraic equations in pn unknowns. Thus, a solution exists if and only if $pn \geq m(n-r) + p(n-r)$, which implies

$$p \geq \frac{m(n-r)}{r}.$$

This proves condition (iv) and completes the proof of the *Theorem*.

Once T is found, matrix W can be determined from eqns. 11a, and matrix G can be determined from eqns. 16a,. As a result, all of the observer (6) and control law (4) parameters are determined and therefore the construction of the observer is complete.

Based on the above development, the design procedure is now summarised as follows:

Design Algorithm

1. Design a stable feedback controller F using any existing state-feedback controller design method.
2. Use condition (iv) of *Theorem* 1 to obtain the order, p, of the observer (6).
3. Choose the $(m \times p)$ elements of matrix K arbitrarily.
4. Choose, arbitrarily, a stable matrix E.
5. Solve Eqn. 18a for matrix T.
6. From eqn. 11a find matrix W.
7. From eqns. 16a, find matrix G.
8. As a result of steps (1)–(7) above, a reduced-order observer of the structure shown in eqn. 6 and a feedback controller of the structure shown in eqn. 4 are obtained.

4 Closed-loop feedback

In the following, an analysis is made regarding the observer-based closed-loop performance of the system. Substituting eqn. 7 in eqn. 4 gives

$$u(t) = Ke(t) + KTx(t) + WCx(t). \tag{19}$$

Hence the following augment closed-loop system is obtained

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BKe(t) + BKTx(t) + BWCx(t)\\ &= (A + BKT + BWC)x(t) + BKe(t)\\ &= (A + BF)x(t) + BKe(t)\end{aligned}$$

and

$$\dot{e}(t) = Ee(t)$$

or

$$\begin{bmatrix}\dot{x}(t)\\ \dot{e}(t)\end{bmatrix} = \begin{bmatrix}A + BF & BK\\ 0 & E\end{bmatrix}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix}. \tag{20}$$

From eqn. 20, it can be seen that the separation principle holds, ie. the eigenvalues of the observer-based closed-loop system are the union of the eigenvalues of the closed-loop system without the observer $\{eig(A+BF)\}$ and the eigenvalues of the observer $eig(E)\}$. Therefore, the controller and observer designs can be performed separately ie. in the same manner as in the design of a full-order Luenberger observer-based control system.

5 Numerical example

Consider the following unstable system comprising 15 states, two inputs and five outputs. The system matrices A, B and C are

$$A = \begin{bmatrix}
3.3 & 0 & -0.6 & -1.5 & -0.3 & 2.2 & 0 & -0.4 & -1 & -0.2 & -3.3 & 0 & 0.6 & 1.5 & 0.3\\
-0.3 & 6 & 0 & -0.6 & 1.5 & -0.2 & 4 & 0 & -0.4 & 1 & 0.3 & -6 & 0 & 0.6 & -1.5\\
-1.2 & 1.5 & 9 & -0.3 & -3 & -0.8 & 1 & 6 & -0.2 & -2 & 1.2 & -1.5 & -9 & 0.3 & 3\\
-2.25 & -0.6 & -2.4 & 3 & 0 & -1.5 & -0.4 & -1.6 & 2 & 0 & 225 & 0.6 & 2.4 & -3 & 0\\
-0.6 & 1.5 & -1.5 & -1.5 & 3.75 & -0.4 & 1 & -1 & -1 & 2.5 & 0.6 & -1.5 & 1.5 & 1.5 & -3.75\\
-1.1 & 0 & 0.2 & 0.5 & 0.1 & -2.2 & 0 & 0.4 & 1 & 0.2 & 1.65 & 0 & -0.3 & -0.75 & -0.15\\
0.1 & -2 & 0 & 0.2 & -0.5 & 0.2 & -4 & 0 & 0.4 & -1 & -0.15 & 3 & 0 & -0.3 & 0.75\\
0.4 & -0.5 & -3 & 0.1 & 1 & 0.8 & -1 & -6 & 0.2 & 2 & -0.6 & 0.75 & 4.5 & -0.15 & -1.5\\
0.75 & 0.2 & 0.8 & -1 & 0 & 1.5 & 0.4 & 1.6 & -2 & 0 & -1.125 & -0.3 & -1.2 & 1.5 & 0\\
0.2 & -0.5 & 0.5 & 0.5 & -1.25 & 0.4 & -1 & 1 & 1 & -2.5 & -0.3 & 0.75 & -0.75 & -0.75 & 1.875\\
1.43 & 0 & -0.26 & -0.65 & -0.13 & -1.1 & 0 & 0.2 & 0.5 & 0.1 & -1.87 & 0 & 0.34 & 0.85 & 0.17\\
-0.13 & 2.6 & 0 & -0.26 & 0.65 & 0.1 & -2 & 0 & 0.2 & -0.5 & 0.17 & -3.4 & 0 & 0.34 & -0.85\\
-0.52 & 0.65 & 3.9 & -0.13 & -1.3 & 0.4 & -0.5 & -3 & 0.1 & 1 & 0.68 & -0.85 & -5.1 & 0.17 & 1.7\\
-0.975 & -0.26 & -1.04 & 1.3 & 0 & 0.75 & 0.2 & 0.8 & -1 & 0 & 1.275 & 0.34 & 1.36 & -1.7 & 0\\
-0.26 & 0.65 & -0.65 & -0.65 & 1.625 & 0.2 & -0.5 & 0.5 & 0.5 & -1.25 & 0.34 & -0.85 & 0.85 & 0.85 & -2.125
\end{bmatrix}$$

$$B = \begin{bmatrix}1 & 0 & 0.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0.3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -0.5 & -1 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}^T \text{ and}$$

$$C = [I_5 \;\; : \;\; 0]$$

The open-loop eigenvalues are:

$$\begin{aligned}\lambda(A) = \{&\underline{4.7594}, -3.499 \pm j4.2421, \underline{3.4326, 2.5368},\\ &-2.5236 \pm j3.0595, -1.865 \pm j2.261, \underline{1.479},\\ &-1.0873 \pm j1.3182, \underline{0.2188}, -0.1609 \pm j0.195\}.\end{aligned}$$

As can be seen, this system is unstable due to the presence of five unstable poles. In the following two studies will be conducted: (i) the design of a fourth-order linear functional to generate the required set of control signals; and (ii) the design of a second-order observer to generate the sixth state of x, ie. to generate x_6. In both studies, the design algorithm presented at the end of Section 3 is followed.

Example 1: Design of a linear functional observer

Step 1: As the system is controllable a linear state feedback control law can be easily derived. Using LQR with the state and control weighting matrices chosen respectively as $Q = 2I_{15}$ and $R = I_2$ the following optimal controller is obtained

$$u(t) = Fx(t)$$

where

$$F = 10^3 \times \begin{bmatrix}-0.2858 & 0.0926 & 1.0888 & 0.1921 & -0.5605 & -0.2116 & 0.0688 & 0.8091 & -0.4160 & 0.1690 & -0.0546 & -0.6439 & -0.1136 & 0.3314 & 0.1433\\ -0.1389 & 0.0209 & 0.6794 & 0.0224 & -0.2577 & -0.1041 & 0.0158 & 0.5044 & -0.1918 & 0.0825 & -0.0125 & -0.4018 & -0.0131 & 0.1526 & 0.0159\end{bmatrix}.$$

The closed-loop eigenvalues are:

$$\begin{aligned}&\lambda(A + BF)\\ &= \{-4.3857 \pm j3.7682, -3.5228 \pm j4.2684, -4.7875,\\ &\quad -2.1884 \pm j2.4354, -1.2292 \pm j1.201, -0.5484,\\ &\quad -1.0054 \pm j0.4762, -3.0437, -2.6762, -2.2079\}.\end{aligned}$$

Note that the point this example seeks to illustrate is the ability of the proposed functional observer to generate any previously designed set of control signals. The design process itself and the quality of the controller are issues of no concern to this paper. Therefore the choice of the state weighting matrix Q and the control weighting matrix R and the subsequently obtained optimal state feedback gain matrix F need not be elaborated on.

Step 2: Condition (iv) of *Theorem* 1 is used to obtain the order of the observer as:

$$p \geq \frac{2 \times (15 - 5)}{5} \geq 4.$$

Step 3: Let, for simplicity, the $(m \times p)$ elements of matrix K be chosen as the first $(m \times p)$ elements of matrix F, i.e.

$$K = 10^3 \times \begin{bmatrix}-0.2858 & 0.0926 & 1.0888 & 0.1921\\ -0.1389 & 0.0209 & 0.6794 & 0.0224\end{bmatrix}.$$

Step 4: Let a stable matrix E be chosen as $E = diag\{-1, -2, -3, -4\}$.

Step 5: Solving Eqn. 18a for matrix T gives

$$T = \begin{bmatrix} 1.0140 & -0.7098 & -6.7497 & -0.2686 & 2.8480 & 1.6463 & -0.9153 & -10.0986 & -0.2129 & 4.1540 & -0.5130 & 0.5423 & 3.6073 & 0.1981 & -1.7535 \\ 0.0179 & -0.8304 & -8.9104 & 0.3222 & 2.4547 & 0.4160 & -1.5917 & -14.7335 & 2.0738 & 5.0191 & 0.4168 & 2.1596 & 3.3583 & 1.1707 & -3.8865 \\ 0.0638 & -0.0529 & -0.4839 & -0.0650 & 0.1862 & 0.1632 & -0.1317 & -0.9181 & -0.0799 & 0.4425 & 0.0081 & 0.0530 & 0.0365 & 0.0168 & -0.0579 \\ 0.1019 & -0.4826 & 0.9500 & 0.0206 & 0.9244 & 0.2225 & 0.5100 & 1.4947 & -0.1179 & -0.9131 & -0.1307 & -0.8189 & 0.1891 & -0.9565 & 1.3182 \end{bmatrix}.$$

Step 6: Solve eqn. 11a for matrix W. As matrix C_1 is an identity matrix, matrix W can be easily obtained as the first five columns of $(F - KT)$, ie.

$$W = \begin{bmatrix} -86.7237 & 116.9888 & 329.3013 & 152.3504 & -354.1538 \\ -44.0753 & -13.5908 & 235.6649 & 22.1126 & -60.6569 \end{bmatrix}.$$

Step 7: Solve eqns. 16a, for matrix G. As matrix C_1 is an identity matrix, matrix G can be easily obtained as the first five columns of $(TA\text{-}ET)$, ie.

$$G = \begin{bmatrix} 3.8801 & -3.3365 & -24.0655 & -1.3754 & 10.6197 \\ 3.3255 & -3.0977 & -39.7296 & 2.7046 & 10.0927 \\ 0.4863 & -0.2790 & -2.8638 & -0.2879 & 1.1126 \\ -0.3080 & -4.2617 & 6.7717 & -3.1935 & 7.3435 \end{bmatrix}.$$

It is easy to check that matrices G, T, E, W and K satisfy conditions (i)–(iii) of *Theorem* 1 and that the eigenvalues of the combined closed-loop system are the union of the eigenvalues of the controller and of the observer, i.e. $\lambda(A_{comb}) = \lambda(A + BF) \cup \lambda(E)$.

Simulation results: When the observer is implemented, the following closed-loop system results:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A + BWC & BK \\ TBWC + GC & E + TBK \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = A_{comb} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$$

$$y(t) = [C0] \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$$

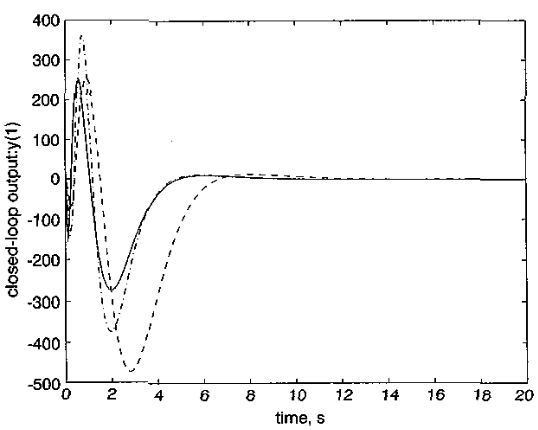

Fig. 1 *Closed-loop output response $y_1(t)$*
—— direct state feedback;
–·–·– full-order observer;
– – – – proposed observer

To simulate the dynamic performance of this closed-loop system, a nonzero initial condition is required. Let us choose, arbitrarily, the following initial conditions

$$x(0) = \begin{bmatrix} 20 & 5 & -1 & 4 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}^T \text{ and}$$

$$z(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

In the following, we report on three simulation studies that have been carried out on this example system. Study one involves the simulation of the output responses when direct state feedback is used. The second study involves simula-

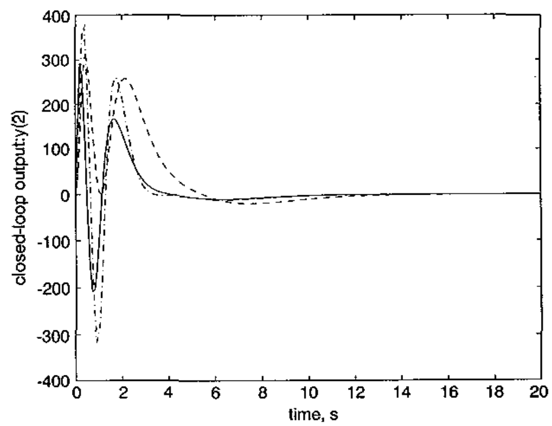

Fig. 2 *Closed-loop output response $y_2(t)$*
—— direct state feedback;
–·–·– full-order observer;
– – – – proposed observer

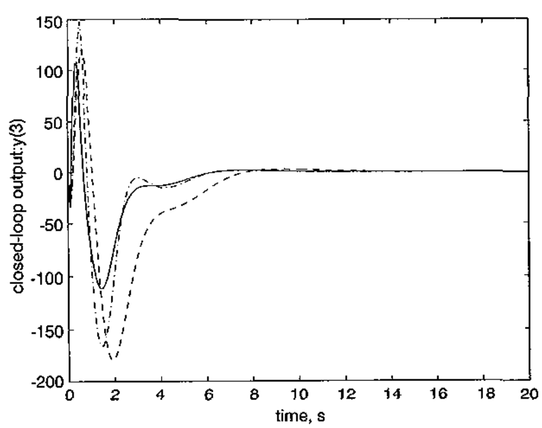

Fig. 3 *Closed-loop output response $y_3(t)$*
—— direct state feedback;
–·–·– full-order observer;
– – – – proposed observer

tion of the output responses when a full-order Luenberger observer is used to generate the entire state vector. The third study involves the Linear functional observer proposed in this paper. In the second study, the same state feedback controller, initial conditions and eigenvalues are used as for the linear functional observer. This is to allow for direct comparisons between the responses of the functional observer, full-order observer and that of the direct state feedback.

Figs. 1–5 show the output responses of the three studies. From these figures, it is quite clear the performance of the linear functional observer is close to that of the full-order observer-based controller. The closeness is demonstrated by two measures; (i) the first overshoot and (ii) the settling time. The first overshoot is quite the same for both but the settling time is slightly shorter for the full-order observer than for the functional observer; about 5 s compared to 7 s. This slight difference is a small price to pay for the reduction in the order of the observer from 15 to four.

Figs. 6–9 show error function $e(t) = z(t) - Tx(t)$ where the fourth-order observer is used to generate the feedback control signal $u(t)$. It is clear from the Figures that convergence takes place in about between 1 s to 5 s from the initiation of the response, which demonstrates the asymptotic convergence property of the linear functional observer.

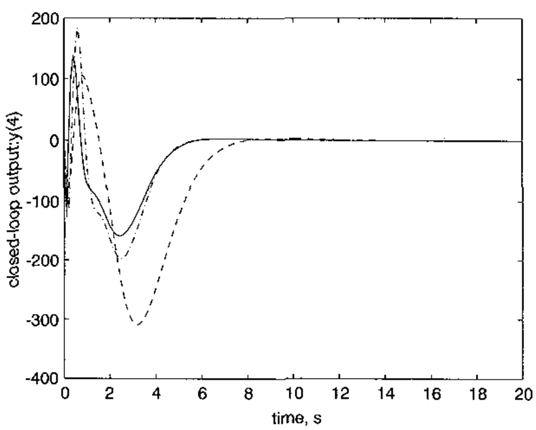

Fig. 4 *Closed-loop output response $y_4(t)$*
—— direct state feedback;
–·–·– full-order observer;
– – – – proposed observer

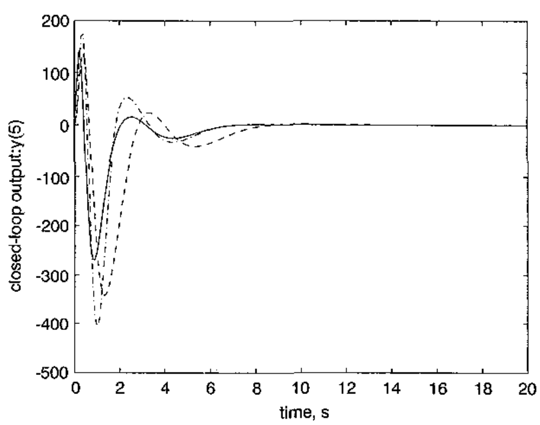

Fig. 5 *Closed-loop output response $y_5(t)$*
—— direct state feedback;
–·–·– full-order observer;
– – – – proposed observer

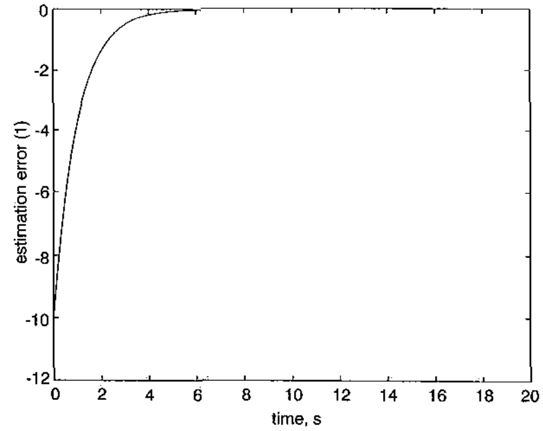

Fig. 6 *Error state estimation $e_1(t)$*

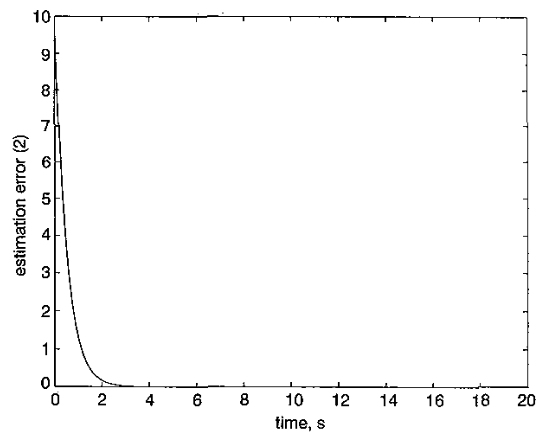

Fig. 7 *Error state estimation $e_2(t)$*

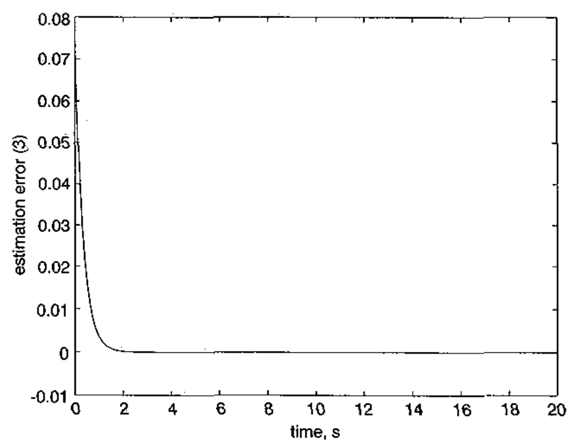

Fig. 8 *Error state estimation $e_3(t)$*

Example 2: Estimation of x_6 (t).

In the following the properties of the proposed observer is further illustrated by using it to estimate a small subset of x. Let us, for simplicity, assume that x_6 (t) is required, for good reason, to be estimated. Then we write eqn. 4 as

$$u(t) = x_6(t) = Fx(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t)$$

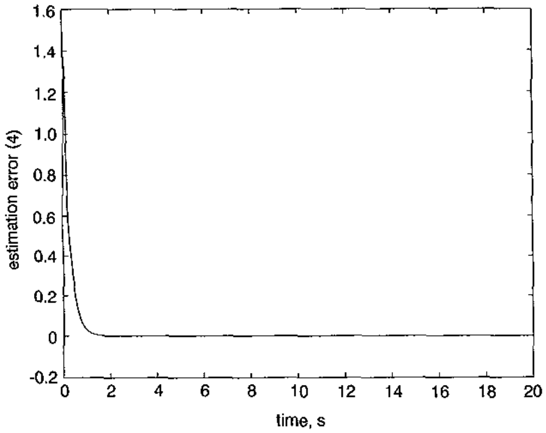

Fig. 9 *Error state estimation* $e_4(t)$

To generate x_6 (t), a second-order observer is required. Implementing the design algorithm of Section 3 gives the following results:

Step 2: Using condition (iv) of Theorem 1,

$$p \geq \frac{10 \times 1}{5} \geq 2.$$

Step 3: Let us now choose the elements of matrix K, arbitrarily, as $K = [1\ 1]$

Step 4: Let $E = diag\ \{-2, -3\}$.

Step 5: Solve Eqn. 18a for matrix T to obtain

$$T = \begin{bmatrix} 28.4766 & -4.5478 & 11.8314 & 25.6563 & 15.2938 & -3.3788 & 0.72 \\ -45.3031 & 7.3078 & -19.1338 & -41.5254 & -24.6587 & 4.3788 & -0.72 \end{bmatrix.$$
$$\begin{matrix} -1.9285 & -4.1902 & -2.4822 & 7.9182 & -1.08 & 3.1927 & 7.0354 & 3.8733 \\ 1.9285 & 4.1902 & 2.4822 & -7.9182 & 1.08 & -3.1927 & -7.0354 & -3.8733 \end{bmatrix}.$$

Step 6: Solve eqn. 11a for matrix W to obtain

$$W = [16.8265 \quad -2.7599 \quad 7.3024 \quad 15.8691 \quad 9.3649]$$

Step 7: Solve eqns. 16a, for matrix G to obtain

$$G = \begin{bmatrix} 72.5047 & -11.2057 & 29.6182 & 64.3634 & 37.9692 \\ -158.8291 & 25.1133 & -66.6495 & -144.9123 & -85.281 \end{bmatrix}.$$

Again, it is easy to check that matrices G, T, E, W and K obtained above satisfy conditions (i)-(iii) of Theorem 1.

Simulation results: The following simulation study was carried out with a control input signal u_1 (t) as shown in Fig. 10. Fig. 11 shows the response of the error state $\tilde{x}_6$ $(t) = x_6$ $(t) - \hat{x}_6$ (t). The initial conditions for the system and observer are taken to be $x(0) = 0$ and $z(t) = [-10 - 20]^T$, respectively. The Figure clearly shows that the asymptotic convergence property of the observer.

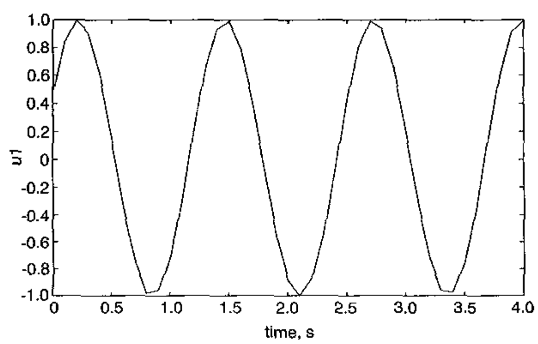

Fig. 10 *Sinusoidal input signal:* u_1

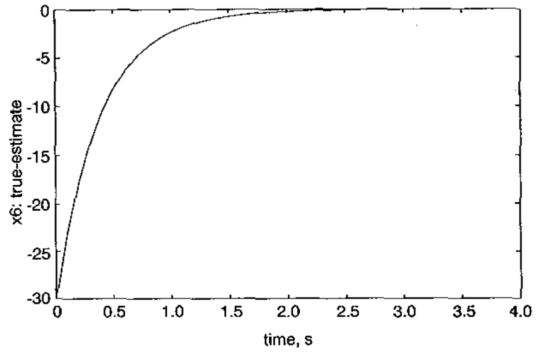

Fig. 11 $\tilde{x} = x_6 - \tilde{x}_6$

6 Conclusions

A new observer for the estimation of multifunctionals of the state vector for use in linear multivariable control systems is presented in this paper. In addition, the observer can serve as an estimator for any small subset of the state vector x. The observer dynamics are related to the ratio of the number of independent output measurements to the number of independent inputs and therefore is suitable for high-order systems with more outputs than inputs. Sufficient conditions for the existence of the observer are given.

The attractive feature of the proposed observer is the simplicity with which the design process can be accomplished. Numerical examples have been presented to illustrate the main properties of the proposed observer. In example 1 a comparison is provided between three control schemes. The schemes are: (i) direct full state feedback controller; (ii) full-order observer based controller; and (iii) the control scheme of this paper. Simulation results of the closed-loop performance of the 15th-order system shows very little difference among the three schemes, despite the fact that the control scheme proposed in this paper has much lower order than the full-order observer; four compared to 15. This demonstrates the usefulness of the proposed functional observer.

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