Nonlinear control approaches

- Part 1: Feedback linearisation
- Part 2: Lyapunov functions revisited
- Part 3: Backstepping control

References: Khalil, ‘Nonlinear Systems’, Krstic ‘Nonlinear and adaptive control design’
Consider the problem of controlling the movement of a robot arm whose nonlinear equations of motion are given by

\[ D(q)\ddot{q} + C(q, \dot{q}) + G(q) + J(q)' f = \tau \in \mathbb{R}^n \]

\[ y = h(q) \in \mathbb{R}^6 \]

- \( q \) is the vector of joint angles for the n-links of the arm
- \( D(\cdot), C(\cdot) \) and \( G(\cdot) \) represent inertial, Coriolis and gravitational forces
- \( J \) is a velocity transformation matrix and \( f \) is the output force vector
- \( \tau \) is the vector of control torques applied to the joints
- \( y \) is the position and orientation vector for the arm, so \( \dot{y} = J(q) \dot{q} \)

In Control Systems 2 we considered linear solutions only
- How would we solve this type of control problem using linear control?

The linear approach is naturally restrictive, and today we will consider the nonlinear alternative of feedback linearisation
- Nonlinear control designers get to use cool names for things like diffeomorphisms...
A simpler looking example...

- Consider a (simpler) nonlinear system of the form
  \[
  \dot{x}_1 = x_2 \\
  \dot{x}_2 = -a \sin x_1 - bx_2 + cu
  \]

- This represents a pendulum system where \(x_1\) is the angle the pendulum makes with vertical and \(u\) is the control torque.

- Now suppose we want to regulate the system to the origin - how can we go about it for these nonlinear system equations?

- A solution: choose \(u = (a/c)\sin x_1 + v/c\)

- State equations become
  \[
  \dot{x}_1 = x_2 \\
  \dot{x}_2 = -bx_2 + v
  \]
  This is a linear system. We know how to solve for \(v\) using linear feedback control!

  \[
  v = -k_1 x_1 - k_2 x_2 \\
  \]
  So,

  \[
  u = \left(\frac{a}{c}\right) \sin x_1 - \left(\frac{1}{c}\right) \left(k_1 x_1 + k_2 x_2\right)
  \]
A second example

• Now consider a system that has a nonlinear state multiplying the control, eg
  \[ \dot{x}_1 = x_2 \]
  \[ \dot{x}_2 = -x_1 + (\cos x_2)u \]

• How can we regulate this system?
  • A solution: choose \( u = \frac{v}{\cos x_2} \) provided \( \cos x_2 \) is nonzero
  • State equations become
    \[ \dot{x}_1 = x_2 \]
    \[ \dot{x}_2 = -x_1 + v \]
    \( v = -k_1 x_1 - k_2 x_2 \)
    \[ \rightarrow u = -\left(\frac{k_1 x_1 + k_2 x_2}{\cos x_2}\right) \]

• This process of using the control to cancel the nonlinearities in the system is known as feedback linearisation
Feedback (input-state) linearisation

- We want to get some idea of the generality of the approach we have used in the previous two examples—surely we cannot use this approach for every nonlinear system?

- We have considered additive and multiplicative nonlinearities. In general, these nonlinearities must affect each state equation in the same way whenever the control appears, i.e., the system is of the form

\[
\dot{x} = Ax + B\gamma(x)(u - \alpha(x))
\]

(A, B) pair are controllable

Invertible in the domain of interest

- The state equation can then be linearised by the following choice of state feedback control

\[
u = \alpha(x) + \frac{\nu}{\gamma(x)}
\]

Standard linear design approaches for \(\nu = -Kx\) can now be applied
A third illustrative example

• How about if we consider a system that is not in the feedback linearisable form we have just encountered? Eg

\[
\begin{align*}
\dot{x}_1 &= a \sin x_2 \\
\dot{x}_2 &= -x_1^2 + u
\end{align*}
\]

• Since we cannot choose a control that cancels out the \( \sin x_2 \) term does this mean that the system is not linearisable by state feedback?

• Not necessarily! Recall there are many different state space representations of the same system, so there may be one for this system that is in the feedback linearisable form we are after...
\[
\begin{align*}
\dot{x}_1 &= a \sin x_2 \\
\dot{x}_2 &= -x_1^2 + u
\end{align*}
\]

• Lets change the variables by the following transformations

\[
\begin{align*}
z_1 &= x_1 \\
z_2 &= a \sin x_2 = \dot{x}_1
\end{align*}
\]

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= a \cos x_2 ( -x_1^2 + u )
\end{align*}
\]

• With the system in this form the nonlinearities can be cancelled by the control

\[
u = x_1^2 + \frac{v}{a \cos x_2}
\]

• This control is well defined for \(-\pi/2 < x_2 < \pi/2\)

• Note we could also express the control in terms of the new \((z)\) variables.
Diffeomorphisms

- If the nonlinear transformations, \( z = T(x) \), which transform \( x \rightarrow z \) coordinates satisfies the following two conditions
  - \( T(.) \) is invertible, i.e. \( x = T^{-1}(z) \) for all \( z \) in the domain of interest
  - The derivatives of \( T(.) \) and \( T^{-1}(.) \) are continuously differentiable, (since we will be requiring continuous derivatives of \( x \) and \( z \) in the state space representations)

then it is known as a **diffeomorphism**

- **Definition**: A nonlinear system
  \[
  \dot{x} = f(x) + G(x)u
  \]

is **feedback linearisable** if there exists a diffeomorphism, \( T \), whose domain contains the origin \((z=0)\) and the change of variables \( z = T(x) \) transforms the system into the form
\[
\dot{x} = Ax + B\gamma(x)(u - \alpha(x))
\]
with \((A,B)\) controllable and \( \gamma(x) \) nonsingular for all \( x \) in the domain of interest.
Input-output linearisation example

- Consider the system
  \[ \dot{x}_1 = a \sin x_2, \quad \dot{x}_2 = -x_1^2 + u, \quad y = x_2 \]

- Using our input-state linearising control after transformation to \( z \) variables as before
  \[ u = x_1^2 + \frac{v}{a \cos x_2} \quad \Rightarrow \quad \dot{z}_1 = z_2, \quad \dot{z}_2 = v, \quad y = \sin^{-1}\left(\frac{z_2}{a}\right) \]

- In this case we may be better choosing \( u = x_1^2 + v \) and not worrying about linearising the state variable \( x_1 \).
  \[ u = x_1^2 + v \quad \Rightarrow \quad \dot{x}_1 = a \sin x_2, \quad \dot{x}_2 = v, \quad y = x_2 \]

- This would allow us to use linear control theory to solve a tracking problem for \( y \).
  - However, this choice is somewhat naive since \( x_1 \) is uncontrollable and may grow unboundedly (consider \( y \to \) constant, \( r \)).
Unanswered questions so far…

• So what should we do when there is a nonlinear output equation, eg. $y = h(x)$?
• And how do we ensure the states don’t explode?
• To investigate these types of systems (e.g. for tracking) as well as to look at what happens when we apply a transformation to the normal feedback linearisable form we need to introduce a few new ideas
  • Lie derivatives
  • Relative degree
  • Zero dynamics (for previous example)
  • Minimum phase

• We will only be considering SISO systems…
Lie derivatives

• Consider the nonlinear SISO system

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]

Note: \(f(x)\) and \(g(x)\) are vectors

• Taking the derivative of the output equation

\[
\frac{dh}{dx} \dot{x} = \frac{dh}{dx} (f(x) + g(x)u) \triangleq L_f h(x) + L_g h(x)u
\]

• The terms \(L_f h(x) \triangleq \frac{dh}{dx} f(x)\) and \(L_g h(x) \triangleq \frac{dh}{dx} g(x)\) are known as the Lie Derivatives. \(L_f h\) represents the derivatives of \(h\) along the system trajectories \(dx/dt = f(x)\).

• They can be concatenated, ie

\[
L_f L_f h(x) = \frac{d(L_f h)}{dx} f(x)
\]

\[
L^k_f h(x) = \frac{d(L^{k-1}_f h)}{dx} f(x)
\]

\[
L_f L_g h(x) = \frac{d(L_g h)}{dx} f(x)
\]
Relative degree

• The relative degree of a system indicates how many integrators exists between the control and the output

• The relative degree of a linear system is easy to establish from the transfer function

\[
\frac{Y(s)}{U(s)} = \frac{b_0 s^m + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}
\]

\[\text{Relative degree} = n - m\]

• In a nonlinear system, we use the Lie derivatives to establish a definition of relative degree

• Formal definition: A nonlinear system is said to have relative degree \(p\) in a region of interest, \(D\), when

\[L_g L_f^i h(x) = 0, \quad i = 1, \ldots, p - 1; \quad L_g L_f^{p-1} h(x) \neq 0 \quad \forall x \in D\]

• Informally: How many times do we have differentiate \(y = h(x)\) before the input appears (ie how many integrators between input and output)?

• Note that this holds for linear systems also…
Relative degree example

• Consider the controlled van der Pol equation
  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2 + u \\
  y &= x_1
  \end{align*}
  \]

• Calculating the derivatives of the output we get
  \[
  \begin{align*}
  \dot{y} &= \dot{x}_1 = x_2 \\
  \ddot{y} &= \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u
  \end{align*}
  \]

• Hence the system has relative degree 2.

• However if we take the output as \( y = x_1 + x_2^2 \):
  \[
  \begin{align*}
  \dot{y} &= \dot{x}_1 + 2x_2 \dot{x}_2 \\
  &= x_2 + 2x_2 \left(-x_1 + \varepsilon(1 - x_1^2)x_2 + u\right)
  \end{align*}
  \]

• Hence the system has relative degree 1 in the real domain excluding \( x_2 = 0 \), ie \( D = \{ x \in \mathbb{R} \mid x_2 \neq 0 \} \)
Lie derivatives and relative degree for input-output linearisation

- Suppose the following $n$th order system has relative degree $p$.

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x)$$

- We know that we can keep calculating derivatives of $y$ that are independent of $u$ up to $p$ times, i.e.

$$\dot{y} = \frac{dh}{dx} \dot{x} = \frac{dh}{dx} (f(x) + g(x)u) = L_f h(x) + L_g h(x)u = L_f h(x)$$

$$\ddot{y} = L_f^2 h(x) + L_g L_f h(x)u = L_f^2 h(x)$$

$$y^{(p)} = L_f^p h(x) + L_g L_f^{p-1} h(x)u$$

- Hence the choice of control $u = \frac{1}{L_g L_f^{p-1} h(x)} (-L_f^p h(x) + v)$ will result in the linearised input-output equation $y^{(p)} = v$. 
• We have the linearised system $y^{(p)}=v$.
• We can use the state vector $\xi = [y, y', y'', ..., y^{(p-1)}]$ and represent the linearised system in the following state space form

$$
\dot{\xi} = 
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
0 \\
\vdots \\
0 \\
1 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
v
\end{bmatrix},

y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}\xi
$$

• If the relative degree of the nonlinear system is $< n$, then the remaining internal nonlinear dynamics (i.e., those not encompassed in the linear equation above) have a state equation of the form

$$
\dot{\eta} = f_0(\eta, \xi) - \frac{L_p h(x)}{L_g L_f^{p-1} h(x)}
$$

• So the entire system is
Transformation to normal form

• If the relative degree of our \( n^{th} \) order system is \( n \) then the mapping to the normal form is given by:

\[
\begin{bmatrix}
\phi_1(x) \\
\vdots \\
\phi_{n-p}(x) \\
h(x) \\
\vdots \\
L_f^{n-1} h(x)
\end{bmatrix} \triangleq \begin{bmatrix} \eta \\ \xi \end{bmatrix}
\]

The \( \phi_i \)'s are choices making \( T \) a diffeomorphism and satisfying:

\[
L_g \phi_i = \frac{d \phi_i}{dx} g(x) = 0
\]

This ensures the \( u \) term cancels out from

\[
\dot{\eta} = \frac{d \phi}{dx} (f + gu)
\]

I.e. \( \eta \) is independent of \( u \)!

• So, given many initial state space representations we can transform into a feedback linearisable form!
Example – from Khalil p542

For the following system find

i) the control \( u \) required for linearised input-output dynamics

ii) the resulting system state matrix, \( z \), for normal form

\[
\begin{align*}
\dot{x}_1 &= -x_1 + \frac{2 + x_3^2}{1 + x_3^2} u \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_1 x_3 + u \\
y &= x_2
\end{align*}
\]

\[
f(x) = \begin{bmatrix} -x_1 \\ x_3 \\ x_1 x_3 \end{bmatrix} \quad g(x) = \begin{bmatrix} \frac{(2 + x_3^2)}{(1 + x_3^2)} \\ 0 \\ 1 \end{bmatrix}
\]

Solution: Start by finding the relative degree

\[
\begin{align*}
\dot{y} &= \dot{x}_2 = x_3 \\
\ddot{y} &= \dot{x}_3 = x_1 x_3 + u
\end{align*}
\]

Control appears after 2 derivatives of output, so relative degree = 2

Now find the Lie derivatives required for input-output linearisation

\[
L_f h(x) = \frac{dh}{dx} f' = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_3 \\ x_1 x_3 \end{bmatrix} = x_3
\]
Example - from Khalil p542

\[ f(x) = \begin{bmatrix} -x_1 \\ x_3 \\ x_1x_3 \end{bmatrix} \quad g(x) = \begin{bmatrix} (2 + x_3^2)/(1 + x_3^2) \\ 0 \\ 1 \end{bmatrix} \quad h(x) = x_2 \quad L_f h = x_3 \]

The remaining Lie derivatives required are

\[ L_f^2 h = \frac{d(L_f h)}{dx} f = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_3 \\ x_1x_3 \end{bmatrix} = x_1x_3 \]

\[ L_g L_f h = \frac{d(L_f h)}{dx} g = 1 \]

So the linearising control is given by

\[ u = \frac{1}{L_g L_f^{-1} h(x)} \left( -L_f^p h(x) + v \right) = -x_1x_3 + v \]

And the transformed system is

\[ z = \begin{bmatrix} \eta \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \phi(x) \\ h \\ L_f h \end{bmatrix} = \begin{bmatrix} \phi(x) \\ x_2 \\ x_3 \end{bmatrix} \]

where

\[ \frac{d\phi}{dx} \frac{g}{dx_1} = \frac{d\phi}{dx_1} \frac{2 + x_3^2}{1 + x_3^2} + \frac{d\phi}{dx_3} = 0 \]

\[ \Rightarrow \phi = -x_1 + x_3 + \arctan x_3 \]

Via separation of variables using \( \phi(0)=0 \)

(may be outside the scope of this course...)

18
Zero dynamics and minimum phase systems

So far

- We have designed an input-output linearised controller.
- However our controller design may render some of the states unobservable (we have called these $\eta$) if the relative degree of the system is less than the system order.
- For good behaviour of the entire system, we want the unobserved states to converge at equilibrium points $(0,0)$.
- The system behaviour for $\dot{\eta} = f_0(\eta,0)$ is called the zero dynamics of the system.
- If the origin $(\eta=0, \xi=0)$ is an equilibrium point of the nonlinear system, then the system is said to be minimum phase if the zero dynamics are asymptotically stable.
- Clearly this is a desirable property for the controlled system!
- Note: A system with no zero dynamics is said by default to be minimum phase.
Example

1. Consider the van der Pol equation:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2 + u \\
y &= x_2
\end{align*}
\]

- This system is already in normal form \((\eta = x_1, \xi = y)\) so no need to transform it.
- Looking at the zero dynamics of the system we get

\[
\dot{x}_1 = 0
\]

- Clearly this does not have an asymptotically stable equilibrium point so the system is not minimum phase!
Tracking

- Consider a system in the normal form, ie
  \[ \dot{\eta} = f_0(\eta, \xi) \]
  \[ \dot{\xi} = A_c \xi + B_c \gamma(x) \left[ u - \alpha(x) \right] \]
  \[ y = C_c \xi \]

- Suppose we now want to design a controller so that the output, \( y \), tracks a reference signal, \( r(t) \).

- We will make some assumptions about the reference
  - \( r(t) \) and its derivatives up to \( r^p \) are available online
  - \( r(t) \) and its derivatives up to \( r^p \) are bounded and piecewise continuous

- Define the following vectors

  \[
  R = \begin{bmatrix}
  r \\
  \vdots \\
  r^{(p-1)}
  \end{bmatrix}
  \]

  \[
  e = \begin{bmatrix}
  \xi_1 - r \\
  \vdots \\
  \xi_p - r^{(p-1)}
  \end{bmatrix} = \xi - R
  \]
\[
e = \begin{bmatrix}
    \xi_1 - r \\
    \vdots \\
    \xi_p - r^{(p-1)}
\end{bmatrix} = \xi - R \quad \quad \quad \quad \quad R = \begin{bmatrix}
    r \\
    \vdots \\
    r^{(p-1)}
\end{bmatrix}
\]

• Now since \( \dot{\xi} = e + R \) we can rewrite the normal form as
  \[
  \dot{\eta} = f_0(\eta, e + R) \\
  \dot{e} + \dot{R} = A_c e + A_c R + B_c \gamma(x)[u - \alpha(x)]
  \]

• But, \( \dot{R} - A_c R = \begin{bmatrix} 0 & \cdots & 0 & r^{(p)} \end{bmatrix}^T = B_c r^{(p)} \)

• So the normal form of the equations becomes:
  \[
  \dot{\eta} = f_0(\eta, e + R) \\
  \dot{e} = A_c e + B_c \left( \gamma(x)[u - \alpha(x)] - r^{(p)} \right)
  \]

• Selecting \( u = \alpha(x) + \frac{1}{\gamma(x)}(v + r^{(p)}) \) and \( \dot{\eta} = f_0(\eta, e + R) \),
  \( \dot{e} = A_c e + B_c v \)

• We can now use linear techniques to design a controller \( \nu = -Ke \)
  to stabilise the second equation, and for minimum phase systems the first equation will be asymptotically stable.
Tracking example

- Let's consider the pendulum equation again, and suppose we want to track a reference, $r(t)$,

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a \sin x_1 - bx_2 + cu \\
y &= x_1
\end{align*}
$$

- The system has relative degree 2, and is already in the normal form. It is also minimum phase by default. To track $r(t)$, we set $e_1 = x_1 - r$, $e_2 = x_2 - r'$ to get

$$
\begin{align*}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= -a \sin x_1 - bx_2 + cu - \ddot{r}
\end{align*}
$$

- The state feedback control is then given by

$$
u = \frac{1}{c} \left( a \sin x_1 + bx_2 + \ddot{r} - k_1 e_1 - k_2 e_2 \right)$$
• Suppose the pendulum parameters are given by $a=c=1$, $b=2$ and we have the following reference

$$r = \cos t, \quad \dot{r} = -\sin t, \quad \ddot{r} = -\cos t$$

• The linearised system equations are given by

$$\dot{e} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

• Designing the feedback gains to have poles at $s=-9, -10$:

```
>> A = [0 1; 0 0]; B = [0 1]';  dclp = [-9 -10];
>> K = acker(A,B,dclp)
K = 90.0000   19.0000
```

Hence the control is

$$u = (\sin x_1 + 2x_2 - \cos t - 90e_1 - 19e_2)$$

Control signal quite large initially so may want to reduce demands on linear control ie use slower poles and smaller K values!
Returning to the robot arm…

• We had the equations

\[ D(q)\ddot{q} + C(q, \dot{q}) + G(q) + J(q)\, f = \tau \in \mathbb{R}^n \]
\[ y = h(q) \in \mathbb{R}^6 \]

• It can be shown using a few of the ideas we have looked at that an input-output feedback linearising controller for this system is:

\[ \tau = D(q)J(q)^\# \left( v - J(q)\dot{q} \right) + C(q, \dot{q}) + G(q) + J(q)\, f \]

where \( J(q)^\# \) is the pseudo-inverse of \( J(q) \)

i.e. \( J(q)^\# = J(q)^{-1} \) for \( n = 6 \) since \( J(q) \) is square.

This results in the output equation

\[ \ddot{y} = v \]
Summary

• We have looked at using the control to linearise a nonlinear system, so that standard linear design tools can then be used
• We considered what sort of systems this approach can be applied to
  - Ie normal (or feedback linearisable) form
• We looked at how to transform a system into this form
  - This was where the Lie derivatives and relative degree of the system come into play
• We considered the concepts of zero phase dynamics and what it means to have a minimum phase nonlinear system
• Finally we looked at how to set up tracking using feedback linearisation