Bayesian Approaches to Track Existence - IPDA and Random Sets

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Abstract – *Most target tracking algorithms implicitly assume that target exists. There are only a few techniques that address the target existence problem along with target tracking. For example, (Integrated Probabilistic Data Association) IPDA filter addresses the target tracking and target existence problems simultaneously and it does so under at most one target assumption. In recent times random sets have been proposed as a general framework for multiple target tracking problem. However, its relationship to well understood existing tracking algorithms like IPDA has not been explored. In this paper, we show that under appropriate conditions random sets provide appropriate mathematical framework for solving the joint target existence and state estimation problem and subsequently show that it results in IPDA under appropriate simplifying assumptions.*

Keywords: Tracking, filtering, estimation, random sets, IPDA, IMMPDA, Target Existence

1 Introduction

Single Target tracking is usually treated as an estimation problem with an implicit assumption that the target exists [1]. A number of Bayesian approaches to multiple target tracking have been proposed that either assume the knowledge of the number of targets is known (e.g., JPDA), or is bounded by a large but known number N [2, 3]. Techniques like MHT [4] are not strictly Bayesian, however, their linkages to Bayesian multi-target tracking has been explored recently in [2] and [5].

In recent times random sets are proposed as a mechanism of developing methods for multi-target tracking by treating the number of targets as a random variable along with the respective target states [6, 5]. Approximations to the random set based solutions in the form of first and second order moments of the multi-target densities have also been proposed [7]. However, realization of random set based multi-target tracking filters has proven to be extremely difficult.

One of the first techniques that addresses the issue of target existence along with the estimation of target states has been proposed almost a decade ago by Musicki and Evans [8]. In the terminology of random sets, IPDA provides an approximate solution to a tracking problem that considers a finite random set Γ of target states whose only instantiations are $\Gamma = \emptyset$ (i.e., no targets are present) or $\Gamma = \{x\}$ (i.e., a single target is present and has state x) where x is the random variable representing the state of the target. The same problem can also be solved in the framework proposed by Stone [2] and Kastella [3] by assuming the maximum number of targets present is large but fixed N. In this paper we focus on establishing fundamental linkages between random set formalism and the well known IPDA algorithm. We derive the IPDA algorithm using the finite set statistics in the framework of random sets and provide insights into the approximations of IPDA and their significance in addressing the target existence issue.

The paper is organized as follows. First we consider the random set formalism in section 2 and obtain the random set model for target existence and its motion along with the associated transition density in section 3.2. We then obtain the random set model for sensor measurements involving clutter and obtain the likelihood function in section 3.3. We then consider the Bayes update equation for obtaining posterior density of the random set representing the target existence and motion model in section 3.4. By imposing simplifying assumptions, we go on to derive the IPDA algorithm in section 4. Some insights are presented in section 5 and conclusions are drawn in section 6

2 Random Set Formalism

Application of random set formalism to multiple object target tracking requires the tools of Finite Set Statistics (FISST). FISST generalizes point estimation theory to finite set estimation [6]. FISST deals with the difficulties associated with multi-source multi-object tracking problems by directly extending the single sensor, single target statistical calculus. The key to using FISST is the concept of belief-mass. Belief mass functions are non-additive generalizations of probability-mass functions and are equivalent to probability mass functions on certain abstract topological spaces. In this paper, we use X_k and Y_k for realizations of random sets Γ_k and Σ_k respectively.

The basic approach is as follows:

- Model the motion of the multi-target systems using a randomly varying finite random set

  $$\Gamma_k = \Phi_k(X_{k-1}, V_{k-1}) \cup B_k(X_{k-1})$$

  where $\Phi_k(X_k - 1, V_k - 1)$ deals with the dynamics of the existing targets (their persistence and death) while $B_k(\cdot)$ deals with the process of target birth.

- Model the multi-sensor multi-target measurement process as a randomly varying finite random set

  $$\Sigma_k = \Sigma'_k \cup \Lambda_k$$

  where Σ'_k models measurements due to targets and $\Lambda_k = \Lambda_k(1) \cup \cdots \cup \Lambda_k(M)$, where each $\Lambda_k(j)$ models an individual clutter source, models measurements due to clutter.

Once the target and measurement models are constructed, then the associated belief masses are obtained as follows:

- The statistics of the finitely varying random state-set Γ_k is described by its belief-mass function

  $$\beta_{\Gamma_{k|k-1}}(S \mid X_{k-1}) = \Pr(\Gamma_k \subseteq S) \tag{1}$$

  This is the total probability of finding all targets in region S at time k, if at time-step $k-1$, they had a multi-target state X_{k-1} . If there is at most one target, then it is the total probability of finding one target, if present, in any region S.

- The statistics of the finitely varying random state-set Σ_k is described by its belief-mass function

  $$\beta_{\Sigma_k}(S \mid X_k) = \beta_{\Sigma'_k \cup \Lambda_k}(S \mid X_k) = P(\Sigma'_k \cup \Lambda_k \subseteq S) \tag{2}$$

  This is is the total probability that all observations in a sensor (or multi-sensor) scan will be found in any given region S, if the target has state X.

By differentiating these belief-mass functions (Generalized Radon-Nikodym theorem), one can obtain, Multi-target Markov densities and Multi-target measurement likelihoods [9]. As belief-mass functions are functions of sets, the derivatives are set derivatives. Using FISST one can obtain the derivatives of belief-mass functions and hence the multi-target Markov densities and multi-target likelihoods as follows:

- The multi-target Markov density is a set derivative of the belief mass function $\beta_{\Gamma_{k|k-1}}(S \mid X_{k-1})$ and can be represented as follows:

  $$f_{k|k-1}(X_k \mid X_{k-1}) = \frac{\delta\beta_{\Gamma_{k|k-1}}(S \mid X_{k-1})}{\delta X_k} \tag{3}$$

- The multi-target measurement likelihood is a set derivative of the belief-mass function $\beta_{\Sigma_k}(S \mid X_k)$ of the corresponding sensor model and is given by

  $$f_{\Sigma_k}(Y_k \mid X_k) = \frac{\delta\beta_{\Sigma_k}(S \mid X_k)}{\delta Y_k} \tag{4}$$

These multi-target Markov transition densities and likelihood functions can then be used in the standard Bayesian nonlinear filtering equations to obtain a recursive method for multi-target posterior density. The general form of the Bayes' recursion is the same as in the case of the single sensor, single target tracking problem and is given by

$$f_{k|k}(X_k|Y^k) = \frac{1}{\Delta} f_{\Sigma_k}(Y_k|X_k) \times \int f_{k|k-1}(X_k|X_{k-1}) f_{k-1|k-1}(X_{k-1}|Y^{k-1}) \delta X_{k-1}$$

We will apply this formalism to solve a simple target tracking problem with target existence uncertainty.

3 Bayesian Filter for Target Tracking with Target Existence Uncertainty

3.1 Problem Description

Here we consider the single target tracking problem with target existence uncertainty. In other words, we wish to consider the problem where, if no targets are present in the scene, then this will continue to be the case. If however, there is one target in the scene then either this target will persist (with probability p_v) or it will vanish (with probability $1 - p_v$). Specifically, we wish to determine if a target exists and if does, its state. This problem, **with at most one target assumption**, was first addressed in literature by Musicki and Evans [8] in 1994 almost in parallel to the introduction of FISST by Mahler [10]. Musicki and Evans [8], provide an approximate solution to this problem under linear Gaussian assumptions and named their techniques as IPDA. This simple target tracking problem has all the essential elements suitable for applying Random set formalism.

In the terminology of random sets, we consider a finite random set Γ_k of target states whose only instantiations are the single event $X_k = \emptyset$ (i.e., no targets are present) or the infinitely numerous events $X_k = \{x_k\}$ (i.e., a single target is present and has state x_k) where x_k is the random variable representing the state of the target. A sensor collects report originating from the target with a probability of detection P_D and in addition, collects reports from clutter objects that are uniformly distributed in space with probability of detection P_{FA} (i.e., probability of detection of clutter sources). We denote measurement random set as Σ_k and its instantiations as Y_k where $|Y_k| = m_k$, i.e., at time k, the sensor collects m_k measurements or reports. The problem is to determine the best estimate of the target state if it exists, given all the reports from the sensor. To solve with problem, we first formulate the motion and measurement models as illustrated in section 2.

3.2 Random Set Model For Target Dynamics with Target Existence Uncertainty

Random set motion models are usually described using

$$\Gamma_k = \Phi_k(X_{k-1}, V_{k-1}) \cup B_k(X_{k-1}) \tag{5}$$

where $B_k(\cdot)$ deals with the process of target birth. For the problem under consideration $X_{k-1} = \emptyset$ or $X_{k-1} = \{x_{k-1}\}$ (i.e., there can be up to one target in the scene and there is no birth process) and define the instantiations of Γ_k as

$$X_k = \emptyset, \quad X_k = \{x_k\} \tag{6}$$

where :

- $X_k = \emptyset$ with probability $p_{\bar{v}} = 1 - p_v$
- $X_k = \{x_k\}$ with probability p_v

Since there is no birth process, the random set motion model could be reduced to

$$\Gamma_k = \Phi_k(X_{k-1}, V_{k-1}) \tag{7}$$

without the birth process. This model encapsulates a random set evolution, where, if no targets are present in the scene then this will continue to be the case. If however, there is one target in the scene then either this target will persist (with probability p_v) or it will vanish (with probability $1 - p_v$). The dynamics described by the random set $\Phi_{k-1}(X_{k-1})$ also covers target existence (persistence) as $\Phi_{k-1}(X_{k-1})$ can take on the empty set as a value. The statistics of the finitely varying random state-set Γ_k is described by its belief-mass function

$$\beta_{\Gamma_{k|k-1}}(S \mid X_{k-1}) = \Pr(\Gamma_k \subseteq S) \tag{8}$$

This is the total probability of finding all targets in region S at time k. If there is at most one target, then it is the total probability of finding the target, if present, in any region S.

If $X_{k-1} = \emptyset$, Γ_k can have only one (set) value, i.e., $\emptyset$ (from the assumption that if the target does not exist at time $k-1$, then it will continue to be the case), the belief measure is given by

$$\beta_{\Gamma_{k|k-1}}(S \mid \emptyset) = \Pr(\Gamma_k = \emptyset \subseteq S) = 1. \tag{9}$$

Writing this in terms of the density function, we have

$$f_{k|k-1}(\emptyset|\emptyset) = \frac{\delta\beta_{\Gamma_{k|k-1}}}{\delta\{x_{k-1}\}}(\emptyset) = 1 \tag{10}$$

On the other hand, if $X_{k-1} = \{x_{k-1}\}$, then Γ_k can take on two (set) values, $\{x_k\}$ or $\emptyset$ with probabilities p_v and $1 - p_v$ respectively. In such a case, the belief measure can be calculated as follows:

$$\begin{aligned}
\beta_{\Gamma_{k|k-1}}(S \mid \{x_{k-1}\}) = \\
& \Pr(\Gamma_k = \emptyset \subseteq S \mid \{x_{k-1}\}) \\
& + \Pr(\Gamma_k = \{x_k\} \subseteq S \mid \{x_{k-1}\}) \\
= \; & 1 - p_v + p_v \int p(x_k|x_{k-1})dx_k
\end{aligned}$$

From equation (10) and (11) the complete Markov transition density for at most one target scenario can be written as

$$\begin{aligned}
f(\emptyset \mid \emptyset) &= 1 & (11)\\
f(\emptyset \mid \{x_{k-1}\}) &= 1 - p_v & (12)\\
f(\{x_k\} \mid \{x_{k-1}\}) &= p_v p(x_k|x_{k-1}) & (13)
\end{aligned}$$

3.3 Random Set Model for Sensor Measurements in Clutter

Let Σ_k be the random set which models the observation set of a sensor which is a union of the random set corresponding to the target Σ'_k and the random set corresponding to clutter Λ_k. First, let us consider the case where there is no clutter.

If there are no clutter objects and if the target exists, the possible realizations for Σ'_k, are

- $\Sigma'_k = \{y_k\}$ where y_k is the measurement originating from the target and takes on any value $y_k \in \mathcal{R}$

- $\Sigma'_k = \emptyset$

due to the fact that the sensor has nonzero probability of detection. Due to the random detection process, some times, even when the target exists, sensor does not detect it. The statistics of this random set are described by the belief measure $\beta_{\Sigma'_k}(S \mid X) = P(\Sigma'_k \subseteq S)$. The belief mass is the total probability that all observations in a sensor (or multi-sensor) scan will be found in any given region S, if the target has state $X_k = \{x_k\}$.

$$\begin{aligned} & \beta_{\Sigma'_k}(S|X_k) \\ = \; & \Pr(\Sigma'_k \subseteq S|\{x_k\}) \\ = \; & \Pr(\Sigma'_k = \emptyset|\{x_k\}) + \Pr(\Sigma'_k \neq \emptyset, \Sigma'_k \in S|\{x_k\}) \end{aligned}$$

Assuming that the detection process and the measurement process are independent and the probability of detection P_D, we have

$$\begin{aligned} \beta_{\Sigma_k}(S|X_k) &= \Pr(\Sigma'_k = \emptyset|\{x_k\}) \\ &\quad + \Pr(\Sigma'_k \neq \emptyset|\{x_k\}) \Pr(\Sigma'_k \in S\{x_k\}) \\ &= (1 - P_D) + (P_D p_{\Sigma'_k}(S \mid \{x_k\})) \\ &= 1 - P_D + P_D p_{\Sigma'_k}(S \mid \{x_k\}) \end{aligned} \quad (14)$$

Let us now take clutter and false alarms into account. The typical observation set Σ will have the general form

$$\Sigma_k = \Sigma'_k \cup \Lambda_k \quad (15)$$

Here Σ'_k is the random set of reports due to the target. It contains either one report or is the empty set as described earlier. The subset Λ_k of observations, on the other hand, models the clutter process. We assume that clutter observations are generated by "clutter objects" which can be modelled in the same way as the targets of actual interest, but whose noise statistics may differ from those of actual target objects.

For a clutter object, a zero probability of detection means that the clutter object is never observed by the sensor. Likewise, a unity probability of detection for a clutter object means that the clutter object always generates a spurious observation. Thus for a clutter object, the probability of detection is P_{FA} (also known as the probability of false alarm). Hence the belief measure of the random set of observations generated by a clutter object is similar to the single target with missed detections,

$$\beta(S) \;=\; 1 - P_{FA} + P_{FA} p_c(S)$$

for some probability measure p_c with spacial density c. It follows that the typical sensor observation-set Σ_k will have the form $\Sigma_k = \Sigma'_k \cup \Lambda_k$ where Σ'_k models the target and $\Lambda_k = \Lambda_k(1) \cup \cdots \cup \Lambda_k(M)$, where each $\Lambda_k(j)$ models an individual clutter source at time k. Assuming that each $\Lambda_k(j)$ has same probability measure p_C and probability of false alarm P_{FA} and that the target reports are independent of clutter, we have,

$$\begin{aligned} & \beta_{\Sigma_k}(S \mid X_k) \\ &= P(\Sigma'_k, \Lambda_k \subseteq S) \\ &= P(\Sigma'_k \subseteq S) P(\Lambda_k \subseteq S) \\ &= P(\Sigma'_k \subseteq S) P(\Lambda_k(1) \subseteq S, \cdots \Lambda_k(M) \subseteq S) \\ &= P(\Sigma'_k \subseteq S) P(\Lambda_k(1) \subseteq S) \cdots P(\Lambda_k(M) \subseteq S) \\ &= \beta_{\Sigma'}(S \mid X_k) \beta_c(S)^M \end{aligned}$$

The derivative of this function will then give us the global likelihood $f_{\Sigma_k}(Y_k \mid X_k)$. However, the belief function is composed of a product of two belief functions. Using the product rule of set derivatives, it can be shown that

$$\begin{aligned} f_{\Sigma_k}(Y_k \mid X_k) &= f_{\Sigma'_k \cup \Lambda_k}(Y_k \mid X_k) \\ &= \sum_{Z_k \subseteq Y_k} f_{\Sigma'_k}(Z_k \mid X_k) f_{\Lambda_k}(Y_k - Z_k) \end{aligned} \quad (16)$$

The global density of the clutter process is

$$f_{\Lambda_k}\{\xi_1, \ldots, \xi_n\} = n! C_{M,n} P_{FA}^n (1 - P_{FA}^{M-n}) c(\xi_1) \cdots c(\xi_n) \quad (17)$$

where $C_{M,n}$ gives the no. of combinations of n out of M. If the clutter is assumed to be distributed uniformly in the surveillance volume V, then $c(\xi_i) = \frac{1}{V}$. Then the global density reduces to

$$f_\Lambda(\{\xi_1, \ldots, \xi_n\}) = n! \left(\frac{1}{V}\right)^n C_{M,n} P_{FA}^n (1 - P_{FA}^{M-n}) \quad (18)$$

If M is large and P_{FA} is small, then Λ may be approximated by a Poission process. In that case, the global density can be further simplified to

$$f_\Lambda(\xi_1, \ldots, \xi_n) = n! \frac{1}{V^n} \frac{\lambda^n e^{-\lambda}}{n!} \quad (19)$$

If there are m_k measurements from clutter, and denoting $f_\Lambda(\xi_1, \ldots, \xi_{m_k})$ by $p_c(m_k)$, the clutter density becomes

$$p_c(m_k) = m_k! \frac{1}{V^{m_k}} \frac{\lambda^{m_k} e^{-\lambda}}{m_k!} \quad (20)$$

This simplifies to

$$p_c(m_k) = \frac{1}{V^{m_k}} \lambda^{m_k} e^{-\lambda} \quad (21)$$

Simple algebraic manipulation of the above equation establishes the following result

$$p_c(m_k - 1) = p_c(m_k) \frac{V}{\lambda} \quad (22)$$

Now the likelihood function from the target measurement process is given by

$$\begin{aligned}\beta_{\Sigma'_k}(S|X_k) &= \Pr(\Sigma'_k \subseteq S) &(23)\\ &= \Pr(\Sigma'_k = \emptyset) + \Pr(\Sigma'_k \neq \emptyset, \Sigma'_k \in S)\\ &= 1 - P_D + P_D p_{\Sigma'_k}(S \mid \{x_k\})\\ &= f(\emptyset \mid X_k) + \int_S f(\{y_k\} \mid X_k)\end{aligned}$$

Which implies that the likelihood function is given by

$$f(\emptyset \mid X_k) = 1 - P_D, \quad (24)$$

$$\begin{aligned}f(\{y_k\} \mid X_k) &= \frac{\partial}{\partial\{y_k\}} P_D p_{\Sigma'}(S \mid \{x_k\})\\ &= P_D p(y_k|x_k) \quad (25)\end{aligned}$$

The overall likelihood function when the target exists, i.e., when $X_k = \{x_k\}$, is obtained by substituting (21), (24), and (25) in (16) and is given by

$$\begin{aligned}& f_\Sigma(Y_k \mid X_k)\\ =\ & f_{\Sigma'}(\emptyset \mid X_k) f_{\Lambda_k}(y_k(1), \ldots y_k(m_k))\\ & + \sum_i f_{\Sigma'}(y_k(i) \mid X_k) f_{\Lambda_k}(Y_k - \{y_k(i)\})\\ =\ & p_c(m_k)(1 - P_D) + p_c(m_{k-1}) \left(P_D \sum_i^{m_k} p(y_k(i)|x) \right)\\ =\ & p_c(m_k) \left(1 - P_D + \frac{P_D V}{\lambda} \sum_i^{m_k} p(y_(i)|x) \right)\end{aligned}$$

We used the result in (22) to obtain the last step. When the target does not exist, i.e., when $X_k \doteq \{\emptyset\}$, the the typical observation set Σ will have the general form

$$\Sigma_k = \Sigma'_k \cup \Lambda_k \quad (26)$$

However, unlike the case where the target exists, Σ'_k takes only one value, i.e., $\Sigma'_k = \{\emptyset\}$. Then it is easy to verify that the likelihood will have no contribution from the target and it is equal to

$$\begin{aligned}f_{\Sigma_k}(Y_k \mid \emptyset) &= f_{\Sigma'_k \cup \Lambda_k}(Y_k \mid \emptyset) &(27)\\ &\doteq f_{\Lambda_k}(Y_k \mid \emptyset) = m_k! \frac{1}{V^{m_k}} \frac{\lambda^{m_k} e^{-\lambda}}{m_k!}\\ &= \frac{1}{V^{m_k}} \lambda^{m_k} e^{-\lambda}\\ &= p_c(m_k)\end{aligned}$$

where the cardinality of set $Y_k = m_k$.

3.4 Bayes Update

The general form of the Bayes' recursion is the same as in the case of the single sensor, single target tracking problem and is given by

$$\begin{aligned}& f_{k|k}(X_k|Y^k) =\\ & \frac{1}{\Delta} f_\Sigma(Y_k|X_k) \times\\ & \int f_{k|k-1}(X_k|X_{k-1}) f_{k-1|k-1}(X_{k-1}|Y^{k-1}) \delta X_{k-1}\\ =\ & \frac{1}{\Delta} f_\Sigma(Y_k|X_k) \{ f_{k|k-1}(X_{k-1}|\emptyset) f_{k-1|k-1}(\emptyset|Y^{k-1}) +\\ & \int f_{k|k-1}(X_k|\{x_{k-1}\}) f_{k-1|k-1}(\{x_{k-1}\}|Y^{k-1}) dX_{k-1} \}\end{aligned} \quad (28)$$

The posterior density $f_{k|k}(X_k \mid Y^k)$ has the form in

$$\begin{aligned}f_{k|k}(\emptyset \mid Y^k) &= \text{posterior probability}\\ &\quad\ \text{that no targets are present}\\ f_{k|k}(\{x_k\} \mid Y^k) &= \text{posterior probability}\\ &\quad\ \text{of one target with state } x_k\end{aligned}$$

$f_{k|k}(X_k \mid Y^k)$, is a probability density in a sense that

$$\begin{aligned}& \int f_{k|k}(X_k \mid Y^k) \delta X\\ & = f_{k|k}(\emptyset \mid Y^k) + \frac{1}{1!} \int f_{k|k}(\{x_k\} \mid Y^k) = 1 \quad (29)\end{aligned}$$

First let us focus on $f_{k|k}(\{x_k\} \mid Y^k) =$

$$\begin{aligned}& \frac{1}{\Delta} f_\Sigma(Y_k|x_k) f_{k|k-1}(\{x_k\}|\emptyset) f_{k-1|k-1}(\emptyset|Y^{k-1})\\ & + \frac{1}{\Delta} f_\Sigma(Y_k|x_k)\\ & \times \int f_{k|k-1}(\{x_k\} \mid \{x_{k-1}\}) f_{k-1|k-1}(\{x_{k-1}\} \mid Y^{k-1})\\ =\ & \frac{p_c(m_k)}{\Delta} \left(1 - P_D + \frac{P_D V}{\lambda} \sum_i p(y_k(i)|x_k) \right)\\ & \left(0 + p_v \int p(x_k \mid x_{k-1}) p(x_{k-1}|Y^{k-1}) dx_{k-1} \right)\\ =\ & \frac{p_c(m_k) p_v}{\Delta} \left(1 - P_D + \frac{P_D V}{\lambda} \sum_i p(y_k(i)|x_k) \right)\\ & \times \int p(x_k \mid x_{k-1}) p(x_{k-1}|Y^{k-1}) dx_{k-1}\end{aligned} \quad (30)$$

It is evident from equation (29) that

$$f_{k|k}(\emptyset \mid Y^k) = 1 - \int f_{k|k}(\{x_k\} \mid Y^k) \quad (31)$$

Thus by deriving posterior for $f_{k|k}(\{x_k\} \mid Y^k)$, we can simultaneously get the posterior for $f_{k|k}(\emptyset \mid Y^k)$. Equations (30) and (31) form the Bayesian filtering equations for the problem of single target tracking with track existence uncertainty. The posterior probability of target existence (persistence) is given by:

$$p_v(\text{posterior}) = \int f_{k|k}(\{x_k\} \mid Y^k) \quad (32)$$

In the next section, we will show that under linear Gaussian assumptions, one can derive IPDA.

4 Derivation of IPDA Filter

Integrated Probabilistic Data Association (IPDA) filter, proposed in [8], is derived based on the work of PDAF [11] by introducing the concept of track existence. However, we develop the same filter from the random set formalism by approximating (30) and (31) and introducing a Markov chain model for the evolution of target persistence. The random set target motion model in (7) encapsulates a model of random set evolution, where, if no targets are present in the scene then this will continue to be the case. If however, there is one target in the scene then either this target will persist (with probability p_v) or it will vanish (with probability $1 - p_v$). These probabilities are defined as as probabilities of target persistence in [9]. These parameters can be directly related to target existence Markov chain used in IPDA [8]. Both existence and persistence model the same thing, i.e., if the target is not present then it will continue to be case and if the target is present then it will continue to be so with finite probability. From this point of view, there is no difference between, the target existence probability defined in [8] and the target persistence probability p_v in [9]. In IPDA, a Markov chain model is used to evolve the target existence (persistence) probability between sensor measurement times. In the original work of IPDA [8], two separate Markov chains were proposed - Markov chain 1 and Markov chain 2. Markov chain 2 deals with observability issue along with the target existence (persistence). Here, we focus only on the Markov chain 1. Owing to this Markov model, all the equations containing track existence probability term in section (3), must replace the static probability of track existence p_v to a dynamic probability of track existence $p_{k|k-1,v}$.

In the original derivation of IPDA, target existence (persistence) is defined as a random variable taking values $\aleph_k$ and $\overline{\aleph}_k$ modelling target existence (persistence) target non-existence (non-persistence). The Markov chain that defines this transition between these states over time is given by

$$\begin{bmatrix} p(\aleph_k) \\ p(\overline{\aleph}_k) \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} p(\aleph_{k-1}) \\ p(\overline{\aleph}_{k-1}) \end{bmatrix} \tag{33}$$

One important fact to be noted is that, in all the works reported on IPDA, the Markov transition probabilities from non-existence to existence is zero, and non-existence to non-existence is 1. This in effect excludes the event that the target can come into existence from a non-existent state, enforcing the equivalence of existence and persistence. Thus probability of target persistence can be equated to probability of target existence. As these probabilities evolve over time according to a Markov chain, we need to define three aspects of target existence. They are:

- Prior probability of target existence and non-existence, $p_{k-1|k-1,v}$ and $p_{k-1|k-1,\overline{v}}$
- Predicted Probability of target existence, $p_{k|k-1,v}$ and $p_{k-1|k,\overline{v}}$
- Posterior probability of target existence, $p_{k|k,v}$ and $p_{k|k,\overline{v}}$

Recognizing $p_{k|k-1,v} = p(\aleph_k)$ and incorporating this Markov chain into the random set formalism, and invoking linear Gausssian assumptions, we derive the IPDA. The Markov transition density is then given by

$$\begin{bmatrix} p_{k|k-1,v} \\ p_{k|k-1,\overline{v}} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & 0 \\ 1-\gamma_{11} & 1 \end{bmatrix} \begin{bmatrix} p_{k-1|k-1,v} \\ p_{k-1|k-1,\overline{v}} \end{bmatrix} \tag{34}$$

Thus there is one free parameter γ_{11} and the predicted probability of target existence can be obtained as

$$\begin{aligned} p_{k|k-1,v} &= \gamma_{11} p_{k-1|k-1,v} + 0 \times p_{k-1|k-1,\overline{v}} \\ &= \gamma_{11} p_{k-1|k-1,v} \end{aligned} \tag{35}$$

and the predicted probability of target non-existence is

$$\begin{aligned} p_{k|k-1,\overline{v}} &= \gamma_{11} p_{k-1|k-1,v} + 1 \times p_{k-1|k-1,\overline{v}} \\ &= (1-\gamma_{11}) p_{k-1|k-1,v} + p_{k-1|k-1,\overline{v}} \end{aligned} \tag{36}$$

As all practical algorithms use some form of gating and consider only the measurements that fall inside the chosen gate, the probability of detection P_D needs to be modified to include the probability of the target originated measurement to fall within the gate, usually denoted by P_G. The over all probability then becomes $P_D P_G$ and is subsequently used in all the equations in place of P_D.

First, let us consider the simplifications that can be introduced into (37): $f_{k|k}(\{x_k\} \mid Y^k)$

$$\begin{aligned} &= \frac{p_c(m_k) p_{k|k-1,v}}{\Delta} \\ &\times \left(1 - P_D P_G + \frac{P_D P_G V}{\lambda} \sum_i p(y_k(i)|x_k)\right) \\ &\times \int p(x_k \mid x_{k-1}) p(x_{k-1}|Y^{k-1}) dx_{k-1} \end{aligned} \tag{37}$$

We can identify the integral as the Chapman-Kolmogorov integral and it solves to a Gaussian density if $p(x_{k-1}|Y^{k-1})$ is Gaussian and the transition density $p(x_k|x_{k-1})$ is also Gaussian. Under these

Gaussianity assumptions, we note that the solution to Chapman-Kolmogorov equation is equivalent to the Kalman Predictor equation and results in a Gaussian density, usually represented by $\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1})$. If, in addition, the sensor is assumed to obtain target originated measurements that are linear in state of the target and is assumed to have been affected by additive white Gaussian noise, then the measurement likelihood

$$p(y_k(i)|x_k) = N(y_k(i); Hx_k, R_k)$$

is also Gaussian where H is the measurement matrix and R_k is the measurement noise covariance. Under these assumptions, the posterior can be further simplified to: $f_{k|k}(X_k|Y^k) =$

$$\begin{aligned} & \frac{p_c(m_k)p_{k|k-1,v}}{\Delta} \\ \times & \left\{1 - P_D P_G + \frac{P_D P_G V}{\lambda}\sum_{i=1}^{m_k}\mathcal{N}(y_k^i; Hx_k, R_k)\right\} \\ \times & \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \end{aligned} \tag{38}$$

Define, $\Lambda_k^i \triangleq \mathcal{N}(y_k^i; \hat{y}_k, S_k)$

$$= |2\pi S_k|^{-\frac{1}{2}} exp\{[y_k^i - \hat{y}_k]^T S_k^{-1}[y_k^i - \hat{y}_k]\}$$

The posterior can be further simplified to

$$\begin{aligned} & f_{k|k}(X_k|Y^k) \\ = & \frac{p_c(m_k)p_{k|k-1,v}}{\Delta}(1 - P_D P_G)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \\ & + \frac{p_c(m_k)p_{k|k-1,v}}{\Delta}\frac{P_D P_G V}{\lambda}\sum_{i=1}^{m_k}\mathcal{N}(y_k^i; Hx_k, R_k) \\ & \times \mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \end{aligned} \tag{39}$$

The updated density can now be written in a simplified form as

$$\begin{aligned} & f_{k|k}(X_k|Y^k) \\ & = \beta_{k,v}(0)\mathcal{N}(x_k; \hat{x}_{k|k-1}, P_{k|k-1}) \\ & \quad + \sum_{i=1}^{m_k}\beta_{k,v}(i)\mathcal{N}(x_k; \hat{x}_{k|k}^i, P_{k|k}^i)\} \end{aligned} \tag{40}$$

where β_i are usually referred to as the data association probabilities and are given by,

$$\beta_{k,\bar{v}}(0) \triangleq \Delta^{-1}p_c(m_k)p_{k|k-1,\bar{v}} \tag{41}$$

$$\beta_{k,v}(0) \triangleq \Delta^{-1}p_c(m_k)p_{k|k-1,v}(1 - P_D P_G) \tag{42}$$

$$\beta_{k,v}(i) \triangleq \Delta^{-1}p_c(m_k)p_{k|k-1,v}\frac{P_D P_G V}{\lambda}\Lambda_k^i \tag{43}$$

From (29), we have

$$\beta_{k,\bar{v}}(0) + \beta_{k,v}(0) + \sum_{i=1}^{m_k}\beta_{k,v}(i) = 1 \tag{44}$$

From this, the normalization factor Δ can be obtained as

$$\begin{aligned} \Delta = & p_c(m_k)p_{k|k-1,\bar{v}} + p_c(m_k)p_{k|k-1,v}(1 - P_D P_G) \\ & + p_c(m_k)p_{k|k-1,v}\frac{P_D P_G V}{\lambda}\Lambda_k^i \\ = & p_c(m_k)\big(p_{k|k-1,\bar{v}} + p_{k|k-1,v}(1 - P_D P_G) \\ & + p_{k|k-1,v}\frac{P_D P_G V}{\lambda}\Lambda_k^i\big) \end{aligned}$$

Recognizing that $p_{k|k-1,\bar{v}} + p_{k|k-1,v} = 1$, and using $\delta_k = P_D P_G - \sum_{i=1}^{m_k}\frac{P_D P_G V}{\lambda}\Lambda_k^i$ we have

$$\Delta = p_c(m_k)(1 - \delta_k p_{k|k-1,v}) \tag{45}$$

Substituting this in (41), (42) and (43), we have

$$\beta_{k,\bar{v}}(0) \triangleq \frac{p_{k|k-1,\bar{v}}}{1 - \delta_k p_{k|k-1,v}} \tag{46}$$

$$\beta_{k,v}(0) \triangleq \frac{(1 - P_D P_G)p_{k|k-1,v}}{1 - \delta_k p_{k|k-1,v}} \tag{47}$$

$$\beta_{k,v}(i) \triangleq \frac{\frac{P_D P_G V}{\lambda}\Lambda_k^i p_{k|k-1,v}}{1 - \delta_k p_{k|k-1,v}} \tag{48}$$

The posterior probability of target existence is the integral of the left hand side of (40)

$$\int f_{k|k}(\{x_k\}|Y^k)dx_k = \beta_{k,v}(0) + \sum_{i=1}^{m_k}\beta_{k,v}(i)$$

$$\text{Thus,}\quad p_{k|k,v} = \frac{1 - \delta_k}{1 - \delta_k p_{k|k-1,v}}p_{k|k-1,v}$$

5 Discussion

Random sets as a formalism to address the problem of simultaneously determining the number of targets has received significant attention in recent years. However, to the best of our knowledge, there are few implementable tracking algorithms that are derived from such a formalism barring the recent work of Mahler [6] on first and second order moments for probability densities on random sets. As the fully general problem of multi-target tracking using random sets is extremely difficult to solve and to visualize, we tried to apply the formalism on a very simple problem where the random set Γ of states has only the following two kinds of instantiations: $\Gamma = \emptyset$ (no targets present) and $\Gamma = \{x\}$ (one target with state x is present). This, along with a target existence (persistence) Markov model and some simplifying assumptions, led us to IPDA.

IMMPDA is another technique that addresses the problem of target existence in an indirect way. It takes the proof by contradiction approach; first it assumes that the target exists and defines the idea of observability, then it argues, that if the probability of target observability is low then the target does not exist. In contrast, IPDA explicitly models the target existence as a random variable and in so doing, it has, in our opinion, made an attempt, albeit by accident, to capture the defining elements of random sets. In this paper, we have shown that, this is indeed the case and we have derived IPDA completely from random set formalism. IPDA has been shown to out perform IMMPDA [12], hence we considered only IPDA in detail. To the best of our knowledge, this is an original derivation of IPDA using the random set formalism. Based on the results obtained in this paper, it is evident that IPDA's extensions to multi-target scenarios will also be firmly based on random set formalism and demands further research.

6 Conclusions

We have adapted the random set formalism to address the problem of simultaneously addressing target existence and target tracking problems. IPDA is one of the first techniques that has addressed this joint problem without exploring any of its linkages to random sets. We have explored the fundamental connections between them and have shown that IPDA can be derived from the random set formalism in a straight forward manner. We conclude that IPDA is one of the first realized filters with firm footing in random sets which explicitly accounts for some of the defining elements of the random set formalism and that Multi-target extensions of IPDA like Joint IPDA, could be one of the first realizable filters of random sets in a multi-target tracking problem with unknown number of targets.

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