7. STRESS ANALYSIS AND STRESS PATHS

7.1 THE MOHR CIRCLE

The discussions in Chapters 2 and 5 were largely concerned with vertical stresses. A more detailed examination of soil behaviour requires a knowledge of stresses in other directions and two or three dimensional analyses become necessary. For the graphical representation of the state of stress on a soil element a very convenient and widely used method is by means of the Mohr circle. In the treatment that follows the stresses in two dimensions only will be considered.

Fig. 7.1(a) shows the normal stresses $\sigma_y$ and $\sigma_x$ and shear stresses $\tau_{xy}$ acting on an element of soil. The normal stress $\sigma$ and shear stress $\tau$ acting on any plane inclined at $\theta$ to the plane on which $\sigma_y$ acts are shown in Fig. 7.1(b). The stresses $\sigma$ and $\tau$ may be expressed in terms of the angle $\theta$ and the other stresses indicated in Fig. 7.1(b). If $a$, $b$ and $c$ represent the sides of the triangle then, for force equilibrium in the direction of $\sigma$:

$$\sigma a = \sigma_x b \sin \theta + \tau_{xy} b \cos \theta + \sigma_y c \cos \theta + \tau_{xy} c \sin \theta$$

$$\sigma = \sigma_x \sin^2 \theta + \tau_{xy} \sin \theta \cos \theta + \sigma_y \cos^2 \theta + \tau_{xy} \cos \theta \sin \theta$$

$$\sigma = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta + \tau_{xy} \sin 2 \theta$$ \hfill (7.1)

Similarly if forces are resolved in the direction of $\tau$

$$\tau a = \sigma_y c \sin \theta - \tau_{xy} c \cos \theta - \sigma_x b \cos \theta + \tau_{xy} b \sin \theta$$

$$\tau = \sigma_y \sin \theta \cos \theta - \tau_{xy} \cos^2 \theta - \sigma_x \sin \theta \cos \theta + \tau_{xy} \sin^2 \theta$$

$$= \frac{\sigma_y - \sigma_x}{2} \sin 2 \theta - \tau_{xy} \cos 2 \theta$$ \hfill (7.2)

Equation (7.1) can be further expressed as follows

$$\sigma - \frac{(\sigma_x + \sigma_y)}{2} = \frac{(\sigma_y - \sigma_x)}{2} \cos 2 \theta + \tau_{xy} \sin 2 \theta$$

This equation can be combined with equation (7.2) to give
Fig. 7.1 Stress at a Point

Fig. 7.2 The Mohr Circle
This is the equation of a circle with a centre at

$$\sigma = \frac{\sigma_x + \sigma_y}{2}, \quad \tau = 0$$

and a radius of

$$\left(\frac{(\sigma_y - \sigma_x)^2}{2} + \tau_{xy}^2\right)^{1/2}$$

This circle known as the Mohr circle is represented in Fig. 7.2. In this diagram point A represents the stresses on the $\sigma_y$ plane and point B represents the stresses on the $\sigma_x$ plane. The shear stresses are considered as negative if they give a couple in the clockwise direction. In geomechanics usage the normal stresses are positive when compressive.

Point C represents the stresses $\tau$ and $\sigma$ on the $\theta$ plane. The location of point C may be found by rotating a radius by an angle equal to $2\theta$ in an anticlockwise direction from the radius through point A.

Alternatively point C may be found by means of the point $O_P$ known as the “origin of planes” or the “pole”. This point is defined as follows: if any line $O_P X$ is drawn through the origin of planes and intersects the other side of the Mohr circle at point X then point X represents the stresses on the plane parallel to $O_P X$. In other words line $O_P A$ in Fig. 7.2 is parallel to the plane on which the stress $\sigma_y$ acts and line $O_P B$ is parallel to the plane on which the stress $\sigma_x$ acts. To find the point on the circle representing the stresses on the $\theta$ plane, line $O_P C$ is drawn parallel to that plane to yield point C. Both of the constructions just described for the location of point C may be verified by means of equations (7.1) and (7.2).

From Fig. 7.2 the major and minor principal stresses $\sigma_1$ and $\sigma_3$ and the inclinations of the planes on which they act may also be determined. A more detailed treatment of the Mohr circle may be found in most books on the mechanics of solids.

**EXAMPLE**

Major and minor principal stresses of 45kN/m$^2$ and 15kN/m$^2$ respectively act on an element of soil where the principal planes are inclined as illustrated in Fig. 7.3(a).
Determine the inclination of the planes on which the maximum shear stresses act.
(b) Determine the inclination of the planes on which the following condition is satisfied
\[ \tau = \pm \sigma \tan 45^\circ \]

(c) On how many planes are shear stresses having a magnitude of 5kN/m\(^2\) acting?

In Fig. 7.3(b) the Mohr circle has been drawn, A and B representing the major and minor principal stresses respectively. By drawing line A O\(_P\) parallel to the major principal plane the origin of planes O\(_P\) may be located.

(a) The maximum shear stress may be calculated from equation (7.3) or it may simply be read from the Mohr circle.

Clearly \[ \tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} \]

\[ = 15 \text{ kN/m}^2 \]

The points of maximum shear stress are represented by C and D. Therefore the planes on which these stresses act are parallel to lines O\(_P\) C and O\(_P\) D respectively. As shown on the figure these planes are inclined at 45\(^\circ\) to the principal planes. This will always be the case regardless of the inclination of the principal planes.

(b) The lines representing the relationship

\[ \tau = \pm \sigma \tan 45^\circ \]

have been drawn in Fig. 7.3(b). Since the circle touches neither of these lines there are no planes on which the relationship holds.

(c) The points on the circle representing a shear stress of 5kN/m\(^2\) are E, F, G and H so there are four planes on which this shear stress acts. These planes are parallel to the lines O\(_P\) E, O\(_P\) F, O\(_P\) G and O\(_P\) H respectively.

### 7.2 STRESS PATHS

When the stresses acting at a point undergo changes, these changes may be conveniently represented on a plot of shear stress against normal stress. Such a situation is illustrated in Fig.
7.4 in which the lower part of the diagram has been omitted for simplicity. The initial major and minor principal stresses are indicated by $\sigma_{li}$ and $\sigma_{3i}$ respectively. Stress changes of $\Delta \sigma_1$ and $\Delta \sigma_3$ have been imposed to give the following final stresses

$$\sigma_{lf} = \sigma_{li} + \Delta \sigma_1$$
$$\sigma_{3f} = \sigma_{3i} + \Delta \sigma_3$$

The initial and final Mohr circles representing these conditions have been drawn in Fig. 7.4. To provide a simple graphical representation of the stress changes from the initial to the final state use has been made of points at the top of the circles. These points A and B, represent the respective circles and if these points only are plotted the circles could easily be drawn should they be needed. The loci of the tops of the Mohr circles is the stress path. The straight line AB is only one of an infinite number of stress paths which indicates they way in which stresses change between the initial and final states. Two other possible stress paths between points A and B have been drawn. More information than a knowledge of $\Delta \sigma_1$ and $\Delta \sigma_3$ would be needed regarding intermediate stress changes before the correct stress path could be drawn.

When stress paths only are plotted then the axes of the diagram are really particular values of the shear stress $\tau$ and normal stress $\sigma$. These values are commonly referred to as $q$ and $p$ where

$$q = \tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2}$$
(7.4)

$$p = \text{mean normal stress} = \frac{\sigma_1 + \sigma_3}{2}$$
(7.5)

If the maximum shear stress $\tau_{\text{max}}$ is expressed in terms of effective stresses instead of total stresses

$$\frac{\sigma'_1 - \sigma'_3}{2} = \frac{(\sigma_1 - u) - (\sigma_3 - u)}{2}$$

$$= \frac{\sigma_1 - \sigma_3}{2} = q$$
Fig. 7.4 Stress Paths

(a) $\Delta \sigma_1 = 0$
\[\Delta \sigma_3 = \text{negative}\]
\[\Delta \sigma_1 = - \Delta \sigma_3\]
\[\Delta \sigma_2 = 0\]
\[\Delta \sigma_1 = \text{positive}\]

(b) constant stress ratio loading
\[K = \sigma_3 / \sigma_1\]
\[K = K_0\]
\[K < 1\]
\[K = 1\]

Fig. 7.5 Examples of Stress Paths
This demonstrates that \( q \) is the same regardless of whether total stresses or effective stresses are being considered. In other words the shear stress is unaffected by pore pressure (this point was also made in section 2.5) Since \( q \) is equal to the radius of the Mohr circle this means that the total and effective Mohr circles must always have the same size.

If the mean normal stress is expressed in terms of effective stresses

\[
p' = \frac{\sigma_1 + \sigma_3}{2}
\]

\[
= \frac{(\sigma_1 - u) + (\sigma_3 - u)}{2}
\]

\[
= \frac{\sigma_1 + \sigma_3 - 2u}{2}
\]

\[
= p - u
\]

This shows (in agreement with the principle of effective stress) that the difference between the total and effective mean normal stresses is equal to the pore pressure. This means that there is not one stress path to consider but two - a total stress path and an effective stress path (see Lambe and Whitman, 1979). Lambe (1967) and Lambe and Marr (1979) have described the use of the stress path method in solving stress-strain problems in soil mechanics.

Some examples of stress paths are shown in Fig. 7.5. Fig. 7.5(a) shows a number of stress paths that start on the \( p \) axis (\( \sigma_1 = \sigma_3 \)), the stress paths going in different directions depending on the relative changes to \( \sigma_1 \) and \( \sigma_3 \). Fig. 7.5(b) shows stress paths for loading under conditions of constant stress ratio (\( \sigma_3/\sigma_1 \)) from an initial zero state of stress. With this type of loading

\[
(q/p) = \frac{(1 - K)}{(1 + K)} \quad (7.6)
\]

where \( K = \sigma_3/\sigma_1 \)

The line marked \( K = 1 \) corresponds to isotropic compression for which the principal stresses (\( \sigma_1 \) and \( \sigma_3 \)) are maintained equal during the loading. The line marked \( K = K_0 \) corresponds to compression under conditions of no lateral strain, as discussed in Chapter 2.
EXAMPLE

Plot the total and effective stress paths for the following stress changes

<table>
<thead>
<tr>
<th></th>
<th>( \sigma_1 ) kN/m²</th>
<th>( \sigma_3 ) kN/m²</th>
<th>( u ) kN/m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>initial state</td>
<td>80</td>
<td>40</td>
<td>20</td>
</tr>
<tr>
<td>intermediate state</td>
<td>140</td>
<td>60</td>
<td>40</td>
</tr>
<tr>
<td>final state</td>
<td>220</td>
<td>60</td>
<td>60</td>
</tr>
</tbody>
</table>

initial state

\[
\begin{align*}
\pi_i &= \frac{\sigma_1 + \sigma_3}{2} = \frac{80 + 40}{2} = 60 \text{kN/m}^2 \\
\pi'_i &= \pi_i - u_i = 60 - 20 = 40 \text{kN/m}^2 \\
q_i &= \frac{\sigma_1 - \sigma_3}{2} = \frac{80 - 40}{2} = 20 \text{kN/m}^2
\end{align*}
\]

These calculations enable the initial points \((\pi_i, q_i)\) and \((\pi'_i, q_i)\) to be plotted as shown in Fig. 7.6.

intermediate state

\[
\begin{align*}
p &= \frac{140 + 60}{2} = 100 \text{kN/m}^2 \\
p' &= 100 - 40 = 60 \text{kN/m}^2 \\
q &= \frac{140 - 60}{2} = 40 \text{kN/m}^2
\end{align*}
\]

Portion of the total and effective stress paths may now be drawn by joining the initial and intermediate points.

final state

\[
\begin{align*}
\pi_f &= \frac{220 + 60}{2} = 140 \text{kN/m}^2
\end{align*}
\]
Fig. 7.6

Fig. 7.7 Undrained Loading of a Soil
\[ p'_f = 140 - 60 = 80 \text{kN/m}^2 \]
\[ q'_f = \frac{220 - 60}{2} = 80 \text{kN/m}^2 \]

This enables the total and effective stress paths to be completed by joining the intermediate and final points as illustrated in Fig. 7.6.

7.3. PORE PRESSURE PARAMETERS

When a soil sample is sealed in a testing apparatus so that water is prevented from moving into or out of the soil, pore pressures develop in the sample when it is subjected to external stress changes. The application of external stresses under these conditions is referred to as undrained loading, since water is unable to drain from the sample.

If water is allowed to drain from the sample and no pore pressure changes are allowed to develop the application of external stresses is referred to as drained loading. It is seen that drained loading involves the process of consolidation. In other words the sample is being consolidated under the externally applied stresses in a drained test.

The stress paths drawn in Fig. 7.6 are clearly for an undrained test since pore pressure changes have developed during the loading as a result of the applied stresses. These pore pressure changes may be calculated from the changes in the major and minor principal stresses by means of an equation that was developed by Skempton. (1954).

In developing this equation a three dimensional state of stress is considered in which \( \sigma_2 \) is always equal to \( \sigma_3 \) (axially symmetric). Three stages of loading are considered in this development and these are illustrated in Fig. 7.7.

Stage I

Initially the soil is consolidated under an all around stress of \( \sigma'_c \). In other words drained loading is applied and at the end of this loading the pore pressure is zero and external stresses of \( \sigma'_c \) exist in all three coordinate directions.

The stress path representing this consolidation stage is given by OP in Fig. 7.8. Since \( \sigma'_c \) is an all around stress the value of q remains at zero throughout the loading. Since the pore pressure is zero, line OP represents both the total and the effective stress paths.
Stage II

After the consolidation in Stage I the soil is subjected to an externally applied all around stress change of $\Delta \sigma_3$ under undrained conditions. During this stage a pore pressure of $\Delta u_a$ develops. The total and effective stress paths are represented in Fig. 7.8 by lines PR and PQ respectively. If $C_s$ and $C_v$ and defined as the compressibilities of the soil skeleton and the pore fluid respectively then the volume change $\Delta V$ of the soil sample is

$$\Delta V = - C_s V \Delta \sigma_3$$

$$\Delta V = - C_s V (\Delta \sigma_3 - \Delta u_a)$$

where $V$ is the total volume of the soil sample. Since the soil mineral particles are relatively incompressible the volume change $\Delta V$ must be the same as the volume change $\Delta V_v$ of the void space $nV$, where $n$ indicates the porosity.

$$\Delta V_v = - C_v n V \Delta u_a$$

$$\Delta V_v = - C_v n V (\Delta \sigma_3 - \Delta u_a)$$

After collecting terms

$$\frac{\Delta u_a}{\Delta \sigma_3} = \left(1 + \frac{nC_v}{C_s}\right)^{-1} = B \quad (7.7)$$

This ratio of the pore pressure change to the change in all around stress is known as the pore pressure parameter $B$. It is clear that $B$ varies from 0 to 1 the zero value applying to a completely dry soil, the value of unity applying to a completely saturated soil.

Stage III

As shown in Fig. 7.7, stage III involves the application under undrained conditions of a stress in one direction only. This stress, somewhat arbitrarily identified by the symbols $(\Delta \sigma_1 - \Delta \sigma_3)$, is referred to as the deviator stress. During this loading a pore pressure $\Delta u_d$ develops. The total and effective stress paths corresponding to this stage are given in Fig.7.8 by lines RT and QS respectively.

The effective stress changes during this stage are

$$\Delta \sigma_1' = (\Delta \sigma_1 - \Delta \sigma_3) - \Delta u_d$$
Fig. 7.8 Stress Paths for the Loading in Fig. 7.7

Fig. 7.9
\[ \Delta \sigma'_2 = \Delta \sigma'_3 = 0 \Rightarrow \Delta u_d = -\Delta u_d \]

If the soil is assumed to behave according to elastic theory, in which the volume change of the soil skeleton is governed by the mean principal effective stress change, then using the same symbols as defined in stage II.

\[
\Delta V = C_s V \frac{1}{3} (\Delta \sigma_1' + 2 \Delta \sigma_3')
\]
\[
= C_s V \frac{1}{3} (\Delta \sigma_1 - \Delta \sigma_3 - 3 \Delta u_d)
\]

and this volume change must equal the volume change of the void space.

\[
\Delta V_v = -C_v n V \Delta u_d
\]
\[
= \Delta V
\]

This leads to the expression

\[
\Delta u_d = \left(1 + \frac{n C_v}{C_s}\right)^{-1} \cdot \frac{1}{3} (\Delta \sigma_1 - \Delta \sigma_3)
\]
\[
= B \cdot \frac{1}{3} (\Delta \sigma_1 - \Delta \sigma_3)
\]

In order to remove the dependence upon elastic theory and to make the expression more generally applicable, Skempton replaced the \(\frac{1}{3}\) by the pore pressure parameter \(A\)

\[
\Delta u_d = B \cdot A \cdot (\Delta \sigma_1 - \Delta \sigma_3) \quad (7.8)
\]

so the total pore pressure change \(u\) throughout all stages of loading is

\[
\Delta u = \Delta u_a + \Delta u_d
\]
\[
= B \left[ \Delta \sigma_3 + A(\Delta \sigma_1 - \Delta \sigma_3) \right] \quad (7.9)
\]

The \(A\) parameter varies with stress and soil type but commonly it lies within the range +1 to -1/2. At failure (discussed in Ch. 8) typical ranges of values for the \(A\) parameter are given in Table 7.1.
TABLE 7.1
Values of Pore Pressure Parameter A at Failure

<table>
<thead>
<tr>
<th>SOIL</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clays of high sensitivity</td>
<td>$+\frac{3}{4}$ to $+\frac{1}{2}$</td>
</tr>
<tr>
<td>Normally consolidated clays</td>
<td>$+\frac{1}{2}$ to $+1$</td>
</tr>
<tr>
<td>Compacted sandy clays</td>
<td>$+\frac{1}{4}$ to $+\frac{3}{4}$</td>
</tr>
<tr>
<td>Lightly over consolidated clays</td>
<td>0 to $+\frac{1}{2}$</td>
</tr>
<tr>
<td>Compacted clay gravels</td>
<td>$-\frac{1}{4}$ to $+\frac{1}{4}$</td>
</tr>
<tr>
<td>Heavily over consolidated clays</td>
<td>$-\frac{1}{2}$ to 0</td>
</tr>
</tbody>
</table>

The pore pressure change $\Delta u$ is sometimes referred to as a dependent pore pressure, which means that the magnitude of $\Delta u$ depends upon the magnitudes of the stresses applied to the soil. This contrasts with independent pore pressures which are not dependent upon applied stresses but are governed by the hydraulic boundary conditions. These independent pore pressures have been discussed in Chapter 4.

EXAMPLE

An undisturbed sample of saturated clay soil is consolidated to an all around stress of 60kN/m$^2$. The following stress changes are then imposed under undrained conditions until the value of $\sigma_1$ reaches 170kN/m$^2$:

$$\Delta \sigma_2 = \Delta \sigma_3 = 0 \text{ and } \Delta \sigma_1 \text{ is positive}$$

The pore pressure parameters for the soil are

$$B = 1.0, \quad A = 0.5$$

Sketch the total and effective stress paths for the undrained loading and calculate the pore pressure in the soil at the end of loading.
At the end of consolidation the effective and total stresses are equal to 60kN/m$^2$. This stage is represented by point R in Fig. 7.9. At the end of loading the change in the major principal stress is

$$\Delta \sigma_1 = 170 - 60 = 110\text{kN/m}^2$$

The pore pressure change may be calculated from equation (7.9)

$$\Delta u = B [\Delta \sigma_3 + A(\Delta \sigma_1 - \Delta \sigma_3)]$$

$$= 1.0 \ [0 + 0.5 (110 - 0)]$$

$$= 55\text{kN/m}^2$$

Since the pore pressure was zero at the beginning of the undrained loading stage the pore pressure at the end of loading is equal to 55kN/m$^2$.

In Fig. 7.9 the total stress Mohr circle at the end of loading has been drawn with

$$\sigma_3 = 60\text{kN/m}^2 \text{ and } \sigma_1 = 170\text{kN/m}^2$$

Clearly the total stress path is represented by line RQ where Q is located at the top of the circle. Alternatively the stress path may have been drawn after calculating the coordinates of point Q

$$p = \frac{\sigma_1 + \sigma_3}{2} = \frac{170 + 60}{2} = 115\text{kN/m}^2$$

$$q = \frac{\sigma_1 - \sigma_3}{2} = \frac{170 - 60}{2} = 55\text{kN/m}^2$$

For the end point P of the effective stress path the coordinates are:

$$q = 55\text{kN/m}^2$$

$$p' = p - u = 115 - 55 = 60\text{kN/m}^2$$

This data enables the effective stress path RP to be drawn.
7.4 PRINCIPAL STRESS PLOT

There are many ways of graphically representing changes in the state of stress on a soil specimen, apart from the $q - p$ diagram discussed in section 7.2. Another way of representing the changes in state of stress is by means of a principal stress plot. That is successive values of $\sigma_1$ are plotted against $\sigma_3$ during loading to identify the total stress path as illustrated in Fig. 7.10. For the effective stress path $\sigma'_1$ is plotted against $\sigma'_3$. The stress paths RT and QS plotted on Fig. 7.10 correspond to the total and effective stress paths respectively that are plotted on Fig. 7.8.

EXAMPLE

(a) On a principal stress plot describe the direction in which lines representing $\Delta q = 0$ should be drawn.

(b) On a principal stress plot describe the direction in which lines representing $\Delta p = 0$ should be drawn.

(c) Loading of a soil sample commences from an initial stress state represented by $q = 10\, \text{kPa}$ and $p = 50\, \text{kPa}$. The loading is such that $\Delta q = 2\Delta p$ and the loading continues until $p = 80\, \text{kPa}$. Plot the stress path on a principal stress plot and determine the values of $\sigma_1$ and $\sigma_3$ at the end of loading.

(a) The lines are parallel to the diagonal line $\sigma_1 = \sigma_3$, sometimes called the space diagonal.

(b) The lines are perpendicular to the space diagonal.

(c) $q = \frac{\sigma_1 - \sigma_3}{2}$

$p = \frac{\sigma_1 + \sigma_3}{2}$

$\therefore \sigma_1 = p + q = 60\, \text{kPa} \text{ initially}$

and $\sigma_3 = p - q = 40\, \text{kPa} \text{ initially}$
Fig. 7.10 Principal Stress Plot

<table>
<thead>
<tr>
<th>$\Delta p$ (kPa)</th>
<th>$\Delta q$ (kPa)</th>
<th>$\Delta \sigma_1$ (kPa)</th>
<th>$\Delta \sigma_3$ (kPa)</th>
<th>$\sigma_1$ (kPa)</th>
<th>$\sigma_3$ (kPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>20</td>
<td>30</td>
<td>-10</td>
<td>90</td>
<td>30</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>60</td>
<td>-20</td>
<td>120</td>
<td>20</td>
</tr>
<tr>
<td>30</td>
<td>60</td>
<td>90</td>
<td>-30</td>
<td>150</td>
<td>10</td>
</tr>
</tbody>
</table>
The stress path which is a straight line is plotted in Fig. 7.11 and the final values of $\sigma_1$ and $\sigma_3$ are given in the table above.
REFERENCES


