The Estimation Entropy of Hidden Markov Process

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We consider a pair of correlated processes \( \{Z_n\}_{n=0}^{\infty} \) and \( \{S_n\}_{n=0}^{\infty} \), where the former is observable and the later is hidden. Based on the history of the observation we try to estimate the current value of the hidden process. The uncertainty in the estimation of \( S_n \) upon past observations \( Z^{n-1} \) is \( H(S_n|Z^{n-1}) \) which is a sequence of \( n \). The limit of this sequence (and its existence) is of practical and theoretical interest. We call this limit estimation entropy, \( \hat{H} = \lim_{n \to \infty} H(S_n|Z^{n-1}) \). An example of a pair of correlated processes is a hidden Markov process defined by a transition matrix \( P \) and an emission matrix \( T \). Using a method similar to [1] for the entropy rate of the hidden Markov process, we obtain an integral expression for the estimation entropy of the hidden Markov process as

\[
\hat{H} = \int_{\Delta} h d\mu,
\]

where \( h \) is the entropy function over \( \Delta \), the probability simplex in \( \mathbb{R}^{|S|} \), and \( \mu \) is the invariant measure of a discrete time Markov process \( \Phi \) defined on \( \Delta \). The Markov process \( \Phi \) at time index \( n \) is defined by \( \Phi_k = Pr(S_n = k|Z^{n-1}) \), where \( \Phi_k, k = 1, 2, ..., |S| \) is the \( k \)-th component of \( \Phi \). Therefore \( \Phi \) represents the state estimation probability mass function which is a sufficient statistics for all past measurements. The transition Kernel for this Markov process can be obtained directly from \( T \) and \( P \) using Baum Equation. We show that for such a Markov process the invariant measure \( \mu \) always exists, therefore the conditional entropy \( H(S_n|Z^{n-1}) \) converges to a number. The invariant measure can be calculated iteratively based on \( T \) and \( P \).