Symmetric Characterization of Finite State Markov Channels

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1. Symmetric discrete memoryless (point-to-point and multiple access) channels.
2. Finite state Markov channels, and extension of symmetry property.
3. Capacity formula for symmetric finite state Markov channels.
4. Comparison with previous capacity results in special cases.
5. Conclusion.
A discrete memoryless channel is defined by a conditional probability function \( \omega(Y|X) \) (single user) or \( \omega(Y|X_1, X_2, \ldots, X_M) \) (multiple access).

\[
\begin{array}{|c|c|c|c|}
\hline
X & Y=1 & Y=2 & Y=3 \\
\hline
1 & .2 & .6 & .2 \\
2 & .7 & .2 & .1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
X_2 & X_1 & Y=1 & Y=2 & Y=3 \\
\hline
1 & 1 & .2 & .6 & .2 \\
1 & 2 & .7 & .2 & .1 \\
2 & 1 & .1 & .3 & .6 \\
2 & 2 & .5 & .2 & .3 \\
\hline
\end{array}
\]

\[ C = \max_{P(X)} I(X, Y) \]

\[ C_{total} = \max_{P(X^M)} I(X_1, X_2, \ldots, X_M, Y) \]
Output symmetry: A DMC is output symmetric if each row is a permutation of each other row and each column is permutation of each other column.

<table>
<thead>
<tr>
<th></th>
<th>Y=1</th>
<th>Y=2</th>
<th>Y=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>X=1</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>X=2</td>
<td>c</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>X=3</td>
<td>b</td>
<td>c</td>
<td>a</td>
</tr>
</tbody>
</table>

Uniform input distribution achieves capacity for symmetric channels.

T-symmetry: A channel is T-symmetric if
\[ T(x) \triangleq \sum_{y \in Y} \omega(y|x) \log \frac{\omega(y|x)}{\sum_{x' \in X} \omega(y|x')} = \text{Cons.} \]
\[ C = \log |X| + \text{Cons.} \]

Example:

<table>
<thead>
<tr>
<th></th>
<th>Y=1</th>
<th>Y=2</th>
<th>Y=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>X=1</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>X=2</td>
<td>p1</td>
<td>p2</td>
<td>1-p1-p2</td>
</tr>
</tbody>
</table>
**Symmetric discrete memoryless multiple access channel**

**Symmetry multiple access channel**: Each row permutation corresponding to transposing source letters $x, x' \in \mathcal{X}$ for all user is the same as a column permutation.

<table>
<thead>
<tr>
<th>$X_2 = 1$</th>
<th>$X_1 = 1$</th>
<th>Y=1</th>
<th>Y=2</th>
<th>Y=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = 1$</td>
<td></td>
<td>a</td>
<td>a</td>
<td>c</td>
</tr>
<tr>
<td>$X_1 = 2$</td>
<td></td>
<td>b</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>$X_1 = 1$</td>
<td></td>
<td>d</td>
<td>d</td>
<td>b</td>
</tr>
<tr>
<td>$X_1 = 2$</td>
<td></td>
<td>c</td>
<td>a</td>
<td>a</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$X_2 = 1$</th>
<th>$X_1 = 1$</th>
<th>Y=1</th>
<th>Y=2</th>
<th>Y=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 = 2$</td>
<td></td>
<td>c</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>$X_1 = 1$</td>
<td></td>
<td>d</td>
<td>d</td>
<td>b</td>
</tr>
<tr>
<td>$X_1 = 2$</td>
<td></td>
<td>b</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>$X_1 = 1$</td>
<td></td>
<td>a</td>
<td>a</td>
<td>c</td>
</tr>
</tbody>
</table>

Uniform distribution for all users archives total capacity.

**T-symmetry**: The marginalization of $T(x_M)$ to each $m$ is a constant $C_m$.

\[
C_{total} = \frac{C_m}{\prod_{l \neq m} |\mathcal{X}|} + \sum_l \log |\mathcal{X}_l|.
\]

Finite state Markov Channels (FSMC)

Basic property of discrete memoryless channels (single user)

\[ P(Y_n | X_n, X_{n-1}, Y_{n-1}) = P(Y_n | X_n) = \omega(Y_n | X_n). \]

Consider channels that can exhibit different conditional probability matrices, eg:

<table>
<thead>
<tr>
<th>( X = 1 )</th>
<th>( X = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = 1 )</td>
<td>.3</td>
</tr>
<tr>
<td>( Y = 2 )</td>
<td>.2</td>
</tr>
<tr>
<td>( Y = 3 )</td>
<td>.5</td>
</tr>
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<tbody>
<tr>
<td>X = 1</td>
<td>.3</td>
<td>.2</td>
<td>.5</td>
</tr>
<tr>
<td>X = 2</td>
<td>.6</td>
<td>.1</td>
<td>.3</td>
</tr>
</tbody>
</table>

S=0

<table>
<thead>
<tr>
<th></th>
<th>Y=1</th>
<th>Y=2</th>
<th>Y=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>X = 1</td>
<td>.1</td>
<td>.6</td>
<td>.3</td>
</tr>
<tr>
<td>X = 2</td>
<td>.2</td>
<td>.4</td>
<td>.4</td>
</tr>
</tbody>
</table>

S=1

- If \( P(Y_n|X_n, X_{n-1}, Y_{n-1}) = P(Y_n|X_n) = \omega_{s_n}(Y_n|X_n). \)
  \( s_n \) a known sequence of time \( \Leftrightarrow \) time varying memoryless channel.

- \( P(Y_n|X_n, S_n, X_{n-1}, Y_{n-1}, S_{n-1}) = P(Y_n|X_n, S_n) = \omega_{s_n}(Y_n|X_n). \)
  \( s_n \) is the realization of a random Markov process \( \Leftrightarrow \) Finite state Markov channel (FSMC).
Finite state Markov Channels (FSMC)

Basic property of discrete memoryless channels (single user)
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<tr>
<td>S=0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>X=1</td>
<td>.3</td>
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<td></td>
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- If \( P(Y_n|X_n, X_n^{n-1}, Y_n^{n-1}) = P(Y_n|X_n) = \omega_{s_n}(Y_n|X_n). \)
  \( s_n \) a known sequence of time \( \iff \) time varying memoryless channel.

- \( P(Y_n|X_n, S_n, X_n^{n-1}, Y_n^{n-1}, S_n^{n-1}) = P(Y_n|X_n, S_n) = \omega_{S_n}(Y_n|X_n). \)
  \( s_n \) is the realization of a random Markov process \( \iff \) Finite state Markov channel (FSMC).

\[
P(Y_n|X_n, X_n^{n-1}, Y_n^{n-1}) = \sum_{S_n} P(S_n|X_n, X_n^{n-1}, Y_n^{n-1})\omega_{S_n}(Y_n|X_n)
\]

For a FSMC, \( P(Y_n|X_n, X_n^{n-1}, Y_n^{n-1}) \) depends on \( X_n^{n-1}, Y_n^{n-1} \), hence the channel has long memory.

\[
C = \lim_{n \to \infty} \frac{1}{n} \max I(X^n; Y^n)
\]
Finite state Markov channel

Defining parameters of a FSMC:
- $\kappa$ is the $K \times K$ transition probability matrix,
- The set of channel conditional probability matrices $\omega_k(Y|X), k = 1, 2, \ldots, K$.

Basic properties of FSMC:
- **A1: Independency of state and input,**
  \[ p(s_n|x_n) = p(s_n) \]  
  \[ (1) \]
- **A2: Sufficient statistics of state,**
  \[ p(s_n|s^{n-1}, x^{n-1}, y^{n-1}) = \kappa(s_n|s_{n-1}) \]  
  \[ (2) \]
- **A3: Memoryless conditioned on state,** For any $n$,
  \[ p(y^n|x^n, s^n) = \prod_{i}^{n} \omega_{s_i}(y_i|x_i) \]  
  \[ (3) \]
Symmetric finite state Markov channel

Symmetric Markov channel: A FSMC is symmetric if for each state the channel is output symmetric and the permutation pattern for all states are the same.

Example:

\[ \omega_1(.|.) = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \quad \omega_2(.|.) = \begin{pmatrix} a' & b' & c' \\ b' & c' & a' \\ c' & a' & b' \end{pmatrix} \quad \omega'_2(.|.) = \begin{pmatrix} a' & b' & c' \\ c' & a' & b' \\ b' & c' & a' \end{pmatrix} \]
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\]

\(\omega_1\) can be represented by the distribution \([a \ b \ c]\) and

\(\phi(., .) : X \times Y \to \mathcal{Z}, |\mathcal{Z}| = |\mathcal{Y}|,\)

\(\phi(1, 1) = \phi(2, 3) = \phi(3, 2) = 1\)

\(\phi(1, 2) = \phi(2, 3) = \phi(3, 3) = 2\)

\(\phi(1, 3) = \phi(2, 1) = \phi(3, 1) = 3\)
Symmetric finite state Markov channel

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**Example:**

\[
\begin{pmatrix}
a & b & c \\
b & c & a \\
c & a & b \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
a' & b' & c' \\
b' & c' & a' \\
c' & a' & b' \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
a' & b' & c' \\
c' & a' & b' \\
b' & c' & a' \\
\end{pmatrix}
\]

\[\omega_2\] can be represented by the distribution \([a' \ b' \ c']\) and \(\phi(., .) : X \times Y \rightarrow Z, |Z| = |Y|,\)

- \(\phi(1, 1) = \phi(2, 3) = \phi(3, 2) = 1\)
- \(\phi(1, 2) = \phi(2, 3) = \phi(3, 3) = 2\)
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\]

\[
\phi(1, 1) = \phi(2, 3) = \phi(3, 2) = 1 \quad \phi(1, 2) = \phi(2, 3) = \phi(3, 3) = 2 \quad \phi(1, 3) = \phi(2, 1) = \phi(3, 1) = 3
\]

Basic properties of symmetric FSMC, besides \(A_1\) to \(A_3\):

\(A4: \text{Symmetric Markov channel,}\) There exists a random process \(Z_n\) such that for any \(n,\)

\[
p(z_n|s_n, x_n) = p(z_n|s_n) = T[s_n, z_n], \quad z_n = \phi(x_n, y_n),
\]
Generalization of results from symmetric DMC to symmetric FSMC

Symmetry for DMC, results in two basic properties:

- There is a random variable \( Z = \phi(X, Y) \) representing noise, such that
  \[
  H(Z) = H(Y|X)
  \]
  \( H(Z) \) is independent of input distribution

- uniform input distribution gives uniform output, so \( H(Y) \) maximizes to \( \log |\mathcal{Y}| \).

\[
C = \max_{p(x)} H(Y) - H(Y|X) = \log |\mathcal{Y}| - H(Z).
\]
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\[
C = \max_{p(x)} H(Y) - H(Y|X) = \log |\mathcal{Y}| - H(Z).
\]

For symmetric FSMC, similarly we show,

- There is a noise process $Z^n$,
  $H(Z^n) = H(Y^n|X^n)$
  $H(Z^n)$ is independent of input distribution

- uninform i.i.d inputs maximizes $H(Y^n)$ to $n \log |\mathcal{Y}|$.

\[
C_n = \max_{p(x^n)} \frac{1}{n}(H(Y^n) - H(Y^n|X^n)) = \log |\mathcal{Y}| - \frac{1}{n}H(Z^n).
\]

\[
C = \lim_{n \to \infty} C_n, \quad C = \log |\mathcal{Y}| - \hat{H}(Z).
\]
For symmetric FSMC, similarly we show,

B1° There is a noise process $Z^n$,

$H(Z^n) = H(Y^n|X^n)$

$H(Z^n)$ is independent of input distribution

B2° uninform i.i.d inputs maximizes $H(Y^n)$ to $n \log |\mathcal{Y}|$.

$C_n = \max_{p(x^n)} \frac{1}{n}(H(Y^n) - H(Y^n|X^n)) = \log |\mathcal{Y}| - \frac{1}{n}H(Z^n)$.  \[\text{Lemma}\]

$C = \lim_{n \to \infty} C_n, \quad C = \log |\mathcal{Y}| - \hat{H}(Z)$.  

---

**Generalization of results from symmetric DMC to symmetric FSMC**
Lemma: A symmetric FSMC with i.i.d inputs is uniquely associated with a hidden Markov process.

**proof**

- \( p(s_{n+1}|s_n, z^n) = p(s_{n+1}|s_n) = \kappa[s_n, s_{n+1}] \). (from A2)
- \( p(z_n|s_n, z^{n-1}) = p(z_n|s_n) = T[s_n, z_n] \). (from i.i.d inputs, and A4)

A K state Markov process

A memoryless channel

Z is a hidden Markov process
The entropy rate of hidden Markov process

\[ \hat{H}(Z) = \lim_{n \to \infty} \frac{1}{n} H(Z^n) \] is the entropy rate of a hidden Markov process, uniquely associated with a symmetric FSMC with i.i.d inputs.

Expressions for entropy rate:

[Blackwell, 1957]

\[ \hat{H}(Z) = E_{\pi} h(\pi \times P \times T), \quad \pi \in \nabla_S \]

The distribution of \( \pi \) needs to be found by solving an integral equation.

New expression,

\[ \hat{H}(Z) = E_{\pi} h(\pi \times T), \quad \pi \in \nabla_S \]

The distribution of \( \pi \) is obtained numerically, using the information-state process.

Theorem: The capacity of symmetric FSMC is
\[ C = \log |\mathcal{Y}| - \hat{H}(Z) \]
where \( \hat{H}(Z) \) is the entropy rate of the associated hidden Markov process.

Proof: For symmetric FSMC, we show,
- **B1**: There is a noise process \( Z^n \),
  \[ H(Z^n) = H(Y^n | X^n) \]
  \( H(Z^n) \) is independent of input distribution
- **B2**: Uniform i.i.d inputs maximizes \( H(Y^n) \) to \( n \log |\mathcal{Y}| \).

**B2**: Uniform i.i.d inputs
\[ p(y_n | y^{n-1}) = \frac{1}{|\mathcal{Y}|}, \quad H(Y_n | Y^{n-1}) = E H(Y_n | y^{n-1}) = \log |\mathcal{Y}|. \]

From output symmetry and uniform \( p(x_n) \):
\[ p(y_n | s_n) = \sum_x p(y_n | s_n, x) p(x) = \frac{1}{|\mathcal{X}|} \sum_x p(y_n | s_n, x) = 1/|\mathcal{Y}|. \]

From i.i.d, \( p(y_n | s_n, y^{n-1}) = p(y_n | s_n) \), thus for any choice of \( y^{n-1} \),
\[ p(y_n | y^{n-1}) = \sum_{s_n} p(y_n | s_n) p(s_n | y^{n-1}) = \frac{1}{|\mathcal{Y}|}. \]
Proof: continued

**B1**: $H(Y^n | X^n) = H(Z^n)$ is independent of input distribution:

Consider matrix $\psi(y_i, x_i) = p(y_i | x_i) = \sum_{s_i} p(s_i) p(y_i | x_i, s_i) = \sum_{s_i} p(s_i) \omega_s(y | x)$.

If $\omega_s$ are output symmetric and have unique permutation pattern, then $\psi$ is output symmetric.

$$
\omega_1 = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \quad \omega_2 = \begin{pmatrix} a' & b' & c' \\ b' & c' & a' \\ c' & a' & b' \end{pmatrix} \quad \psi = \begin{pmatrix} \alpha_1 a + \alpha_2 a' & \alpha_1 b + \alpha_2 b' & \alpha_1 c + \alpha_2 c' \\ \alpha_1 b + \alpha_2 b' & \alpha_1 c + \alpha_2 c' & \alpha_1 a + \alpha_2 a' \\ \alpha_1 c + \alpha_2 c' & \alpha_1 a + \alpha_2 a' & \alpha_1 b + \alpha_2 b' \end{pmatrix}
$$

This results in

$$H(Y_i | X_i) = H(Z_i)$$

$$H(Y_i | X_i, X_{i-1}, Y_{i-1}) = H(Z_i | Z_{i-1})$$

and the expression is independent of $p_{x_i(x_i)}$

By summing over $i$ we get **B1**
Other concepts of symmetry for FSMC

**Uniformly symmetric:** A FSMC is uniformly symmetric if for each state the channel is output symmetric.

**Variable noise symmetric:** If there exist a function $\phi$ such that for $Z_n = \phi(X_n, Y_n)$, $p(Z^n|X^n) = p(Z^n)$ and $Z^n$ is a sufficient statistics for $S_n$.

For uniformly symmetric variable noise channels it is shown

$$C = \log |\mathcal{Y}| - \int_\Delta \sum_{y \in \mathcal{Y}} - \log p(y|x, \pi) \cdot p(y|x, \pi) \mu(d\pi).$$

The proof requires $\sum_y \psi(x, y)$ being constant

$$\omega_1 = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \quad \omega_2 = \begin{pmatrix} a' & b' & c' \\ c' & a' & b' \\ b' & c' & a' \end{pmatrix} \quad \psi = \begin{pmatrix} \alpha_1a + \alpha_2a' & \alpha_1b + \alpha_2b' & \alpha_1c + \alpha_2c' \\ \alpha_1b + \alpha_2c' & \alpha_1c + \alpha_2a' & \alpha_1a + \alpha_2b' \\ \alpha_1c + \alpha_2b' & \alpha_1a + \alpha_2c' & \alpha_1b + \alpha_2a' \end{pmatrix}$$

The uniformly symmetric alone is not enough for $\sum_y \psi(x, y)$ to be constant.

A symmetric Markov channel is variable noise, but the opposite could be invalid.

A. Goldsmith and P. Varaiya, "Capacity, mutual information and coding for finite state Markov channels ", IEEE Trans. IT, May 1996.
Example of symmetric FSMC: The Gilbert-Elliott Channel

\[ \kappa = \begin{pmatrix} 1 - g & g \\ b & 1 - b \end{pmatrix}, \]

\[ \omega_1 = \begin{pmatrix} 1 - p_G & p_G \\ p_G & 1 - p_G \end{pmatrix}, \omega_2 = \begin{pmatrix} 1 - p_B & p_B \\ p_B & 1 - p_B \end{pmatrix} \]

\[ Z = X = Y = \{0, 1\}, \quad \phi(X, Y) = X \oplus Y \quad T = \begin{pmatrix} 1 - p_G & p_G \\ 1 - p_B & p_B \end{pmatrix}. \]

\[ C = 1 - \hat{H}(Z) \]
Previous formulation of capacity of Gilbert-Elliot channel

\[ C = 1 - \int_0^1 h_b(q') \psi(q') dq', \]

\( \psi(.) \) is the limiting distribution of a Markov chain on [0,1],

with transition probabilities

\[ Pr(q_{n+1} = \alpha | q_n = \beta) = \begin{cases} 
1 - \beta, & \alpha = v(0, \beta); \\
\beta, & \alpha = v(1, \beta).
\end{cases} \]

and initial distribution

\[ Pr[q_0 = (gp_G + bp_B)/(g + b)] = 1. \]

\( v(., .) : \mathbb{Z} \times [0, 1] \rightarrow [0, 1] \) is a defined function.

Capacity by the new formulation

\[ C = 1 - \hat{H}(Z), \]

Using the recent formulation of entropy rate of hidden Markov process,

\[ \hat{H}(Z) = \int_\Delta h(\pi \times T)\mu(\pi)d\pi, \]
Capacity by the new formulation

\[ C = 1 - \hat{H}(Z), \]

Using the recent formulation of entropy rate of hidden Markov process,

\[ \hat{H}(Z) = \int_{\Delta} h(\pi \times T)\mu(\pi)d\pi, \]

\[ \hat{H}(Z) = \int_{0}^{1} h_{b}(\zeta(\pi))\mu(\pi)d\pi, \]

\[ \zeta(\pi) = p_{G}(1 - \pi) + p_{B}\pi \]

\[ \mu(.) = \text{The limiting distribution of a Markov chain on } [0,1], \]

with transition probabilities (for a defined function \( f(.,.) : \mathcal{Z} \times [0,1] \rightarrow [0,1] \))

\[ Pr(\pi_{n+1} = \alpha|\pi_{n} = \beta) = \begin{cases} 1 - \zeta(\beta), & \alpha = f(0,\beta); \\ \zeta(\beta), & \alpha = f(1,\beta). \end{cases} \]

and initial distribution

\[ Pr[\pi_{0} = b/(g + b)] = 1. \]
Numerical example

For a Gilbert-Elliot channel with \( g = 0.01, b = 0.03, p_G = 0.02, p_B = 0.6 \), both formulation gives capacity \( C = 0.1625 \).

\[
C = 1 - \int_0^1 h_b(q')\psi(q')dq',
\]
\[
C = 1 - \int_0^1 h_b(\zeta(\pi'))\mu(\pi')d\pi',
\]
Conclusion

- A unified concept of symmetry
- Capacity of symmetric finite state Markov channel can be expressed by the entropy rate of noise.

\[ C = \log|\mathcal{Y}| - \hat{H}(Z) \]

- The noise process can be attributed to symmetric FSMC as a hidden Markov process
- The capacity by the entropy rate of the hidden Markov process matches with known previous results.
Thank You