436-433 Mechanical Systems
Part B: Digital Control
Lecture BL3

• Transforms
  - Laplace transform
  - z-transform
• Discrete transfer functions
Transforms

- For continuous systems we can go from:
  - linear, constant coefficient differential equations describing input-output relationships between time-variables $f(t)$
  - to algebraic relationships between Laplace-transformed variables $F(s)$
  - e.g., the l.c.c. differential equation

\[
\ddot{y}(t) + a_1\dot{y}(t) + a_2\dot{y}(t) + a_3y(t) = b_0\ddot{u}(t) + b_1\dot{u}(t) + b_2u(t) + b_3u(t)
\]

is transformed into the algebraic relationship

\[
Y(s) = H(s)U(s)
\]

where

\[
H(s) = \frac{b_0s^3 + b_1s^2 + b_2s + b_3}{s^3 + a_1s^2 + a_2s + a_3}
\]

is a transfer function
The z-transform

- In the same way, for discrete systems we can go from:
  - linear, constant coefficient difference equations describing input-output relationships between sample sequences \( \{f(k)\} \)
  - to algebraic relationships between z-transformed variables \( F(z) \)
  - e.g., the l.c.c. difference equation

\[
y(k) + a_1 y(k-1) + a_2 y(k-2) + a_3 y(k-3) = b_0 u(k) + b_1 u(k-1) + b_2 u(k-2) + b_3 u(k-3)
\]

is transformed into the algebraic relationship

\[
Y(z) = H(z)U(z)
\]

where

\[
H(z) = \frac{b_0 z^3 + b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}
\]

is a transfer function
Transforms...

- Thus the $z$-transform plays the same role for discrete systems as the Laplace transform does in continuous systems.
- Recall that a deterministic, continuous signal can be 'constructed' using the elementary basis function $e^{st}$:

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

where $X(s)$ is the Laplace transform of the signal $x(t)$.
- Thus, $X(s)$ indicates the 'amount' of the elementary signal $e^{st}$ present in $x(t)$, for a given value of $s$.
- Let's review the basis of transforms and transfer functions in more detail...
The continuous system basis function $e^{st}$

- $e^{st}$ can represent a wide variety of time functions:

\[ s = 0 \text{ generates a constant} \]

\[ y(t) = e^{0.1t} \]
The continuous system basis function $e^{st}$

- $e^{st}$ can represent a wide variety of time functions:

$s = -\sigma$

$y(t) = e^{-\sigma t}$

$s = \text{negative real}$

generates an exponential decay

$s = \text{positive real}$

generates an exponential growth

$s = +\sigma$

$y(t) = e^{\sigma t}$
The continuous system basis function $e^{st}$

- $e^{st}$ can represent a wide variety of time functions:

$s = \text{complex with negative real part}$ generates an exponentially decaying oscillation

$s = \text{complex with positive real part}$ generates an exponentially growing oscillation

$$y(t) = \frac{1}{2} \left[ e^{(-\sigma+j\omega)t} + e^{(-\sigma-j\omega)t} \right]$$

$$y(t) = \frac{1}{2} \left[ e^{(\sigma+j\omega)t} + e^{(\sigma-j\omega)t} \right]$$
The continuous system basis function $e^{st}$

- $e^{st}$ can represent a wide variety of time functions:

  \[ s = \pm j\omega \]

  $s = \text{complex with zero real part generates a simple harmonic oscillation}$

  \[ y(t) = \frac{1}{2} \left[ e^{j\omega t} + e^{j\omega t} \right] \]
The Laplace transform...

- Thus, an arbitrary continuous signal can be 'constructed' from a (possibly infinite) selection of these elementary signal forms, each 'weighted' by the value of the Laplace transform evaluated for the particular value of $s$:

\[ x(t) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} X(s)e^{st} \, ds \]

- The Laplace transform is found from the time-domain signal by the formula:

\[ L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt \]

- Multiplying the signal $x(t)$ by the inverse of the basis function $e^{st}$, and integrating over all time that the signal exists, has the effect of 'filtering out' all those components of $x(t)$ which do not 'look like' $e^{st}$.
Transfer functions for continuous systems

- The **transfer function** of a linear continuous system tells us what happens to signals of the form $e^{st}$

\[
\begin{align*}
    u(t) &= U e^{at} \\
    y(t) &= H(a) U e^{at}
\end{align*}
\]

- Such a signal is simply 're-scaled', its (possibly complex) magnitude being changed by a factor equal to $H(s)$

- The Laplace transform of an arbitrary input signal $u(t)$ tells us the magnitude $U(s)$ of each $e^{st}$ component of the signal

- The product $H(s) U(s)$ thus gives us the magnitude $Y(s)$ of each corresponding component of the response; i.e., the Laplace transform of the response.
• All of this relates to the fact that the homogeneous solutions to continuous, linear, constant-coefficient, ordinary differential equations are of the form $e^{st}$

• The great benefit is that a **differential** input-output relationship

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = b_0 \frac{d^m u(t)}{dt^m} + \cdots + b_{m-1} \frac{du(t)}{dt} + b_m u(t)$$

can be converted to a much-easier-to-solve **algebraic** input-output relationship:

$$(s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n)Y(s) = (b_0 s^m + \cdots + b_{m-1} s + b_m)U(s)$$

That is:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = H(s)$$

• The roots of the numerator polynomial are the **zeros** of the transfer function; the roots of the denominator polynomial are the **poles**
The discrete system basis function $z^k$

- Now consider what happens to a signal of the form $e^{st}$ when it is sampled at times $t_k = kT$. We get a sequence of numbers

$$\{e^{skT}\} = \{(e^{sT})^k\} = \{z^k\}$$

where $k$ is the sample number, and $z$ is a complex number related to $s$ by

$$z = e^{sT}$$

- Again, $z^k$ can represent the same wide variety of sample sequences as $e^{st}$ does for continuous signals.
\[ z = e^{sT} = e^\sigma T e^{j\omega T} = re^{j\theta} \]

**s-plane**

\[ s = 0 \Rightarrow z = 1: \{z^k\} = \{1, 1, 1, \ldots\} \]

\[ s = -\sigma \Rightarrow z = e^{-\sigma T}: \{z^k\} = \text{exponential decay} \]

\[ s = -\sigma \pm j\omega \Rightarrow z = e^{-\sigma T} e^{\pm j\omega T}: \{z^k\} = \text{decaying oscillation} \]

**z-plane**
• Next, consider the discrete version of an arbitrary signal \( x(t) \): the number sequence \( \{x(k)\} \)

• It turns out that this sequence of numbers can be 'constructed' using the formula:

\[
x(k) = \frac{1}{2\pi j} \int X(z) z^k \frac{dz}{z}
\]

\( \text{c.f. } x(t) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} X(s) e^{st} ds \)

for continuous signal

• Thus \( z^k \) plays the same role for sampled data as \( e^{st} \) does for continuous signals. The 'amount' of \( z^k \) present in a sequence, for a given value of \( z \), is given by the \( z \)-transform formula:

\[
Z\{x(k)\} = X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k}
\]

\( \text{c.f. } X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \)

for continuous signal
Example: unit pulse

\[ x(k) = 1 \quad (k = 0) \]
\[ = 0 \quad (k \neq 0) \]
\[ = \delta(k) \]

\[ X(z) = \sum_{k=-\infty}^{\infty} \delta(k) z^{-k} = z^0 = 1 \]

\[ \text{Compare with } x(t) = \delta(t) \]
\[ \text{Laplace transform } X(s) = 1 \]
Example: unit step

\[ x(k) = \begin{cases} 
1 & (k \geq 0) \\
0 & (k < 0) \\
= 1(k) 
\end{cases} \]

\[ X(z) = \sum_{k=-\infty}^{\infty} 1(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \]

Compare with \( x(t) = 1(t) \)
Laplace transform \( X(s) = 1/s \)
**Example: exponential**  \( x(t) = e^{-at} 1(t) \)

- Sample with period \( T \):  \( x(k) = e^{-akT} 1(kT) \)

- Transform:  \( X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=-\infty}^{\infty} x(k) z^{-k} = \sum_{0}^{\infty} e^{-akT} z^{-k} \)

\[
= \sum_{0}^{\infty} \left(e^{-aT} z^{-1}\right)^k \\
= \frac{1}{1 - e^{-aT} z^{-1}} \\
= \frac{z}{z - e^{-aT}}
\]

**Compare with** \( x(t) = e^{-at} 1(t) \)

**Laplace transform**  \( X(s) = 1/(s+a) \)
Shift property of $z$-transform

\[ Z\{x(k)\} = X(z) = \sum_{k=-\infty}^{\infty} x(k)z^{-k} \]

- Then:
  \[ Z\{x(k-1)\} = \sum_{k=-\infty}^{\infty} x(k-1)z^{-k} \]

- Let $k-1 = j$:
  \[ Z\{x(k-1)\} = \sum_{j=-\infty}^{\infty} x(j)z^{-(j+1)} = z^{-1} \sum_{j=-\infty}^{\infty} x(j)z^{-j} \]

i.e., \[ Z\{x(k-1)\} = z^{-1}X(z) \]
Discrete transfer functions

• We have seen that the input and output sequences, \{u(k)\} and \{y(k)\}, of a discrete, linear calculation process (such as the computer program running in a digital controller) will be related by a linear difference equation:

\[ y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \cdots + b_m u(k-m) \]

• Taking the \(z\)-transform:

\[ (1 + a_1 z^{-1} + \cdots + a_n z^{-n})Y(z) = (b_0 + b_1 z^{-1} + \cdots + b_m z^{-m})U(z) \]

• We now have an algebraic relationship between the \(z\)-transforms of the input and output sequences:

\[ Y(z) = H(z)U(z) \]

where \(H(z)\) is the transfer function:

\[ H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} \]
Discrete transfer function...

- Provided $n \geq m$, we can write $H(z)$ as the ratio of polynomials in $z$:

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \cdots + a_n} = \frac{b(z)}{a(z)}$$

- The roots of the polynomial $b(z)$ are the zeros of the transfer function; the roots of $a(z)$ are the poles

- Given the $z$-transform of the response, $Y(z)$, we can find the output sequence $\{y(k)\}$ using a table of inverse transforms, just as one would with Laplace transforms
Relation between time-domain and transfer function representations of system dynamics

• Recall that for a continuous system with a transfer function $G(s)$

\[ Y(s) = G(s)U(s) \]

- if the input is a unit impulse, $u(t) = \delta(t)$, then $U(s) = 1$
- in that case:

\[ Y(s) = G(s) \]

- i.e., the transfer function is the Laplace transform of the unit impulse response:

\[
G(s) = \mathcal{L}\{g(t)\} \\
g(t) = \mathcal{L}^{-1}\{G(s)\}
\]
Transfer function and impulse response for continuous system

\[ G(s) = \frac{1}{s^2 + 0.8s + 1} \]

\[ L \quad g(t) = \frac{5}{\sqrt{21}} e^{-0.4t} \sin\left(\frac{\sqrt{21}}{5} t\right) \]
Relation between time-domain and transfer function representations of system dynamics

- For a discrete system with a transfer function $G(z)$

$$Y(z) = G(z)U(z)$$

- if the input is a unit pulse, $u(k) = \delta(k)$, then $U(z) = 1$
- in that case:

$$Y(z) = G(z)$$

- i.e., the transfer function is the $z$-transform of the unit pulse response:

$$G(z) = Z\{g(k)\}$$

$$g(k) = Z^{-1}\{G(z)\}$$
Transfer function and pulse response for discrete system

Nyquist frequency $\omega_N = \pi/T = 6.28$ rad/s
$\omega_n = 0.16 \omega_N = 1$ rad/s

$\zeta = 0.4$
$\omega_n = 0.16 \omega_N$

$G(z) = \frac{0.3952z}{z^2 - 1.469z + 0.6703}$

$g(k) = \frac{5}{\sqrt{21}} e^{-0.4k} \sin\left(\frac{\sqrt{21}}{5} kT\right)$

Sampling period $T = 0.5$ s