436-433 Mechanical Systems
Part A: State-Space Systems
Lecture AL7

• Pole placement control law design
Conceptual steps in controller design

- We begin by considering the *regulation* problem
  - the task of the controller is to maintain the plant output at a fixed setpoint \( r = 0 \), in the face of disturbance inputs, \( w \).
  - For simplicity we consider SISO systems; the concepts can be generalised to MIMO systems (see 436-405).

- Start with a *plant model*:
  \[
  \begin{align*}
  \dot{x} &= Ax + Bu + Bw \\
  y &=Cx
  \end{align*}
  \]
• Now assume that all of the states are available for feedback

\[ u = -[K_1 \ K_2 \ ... \ K_n]x \] involves \( n \) gains

⇒ enough degrees of freedom to place the \( n \) closed-loop poles at any desired locations in the \( s \)-plane
• If the states are not accessible, we design an estimator (or observer) to obtain an estimate of the entire state vector $\hat{x}$ from measurements of the control input $u$, the plant output $y$, and a model of the plant.

\[
\begin{align*}
\dot{x} & = Ax + Bu + w \\
y & = Cx + w
\end{align*}
\]

We then use the estimated states in the original control law: 
\[u = -K\hat{x}\]
Control law design

• The plant: \[ \dot{x} = Ax + Bu + Bw \]
  \[ y = Cx \]

• The characteristic polynomial of the plant is
  \[ a(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n \]
  \[ = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n) \]

• By use of full state feedback \( u = -Kx \) we aim to place the closed-loop poles (eigenvalues) at specified locations in the \( s \)-plane, viz. \( s_1, s_2, \ldots, s_n \).

• That is, we specify a desired closed-loop char. poly.
  \[ \alpha_c(s) = (s - s_1)(s - s_2) \cdots (s - s_n) \]
  i.e., \( \alpha_c(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n \)
• Desired characteristic polynomial:

\[ \alpha_c(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n \]

• For the closed-loop system:

\[
\dot{x} = Ax + Bu + Bw \\
= Ax - BKx + Bw \\
= (A - BK)x + Bw
\]

• Hence, the closed-loop characteristic polynomial is

\[ a_k(s) = \det(sI - A + BK) = s^n + a_{k1}s^{n-1} + \cdots + a_{kn} \]

The coefficients \( a_{k1}, \ldots, a_{kn} \) involve the \( n \) feedback gains to be determined.

• Our task is to choose the elements of \( K \) such that \( a_k(s) = \alpha_c(s) \)

• The simplest approach is to match the coefficients of \( s \) in \( a_k(s) \) and \( \alpha_c(s) \)
Example: Position regulation of DC servo motor

- Task is to hold motor position at $\theta = 0$ despite fluctuating disturbance torque $\tau$

- Simple model for DC motor with negligible armature inductance (see Exploratory Assignment E1):

$$T(s) \rightarrow \frac{R_a}{K_i} \rightarrow W(s)$$

$$E_a(s) = U(s) \rightarrow \frac{K_m}{T_m s + 1} \rightarrow \frac{1}{s}$$

$$\Theta(s) = Y(s)$$
• Suppose \[ Y(s) = \frac{4}{s(s+2)}[U(s)+W(s)] \]
• That is, \[ \ddot{y}(t) + 2\dot{y}(t) = 4u(t) + 4w(t) \]
• Choose state variables: \[ x_1 = y = \theta, \quad x_2 = \dot{y} = \omega \]
• Then: \[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
4
\end{bmatrix} u + \begin{bmatrix}
0 \\
4
\end{bmatrix} w
\]
\[ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \end{bmatrix} \]
• Specify closed-loop dynamics: \( \zeta = \frac{1}{2}, \quad \omega_n = 4 \)
  i.e., \( s_{1,2} = -2 \pm j2\sqrt{3} \)
• Desired characteristic polynomial:
\[ \alpha_c(s) = (s+2)^2 + (2\sqrt{3})^2 = s^2 + 2(\frac{1}{2})(4)s + 4^2 \]
  i.e., \( \alpha_c(s) = s^2 + 4s + 16 \)
• **Full state feedback:** \( u = -Kx \Rightarrow a_k(s) = \det(sI - A + BK) \)

\[
sI - A + BK = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}
\]

\[
= \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4K_1 & 4K_2 \end{bmatrix}
\]

\[
\therefore a_k(s) = \begin{vmatrix} s & -1 \\ 4K_1 & s+2+4K_2 \end{vmatrix}
\]

i.e., \( a_k(s) = s^2 + (2 + 4K_2)s + 4K_1 \)

• **Compare with desired characteristic polynomial:**

\[ \alpha_c(s) = s^2 + 4s + 16 \]

• **Equate coefficients:**

\[ 2 + 4K_2 = 4 \Rightarrow K_2 = 0.5 \]

\[ 4K_1 = 16 \Rightarrow K_1 = 4 \]

• **Hence:**

\[ K = \begin{bmatrix} 4 & 0.5 \end{bmatrix} \]
Implementation

- **Closed-loop dynamics:**
  \[
  \begin{align*}
  \dot{x} &= (A - BK)x + Bw \\
  y &= Cx \\
  \begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
  \end{bmatrix} &=
  \begin{bmatrix}
  0 & 1 \\
  -16 & -4
  \end{bmatrix}
  \begin{bmatrix}
  x_1 \\
  x_2
  \end{bmatrix} +
  \begin{bmatrix}
  0 \\
  4
  \end{bmatrix}w \\
  y &= \begin{bmatrix}
  1 & 0
  \end{bmatrix}
  \begin{bmatrix}
  x_1 \\
  x_2
  \end{bmatrix}
  \end{align*}
  \]

- **Closed-loop transfer function:**
  \[
  \frac{Y(s)}{W(s)} = C(sI - A + BK)^{-1}B = \frac{4}{s^2 + 4s + 16}
  \]
  dc gain = 0.25; i.e., a unit step disturbance will cause a steady-state position error of 0.25

Note that this is also the command input TF \( \frac{Y(s)}{R(s)} \), so the CL system would have a large steady-state error in response to set-point changes (see later)
Effect of impulse disturbance on open-loop motor, and closed-loop regulated system

**Impulse Response**

From: \( w \)

- **Closed loop**
- **Open loop**

\[ r \rightarrow u \rightarrow \text{Plant} \rightarrow x_1 = y \]

- Plant: \( \frac{4}{G_f w} \)
- Regulator: \( u = -0.5x_1 - 2x_2 - 4 \)

Regulator 'rejects' disturbance
Effect of step disturbance on open-loop motor, and closed-loop regulated system

Step Response

From: w

Closed loop
Open loop

NB: this is also response to unit step position command - large steady-state error

regulator limits response to disturbance
Response calculations with MATLAB

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 4 & 4 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}; \quad D = \begin{bmatrix} 0 & 0 \end{bmatrix}; \]

\( G = \text{ss}(A, B, C, D); \)

\( \text{set}(G, 'InputName', \{ 'w', 'u' \}, 'OutputName', 'y') \)

\( Ga = \text{augstate}(G); \quad \% \text{augment output with states} \)

\( K = \begin{bmatrix} 4 & 0.5 \end{bmatrix}; \)

\( \text{feedin} = [2]; \quad \% \text{feedback to input 'u'} \)

\( \text{feedout} = [2:3]; \quad \% \text{feedback from state outputs of Ga} \)

\( Gca = \text{feedback}(Ga, -K, \text{feedin}, \text{feedout}, +1); \quad \% \text{+ve f.b.} \)
\begin{verbatim}
Gc = Gca(1,:);  % select only output 'y'
set(Gc, 'InputName', {'w', 'r'})
tf(Gc)

Transfer function from input "w" to output "y":
\frac{4}{s^2 + 4s + 16}

Transfer function from input "r" to output "y":
\frac{4}{s^2 + 4s + 16}

% apply impulse to 'w'
% input only
impulse(Gc(:,1), G(:,1))
\end{verbatim}
Benefit of controller canonical form

- Calculation of control-law gains by coefficient matching is tedious for \( n > 3 \)
- The process is simplified if the state equations are in controller canonical form
- Consider general 3rd-order plant from before:
  \[
  \ddot{y} + a_1 \dot{y} + a_2 \dot{y} + a_3 y = b_1 \ddot{u} + b_2 \dot{u} + b_3 u
  \]
- The controller canonical realisation is:

\[
A_c = \begin{bmatrix}
-a_1 & -a_2 & -a_3 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
C_c = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \quad D_c = 0
\]
• The closed-loop system matrix is

\[
A_c - B_c K = A_c - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}
\]

\[
= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & -a_3 - K_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\]

• Note that this is \textit{still in upper companion form}
Control gains by inspection

• Hence, we can write the closed-loop characteristic polynomial by inspection:

\[ a_k(s) = s^3 + (a_1 + K_1)s^2 + (a_2 + K_2)s + (a_3 + K_3) \]

• If the desired characteristic polynomial is

\[ \alpha(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 \]

then equating coefficients gives:

\[
\begin{align*}
K_1 &= -a_1 + \alpha_1 \\
K_2 &= -a_2 + \alpha_2 \\
K_3 &= -a_3 + \alpha_3 
\end{align*}
\]
Simulation diagrams illustrate how controller canonical form is preserved with full state feedback.

Equivalent simulation: