Lecture 1
Multivariable control

- Examples of MIMO systems
- Representation of MIMO systems
- Controllability and observability
- State-space realisations of transfer function matrices

Examples of MIMO systems
Control of aircraft lateral dynamics

\[
\begin{align*}
\dot{\beta} &= \begin{bmatrix} Y, -I, 0 \end{bmatrix} g/U_w \\
\dot{r} &= \begin{bmatrix} N_\phi, N_\gamma, 0 \end{bmatrix} r \\
\dot{p} &= \begin{bmatrix} L_\phi, L_\gamma, 0 \end{bmatrix} p \\
\dot{\phi} &= \begin{bmatrix} 0, 0, 1 \end{bmatrix} \phi
\end{align*}
\]

Ref: Franklin, Powell & Workman, ex. 9.1
Description of MIMO systems

1. Transfer function matrix (TFM), $H(s)$

$$Y(s) = H(s)U(s)$$

e.g. $m = 2$ inputs, $p = 2$ outputs

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}$$

Example: Hot steel rolling mill

$$H(s) = \frac{2.4 \times 10^8}{s^2 + 72s + 90^2} \quad g_x = 10^{-6}$$

$$F_i(s) = \frac{3 \times 10^4 s}{s^2 + 0.125s + 6^2}$$

$$F_j(s) = \frac{10^4}{s + 0.05}$$

$$\delta(s) = \begin{bmatrix} -240 \\ s^2 + 72s + 8100 \end{bmatrix}$$

$$f(s) = \begin{bmatrix} -0.03s \\ s^2 + 0.125s + 36 \end{bmatrix}$$

$$\begin{bmatrix} 0.01 \\ s^2 + 0.125s + 36 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2.4 \times 10^8}{s^2 + 72s + 8100} \\ \frac{3 \times 10^4}{s^2 + 0.125s + 6^2} \end{bmatrix}$$

Ref: Matlab Control System Toolbox
User’s Guide, p. 7-30
2. State-space description \( \{A, B, C, D\} \)

\[
A = \begin{bmatrix}
-0.0558 & -0.9968 & 0.0802 & 0.0415 \\
0.5980 & -0.1150 & -0.0318 & 0 \\
-3.0500 & 0.3880 & -0.4650 & 0 \\
0 & 0.0805 & 1.0000 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0.0729 \\
-4.7500 \\
1.5300 \\
0
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

states={'beta', 'yaw', 'roll', 'phi'};
inputs={'rudder', 'aileron'};
outputs={'yaw rate', 'bank angle'};

e.g. Jet transport model: Mach 0.8, H=40,000 ft

Matlab jetdemo.m

Conversion from state-space to TFM

- Continuous system:
  \[
  \dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
  \]

- Laplace transform, with zero initial conditions:
  \[
  (sI - A)X(s) = BU(s) \\
  Y(s) = CX(s) + DU(s)
  \]

- Hence:
  \[
  X(s) = (sI - A)^{-1}BU(s) \\
  Y(s) = [C(sI - A)^{-1}B + D]U(s) = H(s)U(s)
  \]

- Thus, transfer function matrix is:

  \[
  H(s) = C(sI - A)^{-1}B + D
  \]

Matlab conversion from state-space to TFM

- Example: Jet transport

  \[
  G = ss(A, B, C, D, ... \text{states}, \text{inputs}, \text{outputs});
  \]

  \[
  H = tf(G)
  \]

  % or

  \[
  H = zpk(G)
  \]

H = zpk(G)

Zero/pole/gain from input "rudder" to output...

yaw rate: \(-4.75 (s+0.4981) (s^2 + 0.02379s + 0.2381)\)

  \[
  (s+0.5627) (s+0.007278) (s^2 + 0.06587s + 0.8972)
  \]

  \[
  1.1476 (s-4.44) (s+2.694)
  \]

bank angle: \(-4.75 (s+0.4981) (s^2 + 0.02379s + 0.2381)\)

  \[
  (s+0.5627) (s+0.007278) (s^2 + 0.06587s + 0.8972)
  \]

  \[
  1.1476 (s-4.44) (s+2.694)
  \]

Zero/pole/gain from input "aileron" to output...

yaw rate: \(1.23 (s+0.4484) (s^2 - 0.2024s + 0.7605)\)

  \[
  (s+0.5627) (s+0.007278) (s^2 + 0.06587s + 0.8972)
  \]

  \[
  10.729 (s^2 + 0.2159s + 0.9541)
  \]

bank angle: \(1.23 (s+0.4484) (s^2 - 0.2024s + 0.7605)\)

  \[
  (s+0.5627) (s+0.007278) (s^2 + 0.06587s + 0.8972)
  \]
State-space realisations of transfer function matrix

- Recall that a set of state variables is a minimal, sufficient, statistic: it contains just enough information to allow the calculation of future responses, given future inputs
- For SISO systems, the number of states \((n)\) is equal to the degree of the characteristic polynomial
- For MIMO systems, when the system dynamic model has been derived directly using state variables, the required number of states is obvious
  - e.g. the EoM for aircraft lateral dynamics shown previously

Controllability, observability and minimality

- A realisation \(\{A, B, C\}\) is state controllable iff the \(n \times nm\) controllability matrix \(C_o(A, B)\) has full rank \(n\)
  \[
  C_o = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}
  \]

- A realisation \(\{A, B, C\}\) is state observable iff the \(np \times n\) observability matrix \(O_o(C, A)\) has full rank \(n\)
  \[
  O_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}
  \]

- A minimal realisation is one that has the smallest-size \(A\) matrix for all triples \(\{A, B, C\}\) satisfying
  \[
  C(sI - A)^{-1}B = H(s), \quad \text{a given transfer function}
  \]

- A realisation \(\{A, B, C\}\) is minimal iff it is controllable and observable

Ref: Kailath, *Linear Systems*, Ch. 6
Modal realisation

- If eigenvalues are distinct, can transform to diagonal, modal realisation. E.g., jet transport:

\[ \text{Gm} = \text{canon}(G, \ 'modal') \]

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \beta )</th>
<th>( \text{yaw} )</th>
<th>( \text{roll} )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dutch-roll</td>
<td>(-0.032935)</td>
<td>0.94665</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Roll</td>
<td>(-0.94665)</td>
<td>(-0.032935)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
| Spiral        | 0              | 0               | 0.56265         | \(-0.007277\)

If any row in \( B = 0 \), corresponding mode is uncontrollable.

SIMO controller canonical form

- Transfer function matrix for SIMO system:

\[ H(s) = \frac{b_i(s)}{a_i(s)} \]

- Find least-common-multiple \( d(s) \) of denominators \( a_i(s), i = 1, 2, \ldots, p \)

\[ R(s) = \frac{n_i(s)}{d(s)} \]

- Numerator polynomials: \( n_i(s) = \frac{b_i(s)d(s)}{a_i(s)} \)

SIMO controller canonical form

- Form \( 1/d(s) \) in usual way
- 'Read out' the numerators to form the outputs

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} \]

SIMO controller canonical form

- This is a minimal realisation: the number of states, \( n \), is equal to the degree of the l.c.m. denominator \( d(s) \)
- This realisation is always controllable
- It will be observable if the polynomials \( \{d(s), n_1(s), \ldots, n_p(s)\} \) have no common factors
MISO observer canonical form

- Transfer function matrix for MISO system:
  \[ H(s) = \begin{bmatrix} b_1(s) & \cdots & b_m(s) \\ a_1(s) & \cdots & a_m(s) \end{bmatrix} \]

- Find least-common-multiple \( d(s) \) of denominators \( a_i(s), i = 1,2,\ldots,m \)
  \[ H(s) = \frac{n_1(s) \cdots n_m(s)}{d(s)} \]
  \[ = \frac{n_1s^{n-1} + n_1s^{n-2} + \cdots + n_m}{s^n + d_1s^{n-1} + \cdots + d_n} \]

- Numerator polynomials:
  \[ n_i(s) = \frac{b_i(s)d(s)}{a_i(s)} \]

MISO observer canonical form

- This is a minimal realisation: the number of states, \( n \), is equal to the degree of the l.c.m. denominator \( d(s) \)
- This realisation is always observable
- It will be controllable iff the polynomials \( \{d(s), n_1(s), \ldots, n_m(s)\} \) have no common factors

MISO observer canonical form

- Form \( 1/d(s) \) in usual way
- 'Feed in' the numerators to apply the inputs